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Pricing Dynamic Insurance Risks Using the Principle of Equivalent Utility

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We introduce an expected utility approach to price insurance risks in a dynamic financial market setting. The valuation method is based on comparing the maximal expected utility functions with and without incorporating the insurance product, as in the classical principle of equivalent utility. The pricing mechanism relies heavily on risk preferences and yields two reservation prices—one each for the underwriter and buyer of the contract. The framework is rather general and applies to a number of applications that we extensively analyze. *Key words: Dynamic insurance risks, reservation prices, incomplete markets, expected utility, Hamilton-Jacobi-Bellman equations.*

1. INTRODUCTION

The purpose of this work is to introduce a coherent method for the valuation of insurance risks in a dynamic market setting. Generally, actuaries have analyzed and priced such risks by using methods that rely primarily on static strategies. Due to the ever-increasing complexity of the products introduced daily in insurance markets, it is imperative to find mechanisms that are more sophisticated and able to accommodate the individual features of the inherent insurance risks.

The valuation of dynamic risks has been a fundamental issue in financial markets, primarily in the area of derivative securities. One successful pricing theory is based on a strategy in which one creates a portfolio that accurately replicates the payoff of the product. The risk associated with the financial product is thereby completely eliminated or *hedged*. Thus, one can argue that the value of the product must be the cost of setting up the hedging portfolio. This is the key ingredient of the celebrated Black-Scholes method that has been a landmark in derivative asset pricing (Black & Scholes, 1973). Despite its success, which resulted in the great growth of derivative markets, the Black-Scholes approach breaks down entirely once the fundamental assumptions of the characteristics of the market are removed. These assumptions include completeness of the market, liquidity, absence of transaction costs and trading constraints, constant volatility, and perfectly observable assets, to name a few. For an overview, see Wilmott et al. (1995).

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In the case of *incomplete markets*, there is no universal theory to date that successfully addresses all aspects of pricing, for example, numeraire properties, specification of hedging strategies, and robustness of prices. Various alternative pricing mechanisms have been developed that are strongly oriented towards the specific nature of each market friction. For example, the assumption of constant volatility can be relaxed, and a number of stochastic volatility models have been proposed for valuating and calibrating volatility (Hull & White, 1987; Heston, 1993; Dupire, 1994; Renault & Touzi, 1996). In the case of other frictions, such as transaction costs or trading constraints, imperfect replicating or super-replicating strategies have been introduced that minimize the slippage error in a (model-related) appropriate sense (Leland, 1985; Jouini & Kallal, 1995; Cvitanic & Karatzas, 1996; Karatzas & Kou, 1996; Cvitanic et al., 1999).

A different approach is one that is based on expected utility arguments and produces the so-called *reservation prices*. This methodology is built around the investors' preferences towards the risks that cannot be eliminated due to market frictions. The risk preferences are introduced via utility functionals for the buyer and the writer of the financial claim. To establish the writer's reservation price, for example, one examines her maximal expected utility with and without writing the claim. The compensation at which the writer is indifferent between the two alternative investment opportunities yields his reservation price. The fundamental idea for this approach stems from the basic economic principle of certainty equivalent, but modified and extended to accommodate the dynamic aspects of the market environment. It was introduced by Hodges & Neuberger (1989) for the valuation of European calls in the presence of transaction costs and later extended by Davis et al. (1993). Since then, a substantial body of work has been produced by using either stochastic control methods (Davis & Zariphopoulou, 1995; Constantinides & Zariphopoulou, 1999, 2001; and Barles & Soner 1998) or by using martingale theory arguments (Davis, 1997; Karatzas & Kou, 1996; and Rouge & El Karoui, 2000).

Our purpose herein is to extend an expected utility method, the *principle of equivalent utility*, to price dynamic insurance risks. The motivation to undertake this task comes from the fact that insurance markets are *de facto* incomplete markets. In fact, the risks we want to price are related to uncertainties that do not correspond to fluctuations of a tradable asset; therefore, we are not able to use the classical Black-Scholes analysis and thereby eliminate the relevant risk. This is a fundamental difficulty, and we are going to introduce pricing criteria based on utility arguments in order to overcome it and to construct meaningful prices. Our approach extends the one applied in earlier work in actuarial science (for example, Borch, 1961; Bowers et al., 1997; and Gerber & Pafumi, 1998), in which actuaries use expected utility of terminal wealth to calculate prices in a static setting—the so-called *principle of equivalent utility*.

As will be apparent in subsequent sections, the specification of the reservation price is a formidable task. Indeed, one needs to solve two stochastic optimization problems and to extract the argument that makes their value functions equal. In general, explicit solutions are not readily available. However, it turns out that using exponential utility facilitates the computations and, thus, the specification of the price. Because our purpose is primarily to introduce the principle of equivalent utility in a dynamic setting, our examples use only this class of utility functions.

We start our presentation with the stochastic optimization model of expected utility of terminal wealth. This fundamental model was introduced by Merton (1969 and 1971; or Chapters 4 and 5 in 1992) and subsequently revisited, generalized, and extended by a number of authors. In Section 3, we introduce the principle of expected utility and define the reservation prices of insurance claims. To illustrate the use of expected utility arguments, we start with a claim of fixed expiration time and derive its fair price in the simple case of a complete market. In the absence of market frictions, the price naturally coincides with the Black-Scholes price of the claim and can be directly calculated by replication arguments. Even though, under market completeness, the utility method appears redundant, we choose to present the relevant arguments so that the audience becomes accustomed with the stochastic control framework and the underlying structure of the reservation prices.

In Section 4, we consider liabilities that are payable at a fixed time T and are independent of the underlying risky asset. We begin by regarding a single insured life and calculate the reservation prices for term life insurance; then, we extend that model to one that includes more than one independent life. We next consider pure endowment insurance, and we end the section by modeling insurance risks as diffusion and Poisson processes. In each case, the reservation prices we obtain are calculated via the so-called value functions (optimal expected utility of terminal wealth) that are shown to be solutions of certain non-linear partial differential equations, known as the Hamilton-Jacobi-Bellman equations.

In Section 5, we consider insurance payable at the time of incurrence of the loss, and in Section 6, we look at claims involving a random time, τ, such as the time of death. In both Sections 5 and 6, we parallel the topics from Section 4. In the examples using exponential utility, we find in Sections 4 and 5 that the prices are independent of the risky stock process. Thus, the prices are identical to the ones obtained when investment is limited to the riskless bond. For that reason, our method may appear to be rather complicated—perhaps unnecessarily so. However, in Section 6, we learn the rather interesting fact that when the horizon is random, the prices depend on the parameters of the risky stock process. Also, our method applies to any smooth (increasing and concave) utility function; therefore, our method can be applied to other utility functions, such as power or logarithmic utility. In those cases, the prices will depend on the risky stock process, in contrast with the examples in Sections 4 and 5. In Section 7, we conclude our paper with a summary and suggestions for further research.

2. BACKGROUND RESULTS ON STOCHASTIC OPTIMIZATION AND EXPECTED UTILITY

In this section, we review the fundamental classical model of optimal portfolio management for expected utility of terminal wealth. This model was introduced by Merton in his seminal papers (1969 and 1971; or Chapters 4 and 5 in 1992), and its extensions have attracted great interest both from academics and practitioners. Two main methodologies have been used in the analysis of expected utility models—one relies on martingale techniques and the other uses optimal stochastic control and non-linear partial differential equations. For an overview of the two approaches, see the monograph of Karatzas (1996) and the review papers by Zariphopoulou (1999b, 2001).

Merton's model examines the optimal investment strategies of an individual who, endowed with initial wealth, seeks to maximize her expected utility of terminal wealth, i.e., wealth at the end of a (prespecified) trading horizon. The investor has the opportunity to trade between a riskless bond and a risky stock account. The price of the stock S_s is modeled as a geometric Brownian motion:

$$\begin{cases} dS_s = S_s(\mu\, ds + \sigma\, dB_s), \\ S_t = S > 0. \end{cases} \tag{2.1}$$

The process B_s is a standard Brownian motion on a probability space (Ω, F, P), and the coefficients μ and σ are given positive constants, known, respectively, as the *mean rate of return* and the *volatility coefficient*. It is assumed throughout the analysis that $\mu > r > 0$, in which r is the rate of return of the riskless bond.

The investor is given, at time $t > 0$, an initial endowment $w \geqslant 0$, and she trades dynamically between the two accounts; in other words, the investor chooses the amounts π_s^0 and π_s, $t \leqslant s \leqslant T$ to invest in the bond and the stock account, respectively. The constant $T > 0$ represents the end of the trading horizon. The *total current wealth* satisfies the budget constraint $W_s = \pi_s^0 + \pi_s$ and follows the state dynamics

$$dW_s = rW_s\, ds + (\mu - r)\pi_s\, ds + \sigma\pi_s\, dB_s, \quad t \leqslant s \leqslant T. \tag{2.2}$$

One can easily derive this equation by using the definition of W_s and the dynamics in (2.1); see Merton (1969). Note that the budget constraint, together with the log-normality assumption on the stock dynamics, enables us to eliminate one of the control policies in the controlled wealth diffusion process.

In the absence of any additional risk, for example, a random liability or payoff, the investor seeks to maximize the expected utility of terminal wealth

$$V(w, t) = \sup_{\{\pi_t\} \in \mathcal{A}} E[u(W_T) \mid W_t = w]. \tag{2.3}$$

The set $\mathcal{A}$ is the set of admissible policies, $\{\pi_s\}$, that are $\mathcal{F}_s$-progressively measurable (in which $\mathcal{F}_s$ is the augmentation of $\sigma(W_u\colon t \leqslant u \leqslant s)$) and that satisfy the integrability condition $E\int_t^T \pi_s^2\,ds < +\infty$. The utility function $u: R \to R$ is assumed to be increasing, concave, and smooth.

The solution of (2.3) is known as the *value function*, and it satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} V_t + \max_{\pi}\left[(\mu - r)\pi V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww}\right] + rwV_w = 0, & (2.4) \\ V(w, T) = u(w). & (2.5) \end{cases}$$

The HJB equation is the offspring of the principle of dynamic programming and of stochastic calculus. If it can be shown *a priori* that the value function is smooth ($C^{2,1}(R \times [0, T])$), then results, which are well known by now, yield that the value function equals the unique smooth solution of the HJB equation. Additionally, the optimal policies can be specified via the first-order conditions arising in (2.4). Indeed, the concavity of the utility function u, together with the linearity of the state equation (2.2) with respect to the wealth and the portfolio process, implies that the value function itself inherits this property of concavity. Therefore, the maximum in (2.4) is well defined and achieved at

$$\pi^*(w, t) = -\frac{(\mu - r)}{\sigma^2}\frac{V_w(w, t)}{V_{ww}(w, t)}. \tag{2.6}$$

One can establish that the *optimal policy* is given via (2.6) in a *feedback law* in the following sense: The optimal investment process in the stock account is

$$\Pi_s^* = \pi^*(W_s^*, s) = -\frac{(\mu - r)}{\sigma^2}\frac{V_w(W_s^*, s)}{V_{ww}(W_s^*, s)}, \quad t \leqslant s \leqslant T, \tag{2.7}$$

in which V solves (2.4) and W_s^* is the optimal wealth process solving (2.2) with Π_s^* used for π_s. The above classical optimality results are known as the Verification Theorem (Fleming & Soner, 1993, Chapter 6).

2.1 Remark. Because we assume that the stock price follows a lognormal process, it does not appear as an extra state variable. This is not the case if the dynamics are non-linear (Zariphopoulou, 1999a) or if the volatility is modeled as a correlated process (Zariphopoulou, 2001).

2.2 Remark. If the HJB equation does not have smooth solutions, as is usually the case in incomplete markets, one needs to work with a relaxed class of solutions and to define optimality concepts therein. It turns out that a rich class of weak solutions, which are appropriate for the applications at hand, are the so-called *viscosity solutions*. They were introduced by Crandall & Lions (1983) for first-order non-linear equations and by Lions (1983) for second-order ones. In the context of expected utility models, Zariphopoulou (1992, 1994) first introduced viscosity solutions, which have by now become a standard tool for the analysis of stochastic optimization models in markets with frictions. For an overview, see the review articles by Zariphopoulou (1999b, 2001).

2.3 Remark. Note that in our analysis, the expiration is a fixed time; this is a natural feature in models of asset pricing and portfolio management. However, when we analyze insurance models, this is not necessarily the case, and one may need to consider stochastic horizons, a natural property of random events that affect the entire pricing mechanism. We do this in Section 6 at the expense of increasing the complexity of the model.

2.4 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$. Then, a straightforward, but tedious, calculation shows that

$$V(w, t) = -\frac{1}{\alpha} \exp\left(-\alpha w\, e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T-t) \right).$$

By using (2.7), we find that the optimal investment in the risky asset is

$$\Pi_t^* = \frac{(\mu - r)}{\sigma^2} \frac{e^{-r(T-t)}}{\alpha}.$$

Observe that Π_t^* is not stochastic; in particular, it is independent of wealth. In models of lognormal stock dynamics, such independence from wealth is generally observed in calculations with exponential utility because the absolute risk aversion $-u''(w)/u'(w)$ (Pratt, 1964) is constant (equal to α). Note that as the risk aversion of the decision maker increases, as measured by α, and as the time until expiry increases, the amount of money invested in the risky asset decreases.

For arbitrary utility functions, a closed form solution is not generally available, except in the case of hyperbolic absolute risk aversion (HARA) utilities that are of the form $u(w) = (A + Bw)^\gamma$, for given constants A, B, and $\gamma < 1$. In the general case, the standard way to proceed is to linearize the HJB equation and thereby work with the *dual* function $\tilde{V}(y, t) = \sup_w (V(w, t) - wy)$. Equivalently, one may introduce the transformation $V_w(f(y, t), t) = y$ and determine the function f. The latter turns out to solve a linear partial differential equation that is easy to analyze. For related arguments, see Karatzas & Shreve (1998, Chapter 3).

In the optimization models that we will encounter for valuating insurance risks, such transformations do not linearize the HJB equation, because the market is incomplete. In fact, the technically-oriented reader will recognize that linearization is always compatible with completeness. Because of the technical difficulties that arise for general utility functions, we work examples using only exponential utility whose scaling properties are convenient for the specification of the related value functions. However, we present the HJB equations for general utility functions.

3. RESERVATION PRICES

In this section, we extend the principle of equivalent utility to price dynamic insurance risks. The main ingredient of the pricing methodology is the use of individual risk preferences towards the risks that cannot be eliminated through trading in the financial market. Both parties involved in the insurance claim, namely, the insurer and the buyer, are endowed with a utility function of terminal wealth. We denote both utility functions by u; however, our framework allows for each party to possess a different utility function. For example, one expects that an insurer will be less risk averse than a buyer of insurance, so that in the notation of Example 2.4, $\alpha_{\text{Insurer}} < \alpha_{\text{Buyer}}$. Both parties have the opportunity to invest in a riskless asset and a risky one with the goal of maximizing their expected utility of terminal wealth. The relevant stochastic optimization problem is identical to the one described in the previous section. We introduce an insurance claim that, for the sake of exposition only, is taken to be of European type, namely, it is represented as the insurer's liability or the (potential) buyer's obligation Y_T at expiration T.

For the insurer, the utility-based pricing mechanism relies on considering and subsequently comparing the following possibilities: Either the insurer can choose to accept the risk, receive some premium, and invest in the financial market, or the insurer can choose not to insure the risk and simply invest his wealth in the market with resulting value function V, as in (2.3). The reservation price is defined as the premium at which the insurer is indifferent between these two options; see Definition 3.1 below. Similarly, the buyer of insurance considers two possibilities: Either the buyer can purchase insurance for some premium and invest in the financial market with resulting value function V, as in equation (2.3), or the buyer can retain the risk and invest in the market. The reservation price of the buyer is defined as the premium at which the buyer is indifferent between these two options; see Definition 3.1 below.

If the insurer (the potential buyer of insurance) insures (retains) the risk, then we need to define a value function similar to the one for V in Section 2. As we mentioned earlier, we assume that the insurance liability Y_T is payable at time T by either an individual who has not yet purchased insurance or by an insurance company that has underwritten the liability. We also assume that the liability cannot be traded after its transfer from the buyer to the insurer and before its expiration. In this time horizon, only trading between the two available market assets is permitted. Then, the value function of the agent is defined to be

$$U(w, y, t) = \sup_{\{\pi_t\} \in \mathscr{A}} E[u(W_T - Y_T) \mid W_t = w,\ Y_t = y], \tag{3.1}$$

in which $\mathscr{A}$ is the set of admissible policies for the agent. The process Y_t is the *loss process*, in the sense that it models the cumulative loss incurred up to time t.

If the agent is the insurance company, then U is the value function if the insurance company insures the risk Y_T, while V in (2.3) is the value function if the insurance company does not accept the risk. If the agent is the (potential) buyer of insurance, then U is the value function if the buyer does not buy insurance but instead retains the risk Y_T, while V in (2.3) is the value function if the buyer does buy insurance.

A fundamental assumption is that Y_t is independent of B_t, the Brownian motion that derives the stock price. It is important to observe that the loss process does not represent an asset on which one can trade and thereby create a hedging portfolio. The fact that Y_t is not tradable results in a fundamental difficulty for the specification of the price via classical *arbitrage-free* arguments based on perfect replication. Another important issue is how one defines admissible strategies in $\mathscr{A}$. If one insists on having the constraint $W_T - Y_T \geqslant 0$ a.s., in analogy with the non-negativity constraint of wealth in Merton's problem, then infinite prices may result due to the non-perfect correlation between Y_t and S_t. Such admissibility constraints generate many technical difficulties and, in general, alter the prices in a rather complex way; see the discussion in Constantinides & Zariphopoulou (1999). We do not assume that such constraints are binding herein, but we will deal with this issue in future work; see Young & Zariphopoulou (2001). This is, in fact, one reason we chose to work examples with exponential utility, which is well defined for negative arguments.

3.1 DEFINITION. The (state dependent) *reservation price of the underwriter* (or *insurer*), $P^I(w, y, t)$, is defined as the compensation P^I such that

$$V(w, t) = U(w + P^I, y, t), \tag{3.2}$$

for a given (w, y, t). Similarly, the (state dependent) *reservation price of the buyer*, $P^B(w, y, t)$, is defined as the obligation P^B such that

$$V(w - P^B, t) = U(w, y, t), \tag{3.3}$$

for a given (w, y, t).

Essentially, we equate the value of not insuring the risk with the value of insuring the risk for a price P^I received at time t. One can think of P^I as the minimum premium that the insurer is willing to accept at time t, given the state (w, y), to insure the liability Y_T at time T. Respectively, we equate the value of buying complete insurance for a price of P^B with the value of not insuring the risk. One can think of P^B as the maximum premium that the buyer is willing to pay at time t, given the state (w, y), for insurance against the liability Y_T at time T.

The static analogues of the above prices are presented in Bowers et al. (1997, Equations (1.3.6) and (1.3.1)) in the context of insurance risks. They are the essence of the principle of equivalent utility for pricing insurance; see also Gerber & Pafumi (1998). Similar static criteria are related to the notion of indifference and compensating prices in classical economics. The specification of such prices is much more complex if intermediate trading in a financial market is allowed, and this is the task undertaken herein.

In general, due to the non-linearity of the criteria (3.2) and (3.3) and the market incompleteness, the prices P^I and P^B do not coincide. Moreover, because of the way P^I and P^B are defined, it is inevitable that they depend not only on the liability process but also on the current wealth. In this sense, P^I and P^B are not universal, as opposed to the Black-Scholes price that is independent of the individual's portfolio holdings and depends only on the dynamics of the asset on which the claim is written (as well as the riskless interest rate). Overall, universality is a highly desirable property that is not, in general, valid in incomplete markets. In fact, one may easily verify that if the loss process Y_t and the stock price S_t are not perfectly correlated, the pricing equalities (3.2) and (3.3) cannot be valid for *all* levels of (y, t), if one insists on price functions P^I and P^B that are wealth independent. Besides the dependence on wealth, P^I and P^B are also expected to depend on risk preferences as represented by the utility function; after all, these prices are introduced in order to price risks that cannot be hedged away.

From the above discussion, we see that there are two pitfalls of the pricing mechanism that uses the principle of equivalent utility—the dependence on wealth and on risk aversion. We will see that the latter cannot be eliminated because it is the direct consequence of market incompleteness and risks that cannot be hedged. However, the dependence on wealth can be addressed in two ways—one may use exponential utility or work with universal price bounds. Pratt's measure of absolute risk aversion is independent of wealth for exponential utility (as mentioned in Example 2.4). This property, in turn, yields investment strategies that are independent of wealth, which, together with certain scaling properties, implies wealth-independent reservation prices. We will observe this phenomenon in our examples. An alternative approach is to seek price bounds that are independent of wealth but satisfy (3.2) and (3.3) as inequalities, instead of equalities, for all w.

3.2 DEFINITION. The *universal write price*, $\bar{P}^I(y, t)$, is defined as the <u>minimum</u> price that satisfies

$$V(w, t) \leqslant U(w + \bar{P}^I, y, t), \tag{3.4}$$

for all wealth levels w. Similarly, the universal buy price, $\underline{P}^B(y, t)$, is defined as the <u>maximum</u> price that satisfies

$$V(w - \underline{P}^B, t) \geqslant U(w, y, t)\,, \tag{3.5}$$

for all wealth levels w.

According to the above definition, the insurer (respectively, buyer) should not accept to write, or insure, (respectively, buy) the liability at a price lower (respectively, higher) than $\bar{P}^I$ (respectively, $\underline{P}^B$). Therefore, two agents with the same risk preferences must trade the liability at prices within the above spread. Constantinides & Zariphopoulou (1999) introduced universal prices in markets with transaction costs, and subsequently others used them for different market imperfections (see, for example, Constantinides & Zariphopoulou, 2001; Munk, 2000; and Mazaheri, 2001). We do not pursue this approach in this paper.

We conclude this section by illustrating how the principle of equivalent utility can be used to price dynamic risks that are perfectly correlated with the underlying risky security. This is the well known complete market setting, and the risks can be priced by classical arbitrage arguments which yield the Black-Scholes price. The latter is unique and independent of the current wealth and the individual preferences (Black & Scholes, 1973). Clearly, one expects the reservation prices to be equal to each other and to coincide with the Black-Scholes price. Moreover, the same price satisfies both inequalities (3.4) and (3.5), which reduce to a universal equality among the relevant value functions. Even though the notion of reservation price is redundant in such a perfect market setting, in the calculations below we rederive the Black-Scholes price. We choose to do this in order to familiarize the audience with the involved technical steps and also to have a benchmark case with which to compare when we analyze risks that cannot be priced with traditional methods.

To this end, we assume that the insurer can choose to underwrite a liability Y_T at expiration time T, such that $Y_T = g(S_T)$ for some function $g: R^+ \to R^+$ and S_s given by equation (2.1). We look for a function $h^I(S, t)$ such that

$$V(w, t) = U(w + h^I(S, t), S, t), \forall (w, S, t) \in R \times R^+ \times [0, T], \tag{3.6}$$

in which U here is defined somewhat differently than in equation (3.1), namely,

$$U(w, S, t) = \sup_{\{\pi_t\} \in \mathscr{A}} E[u(W_T - g(S_T)) \mid W_t = w, S_t = S].$$

We define U this way because the liability in this case is a function of the underlying risky asset. Similarly, the buyer of insurance against the liability is willing to pay $h^B(S, t)$ that solves

$$V(w - h^B(S, t), t) = U(w, S, t), \forall (w, S, t) \in R \times R^+ \times [0, T]. \tag{3.7}$$

Note we postulate that the price does not depend on the current wealth level. This independence is not known *a priori* nor can be seen easily from the model in general. On the other hand, one can argue if there exists a function that satisfies both (3.6) and (3.7), then it coincides with the unique Black-Scholes price. In fact, one can easily argue that if the claim is written (respectively, bought) at a price higher (respectively, lower) than the Black-Scholes price, then there exist strategies that create arbitrage opportunities (Musiela & Rutkowski, 1997). Therefore, in the calculations below, it suffices to find a candidate that is wealth independent and to justify that the corresponding pricing equalities hold. As the calculations below demonstrate, this candidate price solves the Cauchy problem (3.10), which has a unique solution (Fleming & Soner, 1993, Chapter 6).

Alternatively, one could start with prices that are wealth dependent and follow the calculations as done below. After more tedious analysis, the candidate price would turn out to satisfy a nonlinear partial differential equation. Using arguments from the theory of viscosity solutions, one could show that the partial differential equation has a unique viscosity solution (Ishii & Lions, 1990), say $H(w, S, t)$. From the terminal data, however, one sees that H is wealth independent, because $H(w, S, T) = g(S_T)$. By the uniqueness of viscosity solution, one concludes that the solution is unique and coincides with the wealth independent solution of (3.10).

In the following calculation, we compute the buyer's price. To this end, the HJB equation for U turns out to be

$$\begin{cases} U_t + \max\limits_{\pi} \left[(\mu - r)\pi U_w + \dfrac{1}{2}\sigma^2\pi^2 U_{ww} + \sigma^2 \pi U_{wS} \right] + \dfrac{1}{2}\sigma^2 S^2 U_{SS} + \mu S U_S + rwU_w = 0, \\ U(w, S, T) = u(w - g(S)), \end{cases}$$

or, equivalently,

$$\begin{cases} U_t - \dfrac{[\sigma^2 S U_{wS} + (\mu - r)U_w]^2}{2\sigma^2 U_{ww}} + \dfrac{1}{2}\sigma^2 S^2 U_{SS} + \mu S U_S + rwU_w = 0, \\ U(w, S, T) = u(w - g(S)). \end{cases} \tag{3.8}$$

Differentiating (3.7) yields

$$-h_t V_w + V_t = U_t, \quad V_w = U_w, \quad V_{ww} = U_{ww}, \quad -h_S V_w = U_S,$$

$$-h_{SS} V_w + h_S^2 V_{ww} = U_S, \quad -h_S V_{ww} = U_{wS},$$

in which $h = h^B$. After inserting these expressions in (3.8) and rearranging terms, we obtain

$$V_w \left[-h_t + rh - \frac{1}{2}\sigma^2 S^2 h_{SS} - rSh_S \right] + \left[V_t - \frac{(\mu - r)^2}{2\sigma^2}\frac{V_w^2}{V_{ww}} + r(w - h(S, t))V_w \right] = 0, \tag{3.9}$$

in which all the derivatives are evaluated at $(w - h(S, t), t)$. Observe that the second bracketed term in (3.9) vanishes because V solves the HJB equation (2.4). Therefore, h^B satisfies

$$\begin{cases} rh^B = h_t^B + \dfrac{1}{2}\sigma^2 S^2 h_{SS}^B + rSh_S^B, \\ h^B(S, T) = g(S), \end{cases} \tag{3.10}$$

the seminal Black-Scholes equation. By using equation (3.6) in place of (3.7), a similar argument shows that the insurer's price, h^I, also satisfies (3.10). The reason that the insurer's and the buyer's prices are equal and independent of the utility function u is that the market is complete.

4. INSURANCE CLAIMS PAYABLE AT EXPIRATION TIME *T*

In this section, we apply the principle of equivalent utility to determine the value of insurance products when payment occurs at a terminal, prespecified time T. We will consider a variety of such products. We will start with the simplest class of insurance claims—static losses of a single life in Section 4.1 and of a group of insured lives in Section 4.2, as in term life insurance payable at time T. In that case, T could be taken to be one year, so that we are considering term life insurance payable at the end of the year of death. Next, we consider the problem of pricing pure endowment insurance on a single life in Section 4.3. Finally, we consider losses modeled as stochastic processes, namely, diffusion and Poisson processes in Sections 4.4 and 4.5, respectively. We derive the reservation prices and compare them with the benchmark premia derived through traditional methods based on present-value arguments.

To specify the reservation prices for insurance, we need to solve stochastic optimization problems of expected utility that are generalizations of Merton's model in a non-trivial way. In fact, the complexity of the payoff results in HJB equations with non-local terms and forms that cannot be manipulated in a straightforward fashion. In our analysis, we derive the associated HJB equation for general utility functions, but we specify the reservation prices only for exponential utility. Generally speaking, under arbitrary choice of preferences, one can establish that the value function of the agent (the buyer or seller of insurance) is a solution to the HJB equation, at least in the viscosity sense. As a matter of fact, one can show that the value function is the unique solution in the class of viscosity solutions. The arguments used to show the viscosity properties are routine adaptations of by-now classical results in the area of stochastic optimization, and we only discuss formally the main steps of the analysis. For a detailed exposition of the use of viscosity solutions for HJB equations arising in asset valuation models, we refer the reader to the review articles of Zariphopoulou (1999b, 2001).

4.1. Single insured life—term life insurance

We start with the case of a claim that pays, at expiration T, a random variable taking the values 0 or 1 with probabilities that depend on the current time t. To be concrete, we consider an individual aged x, who is seeking to buy term life insurance that will pay 1 unit at time T if the individual dies before time T, and 0 otherwise. For the rest of this section, we write (x) to refer to this individual. To this end, denote by ${}_{T-t}q_{x+t}$ the probability that (x) will die before time T given that (x) is alive at time t. Then,

$$
{}_{T-t}q_{x+t} = \frac{F_x(T) - F_x(t)}{1 - F_x(t)},
$$

in which F_x is the cumulative distribution function of the time until death of (x). Other life functions can be obtained similarly. By assuming enough differentiability for the distribution function, we will employ the hazard function, otherwise known as the force of mortality, $\lambda_x(t)$, given by $\lambda_x(t) = f_x(t)/(1 - F_x(t))$, in which f_x is the probability density function of the time until death of (x).

Next, we consider the optimization problem of the seller of the insurance product, as introduced in equation (3.1) and rewritten here for convenience,

$$U(w, t) = \sup_{\{\pi_t\} \in \mathcal{A}} E[u(W_T - Y_T) \mid W_t = w]. \tag{4.1}$$

Recall that the current wealth W_s is defined as in Section 2 and solves (2.2). The expectation is with respect to the product measure $\mathrm{P} \times \mathrm{Q}$, in which $(\Omega', \mathcal{G}, \mathrm{Q})$ is a probability space on which Y_T is defined. Observe that the liability, even though it is indexed by T, is not a function of time (compare, for example, with $Y_T = g(S_T)$ at the end of Section 3). In fact, the liability is a static risk, and this is the reason that U depends explicitly on (w, t), with the dependence on y being absorbed in the measure according to which we calculate the expectation at terminal time.

The classical principle of dynamic programming yields

$$U(w, t) \geqslant E[U(W_{t+h}, t+h) \mid W_t = w]; \tag{4.2}$$

see Fleming & Soner (1993).

We continue with a formal derivation of the HJB equation. Let us assume that $\{\pi_s^*: t \leqslant s \leqslant t+h\}$ is the optimal strategy that the insurer follows. Denote by W_s^* the wealth under $\{\pi_s^*\}$. If the individual $(x+t)$ dies during $[t, t+h]$, in which time is measured from age x, then we are in the certain situation with respect to the insurance risk in the following sense: The insurer will pay 1 at time T for this death and will have to charge $e^{-r(T-t)}$ at time t to cover this payout. Therefore, for this problem, the insurer's value function equals

$$E[V(W_{t+h}^* - e^{-r(T-t-h)}, t+h) \mid W_t = w]$$

multiplied by ${}_hq_{x+t}$, the probability that $(x+t)$ dies during $[t, t+h]$, in which time is measured from age x. If the individual $(x+t)$ survives to time $t+h$, an event that happens with probability ${}_hp_{x+t}$, the insurer's value function is

$$E[U(W_{t+h}^*, t+h) \mid W_t = w \mid W_t = w]$$

multiplied by ${}_hp_{x+t}$.

Therefore,

$$\begin{aligned} U(w, t) \geqslant\ & E[U(W_{t+h}^*, t+h) \mid W_t = w]\, {}_hp_{x+t} \\ & + E[V(W_{t+h}^* - e^{-r(T-t-h)}, t+h) \mid W_t = w]\, {}_hq_{x+t}. \end{aligned} \tag{4.3}$$

By assuming enough regularity conditions and appropriate integrability on the value functions and their derivatives (Bjørk, 1998), we get

$$E[U(W^*_{t+h}, t+h) \mid W_t = w] = U(w,t) + E\left[\int_t^{t+h} \{U_t(W^*_s, s) + (rW^*_s + (\mu - r)\pi^*_s)U_w(W^*_s, s)\}\, ds \mid W_t = w\right] + E\left[\int_t^{t+h} \frac{1}{2}\sigma^2 \pi^{*2}_s U_{ww}(W^*_s, s)\, ds \mid W_t = w\right]. \tag{4.4}$$

We obtain a similar expression for $E[V(W^*_{t+h} - e^{-r(T-t-h)}, t+h) \mid W_t = w]$ that combined with (4.3) and (4.4) yields

$$U(w,t) \geqslant U(w,t)_h p_{x+t} + V(w - e^{-r(T-t)}, t)_h q_{x+t} + E\left[\int_t^{t+h} \{U_t(W^*_s, s) + (rW^*_s + (\mu - r)\pi^*_s)U_w(W^*_s, s) + \frac{1}{2}\sigma^2\pi^{*2}_s U_{ww}(W^*_s, s)\}\, ds \mid W_t = w\right]_h p_{x+t} + E\left[\int_t^{t+h} \{V_t(W^*_s - e^{-r(T-s)}, s) + (rW^*_s + (\mu - r)\pi^*_s)V_w(W^*_s - e^{-r(T-s)}, s)\}\, ds \mid W_t = w\right] \times {}_h q_{x+t} + E\left[\int_t^{t+h} \tfrac{1}{2}\sigma^2\pi^{*2}_s V_{ww}(W^*_s - e^{-r(T-s)}, s)\, ds \mid W_t = w\right]_h q_{x+t}. \tag{4.5}$$

By subtracting $U(w,t)_h p_{x+t}$ from both sides and dividing by h, we obtain

$$U(w,t)\frac{{}_h q_{x+t}}{h} \geqslant V(w - e^{-r(T-t)}, t)\frac{{}_h q_{x+t}}{h} + E\left[\frac{1}{h}\int_t^{t+h} \left\{U_t(W^*_s, s) + (rW^*_s + (\mu - r)\pi^*_s)U_w(W^*_s, s) + \frac{1}{2}\sigma^2\pi^{*2}_s U_{ww}(W^*_s, s)\right\} ds \mid W_t = w\right]_h p_{x+t} + E\left[\frac{1}{h}\int_t^{t+h} \{V_t(W^*_s - e^{-r(T-s)}, s)\right.$$

$$+ (rW_s^* + (\mu - r)\pi_s^*)V_w(W_s^* - e^{-r(T-s)}, s)\} \, ds \mid W_t = w \Bigg]_h q_{x+t}$$

$$+ E\left[\frac{1}{h}\int_t^{t+h} \frac{1}{2}\sigma^2 \pi_s^{*2} V_{ww}(W_s^* - e^{-r(T-s)}, s)\, ds \mid W_t = w\right]_h q_{x+t}. \tag{4.6}$$

By taking the limit as h goes to 0 from the right, we get

$$0 \geqslant \left\{U_t + (rw + (\mu - r)\pi)U_w + \frac{\sigma^2\pi^2}{2} U_{ww}\right\} + \lambda_x(t)[V(w - e^{-r(T-t)}, t) - U(w, t)],$$

which in turn yields, along the optimum, the HJB equation

$$\begin{cases} U_t + \max\limits_{\pi}\left\{(\mu - r)\pi U_w + \dfrac{1}{2}\sigma^2\pi^2 U_{ww}\right\} + rwU_w \\ \quad + \lambda_x(t)[V(w - e^{-r(T-t)}, t) - U(w, t)] = 0, \\ U(w, T) = u(w). \end{cases} \tag{4.7}$$ [1]

By following routine arguments, we can show that the value function U is concave, which implies, as in (2.6), that the above maximum is well defined and achieved at

$$\pi^*(w, s) = -\frac{(\mu - r)}{\sigma^2}\frac{U_w(w, s)}{U_{ww}(w, s)}.$$

Similarly to (2.7), the optimal investment policies are given via the latter function in a feedback form. Specifically, the optimal investment process in the stock account is

$$\Pi_s^* = \pi^*(W_s^*, s) = -\frac{(\mu - r)}{\sigma^2}\frac{U_w(W_s^*, s)}{U_{ww}(W_s^*, s)}, \tag{4.8}$$

in which U solves (4.7) and W_s^* is the optimal wealth solving (2.2) with Π_s^* used for π_s. Therefore, we can rewrite (4.7) as

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[V(w - e^{-r(T-t)}, t) - U(w, t)] = 0, \\ U(w, T) = u(w). \end{cases} \tag{4.9}$$

[1] The reader familiar with Merton's work will recognize that equation (4.7) resembles the equation related to an expected utility maximization problem with discount factor $\lambda_x(t)$ and intermediate utility payoff $\lambda_x(t)V(w - e^{-r(T-t)}, t)$.

One may derive equation (4.7) rigorously by using arguments from the theory of viscosity solutions. We refer the reader to the review article of Zariphopoulou (2001) for a concise exposition.

Next, we calculate the reservation prices for the case of exponential utility.

4.1 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Because of the uniqueness of solutions to the HJB equation, it suffices to construct a candidate solution. To this end, let $U(w, t) = V(w, t)\phi(t)$, with $\phi(T) = 1$ and V as in Example 2.4. Then, (4.9) becomes

$$V_t\phi + V\phi' + rwV_w\phi - \frac{(\mu - r)^2}{2\sigma^2}\frac{V_w^2}{V_{ww}}\phi + \lambda_x(t)\,[V e^{\alpha} - V\phi] = 0.$$

The first, third, and fourth terms cancel because V satisfies the HJB equation (2.4), and we can cancel a factor of V from the remaining three terms to obtain the ordinary differential equation for ϕ:

$$0 = \phi' + \lambda_x(t)[e^{\alpha} - \phi],$$

with boundary condition $\phi(T) = 1$. The solution to this equation is given by

$$\phi(t) = e^{-\int_t^T \lambda_x(s)\,ds} + e^{\alpha}\left[1 - e^{-\int_t^T \lambda_x(s)\,ds}\right] = {}_{T-t}p_{x+t} + e^{\alpha}_{\;T-t}q_{x+t} = M_{Y_T}(\alpha), \qquad (4.10)$$

the moment generating function of Y_T evaluated at α. The reservation prices, P^I and P^B, both equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha}\, e^{-r(T-t)} \ln M_{Y_T}(\alpha), \qquad (4.11)$$

in which the distribution of Y_T depends on the current time t. Note that the price increases with respect to the absolute risk aversion α. As α approaches 0, the price approaches $e^{-r(T-t)}{}_{T-t}q_{x+t}$, the net premium for this risk. Also, note that the price is independent of the risky asset; thus, this price is identical to the one obtained by allowing only investment in the riskless bond.

In this example, the reservation prices of the insurer and the buyer of insurance are equal. In reality, one expects that the insurer's risk aversion will be less than the buyer's, from which it will follow that $P^I < P^B$ because the price increases with respect to α. This spread allows for the trade of insurance.

4.2. Group of insured lives—term life insurance

Suppose the loss payable at time T equals 1 for each of Y_T people who have died from a group of n people alive at time 0. Assume that the n people are all aged x with independent and identically distributed times until death. Then, the number who die in the interval $[t, t+h]$ is distributed according to the Binomial$(n - Y_t, {}_hq_{x+t})$, in which Y_t is the number who have died by time t. One can follow the reasoning in Section 4.1 to show that the value function $U(w, y, t; n)$ given by (3.1) solves the following recursive HJB equation:

$$\begin{cases} U(w, n, t; n) = V(w - n\, e^{-r(T-t)}, t); \\ For \quad y = 0, 1, \ldots, n-1, \\ U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + (n-y)\lambda_x(t)[U(w, y+1, t; n) - U(w, y, t; n)] = 0, \\ U(w, y, T; n) = u(w - y). \end{cases} \tag{4.12}$$

4.2 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient, as in Examples 2.4 and 4.1. From Example 4.1, we know that $U(w, 0, t; 1) = V(w, t)\phi(t)$, in which ϕ is given by (4.10) and V is as in Example 2.4. As in Example 4.1, one can show that

$$U(w, y, t; n) = V(w, t)\, e^{\alpha y}\phi(t)^{n-y}.$$

It follows that both the reservation prices equal

$$P^I(w, y, t; n) = P^B(w, y, t; n) = y\, e^{-r(T-t)} + \frac{n-y}{\alpha}\, e^{-r(T-t)} \ln \phi(t), \tag{4.13}$$

an immediate generalization of the price given in (4.11) for a single life ($y = 0$, $n = 1$). Note that this price contains a riskless provision for the y people who have died by time t and a risk-loaded price for each of the remaining $n - y$ people. The risk-loaded price for each person equals the one in equation (4.11).

4.3. Single insured life—pure endowment insurance

Pure endowment insurance for the period $[0, T]$ pays 1 unit at time T if (x) survives to time T and 0 otherwise. This insurance is the building-block of life annuities; see Bowers et al. (1997). We find the reservation prices for pure endowment insurance in this section, then later use our result when we consider pricing a temporary life annuity in Section 5.3.

In this case, the HJB equation for U reduces to

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[V(w, t) - U(w, t)] = 0, \\ U(w, T) = u(w - 1). \end{cases} \tag{4.14}$$

4.3 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Write $U(w, t) = V(w, t)\eta(t)$, in which V is given in Example 2.4. Then, η solves the first-order differential equation

$$0 = \eta' + \lambda_x(t)[1 - \eta],$$

with boundary condition $\eta(T) = e^{\alpha}$. It follows that η equals

$$\eta(t) = e^{\alpha}{}_{T-t}p_{x+t} + {}_{T-t}q_{x+t} = M_{Y_T}(\alpha), \tag{4.15}$$

and that both reservation prices equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} e^{-r(T-t)} \ln M_{Y_T}(\alpha). \tag{4.16}$$

Again, if the insurer is less risk averse than the buyer, we will have $P^I < P^B$. Compare the expression for η in equation (4.15) with the one for ϕ in equation (4.10). Note the parallel between the two equations in that p and q switch roles in the two equations, which reflects the fact that 1 unit is paid if the person dies under life insurance but lives under pure endowment insurance.

Also, note that the sum of the premiums for life insurance and for pure endowment insurance equals

$$e^{-r(T-t)}\left[1 + \frac{1}{\alpha}\ln\left(1 + \left(e^{\alpha/2} - e^{-\alpha/2}\right)^2 {}_{T-t}p_{x+t}\,{}_{T-t}q_{x+t}\right)\right],$$

which is greater than $e^{-r(T-t)}$, the present value at time t of 1 payable at time T, a direct consequence of the non-linear pricing mechanism induced by market incompleteness.

4.4. Losses modeled as a diffusion process

In this section, we allow losses to follow a diffusion process; that is, Y_t follows the process

$$\begin{cases} dY_s = \theta(Y_s, s)\, ds + \zeta(Y_s, s)\, d\tilde{B}_s, \\ Y_t = y \geqslant 0, \end{cases} \tag{4.17}$$

with $\tilde{B}_s$ a standard Brownian motion independent of B_s, the Brownian motion for the stock process (2.1). Assume that the *drift* and *volatility* coefficients $\theta(y, t)$ and $\zeta(y, t)$ satisfy the usual growth and Lipschitz conditions

$$|\theta(y, t) - \theta(x, t)| + |\zeta(y, t) - \zeta(x, t)| \leqslant K|y - x|,$$

$$|\theta(y, t)|^2 + |\zeta(y, t)|^2 \leqslant K(1 + y)^2,$$

for some positive constant K. These conditions guarantee that a unique solution to equation (4.17) exists (Gihman & Skorohod, 1972, Chapter 6).

A diffusion equation is often used to model insurance losses, especially from the point of view of the insurer. See, among others, Grandell (1990), Asmussen & Taksar (1997) and Højgaard & Taksar (1997).

The corresponding HJB equation for the value function $U(w, y, t)$ reduces

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \theta(y, t) U_y + \dfrac{1}{2} \zeta^2(y, t) U_{yy} = 0, \\ U(w, T, y) = u(w - y). \end{cases} \tag{4.18}$$

4.4 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Also, suppose θ and ζ are independent of y. Then, $U(w, y, t) = V(w, t)\, e^{\alpha y + \psi(t)}$, in which V is as in Example 2.4 and ψ solves the ordinary differential equation

$$\begin{cases} \psi'(t) + \alpha\theta(t) + \dfrac{1}{2} \alpha^2 \zeta^2(t) = 0, \\ \psi(T) = 0. \end{cases}$$

Thus, both reservation prices equal

$$P^I(w, y, t) = P^B(w, y, t) = y\, e^{-r(T-t)} + e^{-r(T-t)} \int_t^T \left(\theta(s) + \frac{1}{2} \alpha \zeta^2(s) \right) ds, \tag{4.19}$$

a term for the loss already incurred and a term for the discounted expected loss plus a loading proportional to the variance of the loss during the period $[t, T]$. That is, the exponential premium principle in this case is a variance premium principle.

4.5 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Also, suppose θ and ζ are given by

$$\theta(y, t) = (n - y)\lambda_x(t)$$

and

$$\zeta(y, t) = \sqrt{(n - y)\lambda_x(t)},$$

in which n is a fixed positive number, $Y_0 = 0$, and λ_x is the hazard function for a person aged x. One can think of this as a limiting case of the model presented in Section 4.2. Specifically, as $n - Y_t$ gets large, one can approximate the Binomial $(n - Y_{t,h} q_{x+t})$ by the

$$\begin{aligned} &\text{Normal}((n - Y_t)_h q_{x+t}, (n - Y_t)_h q_{x+t\,h} p_{x+t}) \\ &\quad = \text{Normal}\left((n - Y_t) \frac{_h q_{x+t}}{h} h, (n - Y_t) \frac{_h q_{x+t\,h} p_{x+t}}{h} h \right) \\ &\quad \approx \text{Normal}((n - Y_t)\lambda_x(t)h, (n - Y_t)\lambda_x(t)h). \end{aligned}$$

Suppose that $\lambda_x(t) \equiv \lambda$, a positive constant. After a fair amount of work, we obtain that

$$U(w, y, t; n) = V(w, t) \exp\left[\alpha \left(n - \frac{2(n - y)}{(2 + \alpha)\, e^{\lambda(T-t)} - \alpha} \right) \right].$$

It follows that both reservation prices equal

$$P^I(w, y, t) = P^B(w, y, t) = y\, e^{-r(T-t)} + (n-y)\, e^{-r(T-t)} \frac{(2+\alpha)(1-e^{-\lambda(T-t)})}{(2+\alpha)-\alpha\, e^{-\lambda(T-t)}},$$

similar in form to the expression given in (4.13) with $(1/\alpha)\ln[(1-e^{-\lambda(T-t)}) + e^{\alpha}\, e^{-\lambda(T-t)}]$ replaced by $(2+\alpha)(1-e^{-\lambda(T-t)})/[(2+\alpha)-\alpha\, e^{-\lambda(T-t)}]$. The two prices are nearly equal if $\alpha^3 \approx 0$ and $e^{-3\lambda(T-t)} \approx 0$.

4.6 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Also, suppose $\theta(y, t) = \theta y$ and $\zeta(y, t) = \zeta y$, for some constants θ and ζ, with $\zeta > 0$. Set $U(w, y, t) = V(w, t)\phi(y, t)$, with $\phi(y, T) = e^{\alpha y}$. Then, by substituting this form of U into (4.18), we learn that ϕ solves

$$\begin{cases} \phi_t + \theta y \phi_y + \dfrac{\zeta^2 y^2}{2}\, \phi_{yy} = 0, \\ \phi(y, T) = e^{\alpha y}. \end{cases}$$

By using standard arguments related to the Feynman-Kac representation (Karatzas & Shreve, 1991, Theorem 5.7.6), we can represent the solution of the above linear partial differential equation in terms of the expectation of the exponential of a diffusion process with generator

$$\mathscr{L} = \theta y \frac{\partial}{\partial y} + \frac{1}{2} \zeta^2 y^2 \frac{\partial^2}{\partial y^2}.$$

It then follows that

$$\phi(y, t) = E[\exp(k_1\, e^{k_2 Z})],$$

in which $k_1 = \alpha y\, e^{(\theta - (\zeta^2/2))(T-t)}$, $k_2 = \zeta\sqrt{T-t}$, and $Z \sim N(0, 1)$. Thus, both reservation prices equal

$$P^I(w, y, t) = P^B(w, y, t) = \frac{1}{\alpha}\, e^{-r(T-t)} E[\exp(k_1\, e^{k_2 Z})] = \frac{1}{\alpha}\, e^{-r(T-t)} M_X(k_1),$$

in which $X = e^{k_2 Z}$ is lognormally distributed and M_X is the moment generating function of X.

4.5. Losses modeled as a Poisson process

In this section, we allow losses to follow a Poisson process; that is, Y_t follows the process

$$\begin{cases} dY_s = L(W_s, Y_s, s)\, dN_s, \\ Y_t = y, \end{cases} \tag{4.20}$$

with N_s a non-homogeneous Poisson process with deterministic parameter $\phi(s)$. We assume that N_s is independent of B_s, the Brownian motion for the stock process (2.1). Also, L is the (random) loss amount at time s, independent of N_s. Note that we allow the loss to depend on the wealth at time s and the losses to date.

The corresponding HJB equation for the value function $U(w, y, t)$ reduces to

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + \phi(t)[EU(w, y + L(w, y, t), t) - U(w, y, t)] = 0, \\ U(w, y, T) = u(w - y). \end{cases} \tag{4.21}$$

See Merton (1992, Section 5.8).

4.7 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient, and suppose that the loss $L(w, y, t)$ is independent of w and y. Set $U(w, y, t) = V(w, t)\, e^{\alpha y}\xi(t)$ with V given in Example 2.4. Then, ξ solves

$$\begin{cases} \xi' + \phi(t)[M_{L(t)}(\alpha) - 1]\xi = 0, \\ \xi(T) = 1. \end{cases}$$

Thus,

$$\xi(t) = \exp\left[\int_t^T \phi(s)[M_{L(s)}(\alpha) - 1]\, ds\right],$$

and both reservation prices equal

$$P^I(w, y, t) = P^B(w, y, t) = y\, e^{-r(T-t)} + \frac{1}{\alpha}\, e^{-r(T-t)} \int_t^T \phi(s)[M_{L(s)}(\alpha) - 1]\, ds. \tag{4.22}$$

There is an interesting relationship between the premia given by equations (4.19) and (4.22) when the expected losses and the variances of the loss during $[t, T]$ are equal. In that case, the Poisson premium in (4.22) is greater than the diffusion premium in (4.19). Indeed, if the expected losses are equal, then

$$\int_t^T \theta(s)\, ds = \int_t^T \phi(s)E[L(s)]\, ds,$$

and if the variances of the loss are equal, then

$$\int_t^T \zeta^2(s)\, ds = \int_t^T \phi(s)E[L^2(s)]\, ds.$$

Thus, the premium in (4.19) becomes

$$y\,e^{-r(T-t)} + e^{-r(T-t)} \int_t^T \left(\theta(s) + \frac{\alpha}{2}\zeta^2(s)\right) ds$$

$$= y\,e^{-r(T-t)} + e^{-r(T-t)} \int_t^T \phi(s)\left(EL(s) + \frac{\alpha}{2}E[L^2(s)]\right) ds$$

$$< y\,e^{-r(T-t)} + e^{-r(T-t)} \int_t^T \phi(s)(M_{L(s)}(\alpha) - 1)\, ds,$$

where the latter is the premium in (4.22). This inequality between the premiums is what one expects because the Poisson process is a jump process and the diffusion process is a continuous one. The latter is, thereby, less risky, and the reservation prices reflect that.

5. INSURANCE PAYABLE AT INCURRENCE—MAXIMIZING EXPECTED UTILITY AT TIME *T*

We assume, as before, that the agent (whether buyer or seller of insurance) seeks to maximize expected utility of wealth at time T. Unlike in Section 4, the insurance now is payable when the loss is incurred. In Sections 5.1 through 5.6, we parallel Sections 4.1 through 4.5.

5.1. Single insured life—term life insurance

Consider again the problem from Section 4.1, but this time the insurance will be paid at the time of death of (x) if (x) dies before time T. We still seek to maximize the expected utility of terminal wealth at time T. By following reasoning similar to that in Section 4.1, the HJB equation for U reduces to

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[V(w-1, t) - U(w, t)] = 0, \\ U(w, T) = u(w). \end{cases} \tag{5.1}$$

Note that this equation is essentially the same one given in (4.9) with $V(w - e^{-r(T-t)}, t)$ replaced by $V(w-1, t)$ because the insurance is payable at the moment of death of (x).

5.1 EXAMPLE. As in Example 4.1, we can derive the price for exponential utility to be

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} e^{-r(T-t)} \ln \psi(t),$$

in which $\psi(t)$ is given by

$$\psi(t) = e^{-\int_t^T \lambda_x(s)\,ds} + \int_t^T \lambda_x(s)\, e^{\alpha e^{r(T-s)} - \int_t^s \lambda_x(u)\,du}\, ds$$

$$= {}_{T-t}p_{x+t} + \int_t^T e^{\alpha e^{r(T-s)}} \lambda_x(s)\, {}_{s-t}p_{x+t}\, ds.$$

As before, the price is independent of the risky asset, and it equals the price we get if we restrict investing only to the riskless bond. The price is also the same whether we are calculating the reservation price of the insurer or of the buyer of insurance, unless the insurer is less risk averse than the buyer. Both of these phenomena occur because we are using exponential utility. Also, note that this price is greater than the one from Example 4.1 because the insurance benefit here is payable when the insured dies, instead of at time T.

5.2. Group of insured lives—term life insurance

Consider again the problem from Section 4.2, but this time the insurance will be paid at the time of death of (x) if (x) dies before time T. We still seek to maximize the expected utility of terminal wealth at time T. By following reasoning similar to that in Section 5.1, one can show that the value function $U(w, y, t; n)$ given by (3.1) solves the following recursive HJB:

$$\begin{cases} U(w, n, t; n) = V(w, t); \\ \text{For} \quad y = 0, 1, \ldots, n-1, \\ U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + (n-y)\lambda_x(t)[U(w-1, y+1, t; n) - U(w, y, t; n)] = 0, \\ U(w, y, T; n) = u(w). \end{cases} \tag{5.2}$$

Note that this equation is parallel to the one given in (4.12) with changes because the insurance benefit is payable at the moment of death.

5.2 EXAMPLE. As in Example 4.2, we can derive the price for exponential utility to be

$$P^I(w, y, t; n) = P^B(w, y, t; n) = \frac{n-y}{\alpha} e^{-r(T-t)} \ln \psi(t),$$

in which $\psi(t)$ is given in Example 5.1. Note that the premium is simply $(n-y)$ times the premium for term insurance on a single life, as we saw in Example 4.2 for term insurance payable at time T. There is no provision for the y lives who have already died because those benefits have been paid.

5.3. Insurance on a single life—temporary life annuity immediate

In this section, we build on the work in Section 4.3 to find the reservation prices for a temporary life annuity that pays 1 unit to (x) at the end of each period for T periods as long as (x) is alive. We assume that T is a positive integer. For $t \in (T-n, T-n+1]$, write $U(w, t; n)$ for U. Then, the HJB equation for the value function U reduces to

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[V(w, t) - U(w, t)] = 0, \\ U(w, T-n; n+1) = U(w-1, T-n; n), \quad \text{for} \quad n = 1, 2, \ldots, T-1, \\ U(w, T; 1) = u(w-1). \end{cases} \tag{5.3}$$

5.3 EXAMPLE. Suppose $u(w) = (1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Then, by using the work in Example 4.3 and induction on n, we obtain that the reservation prices at time 0 equal

$$P^I(w, 0) = P^B(w, 0) = \frac{1}{\alpha} e^{-rT} \ln[q_x + e^{\alpha\, e^{(T-1)r}} p_x q_{x+1} + e^{\alpha(e^{(T-1)r} + e^{(T-2)r})} {}_2p_x q_{x+2} + \cdots + e^{\alpha(e^{(T-1)r} + \cdots + 1)} {}_Tp_x].$$

5.4. Insurance on a single life—temporary continuous life annuity

In this section, we assume that the temporary life annuity is payable continuously at a rate of 1 unit per period for as long as (x) is alive or until time T expires. In this case, we incorporate the loss of 1 unit per period into the wealth equation so that wealth W_s follows

$$\begin{cases} dW_s = [rW_s + (\mu - r)\pi_s - 1]\, ds + \sigma \pi_s\, dB_s, \\ W_t = w. \end{cases}$$

The corresponding HJB equation for U reduces to

$$\begin{cases} U_t + (rw - 1)U_w - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[V(w, t) - U(w, t)] = 0, \\ U(w, T) = u(w). \end{cases} \tag{5.4}$$

5.4 EXAMPLE. Suppose $u(w) = (1/\alpha)\, e^{-\alpha w}$, in which $\alpha > 0$ is the risk aversion coefficient. Let $U(w, t) = V(w, t)\eta(t)$, in which $\eta(T) = 1$. Then, η solves the following first-order differential equation

$$\eta' + [\lambda_x(t) - \alpha\, e^{r(T-t)}]\eta(t) - \lambda_x(t) = 0,$$

so η is given by

$$\begin{aligned} \eta(t) &= \int_t^T \lambda_x(s) \exp\left(\int_t^s [\alpha\, e^{r(T-u)} - \lambda_x(u)]\, du\right) ds + \exp\left(\int_t^T [\alpha\, e^{r(T-s)} - \lambda_x(s)]\, ds\right) \\ &= \int_t^T e^{\int_t^s \alpha\, e^{r(T-u)}\, du} \lambda_x(s)\, {}_{s-t}p_{x+t}\, ds + e^{\int_t^T \alpha\, e^{r(T-s)}\, ds}\, {}_{T-t}p_{x+t}. \end{aligned} \tag{5.5}$$

It follows that the reservation prices equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} e^{-r(T-t)} \ln \eta(t),$$

in which η is given in equation (5.5).

5.5. Losses modeled as a diffusion process

Since we assume that the insurance claims are payable at the time of loss, it makes sense to incorporate those losses into the wealth equation, if possible, as in Section

5.4. Thus, U is given by

$$U(w, y, t) = \sup_{\{\pi_t\}} E[u(W_T) \mid W_t = w,\ Y_t = y], \tag{5.6}$$

in which W_s follows

$$\begin{cases} dW_s = [rW_s + (\mu - r)\pi_s]\, ds + \sigma\pi_s\, dB_s - dY_s, \\ W_t = w, \end{cases} \tag{5.7a}$$

and

$$\begin{cases} dY_s = \theta(W_s, Y_s, s)\, ds + \zeta(W_s, Y_s, s)\, d\tilde{B}_s, \\ Y_t = y, \end{cases} \tag{5.7b}$$

where $\tilde{B}_s$ is a standard Brownian motion, independent of the Brownian motion B_s for the stock process (2.1). Note that we allow θ and ζ to depend on the current wealth W_s, in addition to the loss Y_s.

The corresponding HJB equation for U is

$$\begin{cases} U_t + \max_\pi \left[(\mu - r)\pi U_w + \frac{1}{2}\sigma^2\pi^2 U_{ww} \right] + (rw - \theta(w, y, t))U_w + \theta(w, y, t)U_y \\ \quad + \frac{1}{2}\zeta^2(w, y, t)U_{ww} - \zeta^2(w, y, t)U_{wy} + \frac{1}{2}\zeta^2(w, y, t)U_{yy} = 0, \\ U(w, y, T) = u(w), \end{cases} \tag{5.8}$$

which reduces to

$$\begin{cases} U_t + (rw - \theta(w, y, t))U_w + \theta(w, y, t)U_y - \frac{(\mu - r)^2}{2\sigma^2}\frac{U_w^2}{U_{ww}} \\ \quad + \frac{\zeta_1^2}{2}\zeta^2(w, y, t)U_{ww} - \zeta^2(w, y, t)U_{wy} + \frac{1}{2}\zeta^2(w, y, t)U_{yy} = 0, \\ U(w, y, T) = u(w). \end{cases}$$

5.5 EXAMPLE. Assume the same set up as in Example 4.4; that is, $u(w) = -(1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$, and θ and ζ are independent of w and y. Then, U is independent of y and $U(w, t) = V(w, t)\, e^{\psi(t)}$, in which V is as in Example 2.4. The function ψ solves the ordinary differential equation

$$\begin{cases} \psi'(t) + \alpha\, e^{r(T-t)}\theta(t) + \frac{\alpha^2\, e^{2r(T-t)}}{2}\zeta^2(t) = 0, \\ \psi(T) = 0. \end{cases}$$

Thus, both reservation prices equal

$$P^I(w, t) = P^B(w, t) = e^{-r(T-t)} \int_t^T \left(e^{r(T-s)} \theta(s) + \frac{\alpha\, e^{2r(T-s)}}{2} \zeta^2(s) \right) ds,$$

the discounted expected loss plus a loading proportional to the variance of the loss during the period $[t, T]$. That is, the exponential premium principle in this case is a variance premium principle, as in Example 4.4. Also, this premium is greater than the one in Example 4.4 because insurance is payable at incurrence rather than at the end of the period.

5.6. *Losses modeled as a Poisson process*

As in the previous two sections, because the insurance claims are payable at the time of loss, we incorporate those losses into the wealth equation. The wealth W_s follows

$$\begin{cases} dW_s = [rW_s + (\mu - r)\pi_s]\, ds + \sigma \pi_s\, dB_s - L(W_s, s)\, dN_s, \\ W_t = w, \end{cases} \tag{5.9}$$

in which N_s is a Poisson process with deterministic parameter $\phi(s)$. We assume that N_s is independent of the Brownian motion B_s. Also, L is the (random) loss amount at time s, independent of N_s.

Then, we define the value function by $U(w, t) = \sup_{\{\pi_t\}} E[u(W_T) \mid W_t = w]$, and U's HJB equation reduces to

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \phi(t)\, [EU(w - L(w, t), t) - U(w, t)] = 0, \\ U(w, T) = u(w). \end{cases} \tag{5.10}$$

Compare this equation with equation (4.21).

5.6 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$, and suppose that the loss $L(w, t)$ is independent of w, similar to Example 4.7. Set $U(w, t) = V(w, t)\psi(t)$ with V given in Example 2.4; then, ψ solves

$$\begin{cases} \psi' + \phi(t)[E(\exp\{\alpha L(t)\, e^{r(T-t)}\}) - 1]\psi = 0, \\ \psi(T) = 1. \end{cases}$$

Thus,

$$\psi(t) = \exp\left[\int_t^T \phi(s)[M_{L(s)}(\alpha\, e^{r(T-s)}) - 1]\, ds \right],$$

and both reservation prices equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} e^{-r(T-t)} \int_t^T \phi(s)[M_{L(s)}(\alpha\, e^{r(T-s)}) - 1]\, ds.$$

Note that this premium is greater than the one in Example 4.7 because insurance is payable at incurrence rather than at the end of the period. As at the end of Section 4.5, one can show that when the expected losses and the variances of the loss during $[t, T]$ are equal, the Poisson premium in Example 5.6 is greater than the diffusion premium in Example 5.5.

6. INSURANCE PAYABLE AT INCURRENCE—MAXIMIZING EXPECTED UTILITY AT A RANDOM TIME τ

Throughout Section 6, we assume that decision maker seeks to maximize expected utility of wealth at a random time. For example, if the decision maker is the buyer of insurance, then the random time may be the buyer's time of death τ. This is the point of view we take throughout this section. In Sections 6.1 through 6.4, we parallel Sections 5.1 and 5.4 through 5.6, respectively.

In this case, the value function V without the insurance risk is

$$V(w, t) = \sup_{\{\pi_t\} \in \mathcal{A}} E[u(W_\tau) \mid W_t = w], \tag{6.1}$$

in which the wealth W_s follows the process in (2.2). The HJB equation for V reduces to

$$V_t + rwV_w - \frac{(\mu - r)^2}{2\sigma^2} \frac{V_w^2}{V_{ww}} + \lambda_x(t)[u(w) - V(w, t)] = 0. \tag{6.2a}$$

The boundary condition becomes

$$\lim_{t \to \infty} E\left[e^{-\int_0^t \lambda_x(s)\, ds} V(W_t^*, t)\right] = 0; \tag{6.2b}$$

see Merton (1992, Section 4.6).

6.1 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$, and suppose that $r = 0$. As a trial solution, set $V(w, t) = u(w)\psi(t)$ with the natural transversality condition $\lim_{t \to \infty} e^{-\int_0^t \lambda_x(s)\, ds} \psi(t) < \infty$; the latter follows from the fact that $\lim_{t \to \infty} E[u(W_t^*)] = 0$. The function ψ solves the differential equation

$$\psi'(t) - \left[\lambda_x(t) + \frac{\mu^2}{2\sigma^2}\right]\psi(t) + \lambda_x(t) = 0.$$

It is straightforward to show that

$$\psi(t) = \int_t^\infty e^{-\delta(s-t)} \lambda_x(s)\, {}_{s-t}p_{x+t}\, ds = \bar{A}_{x+t}^{\delta},$$

the net single premium for a whole life insurance of 1 issued to $(x + t)$, payable at moment of death, evaluated at the force of interest $\delta = \mu^2/2\sigma^2$.

6.1. Single insured life—whole life insurance

Paralleling Section 5.1, we consider 1 unit payable to (x) when (x) dies. However, in this section, the insurance is not term but rather whole life. In this case, the value function U with the insurance risk is

$$U(w, t) = \sup_{\{\pi_t\} \in \mathscr{A}} E[u(W_\tau - 1) \mid W_t = w],$$

in which the wealth W_s follows the equation given by (2.2). The HJB equation for U reduces to

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[u(w-1) - U(w, t)] = 0, \\ \lim\limits_{t \to \infty} E\left[e^{-\int_0^t \lambda_x(s)\, ds} U(W_t^*, t) \right] = 0. \end{cases} \tag{6.3}$$

6.2 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$, and suppose that $r = 0$. Then, $U(w, t) = V(w, t)\, e^{\alpha}$, in which V is given in Example 6.1. It follows that the reservation prices both equal

$$P^I(w, t) = P^B(w, t) = 1.$$

It is interesting that the premium is independent of the risk aversion coefficient α.

6.2. Single insured life—continuous life annuity

We parallel Section 5.4 by considering a continuous life annuity payable until (x) dies. We maximize expected utility at the random time of death of (x), τ. The HJB equation for U reduces to

$$\begin{cases} U_t + (rw - 1)U_w - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \lambda_x(t)[u(w) - U(w, t)] = 0, \\ \lim\limits_{t \to \infty} E\left[e^{-\int_0^t \lambda_x(s)\, ds} U(W_t^*, t) \right] = 0. \end{cases} \tag{6.4}$$

Compare this equation with the one given in (5.4).

6.3 EXAMPLE. Suppose $u(w) = -(1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$, and suppose that $r = 0$. Then, set $U(w, t) = u(w)\eta(t)$. The function η solves the differential equation

$$0 = \eta'(t) - [\lambda_x(t) + \delta - \alpha]\eta(t) + \lambda_x(t),$$

in which $\delta = \mu^2/2\sigma^2$, as in Example 6.1, so that η equals

$$\eta(t) = \int_t^\infty e^{-(\delta - \alpha)(s - t)} \lambda_x(s)\, {}_{s-t}p_{x+t}\, ds = \bar{A}_{x+t}^{\delta - \alpha}.$$

The reservation prices equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} \ln\left(\frac{\bar{A}^{\delta - \alpha}_{x+t}}{\bar{A}^{\delta}_{x+t}}\right).$$

Note that the reservation prices depend on the parameters of the stock process through δ, unlike the examples in Sections 4 and 5. To understand why this occurs, consider the expansion of this premium up to second-order moments:

$$P(w, t) \approx E(\tau_{x+t}) + \frac{\alpha - 2\delta}{2} Var(\tau_{x+t}),$$

in which τ_{x+t} is the time of death of $(x+t)$, where time is measured from age x. The parameter δ affects the price only through the higher-order terms. Thus, as the horizon becomes less random, that is, as the variance of τ_{x+t} decreases, then the premium approaches the present value of the annuity at the risk-free rate, $r = 0$.

If we compare this premium after setting $\delta = 0$ with the one in Example 5.4 after setting $r = 0$ and $T - t = \mathring{e}_{x+t}$, the expected future years lived of $(x+t)$, then the latter is less than the former. Indeed,

$$P_T = \frac{1}{\alpha} \ln(\bar{A}^{-\alpha}_{x+t:\overline{\mathring{e}_{x+t}}|}) < \frac{1}{\alpha} \ln(\bar{A}^{-\alpha}_{x+t}) = P_\tau,$$

because the "force of interest," $-\alpha$, is negative. One expects the premium to be higher in the case of the random horizon because of the greater uncertainty in that case.

6.3. Losses modeled as a diffusion process

Paralleling Section 5.5, we incorporate the losses into the wealth equation, but we assume that the expected utility is maximized at the random time of death τ. Thus, U is given by

$$U(w, y, t) = \sup_{\{\pi_t\} \in \mathscr{A}} E[u(W_\tau) | W_t = w,\ Y_t = y], \tag{6.5}$$

in which W_s and Y_s follow the processes given in (5.7a) and (5.7b), respectively.
The corresponding HJB equation for U reduces to

$$\begin{cases} U_t + (rw - \theta(w, y, t))U_w + \theta(w, y, t)U_y - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \dfrac{1}{2}\zeta^2(w, y, t)U_{ww} \\ \quad - \zeta^2(w, y, t)U_{wy} + \dfrac{1}{2}\zeta^2(w, y, t)U_{yy} + \lambda_x(t)[u(w) - U(w, y, t)] = 0, \\ \lim_{t \to \infty} E\left[e^{-\int_0^t \lambda_x(s)\, ds} U(W_t^*, t)\right] = 0. \end{cases} \tag{6.6}$$

6.4 EXAMPLE. Suppose $u(w) = -(1/\alpha)e^{-\alpha w}$, for some $\alpha > 0$, and suppose that $r = 0$, $\theta(w, y, t) = \theta(t)$, and $\zeta\ (w, y, t) = \zeta(t)$. Then, by using a similar calculation as in Examples 6.1 through 6.3,

$$U(w, t) = u(w) \int_t^\infty e^{-\int_t^s (\delta - \alpha\theta(u) - \alpha^2\zeta^2(u)/2)\,du} \lambda_x(s)\,{}_{s-t}p_{x+t}\,ds,$$

in which $\delta = \mu^2/2\sigma^2$. It follows that the reservation prices equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} \ln\left[\frac{\int_t^\infty e^{-\int_t^s (\delta - \alpha\theta(u) - \alpha^2\zeta^2(u)/2)\,du} \lambda_x(s)\,{}_{s-t}p_{x+t}\,ds}{\bar{A}^{\delta}_{x+t}}\right].$$

Note that the reservation prices depend on the parameters of the stock process through δ, unlike the examples in Sections 4 and 5. To understand why this occurs, consider the expansion of this premium up to second-order moments:

$$P(w, t) \approx E\left(\int_t^{\tau_{x+t}} \left[\theta(s) + \frac{\alpha}{2}\zeta^2(s)\right] ds\right) + \frac{\alpha}{2} Var\left(\int_t^{\tau_{x+t}} \theta(s)\,ds\right)$$
$$- \delta\, Cov\left(\tau_{x+t}, \int_t^{\tau_{x+t}} \theta(s)\,ds\right),$$

in which τ_{x+t} is the time of death of $(x+t)$, where time is measured from age x. The parameter δ affects the price only through the higher-order terms. Thus, as the horizon becomes less random, that is, as the variance of τ_{x+t} decreases, then the premium approaches the premium in Example 5.5 with $r = 0$ and $T = E(\tau_{x+t})$.

If we compare this premium after setting $\delta = 0$ with the one in Example 5.5 after setting $r = 0$ and $T - t = \mathring{e}_{x+t}$, the expected future years lived of $(x+t)$, then the latter is less than the former. Indeed,

$$P_T = \int_t^{\mathring{e}_{x+t}+t} \left(\theta(s) + \frac{\alpha\zeta^2(s)}{2}\right) ds < \frac{1}{\alpha} \ln\left(E\left[e^{\int_t^\tau (\alpha\theta(u) + \alpha^2\zeta^2(u)/2)\,du}\right]\right) = P_\tau,$$

by Jensen's inequality because $g(x) = e^x$ is convex. One expects the premium to be higher in the case of the random horizon because of the greater uncertainty in that case.

6.4. Losses modeled as a Poisson process

Paralleling Section 5.6, we assume that wealth W_s follows (5.9), but U is given by

$$U(w, t) = \sup_{\{\pi_t\} \in \mathcal{A}} E[u(W_\tau) \mid W_t = w].$$

In this case, the HJB equation U reduces

$$\begin{cases} U_t + rwU_w - \dfrac{(\mu - r)^2}{2\sigma^2} \dfrac{U_w^2}{U_{ww}} + \phi(t)[EU(w - L(w, t), t) - U(w, t)] \\ \quad + \lambda_x(t)[u(w) - U(w, t)] = 0, \\ \lim_{t\to\infty} E\left[e^{-\int_0^t \lambda_x(s)\,ds} U(W_t^*, t)\right] = 0. \end{cases} \tag{6.7}$$

6.5 EXAMPLE. Suppose $u(w) = (1/\alpha)\, e^{-\alpha w}$, for some $\alpha > 0$, and suppose that $r = 0$ and $L(w, t) = L(t)$. Then, as in the previous few examples,

$$U(w, t) = u(w) \int_t^{\infty} e^{-\int_t^s (\delta - \phi(u)[M_{L(u)}(\alpha) - 1])\, du} \lambda_x(s)_{s-t}p_{x+t}\, ds,$$

in which $\delta = \mu^2/2\sigma^2$. It follows that the reservation prices equal

$$P^I(w, t) = P^B(w, t) = \frac{1}{\alpha} \ln\left[\frac{\int_t^{\infty} e^{-\int_t^s (\delta - \phi(u)[M_{L(u)}(\alpha) - 1])\, du} \lambda_x(s)_{s-t}p_{x+t}\, ds}{\bar{A}^{\delta}_{x+t}}\right].$$

If the expected loss in the diffusion case, Example 6.4, equals the expected loss here, as well as the variances of the loss, then the Poisson premium is greater than the diffusion premium, as before. Note that the prices depend on the parameters of the stock process, unlike the examples in Sections 4 and 5. To understand why this occurs, consider the expansion of this premium up to second-order moments:

$$P(w, t) \approx \frac{1}{\alpha} E\left(\int_t^{\tau_{x+t}} \phi(s)[M_{L(s)}(\alpha) - 1]\, ds\right) + \frac{1}{2\alpha} Var\left(\int_t^{\tau_{x+t}} \phi(s)[M_{L(s)}(\alpha) - 1]\, ds\right)$$
$$- \frac{\delta}{\alpha} Cov\left(\tau_{x+t}, \int_t^{\tau_{x+t}} \phi(s)[M_{L(s)}(\alpha) - 1]\, ds\right),$$

in which τ_{x+t} is the time of death of $(x + t)$, where time is measured from age x. The parameter δ affects the price only through the higher-order terms. Thus, as the horizon becomes less random, that is, as the variance of τ_{x+t} decreases, then the premium approaches the premium in Example 5.6 with $r = 0$ and $T = E(\tau_{x+t})$.

If we compare this premium after setting $\delta = 0$ with the one in Example 5.6 after setting $r = 0$ and $T - t = \mathring{e}_{x+t}$, the expected future years lived of $(x + t)$, then the latter is less than the former. Indeed,

$$P_T = \frac{1}{\alpha} \int_t^{\mathring{e}_{x+t}+t} \phi(u)(M_{L(u)}(\alpha) - 1)\, ds < \frac{1}{\alpha} \ln\left(E\left[e^{\int_t^{\tau} \phi(u)(M_{L(u)}(\alpha) - 1)\, du}\right]\right) = P_{\tau},$$

by Jensen's inequality because $g(x) = e^x$ is convex. One expects the premium to be higher in the case of the random horizon because of the greater uncertainty in that case.

7. SUMMARY AND FUTURE RESEARCH

We showed how one can apply the principle of equivalent utility in a case for which one invests in a risky asset, as well as a riskless asset. Generally, actuaries have applied the principle of equivalent utility by assuming one invests in a riskless asset.

This static-ness masks the stochastic nature of the assets and liabilities, while our method allows for pricing risks in a dynamic framework in which assets and liabilities are represented by stochastic processes.

In the examples using exponential utility, we find in Sections 4 and 5 that the prices are independent of the risky stock process. Thus, the prices are identical to the ones obtained when investment is limited to the riskless bond. However, in Section 6, we learn the interesting fact that when the horizon is random, the prices depend on the parameters of the risky stock process, but only through second- and higher-order terms. Also, our partial differential equations allow any smooth (increasing and concave) utility function; therefore, one can apply our method to other utility functions, such as power or logarithmic utility. In those cases, the prices depend on the risky stock process, in contrast with the examples in Sections 4 and 5.

An intuitively pleasing result that we obtained repeatedly in our examples is that if the losses are modeled as a Poisson process, then the reservation prices are higher than when losses are modeled as a diffusion process with the same expected loss and same variance of loss. One expects this result because of the jump nature of a Poisson process versus the continuous nature of a diffusion process.

Throughout our examples, we found that the reservation prices of the insurer and of the buyer of insurance are equal, an artifact of using exponential utility. If the absolute risk aversion of the insurer is less than the one of the buyer, then we will have $P^I < P^B$. This gives the insurance market a spread in which to trade. On the other hand, if one uses this approach to price derivatives in markets with imperfections, such as transaction costs (Hodges & Neuberger, 1989), one gets a spread under exponential utility because both the seller and buyer of the derivative contract have simultaneous payoffs from the contract that are equal in absolute value but with opposite signs. In contrast, in our insurance model, either the insurer or the insured bear all the risk. This results in the absence of any spreads under exponential utility if the insurer and insured have the same absolute risk aversion.

In future work, we plan to extend our model by considering consumption during the interval $[0, T]$ or during the random lifetime $[0, \tau]$, in addition to wealth at the end of the period. Such consumption could represent "ordinary" consumption if we take an individual's viewpoint. If we take the viewpoint of an insurer, then consumption could represent dividends paid to shareholders or salaries paid to employees. Also, in future work we will calculate reservation prices for more exotic insurance products, such as annuities with options tied to a financial market.

We plan to consider the case in which premium is paid continuously during the interval $[0, T]$ or $[0, \tau]$. We will also allow for independent income sources, say, from a job if we take the buyer's point of view. Finally, we will consider stock price processes more general than geometric Brownian motion, as in Zariphopoulou (1999a).

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