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BOUNDS ON DERIVATIVE PRICES IN AN INTERTEMPORAL SETTING WITH PROPORTIONAL TRANSACTION COSTS AND MULTIPLE SECURITIES

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The observed discrepancies of derivative prices from their theoretical, arbitrage-free values are examined in the presence of transaction costs. Analytic upper and lower bounds on the reservation write and purchase prices, respectively, are obtained when an investor's preferences exhibit constant relative risk aversion between zero and one. The economy consists of multiple primary securities with stationary returns, a constant rate of interest, and any number of American or European derivatives with, possibly, path-dependent arbitrary payoffs.

Key Words: derivative pricing, transaction costs, multiple securities, American claims, exotic options, utility maximization, volatility smile

1. INTRODUCTION

The paper examines the role of one market imperfection, namely, *transaction costs*, on the observed discrepancies of derivative prices from their theoretical, arbitrage-free values. We derive an upper bound on the reservation write price of a derivative and a lower bound on its reservation purchase price. We then discuss the equilibrium implications of transaction costs on the transaction prices of derivative securities.

The market setup is general. The economy consists of N positive-net-supply primary securities, a riskless short-term bond with constant rate of interest, and any number of exotic derivatives of a wide range. The derivatives may be of European or of American type; they may have different expiration dates but all expire at or prior to some given time. Each derivative may also have a path-dependent payoff at one or more, exogenously given, random stopping times. Examples of exotic derivatives to which the bounds apply are packages, nonstandard American options, compound options, barrier options, lookback options, Asian options, and options involving several assets.

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The main assumption throughout is that the prices of the primary securities have i.i.d. returns and rebalancing between the various accounts is charged with transaction costs. Trading may take place in discrete or in continuous time and the transaction costs may be proportional or quasi-linear. The restrictions on the prices of derivatives are obtained from the requirement that, in equilibrium, an investor cannot increase her expected utility by further trade. The investor is endowed with a utility function with constant relative risk aversion coefficient between zero and one. No restrictions are imposed on the preferences of other investors in the economy.

In order to better demonstrate the valuation approach and to explain the methodology, we concentrate the analysis on models in a market with proportional transaction costs with multiple primary securities with prices modeled as log-normal price processes. Subsequently, we look at a more general structure of transaction fees and discuss their implications on the derivative prices in a discrete-time framework.

Our results apply to options traded in *high-volume markets* (exchanges and over-the-counter markets) in which it is plausible to assume that there are numerous competitive buyers and sellers with preferences satisfying the rather weak assumptions made in this paper. The results may be inapplicable to the pricing of illiquid customized derivatives traded between financial institutions and their clients.

The paper is organized as follows. In Section 2, we define the economy of N positive-net-supply risky primary securities and a riskless short-term bond, in a continuous-time framework and with price processes being modeled as diffusions. We also introduce the utility maximization problem of a single investor whose preferences are described via a constant relative risk aversion function. In Section 3, we introduce the M exotic derivatives that are available for trading and whose reservation prices we would like to specify. The main result is then derived in Theorem 3.1 which states that if the payoff of a derivative related (unit-cost) portfolio satisfies certain growth conditions, namely inequalities (3.1) and (3.2), then the investor may trade and increase her expected utility. Therefore, in equilibrium, a payoff that satisfies the condition (3.1) must also satisfy the complement to condition (3.2), stated as inequality (3.5).

In Section 4, we relax the assumption of proportional transaction costs to nonlinear fees and we examine the derivative valuation problem in a discrete-time setting. In Section 5, we discuss various applications of the derived price bounds. We illustrate the bounds in the case that the stock price distribution is log-normal and address the observed discrepancies in the transaction prices of exchange-traded index options from their Black–Scholes (1973) value. Such discrepancies are known as the implied volatility smile. It turns out that the bounds impose tight restrictions on the *slope* of the volatility smile. Discussion on the case of more general price processes, for example mixed diffusion/Poisson processes, is presented together with concluding remarks and future research directions.

In a related paper, Constantinides and Zariphopoulou (1999) derived an upper bound on the reservation write price of a European call option for, relatively, general individual preferences. Their main assumptions were that the utility function is bounded by two other utility functions, each with a (common) constant relative-risk-aversion coefficient that lies between zero and one, that there is only one positive-net-supply primary security, and, finally, that the price is a geometric Brownian motion. The present paper restricts the utility function to the subset of utility functions with constant relative-risk-aversion coefficient between zero and one. However, it generalizes the work of Constantinides and Zariphopoulou in various ways. In fact, it allows for multiple primary securities, as well as for multiple derivatives. The latter may be of general type: European, or American, path-dependent and, in general, of arbitrarily specified payoff. Also, the primary prices may be modeled as jump-diffusions processes. Finally, the existing results are considerably extended in the context of discrete-time trading for primary prices of stationary returns and for quasi-linear transaction costs, departing from the standard case of proportional ones.

In another related paper, Constantinides and Perrakis (2001) derived upper bounds on the reservation write price of European call and put options and lower bounds on the reservation purchase price of these derivatives in a multiperiod economy with proportional transaction costs, general individual preferences, and arbitrarily small holding of the derivative. The present paper allows for multiple primary securities and derivatives, allows derivatives that are path dependent and have arbitrary payoffs, and allows for finite holding of the derivatives.

A review of the literature on pricing derivative securities in markets with transaction costs is provided in Zariphopoulou (2000).

2. THE CONTINUOUS-TIME MODEL

We consider an economy with $N+1$ securities, a bond with price B_t, and N stocks with prices S_t^i, $1 \le i \le N$, at date $t \ge 0$. The bond pays no coupons, is default free, and has price dynamics

$$B_t = e^{rt} B_0, \qquad t \ge 0, \tag{2.1}$$

where r is the constant rate of interest.

We denote by $\widetilde{W}(t) = (W_t^1, \dots, W_t^N)^T$ an N-dimensional standard Brownian motion that generates the filtration $\mathcal{F}_t$ on a fixed, complete probability space $(\Omega, \mathcal{F}, P)$. The stock prices are diffusion processes given by

$$S_t^i = S_0^i + \int_0^t \mu^i S_u^i \, du + \int_0^t S_u^i \sum_{j=1}^N \sigma_{ij} \, dW_u^j, \qquad 1 \le i \le N, \tag{2.2}$$

where $\vec{\mu} = (\mu^1, \dots, \mu^N)^T$ is the mean rate of return vector and $\sigma = \{\sigma_{ij}\}$, $1 \le i, j \le N$ is the volatility matrix for $t \ge 0$; $S_0^i = S_0 \ge 0$ for $1 \le i \le N$. The values $\mu^1, \dots, \mu^N$ and σ_{ij}, $1 \le i, j \le N$ are assumed to be positive constants.

The investor holds x_t dollars of the bond and y_t^i dollars of the stock S^i at date t, for $1 \le i \le N$. We consider pairs of right-continuous with left limits (CADLAG) nondecreasing processes $(L_t^i, \widetilde{L}_t^i)$, $1 \le i \le N$, such that L_t^i represents the cumulative dollar amount transferred from the bond account into the ith stock account and $\widetilde{L}_t^i$ the cumulative dollar amount transferred from the ith stock account to the bond account. By convention, we set $L_0^i = \widetilde{L}_0^i = 0$, $1 \le i \le N$.

The stock account processes $\vec{y}_t = (y_t^1, \dots, y_t^N)$ are given, for $y_0^i = y^i$, $1 \le i \le N$, by

$$y_t^i = y_0^i + \int_0^t \mu^i y_u^i \, du + \int_0^t y_u^i \sum_{j=1}^N \sigma_{ij} \, dW_u^j + \int_0^t dL_u^i - \int_0^t d\widetilde{L}_u^i. \tag{2.3}$$

Transfers between the stock and the bond accounts incur proportional transaction costs. In particular, the cumulative transfer L_t^i into the ith stock account reduces the bond account by $\beta^i L_t^i$ and the cumulative transfer $\widetilde{L}_t^i$ out of the ith stock account increases the bond account by $\alpha^i M_t^i$, where $0 < \alpha^i < 1 < \beta^i$.

The investor consumes at the rate c_t dollars out of the bond account. There are no transaction costs in transfers from the bond account into the consumption good.

The bond account process is

$$x_t = x + \int_0^t \{rx_u - c_u\}\, du - \sum_{i=1}^N \beta^i L_t^i + \sum_{i=1}^N \alpha^i \widetilde{L}_t^i, \tag{2.4}$$

with $x_0 = x$. The integral represents the accumulation of interest and the drain due to consumption. The last two terms represent the cumulative transfers between the stock and bond accounts, net of transaction costs. Note that in general, transactions may be allowed between stock accounts. It is only to simplify the exposition that we assume that all transactions take place directly through the bond account.

A policy is an $\mathcal{F}_t$-progressively measurable triple $(c_t, L_t^i, \widetilde{L}_t^i)$, $1 \le i \le N$ and $t \ge 0$. We restrict our attention to the set of admissible policies $\mathcal{A}$ such that

$$\begin{cases} c_t > 0 \quad \text{and} \quad E\displaystyle\int_0^t c_u\, du < \infty \quad \text{a.s. for } t \ge 0, & (2.5) \\ w_t = x_t + \displaystyle\sum_{i=1}^N \binom{\alpha^i}{\beta^i} y_t^i \ge 0 \quad \text{a.s. for } t \ge 0, & (2.5') \end{cases}$$

where, for the rest of the presentation, we adopt the notation

$$\binom{\alpha}{\beta} z = \begin{cases} \alpha z & \text{if } z \ge 0 \\ \beta z & \text{if } z < 0. \end{cases} \tag{2.6}$$

We refer to w_t as the net worth. It represents the investor's bond holdings if the investor were to transfer the holdings from the stock accounts into the bond account, incurring in the process the transaction costs.

The investor has von Neumann–Morgenstern preferences $E\big[\int_0^{+\infty} e^{-\rho t}(c_t^\gamma/\gamma)\, dt\big]$ over the consumption stream $\{c_t, t \ge 0\}$, where ρ is the *subjective discount rate* and $1-\gamma$ is the *relative risk aversion coefficient* with $0 < \gamma < 1$.

Given the initial endowment $(x, y^1, \dots, y^N)$ on

$$\overline{D} = \left\{ (x, y^1, \dots, y^N) \in \mathbb{R}^{N+1} : x + \sum_{i=1}^N \binom{\alpha^i}{\beta^i} y^i \ge 0 \right\},$$

we define the value function V as

$$V(x, y^1, \dots, y^N) = \sup_{(c, L, \widetilde{L}) \in \mathcal{A}} E\Bigg[\int_0^{+\infty} e^{-\rho t} \frac{c_t^\gamma}{\gamma}\, dt / x_0 = x, \quad y_0^1 = y^1, \dots, y_0^N = y^N \Bigg]. \tag{2.7}$$

In order to guarantee that the value function is well defined, we assume that

$$\rho > r\gamma + \frac{\gamma}{2(1-\gamma)} (\vec{\mu} - r\,\mathbb{1})(\sigma\sigma^T)^{-1}(\vec{\mu} - r\,\mathbb{1})^T. \tag{2.8}$$

The above condition yields that the value function in the absence of transaction costs, which corresponds to $\alpha^i = \beta^i = 1$, $1 \le i \le N$, and $U(c) = c^\gamma/\gamma$, is finite, and therefore, all value functions with $0 \le \alpha^i < 1$, $\beta^i \ge 1$ are finite (e.g., see Merton 1973).[1]

[1] The growth condition (2.8) was subsequently relaxed by Shreve and Soner (1994) in the case of only two securities being available, a bond and a stock. Their conditions (stated in Theorem 2.1) provide a useful insight for the value function and the nature of the optimal trading policies, but similar conditions have not been explored to date for the case of multiple risky securities as in the model introduced herein.

Models of optimal investment and consumption with transaction costs and with two securities, a bond and a stock, were introduced by Constantinides (1979, 1986) and subsequently studied extensively by others. Note that in these models, because there is a single risky asset, the value function depends only on two state variables, say (x, y) with x and y corresponding to the holdings in the bond and the stock accounts.

Using the special form of the power utility functions, Davis and Norman (1990) obtained a closed form expression for the value function employing the homogeneity of the problem. They also showed that the optimal policy confines the investor's portfolio to a certain wedge-shaped region in the wealth plane and they provided an algorithm and numerical computations for the optimal investment rules. The same class of utility functions was later further explored by Shreve and Soner (1994) who relaxed some of the technical assumptions on the market parameters of Davis and Norman and provided further results related to the regularity of the value function and the location of the exercise boundaries. The case of general utilities was examined through numerical methods by Tourin and Zariphopoulou (1994) (see also Tourin and Zariphopoulou 1995 and 1997) who built a coherent class of approximation schemes for investment models with transaction costs.

A straightforward argument along the lines of Constantinides (1979) shows that the value function is increasing and jointly concave in (x, y); it can also be shown that it is uniformly continuous (see Tourin and Zariphopoulou 1994). Furthermore, the value function is expected to solve the Hamilton–Jacobi–Bellman (HJB) equation associated with the two-dimensional analogue of the stochastic control problem (2.7).

The HJB equation turns out to be a Variational Inequality with gradient constraints given by

$$\min\left[\mathcal{L}V,\ \beta\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y},\ -\alpha\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}\right]=0, \tag{2.9}$$

where the differential operator $\mathcal{L}$ is

$$\mathcal{L}V=\rho V-\frac{1}{2}\sigma^2y^2\frac{\partial^2 V}{\partial y^2}-\mu y\frac{\partial V}{\partial y}-rx\frac{\partial V}{\partial x}-\max_{c\geq 0}\left\{-c\frac{\partial V}{\partial x}+U(c)\right\}. \tag{2.10}$$

The following theorem can be found in Davis and Norman (1990) and Shreve and Soner (1994). The conditions on the market coefficients, stated below, were introduced respectively, by Davis and Norman and Shreve and Soner.

THEOREM 2.1. *Assume that there are only two securities in the market, a bond and a stock, and that the interest rate r, the mean rate of return μ, and the volatility σ satisfy the growth conditions $\rho > r\gamma+\gamma(\mu-r)^2/2\sigma^2(1-\gamma)$ or $\rho > r\gamma+\gamma^2(\mu-r)^2/2\sigma^2(1-\gamma)^2$. Then the value function V is the unique concave $C^2((0,+\infty)\times(0,+\infty))$ solution of the Variational Inequality (2.9) associated to the differential operator (2.10).*

Using classical results from the theory of singular stochastic control (e.g., see Fleming and Soner 1993, Chap. 5), the following well-known Dynamic Programming Principle (DPP) holds.

PROPOSITION 2.1. *Let τ be a stopping time, that is, a nonnegative random variable such that the event $\{\tau \leq t\}$ is in $\mathcal{F}_t$, for each $t \geq 0$. Then*

$$V(x,y)=\sup_{\mathcal{A}} E\left[\int_0^{\tau} e^{-\rho s}\frac{c_s^{\gamma}}{\gamma}\,ds+e^{-\rho\tau}V(x_\tau,y_\tau)/x_0=x,\,y_0=y\right], \tag{2.11}$$

where x_τ and y_τ are the controlled state processes at time τ.

When there are multiple securities available and traded in the market, as in the case we study herein, a number of the above arguments can be extended in a direct, albeit tedious, way.

Below we state some elementary properties of the value function together with the multidimensional analogue of the HJB equation (2.9).

PROPOSITION 2.2. *The value function V, given in (2.7), is increasing and jointly concave in $(x, y^1, \dots, y^N)$, uniformly continuous on $\overline{D}$, and homogeneous of degree γ.*

The proof follows along similar arguments as in Proposition 2.2 of Tourin and Zariphopoulou (1994) and therefore it is omitted.

The associated HJB equation turns out to be the multidimensional Variational Inequality with gradient constraints

$$\begin{aligned}&\min\left\{\widetilde{\mathcal{L}}V,\ \min_{1\le i\le N}\left\{\beta^1\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y^1},\dots,\beta^i\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y^i},\dots,\beta^N\frac{\partial V}{\partial x}-\frac{\partial V}{\partial y^N}\right\},\right.\\ &\left.\min_{1\le i\le N}\left\{-\alpha^1\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y^1},\dots,-\alpha^i\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y^i},\dots,-\alpha^N\frac{\partial V}{\partial x}+\frac{\partial V}{\partial y^N}\right\}\right\}=0, \qquad (2.12)\end{aligned}$$

where the differential operator $\widetilde{\mathcal{L}}$ is

$$\widetilde{\mathcal{L}}V=\rho V-\frac{1}{2}\operatorname{tr}\big((\sigma\sigma^T),D^2V\big)-\langle\vec{\mu},DV\rangle-rx\frac{\partial V}{\partial x}-\max_{c\ge 0}\left\{-c\frac{\partial V}{\partial x}+\frac{c^\gamma}{\gamma}\right\} \qquad (2.13)$$

with D^2V and DV being, respectively, the second-order derivative matrix and the gradient vector of V.

The value function is also expected to satisfy the DPP stated in Proposition 2.1. Note that even though the DPP is a natural consequence of the Markovian nature of the controlled state processes involved, as well as the optimality of the relevant controls, the multidimensional analogue of (2.11) has not been established yet. We do not attempt to establish it either because such an effort would be beyond the interests of the audience; rather, we introduce it formally as our main assumption.

ASSUMPTION 2.1. *The value function V satisfies the Dynamic Programming Principle*

$$\begin{aligned}V(x,y^1,\dots,y^N)=\sup_{\mathcal{A}}E\bigg[\int_0^\tau e^{-\rho s}\frac{c_s^\gamma}{\gamma}\,ds+e^{-\rho\tau}V(x_\tau,y_\tau^1,\dots,y_\tau^N)/x_0\\ =x,y_0^1=y^1,\dots,y_0^N=y^N\bigg],\end{aligned} \qquad (2.14)$$

where τ is a stopping time and $(x_\tau, y_\tau^1, \dots, y_\tau^N)$ is the vector of the controlled state processes at time τ. Moreover, the above supremum is obtained at the optimum vector of control processes, say $(c_t^*, L_t^{1,*}, \dots, L_t^{N,*}, \widetilde{L}_t^{1,*}, \dots, L_t^{N,*})$; that is,

$$V(x,y^1,\dots,y^N)=E\bigg[\int_0^\tau e^{-\rho s}\frac{(c_s^*)^\gamma}{\gamma}\,ds+e^{-\rho\tau}V(\tilde{x}_\tau^*,\tilde{y}_\tau^{1,*},\dots,\tilde{y}_\tau^{N,*})\bigg], \qquad (2.15)$$

where $\tilde{x}_\tau^*, \tilde{y}_\tau^{1,*}, \dots, \tilde{y}_\tau^{N,*}$ are the state processes, given by (2.4) and (2.3), at time τ with the optimal policies $(c_s^*, L_s^{1,*}, \dots, L_s^{N,*}, \widetilde{L}_s^{1,*}, \dots, \widetilde{L}_s^{N,*})$, $0 \le s \le \tau$, being used.

3. BOUNDS ON THE PRICES OF DERIVATIVES

We now enrich the investment opportunity set by introducing M, $M \geq 1$, derivatives at initial time $t = 0$. The derivatives are allowed to be of European or American type. They may have different expiration dates, but all expire at or prior to some given time T, with $T > 0$. Each derivative may also have a path-dependent payoff at one or more, exogenously given, random stopping times.

We assume that the M derivatives may be bought or sold only at time zero. Neither these M derivatives nor additional derivatives may be traded thereafter.[2] Whereas these assumptions are counterfactual, they are sufficient for our goal at hand: the derivation of an upper bound on the reservation write price of derivatives and the derivation of a lower bound on the reservation purchase price of derivatives. If we were to relax these assumptions and allow these and other derivatives to be traded at later times, one may be able to tighten our bounds. In any case, by relaxing these assumptions, the bounds that we derive in this paper remain valid.

Next, we consider a portfolio of long and short positions in the M derivatives and in the $N+1$ primary securities. The portfolio is normalized so that the cost, net of transaction costs, in setting it up is one. The portfolio is managed according to some specified self-financing and nonanticipating strategy. We denote by $h(\omega; \tau, T)$ the value of the component assets in the portfolio when they are converted into cash at the liquidation time $\tau \leq T$.

Generally speaking, the value $h(\omega; \tau, T)$ depends on the specific nature of the derivatives involved, as well as on the possibly random liquidation time τ and the trading horizon $[0, T]$. Note that the stopping time τ is exogenously specified and is not necessarily given by the optimal exercise time of any relevant derivative security. Moreover, we use ω in the argument of h in order to incorporate possible path dependence of various derivative components. At this point, we choose to keep h general enough and we present concrete applications in Section 5.

The next theorem contains the main results of our work. To ease the presentation, we assume that the initial endowment is entirely invested in the bond account.

THEOREM 3.1. *Assume that the initial endowment of the investor is $(x, 0, \ldots, 0)$, with $x > 0$ and that the investor's utility without the derivatives is $V(x, 0, \ldots, 0)$. Moreover, assume that the investor faces the opportunity of investing in a unit cost portfolio with payoff $h(\omega; \tau, T)$ such that*

$$h(\omega; \tau, T) \geq 0 \quad a.s. \tag{3.1}$$

and

$$E\left[e^{-\rho\tau}(h(\omega; \tau, T))^{\gamma} - 1/\mathcal{F}_0\right] > 0. \tag{3.2}$$

Then, there exists a policy under which the investor's utility is higher than $V(x, 0, \ldots, 0)$ if the investor chooses to invest λx in this portfolio, where $\lambda \in (0, 1)$.

Proof. First recall that the investor's preferences are modeled via a power function which, together with the special form of the state processes $(x_t, y_t^1, \ldots, y_t^N)$, results in V being concave and homogeneous of degree γ. Let us also assume that the optimal

[2] The choice of introducing the derivatives at $t = 0$ instead of at a future time $t > 0$ is done only for the sake of simplicity. The key assumption is that all derivatives are introduced at the same time.

policy for the point $(x, 0, \ldots, 0)$ is given by $(c_t^*, L_t^{1,*}, \ldots, L_t^{N,*}, \widetilde{L}_t^{1,*}, \ldots, \widetilde{L}_t^{N,*})$ for $0 \le t \le T$. Then, Assumption 2.1 yields that

$$V(x, 0, \ldots, 0) = E\left[\int_0^\tau e^{-\rho s} \frac{(c_s^*))^\gamma}{\gamma} ds + e^{-\rho\tau} V(x_\tau^*, y_\tau^{1,*}, \ldots, y_\tau^{N,*})/x_0 = x, y_0^1 = \cdots = y_0^N = 0\right], \tag{3.3}$$

where $(x_\tau^*, y_\tau^{1,*}, \ldots, y_\tau^{N,*})$ are the optimal state trajectories starting at $(x, 0, \ldots, 0)$ and using the above optimal strategies.

Let $\lambda \in (0, 1)$ be fixed and consider the following admissible, but in general *suboptimal*, policy. At initial time t, split the available wealth x into two components, namely $x_1 = (1-\lambda)x$ and $x_2 = \lambda x$. Consider the points $(x_1, 0, \ldots, 0)$ and $(x_2, 0, \ldots, 0)$ on $\overline{D}$, and follow the policy $((1-\lambda)c_t^*, (1-\lambda)L_t^{1,*}, \ldots, (1-\lambda)L_t^{N,*}, (1-\lambda)\widetilde{L}_t^{1,*}, \ldots, (1-\lambda)\widetilde{L}_t^{N,*})$ for the first component $(x_1, 0, \ldots, 0)$. Moreover, invest the rest of the initial wealth, $x_2 = \lambda x$, in the portfolio. This policy will result in a payoff of $\lambda x h(\omega; \tau, T)$ dollars.

Denote by $J(x, 0, \ldots, 0)$ the utility coming from the above investment and consumption plan. We claim that

$$J(x, 0, \ldots, 0) \ge V(x, 0, \ldots, 0). \tag{3.4}$$

To this end, observe that at the random time $\tau \le T$, the above policy yields the payoff

$$J(x, 0, \ldots, 0) = E\left[\int_0^\tau e^{-\rho s} \frac{\big((1-\lambda)c_s^*\big)^\gamma}{\gamma} ds + e^{-\rho\tau} V((1-\lambda)x_\tau^* + \lambda x h(\omega; \tau, T), (1-\lambda)y_\tau^{1,*}, \ldots, (1-\lambda)y_\tau^{N,*})/x_0 = x, y_0^1 = \cdots = y_0^N = 0\right].$$

Using the concavity of the value function V, the above equality implies

$$J(x,0,\ldots,0) \ge (1-\lambda)E\left[\int_0^\tau e^{-\rho s} \frac{(c_s^*)^\gamma}{\gamma} ds + e^{-\rho\tau} V(x_\tau^*, y_\tau^{1,*}, \ldots, y_\tau^{N,*})/x_0 = x, y_0^1 = \cdots = y_0^N = 0\right] + \lambda E\big[e^{-\rho\tau} V(xh(\omega;\tau,T), 0, \ldots, 0)/\mathcal{F}_0\big].$$

Using (3.3), the above inequality becomes

$$J(x,0,\ldots,0) \ge (1-\lambda)V(x,0,\ldots,0) + \lambda E\big[e^{-\rho\tau} V(xh(\omega;\tau,T),0,\ldots,0)/\mathcal{F}_0\big].$$

Using the fact that V is homogeneous of degree γ, the above inequality yields

$$\begin{aligned} J(x,0,\ldots,0) &\ge (1-\lambda)V(x,0,\ldots,0) + \lambda V(x,0,\ldots,0)E\big[e^{-\rho\tau}(h(\omega;\tau,T))^\gamma/\mathcal{F}_0\big] \\ &= V(x,0,\ldots,0)\big(1+\lambda E\big[e^{-\rho\tau}(h(\omega;\tau,T))^\gamma - 1/\mathcal{F}_0\big]\big) \\ &> V(x,0,\ldots,0), \end{aligned}$$

where we used that $\lambda > 0$ and that h satisfies the inequality (3.2). □

The important implication of the above theorem is that, in equilibrium, the payoff $h(\omega; \tau, T)$ must satisfy the restriction

$$E[e^{-\rho\tau}(h(\omega; \tau, T))^\gamma - 1/\mathcal{F}_0] \le 0. \tag{3.5}$$

4. MODELS OF GENERAL TRANSACTION COSTS

In this section, we derive bounds on derivative prices under a class of convex transaction costs that are more general than the proportional transaction costs studied above. Our generalization of transaction costs does not involve a fixed-cost component because with a fixed-cost component the value function is no longer homogeneous and concave in the asset holdings, two crucial properties employed in the proof of Theorem 3.1. Without a fixed-cost component in the transaction cost structure, the powerful method of impulse control is inapplicable and the formulation and study of the problem becomes cumbersome.[3] We bypass this difficulty by recasting the continuous-time model as a discrete-time model.

We consider an infinite-horizon economy defined over discrete dates $t = 0, 1, \ldots$; the present date is denoted by $t = 0$. There is one riskless security, the bond, and N risky securities, the stocks.

The bond is default free, pays no coupons, and has constant one-plus-rate-of-return R_0. The bond can be held long or short by the investors and the trading in the bond does not incur transaction costs. Essentially we assume that there is unlimited borrowing and lending at the constant rate of interest $R_0 - 1$ per time period.

The stocks pay no dividends. Their prices at date t are represented by the $N \times 1$ vector $\mathcal{S}(t+1)$. The price dynamics are

$$\mathcal{S}(t+1) = \mathbf{R}(t+1)\mathcal{S}(t), \tag{4.1}$$

where $\mathbf{R}(t+1)$ is an $N \times N$ diagonal matrix with diagonal elements the random variables $R_n(t+1)$, $n = 1, \ldots, N$, representing the return on the stocks from date t to $t+1$.

We make the following assumptions:

1. The initial stock prices are nonnegative $S_n(0) \geq 0$, $n = 1, \ldots, N$.
2. The stock returns are nonnegative; that is, $R_n(t+1) \geq 0$ for all n and t.
3. The system of stock and bond prices does not admit arbitrage opportunities in the absence of transaction costs.
4. The matrices $\mathbf{R}(t)$, $t = 1, 2, \ldots$ are identically and independently distributed, with the distribution known to the investor.

Next, we consider an investor who enters at date t with dollar holdings in the bond account denoted by the scalar $x(t)$ and dollar holdings in the N stock accounts denoted by the N-entry vector $y(t)$. The investor makes investment decisions at sequential times, rebalances the entire portfolio, and consumes out of the bond account. We denote by $v(\cdot) = (v_1(\cdot), \ldots, v_N(\cdot))$ the vector of account transfers in the sense that $v_i(s)$, $1 \leq i \leq N$ and $s = t + n, n \geq 1$ represents the number of units of account by which the investor increases (decreases) his holdings in the ith asset at time s. After each transaction, the investor's holdings of the N assets are $y(s) + v(s)$ with $v(s)I$ being the total amount transferred to them and I being an N-dimensional unit column vector. The amount $v(s)I$ is entirely subtracted from the riskless asset. The underlying assumption is that

[3] Bensoussan and Lions (1973, 1975) introduced the optimal control technique of impulse control. Constantinides and Richard (1978) presented an early application of impulse control in proving the existence and addressing the form of the optimal inventory policy and cash management policy in a continuous-time model with fixed and proportional transaction costs. Korn (1998) presented a more recent application in portfolio optimization. Bielecki and Pliska (2000) applied impulse control in the study of asset management with a transaction cost structure that represents a generalization of the class of fixed and proportional costs.

any transfer is charged with transaction fees that depend on the level of transactions via a nonlinear cost functional $T : \widetilde{N} \to \mathbb{R}$, where $\widetilde{N}$ is the space of N vectors. We introduce the following assumptions:

5. T satisfies $T(0) = 0$ and $|T(v) - T(\tilde{v})| \le \|v - \tilde{v}\|$.
6. The mapping $v \to T(v)$ is convex and for positive scalars λ, $T(\lambda v) = \lambda T(v)$.

EXAMPLE 4.1. The transaction cost functional is defined as

$$T(v(s)) = k_1\Big|\sum_{i=1}^{m} v_i(s)\Big| + k_2\Big|\sum_{i=m+1}^{j} v_i(s)\Big| + \cdots + k_p\Big|\sum_{\ell}^{N} v_i(s)\Big|;$$
$$1 < j \le \cdots \le \ell < N,$$

where $k_1, k_2, \ldots, k_p \ge 0$. Essentially, we assume that transfers between the pth and qth stocks incur no transaction costs if $1 \le p \le i$ and $1 \le q \le i$; but transfers between the pth and qth stocks incur proportional transaction costs at the rate k_1, if $1 \le p \le i$ and $i < q$. This example captures the notion that, whereas the investor is generally charged proportional transaction costs in transferring funds between two unrelated mutual funds p and q, the fee is often waived if the two funds belong to the same mutual fund company.

The investor consumes $c(t)$ dollars out of the bond account at date t. There are no transaction costs in transfers from the bond account into the consumption good.

Given the consumption and investment decisions $(c(t), \mathbf{v}(t))$ at date t, the bond account dynamics are

$$x(t+1) = R_0(x(t) - c(t) - v(t)I - T(v(t))) \tag{4.2}$$

and the stock accounts dynamics are

$$\mathbf{y}(t+1) = \mathbf{R}(t+1) \cdot (\mathbf{y}(t) + v(t)). \tag{4.3}$$

The investor chooses consumption and investment policies $\{c(t), \mathbf{v}(t); t = 0, 1, \ldots\}$, which are processes measurable with respect to the information up to time t. Moreover, these processes satisfy the state and control constraints, for $t = 0, 1, \ldots$

$$c(t) \ge 0 \quad \text{and} \quad w(t) = x(t) + \mathbf{y}(t)I - T(-\mathbf{y}(t)) \ge 0 \quad \text{a.s.,} \tag{4.4}$$

where $x(t)$ and $\mathbf{y}(t)$ are given by (4.2) and (4.3) using the policies $\{c(t), \mathbf{v}(t)\}$. We denote by $\mathcal{A}$ the set of admissible policies. We refer to $w(t)$ as the net worth.

The investor's utility, at date $t = 0$, of the consumption stream $\{c(t), t = 0, 1, \ldots\}$ is the date-zero expectation of $\sum_{t=0}^{+\infty} \delta^t (c(t))^\gamma$. The subjective discount factor δ is constrained as $0 < \delta < 1$ and the relative risk aversion coefficient $1 - \gamma$ is constrained as $0 < 1 - \gamma < 1$.

Given the initial endowment $x(0) = x$, $\mathbf{y}(0) = \mathbf{y}$ such that the initial net worth satisfies $w(0) = x + \mathbf{y}I - T(-\mathbf{y}) \ge 0$, the investor's optimization problem is to maximize, over all admissible policies, the above payoff, $E_0\big[\sum_{t=0}^{\infty} \delta^t (c(t))^\gamma\big]$, where E_0 denotes the date-zero expectation. The value function $V(x, \mathbf{y})$ is then defined as

$$V(x, \mathbf{y}) = \sup_{\{c, \mathbf{v}\} \in \mathcal{A}} E_0\left[\sum_{t=0}^{\infty} \delta^t (c(t))^\gamma\right]. \tag{4.5}$$

The value function is bounded from below by zero since $c(t) \ge 0$ and $0 \le \gamma < 1$ and it is also bounded from above by the corresponding value function of the zero-transaction-costs problem. We assume that the subjective discount factor δ is sufficiently small such that the latter value function is well defined. This, in turn, will guarantee that the value function V is everywhere finite as well.

In Constantinides (1979), the value function in (4.5) was extensively analyzed under the aforeintroduced Assumptions 1 to 6. We refer the reader to the arguments used in Constantinides (1979) which yield the following properties.

PROPOSITION 4.1. *The value function V is homogeneous of degree γ and a jointly concave function of (x, y).*

Next, we enrich our investment opportunity set by introducing M, $M \geq 1$, derivative securities at initial time $t = 0$. These derivatives have the same general characteristics as the ones introduced in Section 3. We also impose the same restrictions as before with regard to when we may trade these derivatives. We also consider a portfolio of long and short positions in the M derivatives and in the $N+1$ available primary securities. Trading takes place at discrete intervals and the cost of the portfolio is normalized to one.

The portfolio is managed according to a specific self-financing and nonanticipating strategy at times t_i, $i = 1, \ldots, N$ with $t_N = T$. We denote by $h(\omega; \tau, T)$ the value of the component assets in the portfolio when they are converted into cash at liquidation time $\tau \leq T$. The value $h(\omega; \tau, T)$ depends on the specific characteristics of the derivatives as well as the possibly random liquidation time τ. The latter is exogenously given and may take the values t_i, $i = 1, \ldots, N$, according to some specified distribution. For example, in the case of early exercise instruments, τ may be the optimal exercise time of the same American instrument but in the absence of transaction costs. Finally, we use the ω argument in h to denote possible path dependence.

The next theorem contains the main results for the bounds on the prices of the derivatives. For simplicity, we assume that the initial endowment is entirely invested in the bond account; that is, $\mathbf{y}(0) = \vec{0}$.

THEOREM 4.1. *Assume that the initial endowment of the investor is* $(x, 0, \ldots, 0)$*, with* $x > 0$*, and that his utility without the available derivatives is* $V(x, 0, \ldots, 0)$*. Also, assume that the investor faces the opportunity of investing in a unit cost portfolio with payoff* $h(\omega; \tau, T)$ *such that*

$$h(\omega; \tau, T) \geq 0 \tag{4.6}$$

and

$$E_0\left[\delta^\tau (h(\omega; \tau, T))^\gamma - 1\right] > 0. \tag{4.7}$$

Then there exists a policy under which his utility is higher than $V(x, 0, \ldots, 0)$ *if the investor chooses to invest* λx *in this portfolio, with* $\lambda \in (0, 1)$.

The proof, available from the authors upon request, parallels the proof of Theorem 3.1. The important implication of the proposition is that, in equilibrium, the payoff $h(\omega; \tau, T)$ must satisfy the restriction

$$E_0[\delta^\tau (h(\omega; \tau, T))^\gamma - 1] \leq 0. \tag{4.8}$$

5. APPLICATIONS AND EXTENSIONS

5.1. Bounds on the Implied Volatility Smile

We derive an upper bound on the reservation purchase price and a lower bound on the reservation write price of a vertical call (or, put) spread consisting of two T-expiration European call (or put) options, struck, respectively, at prices K_1 and K_2 with $K_1 < K_2$. We then translate these bounds into the language of implied volatility as upper and lower bounds on the slope of the volatility smile.

Typically, we observe that the implied volatility is a decreasing function of the ratio $\frac{K}{S}$, where K is the exercise price. We obtain an upper bound on the absolute value of the slope which can be attributed to transaction costs or, more generally, to illiquidity over the term of the options. To this end, we consider a portfolio at initial time $t = 0$, which consists of writing n call options struck at K_1, buying n call options struck at K_2, and investing $ne^{-rT}(K_2 - K_1)$ in the riskless asset.

To keep the notation simple, we denote by $C(K_i)$, $i = 1, 2$, the prices of the European calls with expiration at time T and strike prices K_i, $i = 1, 2$. The cost of the above portfolio is $n[C(K_2) - C(K_1) + e^{-rT}(K_2 - K_1)]$ and is nonnegative for $n > 0$, by the absence of arbitrage. We assume that the cost of the portfolio is strictly positive and we normalize it to one by setting

$$n = [C(K_2) - C(K_1) + e^{-rT}(K_2 - K_1)]^{-1}. \tag{5.1}$$

The payoff at date T is nonnegative and is given by

$$h(\omega; T) = n\{(S_T - K_2)^+ - (S_T - K_1)^+ + (K_2 - K_1)\}. \tag{5.2}$$

Theorem 3.1 restricts the prices of the call options as

$$E[e^{-\rho T}(h(\omega; T))^\gamma - 1/\mathcal{F}_0] \leq 0.$$

Combining the above inequality with (5.1) and (5.2) yields that the prices $C(K_1)$ and $C(K_2)$ must satisfy

$$\begin{aligned} e^{-\rho T} E\big[\big\{(S_T - K_2)^+ - (S_T - K_1)^+ + (K_2 - K_1)\big\}^\gamma/\mathcal{F}_0\big] \\ \leq \{C(K_2) - C(K_1) + e^{-rT}(K_2 - K_1)\}^\gamma. \end{aligned} \tag{5.3}$$

Next, we look at the probability of finishing below the lowest strike price K_1, and we observe that

$$\begin{aligned} E[\{(S_T - K_2)^+ - (S_T - K_1)^+ + (K_2 - K_1)\}^\gamma/\mathcal{F}_0] \\ \geq (K_2 - K_1)^\gamma P(S_T < K_1/\mathcal{F}_0). \end{aligned} \tag{5.4}$$

Therefore, if we introduce the differences δK and $\delta C(K)$ defined by $\delta K = K_2 - K_1$ and $\delta C(K) = C(K_2) - C(K_1)$, the above inequality, together with (5.3), yields

$$\frac{\delta C(K)}{\delta K} \geq \{e^{-\rho T} P(S_T < K_1/\mathcal{F}_0)\}^{1/\gamma} - e^{-rT}. \tag{5.5}$$

For simplicity, we drop the $\mathcal{F}_0$-notation in the above cumulative probability.

Next, we translate this bound into the language of implied volatility. The implied volatility, $\hat{\sigma}(K/S)$, is defined implicitly through the equation

$$C_{B-S}(S; T, K, \hat{\sigma}(K/S)) = C(S; T, K), \tag{5.6}$$

where $C_{B-S}(\cdot)$ denotes the Black–Scholes price of a call (or put) option and $C(S; T, K)$ is the actual market price of the option. The slope of the implied volatility, as a function of K/S is obtained by implicit differentiation of equation (5.6) with respect to K:

$$\frac{\partial C_{B-S}}{\partial K} + \frac{\Lambda}{S}\frac{d\hat{\sigma}(K/S)}{d(K/S)} = \frac{\partial C}{\partial K}$$

or

$$\frac{d\hat{\sigma}(K/S)}{d(K/S)} = \frac{S}{\Lambda}\left[\frac{\partial C}{\partial K} - \frac{\partial C_{B-S}}{\partial K}\right], \tag{5.7}$$

where $\Lambda \equiv \partial C_{B-S}/\partial\hat{\sigma}$.

We combine equations (5.5) and (5.7), interpreting equation (5.5) in the limit as $\delta K \to 0$, and obtain

$$\frac{d\hat{\sigma}(K/S)}{d(K/S)} \geq \frac{S}{\Lambda}\left[\left\{e^{-\rho T} P(S_T < K)\right\}^{1/\gamma} - \frac{\partial C_{B-S}}{\partial K} - e^{-rT}\right]. \tag{5.8}$$

The right-hand side, which is negative, provides a lower bound on the volatility skew, $d\hat{\sigma}(K/S)/d(K/S)$. We may further simplify the bound by recalling that

$$\frac{\partial C_{B-S}}{\partial K} + e^{-rT} = e^{-rT}(1 - \mathcal{N}(d_2)) = Q_{B-S}, \tag{5.9}$$

where Q_{B-S} is the Black–Scholes price of a cash-or-nothing binary put option with payoff of one dollar if $S_T < K$.[4] Then we may rewrite equation (5.8) as

$$\frac{d\hat{\sigma}(K/S)}{d(K/S)} \geq \frac{S}{\Lambda}\left[\left\{e^{-\rho T} P(S_T < K)\right\}^{1/\gamma} - Q_{B-S}\right]. \tag{5.10}$$

REMARK 5.1. The bound in equation (5.10) was obtained by writing n vertical call spreads and investing $ne^{-rT}(K_2 - K_1)$ in the riskless asset. An alternative procedure is to buy n vertical put spreads. We proceed as before and arrive at the identical bound, equation (5.10).

Next, we obtain an upper bound on the slope of the volatility smile by considering a project that consists of buying n calls struck at K_1 and writing n calls struck at K_2 with $K_1 < K_2$. We set $n = [C(K_1) - C(K_2)]^{-1}$.

The payoff at date T is nonnegative and is given by

$$h(\omega; T) = n\{(S_T - K_1)^+ - (S_T - K_2)^+\}. \tag{5.11}$$

We apply the result in Theorem 3.1, retrace our earlier steps, and finally obtain the upper bound

$$\frac{d\hat{\sigma}(K/S)}{d(K/S)} \leq \frac{S}{\Lambda}\left[e^{-rT}\mathcal{N}(d_2) - \left\{e^{-\rho T} P(S_T > K)\right\}^{1/\gamma}\right]. \tag{5.12}$$

In Figure 5.1, the solid line illustrates the slope of the volatility smile on the S&P 500 index on March 19, 1999, implied by one-month options, as a function of the ratio K/S. The dotted lines illustrate the lower bound stated in equation (5.10) and the upper bound stated in equation (5.12). Finally, the grey lines illustrate non arbitrage bounds. The interest rate is taken to be $r = 5.5\%$, the volatility is taken to be $\sigma = 21.8\%$, matching the observed at-the-money volatility, $\gamma = 0.2$, $\mu = 12\%$, and $\rho = 0.03$. The figure indicates that the preference-dependent bounds are considerably tighter than the non arbitrage bounds. The observed slope of the volatility smile is within the preference-dependent bounds. The same conclusion applies to six-month options, illustrated in Figure 5.2.

[4] $\mathcal{N}$ is the cumulative normal distribution and d_2 the familiar argument.

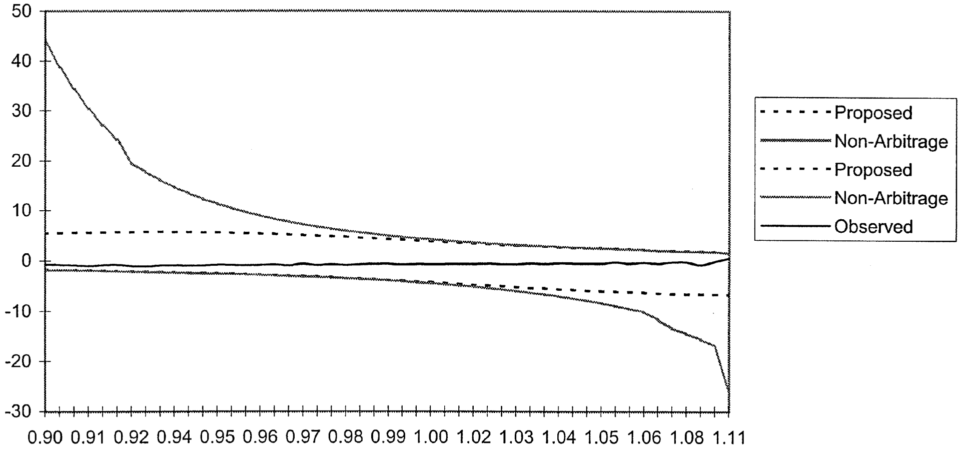

FIGURE 5.1 Bounds on the slope of the smile on 1-month options with ATM vol. 21.8% and RRA coefficient 0.8.

5.2. A Lower Bound on the Reservation Purchase Price of an American Put

We illustrate the applicability of Theorem 3.1 to early exercise claims by deriving a lower bound on the reservation purchase price of American puts. We consider a project that consists of simply purchasing p^{-1} American put options struck at K and expiring at date T, where $p = p(S_0, 0)$ is the purchase price of the put at time $t = 0$.

The payoff at an arbitrary random stopping time τ, $\tau \le T$, is given by $h(\omega; \tau, T) = (K - S_\tau)^+$. We apply Theorem 3.1 and obtain, after rearranging terms, the following lower bound on the reservation purchase price of the put:

$$p \ge \left\{ \max_{\tau \le T} E[e^{-\rho\tau}((K - S_\tau)^+)^\gamma / \mathcal{F}_0] \right\}^{1/\gamma}. \tag{5.13}$$

One may solve numerically for the bound.

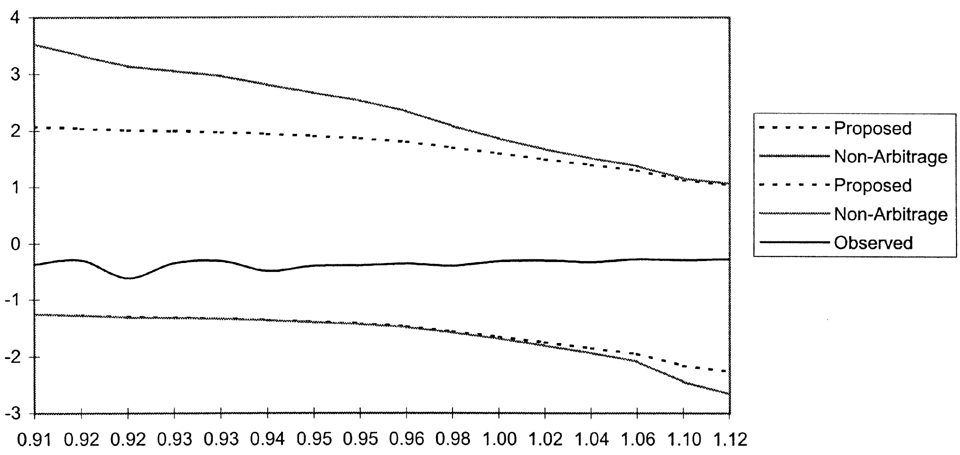

FIGURE 5.2 Bounds on the slope of the smile on 6-month options with ATM vol. 24.3% and RRA coefficient 0.8.

We present below a closed-form expression for the above bound by constructing an explicit solution to the optimal stopping time problem that appears in the right-hand side of the latter inequality. We define the function $f(S,t)$ as

$$f(S,t) = \sup_{\tau \geq t} E\big[e^{-\rho\tau}((K - S_\tau)^+)^\gamma / S_t = S\big]. \tag{5.14}$$

Using standard arguments from the analysis of optimal stopping problems, one readily gets that the above function is given by

$$f(S,t) = e^{-\rho t}(K - \overline{S})^\gamma \left(\frac{S}{\overline{S}}\right)^{-n}, \qquad S \geq \overline{S}, \tag{5.15}$$

where

$$n = \frac{(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\rho\sigma^2}}{\sigma^2} > 0$$

and

$$\overline{S} = \left(\frac{n}{n+\gamma}\right) K < K.$$

Evaluating (5.15) at $t = 0$ and using (5.13) yields the following lower bound on the purchase price of the perpetual put is

$$p(S_0, 0) \geq (K - \overline{S})\left(\frac{\overline{S}}{S_0}\right)^{n/\gamma}, \qquad S_0 \geq \overline{S}.$$

5.3. Alternative Price Processes

One could extend the existing results to the case that the prices of primary securities are modeled as more general processes. For example, an alternative price process for a risky security could be a diffusion process with a Poisson component. To simplify the exposition, we assume for the moment that there is a single risky security whose price S_t, $t \geq 0$ is given by the mixed-diffusion process $dS_t = \mu S_t dt + \sigma S_t dW_t + S_t dP_t$. We also assume that the transaction costs are proportional to the transfers, as in Section 3.

The processes W_t and P_t are, respectively, a standard Brownian motion and a Poisson process. The state trajectories x_t and y_t satisfy, respectively, (2.4) for $N = 1$, and

$$dy_t = \mu y_t dt + \sigma y_t dW_t + y_t dP_t + dL_t - d\widetilde{L}_t. \tag{5.16}$$

Because the individual preferences are modeled by a homogeneous of degree γ utility function and because the state wealth processes (2.4) and (5.16) preserve the linear dynamics with respect to the states x_t and y_t, as well as the relevant controls c_t, L_t, and $\widetilde{L}_t$, the value function also turns out to be a homogeneous function of the same degree. Using in turn the Dynamic Programming Principle—the generalized version of (3.3) to incorporate the jump component—one could reproduce the arguments used in the proof of Theorem 3.1 and derive the analogous desired pricing condition.

Observe that the assumption regarding the Poisson nature of the alternative price process can be easily replaced by any other jump Markov process. The underlying consideration remains that the state dynamics and the transaction cost functional are such that the homogeneity is preserved and also the probabilistic structure leads to a Markovian behavior of the value function so that the Dynamic Programming Principle holds.

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