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An example of indifference prices under exponential preferences

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Abstract. The aim herein is to analyze utility-based prices and hedging strategies. The analysis is based on an explicitly solved example of a European claim written on a nontraded asset, in a model where risk preferences are exponential, and the traded and nontraded asset are diffusion processes with, respectively, lognormal and arbitrary dynamics. Our results show that a nonlinear pricing rule emerges with certainty equivalent characteristics, yielding the price as a nonlinear expectation of the derivative's payoff under the appropriate pricing measure. The latter is a martingale measure that minimizes its relative to the historical measure entropy.

Key words: Incomplete markets, indifference prices, nonlinear asset pricing

JEL Classification: C61, G11, G13

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1 Introduction

The purpose of this paper is to provide new insights and ideas for pricing and hedging in incomplete markets. Incompleteness is generated by nontraded assets and the underlying problem is how to price and hedge derivatives that are written on such securities. The level of the nontraded assets can be fully observed across time but it is not feasible to create a perfectly replicating portfolio. Therefore, the market

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is incomplete and alternatives to the arbitrage pricing must be developed in order to specify the appropriate price concept and to define the related risk management.

A popular by now pricing methodology is based on utility maximization criteria which produce the so called *indifference prices* via the related optimal investment opportunities with and without the derivative at hand. The underlying idea aims at incorporating an investor's attitude towards the risks that cannot be eliminated.

So far, indifference prices have been studied either through the underlying (primary) expected utility problems or through their dual counterparts (see, respectively, Davis et al. 1993 and Rouge and El Karoui 2000, and other references listed in the bibliography). The first approach relies on the theories of stochastic control and nonlinear partial differential equations. The duality approach concentrates on certain measures and entropic criteria arising in relevant reduced optimization problems. In Markovian settings, one may readily work across the two methodologies and produce equivalent results.

So far, prices have been typically represented as solutions to simpler optimization problems or to quasilinear pdes. However, despite the existing volume of work in this direction, the available price formulae often appear as mere technical outputs with no intuitive value. Indeed, no price formula enticing the elegant and, at the same time, simple representation of the price as expectation, under the risk neutral measure, of the derivative's payoff has been produced.

The goal herein is not to reinvent techniques for the solution of the underlying problems but, rather, using an explicit example, to expose some fundamental ingredients and intuitive elements of the indifference valuation theory. We consider a market environment with lognormal dynamics for the stock and general diffusion dynamics for the (correlated) nontraded asset. We establish that the indifference price of a European claim, written exclusively on the nontraded asset, is given as a nonlinear functional of the payoff represented solely in terms of the risk aversion, the correlation and the pricing measure. The nonlinearities of the pricing functional resemble the ones appearing in traditional static certainty equivalent valuation rules. However, it is interesting to note that we do not encounter a naive extension of this pricing device but rather a conditional dynamic analogue of it. The pricing measure is independent of risk preferences and, among all martingale measures, it has the minimal, with respect to the historical measure, entropy.

2 The indifference price

We assume a dynamic market environment with two risky assets, namely a stock that can be traded and a nontraded asset on which a European claim is written. We model the assets as diffusion processes denoted by S and Y, respectively.

The stock price is a lognormal diffusion satisfying

$$dS_s = \mu S_s ds + \sigma S_s dW_s^1, \quad t \le s, \tag{1}$$

with $S_t = S > 0$. The level of the nontraded asset is given by

$$dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s, \quad t \le s, \tag{2}$$

with $Y_t = y \in R$.

The processes W_s^1 and W_s are standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$,where $\mathcal{F}_s$ is the augmented σ-algebra generated by $\left(W_u^1, W_u,\ 0 \le u \le s\right)$. The Brownian motions are correlated with correlation $\rho \in (-1, 1)$. Assumptions on the drift and diffusion coefficients $b(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are such that the above equation has a unique strong solution.

We also assume that a riskless bond $B = 1$ with maturity T is available for trading, yielding constant interest rate $r = 0$. The case $r \neq 0$ can be treated using standard arguments. The derivative to be priced is of European type with the payoff at T of the form $G = g\left(Y_T\right)$, where the function g is bounded.

The valuation method used herein is based on the comparison of maximal expected utilities corresponding to investment opportunities with and without involving the derivative. In both situations, trading occurs in the time horizon $[t, T]$, $0 \le t \le T$, and only between the two traded assets, i.e., the riskless bond B and the risky asset S. The investor starts, at time t, with initial endowment x and rebalances his portfolio holdings by dynamically choosing the investment allocations, say π_s^0 and π_s, $t \le s \le T$, in the bond and the risky asset, respectively. It is assumed throughout that no intermediate consumption nor infusion of exogenous funds are allowed. The current wealth $X_s = \pi_s^0 + \pi_s$ satisfies the controlled diffusion equation

$$dX_s = \mu \pi_s ds + \sigma \pi_s dW_s^1, \quad t \le s \le T, \tag{3}$$

with $X_t = x \in R$ (see, for example, Merton 1969). It is worth noticing that the specific model assumptions enable us to work with a single state X_s and control variable π_s, respectively. The latter is deemed admissible if it is $\mathcal{F}_s$-progressively measurable and satisfies the integrability condition $E \int_t^T \pi_s^2 ds < \infty$. The set of admissible controls, also referred to as policies, is denoted by $\mathcal{Z}$.

The individual risk preferences are modelled via an exponential utility function

$$U(x) = -e^{-\gamma x}, \ \ \gamma > 0\ . \tag{4}$$

Next, we consider two expected utility problems via which the indifference price of the writer will be defined. The first problem arises in the classical Merton model of optimal investment, namely

$$V\left(x, t\right) = \sup_{\mathcal{Z}} E\left(-e^{-\gamma X_T} \left| X_t = x\right.\right). \tag{5}$$

The investor seeks to maximize the expected utility of terminal wealth without taking into account the claim G. The second problem corresponds to the situation in which the derivative G is written at time t and no trading of the asset Y is allowed in the horizon $[t, T]$. The writer's maximal expected utility (value function) of terminal wealth, denoted by u^w, is

$$u^w\left(x, y, t\right) = \sup_{\mathcal{Z}} E\left(-e^{-\gamma\left(X_T - G\right)} \left| X_t = x, Y_t = y\right.\right). \tag{6}$$

Due to the choice of exponential preferences and the absence of trading constraints, one may directly define the buyer's value function and proceed with intuitively clear parity relations. For this, we do not consider the buyer's price.

Definition 1 *(cf., Hodges and Neuberger 1989). The indifference writer's price of the European claim $G = g(Y_T)$, is defined as the function $h^w \equiv h^w(x, y, t)$, such that the investor is indifferent towards the following two scenarios: optimize the expected utility without employing the derivative and optimize it taking into account, on one hand, the liability $G = g(Y_T)$ at expiration T, and on the other, the compensation $h^w(x, y, t)$ at time of inscription t. Therefore,*

$$V(x, t) = u^w(x + h^w(x, y, t), y, t) \tag{7}$$

with V and u^w defined in (5) and (6), respectively.

To ease the presentation we skip the $w-$notation.

In what follows, we construct the writer's indifference price by first calculating the value functions (5) and (6) and, subsequently, using the pricing condition (7). To this end, we recall (see Merton 1969) that

$$V(x,t) = -e^{-(\gamma x + \frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t))}. \tag{8}$$

We next compute the writer's value function u. The arguments follow closely the ones in Zariphopoulou (2001b) and we refer the reader to the latter paper for the rigorous arguments and verification results.

Theorem 2 *The writer's value function u is given by*

$$u(x,y,t) = -e^{-\gamma x}\left(E_{\mathbb{Q}}\left(e^{\gamma(1-\rho^2)g(Y_T) - \frac{1}{2}(1-\rho^2)\frac{\mu^2}{\sigma^2}(T-t)}\,|Y_t = y\right)\right)^{\frac{1}{1-\rho^2}}, \tag{9}$$

for $(x,y,t) \in R \times R \times [0,T]$, with

$$\mathbb{Q}(A) = E_{\mathbb{P}}\left(\exp\left(-\frac{\mu}{\sigma}W_T^1 - \frac{1}{2}\frac{\mu^2}{\sigma^2}T\right)I_A\right), \; A \in \mathcal{F}_T. \tag{10}$$

The above measure is a martingale measure that has the minimal, relative to $\mathbb{P}$, entropy, i.e.

$$\min_Q \mathcal{H}(Q\,|\mathbb{P}) = \min_Q E_{\mathbb{P}}\left(\frac{dQ}{d\mathbb{P}}\ln\frac{dQ}{d\mathbb{P}}\right) = \mathcal{H}(\mathbb{Q}\,|\mathbb{P}) \tag{11}$$

over all martingale measures Q.

Proof First consider the HJB equation satisfied by u, namely,

$$u_t + \max_\pi\left(\frac{1}{2}\sigma^2\pi^2 u_{xx} + \pi\left(\rho\sigma a(y,t)u_{xy} + \mu u_x\right)\right) + \tag{12}$$
$$+\frac{1}{2}a^2(y,t)u_{yy} + b(y,t)u_y = 0.$$

Using the scaling properties of the utility function and the structure of the controlled state dynamics, we postulate a solution of the form

$$u(x,y,t) = -e^{-\gamma x}F(y,t). \tag{13}$$

Substituting in (12), we deduce that F solves a quasilinear equation. The latter can be linearized via a power transformation, in the sense,

$$F(y,t) = v(y,t)^{\frac{1}{1-\rho^2}} \tag{14}$$

with v solving the linear partial differential equation

$$v_t + \frac{1}{2}a^2(y,t)v_{yy} + \Big(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\Big)\, v_y = \frac{1}{2}\left(1-\rho^2\right)\frac{\mu^2}{\sigma^2}v \tag{15}$$

with terminal condition $v(y,T) = e^{\gamma(1-\rho^2)g(y)}$, for $(y,t) \in R \times [0,T]$.

Observe that under the measure defined in (10), the process $\tilde{W}_s = W_s + \rho\frac{\mu}{\sigma}s$, $0 \le s \le T$ is a Brownian motion and the dynamics of Y are given by

$$dY_s = \Big(b(Y_s,s) - \rho\frac{\mu}{\sigma}a(Y_s,s)\Big)\, ds + a(Y_s,s)d\tilde{W}_s, \quad t \le s \tag{16}$$

with $Y_t = y \in R$.

Using the Feynman-Kac representation of solutions to (15), we deduce that

$$v(y,t) = E_{\mathbb{Q}}(e^{\gamma(1-\rho^2)g(Y_T) - \frac{1}{2}(1-\rho^2)\frac{\mu^2}{\sigma^2}(T-t)}\,|Y_t = y). \tag{17}$$

Combining (13), (14) and (17) yields the claimed value function formula (9).

It remains to show that the pricing measure $\mathbb{Q}$ is a martingale measure and minimizes the entropy relative to the historical measure $\mathbb{P}$. These facts are already well established; for example, we refer the reader to Frittelli (2000a) for the detailed arguments. □

We are now ready to derive a closed form formula for the writer's indifference price.

Theorem 3 *Assume exponential preferences and that the dynamics of the traded and nontraded asset are given respectively by (1) and (2). Then, the writer's indifference price of a European claim $G = g(Y_T)$ is given by*

$$h(y,t) = \frac{1}{\gamma(1-\rho^2)}\ln E_{\mathbb{Q}}(e^{\gamma(1-\rho^2)g(Y_T)}\,|Y_t = y) \tag{18}$$

with $\mathbb{Q}$ defined in (10).

The proof follows directly from the pricing equality (7) and the value function representations (8) and (9).

The above pricing formula brings out important ingredients of the utility-based valuation approach. We first observe that, in contradistinction to existing methodologies in incomplete markets, the price is not given in terms of the payoff's expectation under a suitably chosen measure. Note that such representations may involve pricing measures dependent on the payoff, an unnatural pricing ingredient.

The pricing mechanism herein is nonlinear yielding the price in terms of a *conditional nonlinear expectation*

$$h(y,t) = \mathcal{E}_{\mathbb{Q}}(g(Y_T)\,|Y_t = y).$$

The form of the pricing functional $\mathcal{E}$, following directly from (18), shows that the utility based mechanism distorts the original payoff. This is a direct and natural consequence of the role of risk preferences in the valuation approach. The distortion has insurance type certainty equivalent characteristics and its specific form reflects the choice of exponential utility. However, the presence of the coefficient $\gamma\left(1-\rho^2\right)$, which depends exclusively on the risk aversion and the conditional variance, indicates that the pricing formula is not the direct dynamic analogue of the standard actuarial pricing device. The involved pricing measure is the martingale measure that minimizes the relative to the historical measure entropy. A pleasing observation is its independence of risk aversion. Neither the pricing functional nor the pricing measure depends on the specific payoff.

Reverting to the nonlinear nature of (18), we mention that nonlinear pricing structures have been produced in Frittelli (2000b), Rouge and El Karoui (2000) and others. But, to our knowledge, a price formula similar to (18) is new.

One should not forget however, that we were able to derive explicit formulae for the involved value functions, and subsequently for the prices, because of the underlying model assumptions. Namely, the dynamics of the traded asset are lognormal and the payoff of the derivative does not depend on the traded asset. If either assumption is violated, one cannot linearize the relevant equation and the indifference price cannot have the above closed form. For example, if the dynamics of the traded asset are nonlinear, the value function of the corresponding Merton's problem depends on two state variables, the wealth and the stock price. As a matter of fact, it is given by $V(x,S,t)=-e^{-\gamma x}G(S,t)$ where G solves a linear partial differential equation (see Zariphopoulou 1999). The writer's value function will in turn be a function of three variables, namely, x, S and the level of the nontraded asset. This is also the case when the European payoff depends on both the stock price and the level of the nontraded asset even if the dynamics of the traded asset remain lognormal. For both models, one can easily derive a quasilinear equation for the price either by duality methods or just from the primal problem (see, Sircar and Zariphopoulou 2002; Musiela and Zariphopoulou 2002). However, no closed form solution is available due to the asymmetries of the involved gradient nonlinearities.

In order to obtain a viable pricing scheme, one needs to extend the nonlinear indifference pricing formula (18) to more complex algorithms that can accommodate general market situations. From a work in progress of the authors (see Musiela and Zariphopoulou 2003), an iterative nonlinear probabilistic algorithm seems to emerge as the appropriate pricing device for claims of arbitrary payoffs in market models of high dimension and of not necessarily Markovian nature.

Finally, we note that the assumption of no intermediate consumption was made only for computational ease. One may easily verify that if intermediate consumption is allowed for all involved models and utility from consumption is of exponential type, the indifference prices are still given by the above formulae. This is a direct consequence of the scaling properties of the exponential function and the behavior of the HJB equation with respect to the gradient of its solution.

3 Payoff decomposition and price representation

In this section, we provide a comparative analysis of the pricing methodology based on the concept of indifference with the arbitrage free pricing approach of a nested complete Black and Scholes framework. We concentrate on the following two cornerstones of the classical theory, namely, the martingale representation theorem and the related payoff decomposition and the price representation. Recall that in complete models both payoff decomposition and price calculation are done under the unique risk neutral martingale measure. In our framework, the minimal relative entropy martingale measure $\mathbb{Q}$, defined in (10), is used for the price calculation.

In a complete model setting, the price is essentially equal to what it costs to manufacture the option payoff. In other words, in view of the martingale representation theorem, the payoff is equal to the price plus the proceeds from trading the stock and the bond, due to the execution of the self-financing and replicating strategy. Consequently, all risk can be hedged completely by taking positions in the market, with the price being uniquely determined.

In incomplete models, however, not all risk can be hedged. The 'total risk' contains both, hedgeable and unhedgeable components. As a result, one would expect the payoff to be decomposed as a sum of the following three components: the price plus the wealth generated by the hedge execution plus the accumulated residual risk. This section provides such a decomposition under the historical measure $\mathbb{P}$. As expected, when the correlation increases to 1, the residual risk decreases to 0, and the decomposition converges to the one of the Black and Scholes model.

Note that the historical measure $\mathbb{P}$ plays an important role in our analysis, in contrast to the case of complete models, where the pricing and risk management are carried out under the unique risk neutral measure. The historical data are used to identify the appropriate model for the dynamics of the nontraded asset. The correlation between the traded and nontraded asset is also estimated historically. Finally, specification of the parameter $\frac{\mu}{\sigma}$, which is in fact well known to the funds management industry and often referred to as *Sharpe ratio,* depends entirely on the assessment of the actual market conditions.

We begin with some auxiliary results. To this end, we consider a partial differential equation that the indifference pricing function $h\left(y,t\right)$ satisfies. It follows from Theorem 3 that

$$h\left(y,t\right)=\frac{1}{\gamma\left(1-\rho^{2}\right)}\ln w\left(y,t\right) \tag{19}$$

with w being the solution to the Cauchy problem

$$w_{t}+\frac{1}{2}a^{2}\left(y,t\right)w_{yy}+\Big(b\left(y,t\right)-\rho\frac{\mu}{\sigma}a\left(y,t\right)\Big)\,w_{y}=0, \tag{20}$$

with $w\left(y,T\right)=e^{\gamma\left(1-\rho^{2}\right)g(y)}$. Consequently, h solves the *quasilinear* equation

$$h_{t}+\frac{1}{2}a^{2}(y,t)h_{yy}+\Big(b(y,t)-\rho\frac{\mu}{\sigma}a(y,t)\Big)\,h_{y}+\frac{1}{2}\gamma(1-\rho^{2})a^{2}(y,t)h_{y}^{2}=0 \tag{21}$$

with $h(y,T) = g(y)$. The classical regularity results yield that $h \in C^{2,1}(R\times[0,T])$ (see, for example, Pham 2002).

Next, we introduce the *price process*

$$H_s = h(Y_s, s) \;\; t \le s \le T. \tag{22}$$

The following results are a direct consequence of (21) and stochastic calculus.

Proposition 4 *The indifference price process H_s satisfies*

$$dH_s = -\frac{1}{2}\gamma(1-\rho^2)a^2(Y_s,s)h_y^2(Y_s,s)ds + \rho\frac{\mu}{\sigma}a(Y_s,s)h_y(Y_s,s)ds \\ +a(Y_s,s)h_y(Y_s,s)dW_s. \tag{23}$$

Because the indifference price is extracted from the arguments of the relevant value functions (see (5), (6) and (7)), we expect the price process to be directly related to the optimally controlled state wealth process with and without employing the derivative contract. So we consider the writer's optimal wealth process X_s^*, $t \le s \le T$ evaluated at the optimal portfolio process Π_s^*, $t \le s \le T$. The optimal control is provided in the feedback form

$$\pi^*(x,y,t) = \rho\frac{a(y,t)}{\sigma}h_y(y,t) + \frac{1}{\gamma}\frac{\mu}{\sigma^2}. \tag{24}$$

Therefore,

$$\Pi_s^* = \rho\frac{a(Y_s,s)}{\sigma}h_y(Y_s,s) + \frac{1}{\gamma}\frac{\mu}{\sigma^2}, \tag{25}$$

with its optimality following from the regularity properties of the value function u and classical verification results (see, for example, Zariphopoulou 2001a). The wealth Eq. (3) at optimum becomes

$$dX_s^* = \mu\Pi_s^* ds + \sigma\Pi_s^* dW_s^1, \;\; t \le s \le T, \tag{26}$$

with initial condition $X_t^* = x + h(y,t)$, reflecting the compensation received at the contract's inscription. Respectively, the optimal wealth process $X_s^{0,*}$, $t \le s \le T$, of the classical Merton problem (5) is given by

$$dX_s^{0,*} = \mu\Pi_s^{0,*}ds + \sigma\Pi_s^{0,*}dW_s^1, \;\; t \le s \le T, \tag{27}$$

with $\Pi_s^{0,*} = \frac{1}{\gamma}\frac{\mu}{\sigma^2}$ and initial condition $X_t^{0,*} = x$. It may be derived directly from the writer's optimization problem for the degenerate payoff $G \equiv 0$. In fact, one can see that in this case, $h \equiv 0$ is the unique solution to (7) and Π_s^* in (25) reduces to $\Pi_s^{0,*}$.

Definition 5 *Let H_s, X_s^* and $X_s^{0,*}$ be given, respectively, by (22), (26) and (27). We define the residual optimal wealth process,*

$$L_s = X_s^* - X_s^{0,*}, \; t \le s \le T, \;\; L_t = h(y,t)$$

and the residual risk process

$$R_s = L_s - H_s, \;\; t \le s \le T, \quad R_t = 0.$$

A key observation, justifying calling R_s the residual risk is that, under market completeness, $R_s = 0$ for all $t \leq s \leq T$. In this case, the residual wealth process reduces naturally to the derivative price process, and represents the wealth that needs to be put aside in order to hedge the derivative liability in (6).

The dynamics of the process L_s follow from (25), (26) and (27), namely

$$\begin{aligned} dL_s &= \mu \left(\Pi_s^* - \Pi_s^{0,*}\right) ds + \sigma \left(\Pi_s^* - \Pi_s^{0,*}\right) dW_s^1 \\ &= \frac{\rho}{\sigma} a\left(Y_s, s\right) h_y\left(Y_s, s\right) \left(\mu ds + \sigma dW_s^1\right). \end{aligned} \tag{28}$$

Hence, L is a *local martingale* under the measure $\mathbb{P}$ and a martingale subject to the appropriate integrability conditions.

Comparison of the above with the price dynamics in Proposition 4 yields

$$\begin{aligned} dR_s = dL_s - dH_s &= -\sqrt{1-\rho^2} a\left(Y_s, s\right) h_y\left(Y_s, s\right) dW_s^{\perp} \\ &+ \frac{1}{2}\gamma\left(1-\rho^2\right) a^2\left(Y_s, s\right) h_y^2\left(Y_s, s\right) ds, \end{aligned} \tag{29}$$

where the process $W^{\perp}$ is defined by

$$W_s^{\perp} = \frac{1}{\sqrt{1-\rho^2}} W_s - \frac{\rho}{\sqrt{1-\rho^2}} W_s^1, \quad t \leq s \leq T.$$

Clearly $W^{\perp}$ is a Brownian motion orthogonal to W^1 and as such should naturally be linked to the *unhedgeable risk components*.

Proposition 6 *The preference-adjusted exponential of the residual risk process*

$$Z_s = -e^{-\gamma R_s} \quad t \leq s \leq T, \quad Z_t = 1$$

is a local martingale (and a martingale under the appropriate integrability conditions) under the historical measure $\mathbb{P}$. Therefore, the expected utility under the historical measure of the residual risk remains constant.

Proof Combining the definition of Z and the dynamics of R as in (29) yields

$$dZ_s = Z_s \gamma \sqrt{(1-\rho^2)} a(Y_s, s) h_y(Y_s, s) dW_s^{\perp},$$

and the (local) martingale property follows. Moreover, for the exponential utility (4) we get

$$E_{\mathbb{P}}(U(R_s)) = U(0) = -1, \quad t \leq s \leq T. \tag{30}$$

□

The following theorem provides the optimal payoff decomposition and the hedging strategies. We recall that throughout the analysis, the interest rate is assumed to be zero and therefore, no presence of the bond price B is expected in the payoff formula.

Theorem 7 *The payoff $G = g(Y_T)$ admits the following decomposition*

$$g(Y_T) = h(Y_t, t) + \int_t^T \frac{\rho}{\sigma} a(Y_s, s)\, h_y(Y_s, s)\, \frac{dS_s}{S_s} \qquad (31)$$
$$+\sqrt{1-\rho^2} \int_t^T a(Y_s, s)\, h_y(Y_s, s)\, dW_s^{\perp}$$
$$-\frac{1}{2}\gamma\left(1-\rho^2\right) \int_t^T a^2(Y_s, s)\, h_y^2(Y_s, s)\, ds.$$

Proof Integrating (29) yields

$$g(Y_T) = L_T + \sqrt{1-\rho^2} \int_t^T a(Y_s, s)\, h_y(Y_s, s)\, dW_s^{\perp}$$
$$-\frac{1}{2}\gamma\left(1-\rho^2\right) \int_t^T a^2(Y_s, s)\, h_y^2(Y_s, s)\, ds.$$

Moreover, using (28) we get that

$$L_T = h(Y_t, t) + \int_t^T \frac{\rho}{\sigma} a(Y_s, s)\, h_y(Y_s, s)\, \frac{dS_s}{S_s}$$

and hence the statement follows. □

The first term in (31) is the indifference price. The integrand in the second represents the hedge one should put into the traded asset. Indeed, $\Pi_s^* - \Pi_s^{0,*}$ is the optimal residual amount invested into the traded asset due to the presence of an option. Hence,

$$\frac{\Pi_s^* - \Pi_s^{0,*}}{S_s} = \rho \frac{a(Y_s, s)}{\sigma S_s} h_y(Y_s, s) \qquad (32)$$

is the *optimal number of shares* of a correlated asset to be held in the portfolio. The last two terms quantify the risk that cannot be hedged. When $\rho = 0$ there is no distortion, the pricing takes place under the historical measure, and the optimal policy is the same as in the classical Merton problem. Also, when $\rho = 1, b(y,t) = \mu y$ and $a(y,t) = \sigma y$, the integrand in the second term reduces to the usual delta of the Black-Scholes model.

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