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A valuation algorithm for indifference prices in incomplete markets

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Abstract. A probabilistic iterative algorithm is constructed for indifference prices of claims in a multiperiod incomplete model. At each time step, a nonlinear pricing functional is applied that isolates and prices separately the two types of risk. It is represented solely in terms of risk aversion and the pricing measure, a martingale measure that preserves the conditional distribution of unhedged risks, given the hedgeable ones, from their historical counterparts.

Key words: Incomplete markets, indifference prices, nonlinear pricing algorithm

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1 Introduction

The aim herein is to build a probabilistic pricing algorithm for indifference prices and to offer new insights for the utility based valuation methodology. In recent years, this approach has gained considerable ground as a number of new applications and results appear to support more and more its potential contribution. Generally speaking, the indifference price is specified through two stochastic optimization problems of expected utility for the plain investor and for the writer, or the buyer, of the claim. Existing price representations are the direct outcome of the specific technique used to solve the underlying optimization problems.

A popular method is based on duality principles which reduce the primary problems to simpler ones, exposing at the same time certain measures that characterize the dual solutions. Its appeal comes both from its elegance and applicability in non

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Markovian settings (see, for example, Rouge and El Karoui 2000). An alternative approach is based on the analysis of the primary optimization problems via their Hamilton-Jacobi-Bellman equations (Davis et al. 1993) and the quasilinear partial differential equations that indifference prices turn out to satisfy (Musiela and Zariphopoulou 2002, 2004). In Markovian settings, this approach has yield robustness and approximation results for the specification of price spreads and convergence of numerical schemes. Additional references for both methodologies are listed in the bibliography section.

Despite the emphasis that has been given to the technical elements and the specific methodological advantages of each approach, a satisfactory understanding of the economic nature of indifference prices is still limited. Indeed, a constitutive analogue of the arbitrage free theory, and its ultimate alignment with classical asset equilibrium concepts are still lacking. Towards the first direction, a lot of insight for the measurement of unhedgeable risks has been gained from the recent work on coherent risk measures, originated by Artzner et al. (1999). Besides its generality and intuitive character, their theory exposes the importance of nonlinear valuation structures when the assumption of market completeness is abandoned. In many aspects, there is a lot of common ground between this direction and our work, with the main difference being that the aforementioned approach is oriented towards quantifying risks rather than producing a concept of value.

Aiming at a deeper understanding of the nature of indifference prices, it is imperative that one reexamines their representation. So far, indifference prices have been primarily represented either as solutions to new optimization problems or, in certain special cases, through explicit formulae. Departing from such representation, an iterative integrated pricing procedure was proposed by Smith and McCardle (1998). Their basic idea of cash flow valuation is to use subjective believes and risk preferences to determine the cash flow values, conditional on the occurrence of a particular market state, and then use the risk neutral valuation procedure to evaluate these market-state contingent cash flows. The proposed scheme depends heavily on the use of two different pricing measures, an element that makes the general economic structure of their result non transparent.

Our work contributes in further exploring the indifference pricing mechanism through a representation of prices as iterative output of a probabilistic valuation scheme. The scheme is essentially nonlinear and, at each time step, has the ability to identify, isolate and price the unhedgeable risks. This is executed via a sequential pricing procedure that combines actuarial and arbitrage type arguments. Despite the underlying highly nonlinear nature of the utility based valuation, the algorithm yields prices that preserve the time consistency (semigroup) property, a rather indispensable ingredient of any viable pricing theory.

Throughout the scheme iterations, a single pricing measure is being used. This measure is chosen under a natural and intuitively pleasing criterion, namely, as a martingale measure that does not alter the conditional distribution of the unhedgeable risks, given the hedgeable ones, from their historical counterparts.

The entire analysis is based on the assumption of exponential preferences. This has various advantages. Due to scaling properties, this class of utilities yields prices that are wealth independent, a feature desirable and natural in many situations.

Moreover, exponential utilities allow for arbitrary wealth levels and, therefore, stringent non negativity admissibility constraints are not binding. Such constraints, emerging, for example, when power utilities are being used, distort significantly the class of admissible strategies and the size of the solvency domain, and may produce prices of little economic value (see, among others, the discussion in Constantinides and Zariphopoulou 1999 and references therein).

2 Price representation in one period model

Consider a one-period model in a market environment with one riskless and two risky assets. Only one of the risky assets is traded. For simplicity, assume zero interest rate. The current values of the traded and nontraded risky assets are denoted, respectively, by S_0 and Y_0. At the end of the period T, the value of the traded asset is S_T with $S_T = S_0\xi$, $\xi = \xi^d, \xi^u$ and $0 < \xi^d < 1 < \xi^u$. Similarly, the value of the nontraded asset Y_T satisfies $Y_T = Y_0\eta$, $\eta = \eta^d, \eta^u$,with $\eta^d < \eta^u$.

Denote by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ the probability space and by $\mathbb{P}$ the historical probability measure on the $\sigma-$algebra $\mathcal{F} = 2^{\Omega}$ of all subsets of Ω. For each $i = 1, 2, 3, 4$ let $p_i = \mathbb{P}\{\omega_i\} > 0$. The random variables S_T and Y_T can then be written as

$$\begin{array}{ll} S_T(\omega_1) = S_0\xi^u,\ Y_T(\omega_1) = Y_0\eta^u & S_T(\omega_3) = S_0\xi^d,\ Y_T(\omega_3) = Y_0\eta^u \\ S_T(\omega_2) = S_0\xi^u,\ Y_T(\omega_2) = Y_0\eta^d & S_T(\omega_4) = S_0\xi^d,\ Y_T(\omega_4) = Y_0\eta^d. \end{array}$$

Consider a portfolio consisting of α shares of stock and the amount β invested in the riskless asset. Its current value $X_0 = x$ is equal to $\beta + \alpha S_0 = x$. Its wealth X_T, at the end of the period $[0, T]$, is given by

$$X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0).$$

Now introduce a claim settling at time T and yielding payoff $C_T \in \mathcal{F}$. In pricing of C_T, we need to specify our risk preferences. We choose to work with exponential utility of the form

$$U(x) = -e^{-\gamma x},\ x \in \mathbb{R} \text{ and } \gamma > 0.$$

Optimality of investments, which will ultimately yield the indifference price of C_T, is examined via the value function

$$V^{C_T}(x) = \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T)}\right) = e^{-\gamma x} \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma\alpha(S_T - S_0) + \gamma C_T}\right). \tag{1}$$

Below, we recall the definition of indifference prices.

Definition 1 *The indifference price of the claim $C_T = c(S_T, Y_T)$ is defined as the amount $\nu(C_T)$ for which the two value functions V^{C_T} and V^0, defined in (1) and corresponding, respectively, to the claims C_T and 0 coincide. Namely, $\nu(C_T)$ is the amount which satisfies*

$$V^0(x) = V^{C_T}(x + \nu(C_T)), \tag{2}$$

for all initial wealth levels x.

Looking at the classical *arbitrage free* pricing theory, we recall that derivative valuation has two fundamental components which do not depend on specific model assumptions. Namely, the price is obtained as a *linear* functional of the (discounted) payoff representable via the (unique) *risk neutral* equivalent martingale measure.

Our goal is to understand how these two components, namely, the linear valuation operator and the risk neutral pricing measure change when markets become incomplete. In the context of pricing by *indifference*, we will look for a *valuation functional* and a naturally related with it *pricing measure* under which the price is given as

$$\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T). \tag{3}$$

Before we determine the fundamental features that $\mathcal{E}$ and $\mathbb{Q}$ should have, let us look at some representative cases.

Examples

i) We first consider a claim of the form $C_T = c\,(S_T)$. Naturally, an indifference price of the above form (3) must coincide with the arbitrage free price, for there is no risk that cannot be hedged. In fact, one can construct a nested complete one-period binomial model and represent the price as

$$\nu(c(S_T)) = E_{\mathbb{Q}^*}(c(S_T)) \tag{4}$$

with $\mathbb{Q}^*$ being the relevant risk neutral measure. The indifference price mechanism reduces to the arbitrage free one and any effect on preferences dissipates.

ii) Next, we look at a claim of the form $C_T = c\,(Y_T)$ and assume for simplicity that the random variables S_T and Y_T are independent under the measure $\mathbb{P}$. Intuitively, the presence of the traded asset should *not* affect the price. Indeed, working directly with the value function (1) and the definition (2) it is straightforward to deduce that

$$\nu(c(Y_T)) = \frac{1}{\gamma} \log E_{\mathbb{P}}(e^{\gamma c(Y_T)}). \tag{5}$$

The indifference price coincides with the classical actuarial valuation principle, the so-called *certainty equivalent* value which is nonlinear in the payoff and the involved measure remains the historical one. Observe that (5) cannot be put in the form of (4), unless we allow for dependence of the measure on the payoff and/or the risk preference, deducing, in this case, a rather unnatural price representation.

iii) We finally examine a claim of the form $C_T = c_1\,(S_T) + c_2\,(Y_T)$. One could be, wrongly, tempted to price C_T by first pricing $c_1\,(S_T)$ by arbitrage, next pricing $c_2\,(Y_T)$ by certainty equivalent, and adding the results. Intuitively, this should work when S_T and Y_T are independent. However, this cannot possibly work under strong dependence between the two variables, for example, when Y_T is a function of S_T. In general,

$$\nu(c_1(S_T) + c_2(Y_T)) \neq E_{\mathbb{Q}^*}(c_1(S_T)) + \frac{1}{\gamma} \log E_{\mathbb{P}}(e^{\gamma c_2(Y_T)}). \tag{6}$$

The above illustrative examples indicate certain fundamental characteristics the $\mathcal{E}$ and $\mathbb{Q}$ should have. We first realize that a *nonlinear* valuation functional must be sought. Clearly, any effort to represent indifference prices as expected payoffs under an appropriately chosen measure must be abandoned. Indeed, *no linear pricing mechanism can be compatible with the concept of utility based valuation.* Note that this fundamental observation comes in contradistinction to the central direction of existing approaches in incomplete markets that yield prices as expected payoffs under an optimally chosen measure. There is no justification nor intuition why such a choice of measure must be performed and why prices must preserve a linear structure. An immediate flow of this approach is model dependence and numeraire inconsistency.

We also see that risk preferences may affect the valuation device given their inherent role in price specification. However, intuitively speaking, we should not expect any dependence on risk preferences to the pricing measure. Finally, no dependence on the specific payoff should be allowed on neither the pricing measure nor the valuation device.

The next Proposition yields the indifference price in the desired form (3).

Proposition 2 *Let $\mathbb{Q}$ be a measure under which the traded asset is a martingale and, at the same time, the conditional distribution of the nontraded asset, given the traded one, is preserved with respect to the historical measure $\mathbb{P}$, i.e.*

$$\mathbb{Q}(Y_T\,|S_T\,) = \mathbb{P}(Y_T\,|S_T\,). \tag{7}$$

Let $C_T = c(S_T, Y_T)$ be the claim to be priced under exponential preferences with risk aversion coefficient γ. Then, the indifference price of C_T is given by

$$\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\,|S_T\,\right)\right). \tag{8}$$

Proof We prove the above result by constructing the indifference price via its definition (2). We start with the specification of the value functions V^0 and V^{C_T}. We represent the payoff C_T as a random variable defined on Ω with values $C_T\,(\omega_i) = c_i \in \mathbb{R}$, for $i = 1, 2, 3, 4$. Elementary transformations lead to

$$V^{C_T}\,(x) = e^{-\gamma x}\sup_{\alpha}\Big(-e^{-\gamma\alpha S_0(\xi^u-1)}\left(e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2\right)$$

$$-e^{-\gamma\alpha S_0\left(\xi^d-1\right)}\left(e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4\right)\Big).$$

Further straightforward, albeit tedious, calculations yield

$$V^{C_T}\,(x) = -e^{-\gamma x}\frac{1}{q^q\,(1-q)^{1-q}}\left(e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2\right)^q\left(e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4\right)^{1-q}, \tag{9}$$

where

$$q = \frac{1-\xi^d}{\xi^u-\xi^d}. \tag{10}$$

For $C_T = 0$, the value function takes the form

$$V^0(x) = -e^{-\gamma x}\left(\frac{p_1+p_2}{q}\right)^q\left(\frac{p_3+p_4}{1-q}\right)^{1-q}. \tag{11}$$

From the definition of the indifference price (2) and the representations (9), (11) of the relevant value functions, it follows that

$$\nu(C_T) = q\left(\frac{1}{\gamma}\log\frac{e^{\gamma c_1}p_1+e^{\gamma c_2}p_2}{p_1+p_2}\right)+(1-q)\left(\frac{1}{\gamma}\log\frac{e^{\gamma c_3}p_3+e^{\gamma c_4}p_4}{p_3+p_4}\right). \tag{12}$$

We next show that the above price admits the claimed probabilistic representation (8).

We start with the specification of the pricing measure defined in (7). Elementary calculations yield that a martingale measure Q, with $Q\{\omega_i\} = q_i > 0$ needs to satisfy

$$q_1 + q_2 = q \tag{13}$$

with q as in (10). We now consider the terms involving the historical probabilities in (12) and we note that they can be actually written in terms of the *conditional historical* expectations, namely,

$$\frac{e^{\gamma c_1}p_1+e^{\gamma c_2}p_2}{p_1+p_2} = E_{\mathbb{P}}\left(e^{\gamma C_T}\,|S_T = S_0\xi^u\right)$$

and

$$\frac{e^{\gamma c_3}p_3+e^{\gamma c_4}p_4}{p_3+p_4} = E_{\mathbb{P}}\left(e^{\gamma C_T}\,|S_T = S_0\xi^d\right).$$

It is important to observe that conditioning is taken with respect to the *terminal* values of the traded asset.

Next, we specify the martingale measure satisfying (7). For this, we denote (with a slight abuse of notation) by q_1, q_2, q_3, q_4 the elementary probabilities of the sought measure $\mathbb{Q}$.To compute q_1, we look at the conditional historical probability of $\{Y_T = Y_0\eta^u\}$, given $\{S_T = S_0\xi^u\}$, and we impose (7), yielding

$$\frac{p_1}{p_1+p_2} = \frac{q_1}{q}.$$

The probabilities q_2, q_3 and q_4, computed in a similar manner, are written below in a concise form

$$q_i = q\frac{p_i}{p_1+p_2},\ i=1,2 \ \text{ and } \ q_i = (1-q)\frac{p_i}{p_3+p_4}\ i=3,4.$$

Therefore,

$$\frac{1}{\gamma}\log E_{\mathbb{P}}(e^{\gamma C_T}\,|S_T) = \frac{1}{\gamma}\log E_{\mathbb{Q}}(e^{\gamma C_T}\,|S_T)$$

and using (12) we conclude. □

We next discuss the key ingredients and highlight the intuitively natural features of the probabilistic pricing formula (8).

Interpretation of the indifference price. Valuation is done via a *two-step nonlinear* procedure and under a single pricing measure.

i) *Valuation procedure* In the first step, risk preferences are injected into the valuation process. The original derivative payoff is being distorted to the *preference adjusted* payoff

$$\tilde{C}_T = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \,|S_T \right).$$

This new payoff has actuarial type characteristics and reflects the weight that risk aversion carries in the utility based methodology. However, certainty equivalent is not applied in a naive way. Indeed, we do not consider any classical actuarial type functional,

$$\tilde{C}_T \neq \frac{1}{\gamma} \log E_{\mathbb{P}} \left(e^{\gamma C_T} \right) \text{ and } \tilde{C}_T \neq \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \right).$$

Rather, the indifference price mechanism creates a conditional certainty equivalent of the payoff, under the pricing measure $\mathbb{Q}$.

In the second step, the pricing procedure picks up arbitrage free pricing characteristics. It prices the preference adjusted payoff $\tilde{C}_T$, dependent only on S_T, through an arbitrage free method. The same pricing measure is being used in both steps.

The price is then given by

$$\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T) = E_{\mathbb{Q}}(\tilde{C}_T).$$

It is important to observe that the two steps are *not* interchangeable and of entirely different nature. The first step prices in a nonlinear way as opposed to the second step that uses linear, arbitrage free, valuation principles. In a sense, this is entirely justifiable: *the unhedgeable risks are identified, isolated and priced in the first step and, thus, the remaining risks become hedgeable. One should then use a nonlinear valuation device for the unhedgeable risks and, linear pricing for the hedgeable ones.* A natural consequence of this is that risk preferences enter exclusively in the conditional certainty equivalent term, the only term related to unhedgeable risks. Both steps are payoff independent.

ii) *Pricing measure* One pricing measure is used throughout. Its essential role is not to alter the conditional distribution of risks, given the ones we can trade, from their respective historical values.

Naturally, there is no dependence on the payoff. The most interesting part however is its independence on risk preferences. This universality is expected and quite pleasing. It follows from the way we identified the pricing measure, via (7), a selection criterion that is naturally independent of any risk attitude. The distorted payoff $\tilde{C}_T$ is *invariant* when seen through the historical and the pricing measure,

$$\tilde{C}_T = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \,|S_T \right) = \frac{1}{\gamma} \log E_{\mathbb{P}} \left(e^{\gamma C_T} \,|S_T \right).$$

The following result highlights an important property of the indifference price operator. We see that any hedgeable risk is automatically scaled out from the nonlinear part of the pricing rule and it is priced directly by arbitrage. Hedgeable risks do not generate conditional certainty equivalent payoffs. In this sense, we say that the pricing operator has the property of *translation invariance with respect to hedgeable risks*.

Note that this property is stronger, and more intuitive, than requiring mere invariance with respect to constant risks.

Corollary 3 *The indifference pricing operator is translation invariant with respect to hedgeable risks, i.e. if* $C_T = c_1(S_T) + c_2(S_T, Y_T)$, *then*

$$\begin{aligned}\nu(c_1(S_T) + c_2(S_T, Y_T)) &= \mathcal{E}_{\mathbb{Q}}(c_1(S_T) + c_2(S_T, Y_T)) \\ &= E_{\mathbb{Q}}\left(c_1\left(S_T\right)\right) + \nu(c_2\left(S_T, Y_T\right)). \qquad (14)\end{aligned}$$

The proof follows from (12) directly.

We conclude this section by looking at two extreme cases of (14).

Special cases

i) Let Y_T depend functionally on S_T. Then, the payoff c_2 is measurable with respect to S_T and

$$\tilde{c}_2(S_T, Y_T) = \frac{1}{\gamma} \log E_{\mathbb{Q}}(e^{\gamma c_2(S_T, Y_T)} \,|S_T) = c_2(S_T, Y_T).$$

In turn, the second term in (14), yields the expectation under $\mathbb{Q}$ of the original claim c_2, and $\mathbb{Q}$ reduces to the nested risk neutral measure $\mathbb{Q}^*$. The indifference price of $c_1 + c_2$ is given by the classical arbitrage free price, namely,

$$\nu(c_1(S_T) + c_2(S_T, Y_T)) = E_{\mathbb{Q}^*}(c_1(S_T) + c_2(S_T, Y_T)).$$

ii) Let Y_T and S_T be independent under $\mathbb{P}$ and c_2 to depend only on Y_T. Then,

$$\tilde{c}_2(Y_T) = \frac{1}{\gamma} \log E_{\mathbb{Q}}\left(e^{\gamma c_2(Y_T)} \,|S_T\right) = \frac{1}{\gamma} \log E_{\mathbb{P}}\left(e^{\gamma c_2(Y_T)}\right).$$

The conditional certainty equivalent term becomes the traditional actuarial certainty equivalent value and the indifference price of $c_1 + c_2$ reduces to

$$\nu(c_1(S_T) + c_2(S_T, Y_T)) = E_{\mathbb{Q}^*}(c_1(S_T)) + \frac{1}{\gamma} \log E_{\mathbb{P}}\left(e^{\gamma c_2(Y_T)}\right).$$

3 The multiperiod model and the pricing algorithm

We construct a general pricing algorithm in a multiperiod incomplete market environment. We aim at representing the indifference price $\nu_t(C_T)$ in a natural valuation format that will preserve the intuitive and universal characteristics of the single period case namely, a single pricing measure and a nonlinear structure that isolates and prices effectively the two types of risk. Departing from the static case, however, in order to develop a meaningful valuation algorithm, we should also aim at constructing a pricing scheme with the fundamental property of *time consistency,*

$$\nu_t(C_T) = \nu_t(\nu_s(C_T)) \text{ for } t \leq s \leq T. \tag{15}$$

This property states that the current price can serve as a new payoff that is in turn priced, moving backwards in time, via the time-adjusted price operator. This is a fundamental feature of arbitrage free prices and, clearly, indispensable in incomplete markets as well. No pricing algorithm can be viable if such semigroup price property fails.

Given the inherent nonlinearities of the indifference pricing mechanism, we stress that, a priori, it is not at all obvious what are the multiperiod analogues of (7) and (8), and why the semigroup price property (15) must follow. We will see that there is a delicate interplay among optimality of policies, semigroup properties of the involved value functions, the nature of the pricing measure and the structure of the valuation scheme that ultimately yields the desired pricing ingredients.

We start with the probabilistic set-up of the multiperiod model. We denote by $S_t, t = 0, 1, ..., T$ the values of the traded risky asset. It is assumed throughout that $S_t > 0$. We define the random variables ξ via

$$\xi_{t+1} = \frac{S_{t+1}}{S_t},\ \xi_{t+1} = \xi_{t+1}^d, \xi_{t+1}^u \text{ with } 0 < \xi_{t+1}^d < 1 < \xi_{t+1}^u.$$

The second traded asset is riskless and is assumed to yield zero interest rate. We also denote by Y_t, $t = 0, 1, ..., T$ the values of a nontraded asset and we introduce the random variables η satisfying

$$\eta_{t+1} = \frac{Y_{t+1}}{Y_t},\ \eta_{t+1} = \eta_{t+1}^d, \eta_{t+1}^u \text{ with } \eta_{t+1}^d < \eta_{t+1}^u.$$

We then view $\{S_t, Y_t : t = 0, 1, ..., T\}$ as a two-dimensional stochastic process defined on the probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$, where the filtration $\mathcal{F}_t$ is generated by the random variables S_s, Y_s, $s = 1, ..., t$, or, equivalently, by the random variables ξ_s, η_s, $s = 1, ..., t$. We denote by $\mathcal{F}_t^S$ and $\mathcal{F}_t^Y$ the filtrations generated respectively by the random variables S_s and Y_s, $s = 1, ..., t$.

The real (historical) probability measure on Ω and $\mathcal{F}_T$ is denoted by $\mathbb{P}$. We assume that it satisfies for all $t = 0, 1, ..., T-1$

$$\mathbb{P}\left(\xi_{t+1} \left| \mathcal{F}_t \right.\right) = \mathbb{P}\left(\xi_{t+1} \left| \mathcal{F}_t^S \right.\right). \tag{16}$$

The above condition characterizes a model in which the movements of the traded asset are not affected by the dynamics of the nontraded one. This is, for example, the case when the stock dynamics depend in a nonlinear way on the stock levels. However, this condition fails if the traded stock has stochastic volatility and the latter plays the role of the nontraded asset (see, Musiela and Zariphopoulou 2003).

To facilitate the analysis, we provide the multiperiod definition of indifference prices and related quantities. To this end, using notation compatible with the single period case, we let X_s, $s=1,...,T$ represent the wealth process associated with a multiperiod self-financing portfolio, with $X_0 = x$ being the initial wealth. We will denote by α_s, $s=1,2,...,T$ the number of shares of the traded asset held in this portfolio over the time period $[s-1,s]$. Then, $\triangle X_s = X_s - X_{s-1} = \alpha_s \triangle S_s$ and hence, $X_T = x + \sum_{s=1}^{T} \alpha_s \triangle S_s$.

Consider a claim C_T expiring at time $T \geq 0$. The value function $V^{C_T}(X_t, t; T)$, corresponding to a short position in C_T is defined as the dynamic analogue of (1)

$$V^{C_T}(X_t, t; T) = \sup_{\alpha_{t+1},\dots,\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T)}\left|\mathcal{F}_t\right.\right). \tag{17}$$

The *indifference price* $\nu_t(C_T)$ is an $\mathcal{F}_t$-measurable random variable such

$$V^0(X_t, t; T) = V^{C_T}(X_t + \nu_t(C_T), t; T) \tag{18}$$

holds $\mathbb{P}$ a.s.

Let Z_s, $0 \leq s \leq T$ be an $\mathcal{F}_s$-adapted process and Q a martingale measure. For each $0 \leq t \leq s$, define the *iterative functional*

$$\mathcal{E}_Q^{(t,s)}(Z_s) = \mathcal{E}_Q^{(t,s-1)}(\mathcal{E}_Q^{(s-1,s)}(Z_s)) \tag{19}$$

where

$$\mathcal{E}_Q^{(s-1,s)}(Z_s) = E_Q\left(\frac{1}{\gamma}\log E_Q\left(e^{\gamma Z_s}\left|\mathcal{F}_{s-1} \vee \mathcal{F}_s^S\right.\right)\left|\mathcal{F}_{s-1}\right.\right) \tag{20}$$

and

$$\mathcal{E}_Q^{(s,s)}(Z_s) = Z_s.$$

We caution the reader that for $t < s-1$,

$$\mathcal{E}_Q^{(t,\,s)}(Z_s) \neq E_Q\left(\frac{1}{\gamma}\log E_Q\left(Z_s\left|\mathcal{F}_t \vee \mathcal{F}_s^S\right.\right)\left|\mathcal{F}_t\right.\right). \tag{21}$$

Lemma 4 *For $0 \leq t \leq s$, $\mathcal{E}_Q^{(t,s)}(Z_s)$ is an $\mathcal{F}_t$ adapted process. Moreover, for $t \leq s \leq T$,*

$$\mathcal{E}_Q^{(t,T)}(Z_s) = \mathcal{E}_Q^{(t,s)}(Z_s),$$

$$\mathcal{E}_Q^{(t,s)}(Z_s) = \mathcal{E}_Q^{(t,t+1)}(...\mathcal{E}_Q^{(s-2,s-1)}(\mathcal{E}_Q^{(s-1,s)}(Z_s))),$$

$$\mathcal{E}_Q^{(t,s)}(Z_s) = \mathcal{E}_Q^{(t,s-t)}(\mathcal{E}_Q^{(s-t,s)}(Z_s)).$$

The theorem below presents the *multiperiod pricing algorithm*, the main result of this section.

Theorem 5 *Let $\mathbb{Q}$ be a martingale measure satisfying, for $t=0,1,...,T$,*

$$\mathbb{Q}(\eta_{t+1}\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.) = \mathbb{P}(\eta_{t+1}\left|\mathcal{F}_t \vee \mathcal{F}_{t+1}^S\right.) \tag{22}$$

i) The indifference price $\nu_t(C_T)$, defined in (18), satisfies

$$\nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{(t,\, t+1)}(\nu_{t+1}(C_T)), \tag{23}$$

$$\nu_T\left(C_T\right) = C_T,$$

with $\mathcal{E}_{\mathbb{Q}}^{(t,t+1)}$ defined in (20) for $Q = \mathbb{Q}$.

ii) The indifference price process is given by

$$\nu_t(C_T) = \mathcal{E}_{\mathbb{Q}}^{(t,\, T)}(C_T) \tag{24}$$

with $\mathcal{E}_{\mathbb{Q}}^{(t,\, T)}$ defined in (19) for $Q = \mathbb{Q}$.

iii) The pricing algorithm is consistent across time in that, for $0 \leq t \leq s \leq T$, the semigroup property

$$\nu_t\left(C_T\right) = \mathcal{E}_{\mathbb{Q}}^{(t,\, s)}(\mathcal{E}_{\mathbb{Q}}^{(s,\, T)}\left(C_T\right)) = \mathcal{E}_{\mathbb{Q}}^{(t,\, s)}((\nu_s\left(C_T\right)) = \nu_t(\mathcal{E}_{\mathbb{Q}}^{(s,\, T)}(C_T)) \tag{25}$$

holds.

Proof We are going to establish (23) for $t = T - 1$ and (25) for $t = T - 2$, $s = T - 1$. The rest of the proof follows by the above lemma and routine, albeit tedious, induction arguments. We start by showing that

$$\nu_{T-1}\left(C_T\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\left|\mathcal{F}_{T-1} \vee \mathcal{F}_T^S\right.\right)\left|\mathcal{F}_{T-1}\right.\right). \tag{26}$$

For this, we first consider the value function $V^{C_T}(X_{T-1}, T-1; T)$, given by

$$V^{C_T}(X_{T-1}, T-1; T) = \sup_{\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T)}\left|\mathcal{F}_{T-1}\right.\right)$$

$$= \sup_{\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_{T-1}+\alpha_T \triangle S_T - C_T)}\left|\mathcal{F}_{T-1}\right.\right).$$

Imitating the one period calculations, presented in the proof of Proposition 2 and appropriately modified to accommodate the conditioning on $\mathcal{F}_{T-1}$, we obtain

$$V^{C_T}(X_{T-1}, T-1; T) = -e^{-\gamma X_{T-1} - h_{T-1} + \gamma \lambda_{T-1}(C_T)} \tag{27}$$

where

$$h_{T-1} = q_T \log\frac{q_T}{\mathbb{P}\left(A_T\left|\mathcal{F}_{T-1}\right.\right)} + (1 - q_T)\log\frac{1 - q_T}{1 - \mathbb{P}\left(A_T\left|\mathcal{F}_{T-1}\right.\right)}, \tag{28}$$

$A_T = \{\omega : \xi_T(\omega) = \xi_T^u\}$ and

$$\lambda_{T-1}\left(C_T\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\left|\mathcal{F}_{T-1} \vee \mathcal{F}_T^S\right.\right)\left|\mathcal{F}_{T-1}\right.\right). \tag{29}$$

Similarly, we deduce that

$$V^0(X_{T-1}, T-1; T) = -e^{-\gamma X_{T-1} - h_{T-1}}$$

which, in view of (27) and (18) readily enables us to identify $\lambda_{T-1}(C_T)$ with $\nu_{T-1}(C_T)$ and, thus, proving the validity of (26).

Next we establish the semigroup property (25) for two time steps, i.e.

$$\nu_{T-2}(C_T) = \mathcal{E}_{\mathbb{Q}}^{(T-2,T-1)}(\nu_{T-1}(C)), \tag{30}$$

with $\nu_{T-1}(C_T)$ given by (26).

We first construct $\nu_{T-2}(C_T)$ directly via the pricing formula (18).

To this end, we compute $V^{C_T}(X_{T-2}, T-2; T)$ defined as

$$V^{C_T}(X_{T-2}, T-2; T) = \sup_{\alpha_{T-1}, \alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_T - C_T)} \,|\mathcal{F}_{T-2}\right).$$

Using direct arguments, we see that it can be written in terms of a single period value function, from $T-2$ to $T-1$, namely,

$$\begin{aligned} &V^{C_T}(X_{T-2}, T-2; T) \\ &= \sup_{\alpha_{T-1}} E_{\mathbb{P}}\left(\sup_{\alpha_T} E_{\mathbb{P}}\left(-e^{-\gamma(X_{T-1} + \alpha_T \triangle S_T - C_T)} \,|\mathcal{F}_{T-1}\right) |\mathcal{F}_{T-2}\right) \end{aligned}$$

and, in turn, as

$$V^{C_T}(X_{T-2}, T-2; T) = \sup_{\alpha_{T-1}} E_{\mathbb{P}}\left(-e^{-\gamma X_{T-1} - h_{T-1} + \gamma \nu_{T-1}(C_T)} \,|\mathcal{F}_{T-2}\right)$$

$$= \sup_{\alpha_{T-1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{T-1} - (\nu_{T-1}(G) - \frac{1}{\gamma} h_{T-1}))} \,|\mathcal{F}_{T-2}\right)$$

with h_{T-1} given by (28).

We note that the last expression corresponds to the calculation of the value function in a single period binomial case, in the time interval $[T-2, T-1]$, under conditional, on $\mathcal{F}_{T-2}$, distribution. Following arguments used for the derivation of (27) , we obtain

$$V^{C_{T-1}}(X_{T-2}, T-2; T-1) = e^{-\gamma X_{T-2} - h_{T-2} + \gamma \mu_{T-2}(C_{T-1})},$$

where

$$h_{T-2} = q_{T-1} \log \frac{q_{T-1}}{\mathbb{P}(A_{T-1} \,|\mathcal{F}_{T-2})} + (1 - q_{T-1}) \log \frac{1 - q_{T-1}}{1 - \mathbb{P}(A_{T-1} \,|\mathcal{F}_{T-2})}, \tag{31}$$

$A_{T-1} = \{\omega : \xi_{T-1}(\omega) = \xi^u_{T-1}\}$ and

$$\mu_{T-2}(C_{T-1}) = E_{\mathbb{Q}}\left(\frac{1}{\gamma} \log E_{\mathbb{Q}}\left(e^{\gamma\left(\nu_{T-1}(C_T) - \frac{1}{\gamma} h_{T-1}\right)} \,|\mathcal{F}_{T-2} \vee \mathcal{F}^S_{T-1}\right) |\mathcal{F}_{T-2}\right).$$

Next, we observe that, because of (16), $h_{T-1} \in \mathcal{F}^S_{T-1}$. Then, Corollary 3 yields that the above (single time step) price functional μ is translation invariant with respect to the risks that can be hedged. Therefore,

$$\mu_{T-2}(C_{T-1}) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\nu_{T-1}(C_T)}\left|\mathcal{F}_{T-2}\vee\mathcal{F}^S_{T-1}\right.\right)|\mathcal{F}_{T-2}\right) +$$

$$+E_{\mathbb{Q}}\left(-h_{T-1}\left|\mathcal{F}_{T-2}\right.\right).$$

This in turn yields

$$V^{C_T}(X_{T-2}, T-2; T) = -e^{-\gamma X_{T-2} - E_{\mathbb{Q}}(h_{T-2}+h_{T-1}|\mathcal{F}_{T-2}) + \gamma\lambda_{T-2}(C_T)}, \quad (32)$$

with $\nu_{T-1}(C_T)$ given by (26) and

$$\lambda_{T-2}\left(C_T\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\nu_{T-1}(C_T)}\left|\mathcal{F}_{T-2}\vee\mathcal{F}^S_{T-1}\right.\right)|\mathcal{F}_{T-2}\right).$$

Observe that

$$\lambda_{T-2}\left(C_T\right) = \mathcal{E}_{\mathbb{Q}}^{(T-2,T-1)}(\nu_{T-1}(C_T)).$$

Similarly, we deduce that

$$V^0(X_{T-2}, T-2; T) = -e^{-\gamma X_{T-2} - E_{\mathbb{Q}}(h_{T-2}+h_{T-1}|\mathcal{F}_{T-2})} \quad (33)$$

with h_{T-1} and h_{T-2} given respectively by (28) and (31). Combining the above equality, (32) and the definition of the indifference price (18) yields

$$\nu_{T-2}(C_T) = \lambda_{T-2}\left(C_T\right) = \mathcal{E}_{\mathbb{Q}}^{(T-2,T-1)}(\nu_{T-1}(C_T))$$

and (30) is proved. □

Proposition 6 *The indifference price process $\nu_t(C_T)$ is a supermartingale under $\mathbb{Q}$.*

The claim is an immediate consequence of Jensen's inequality for concave functions.

Interpretation of the pricing algorithm. Valuation is done via an iterative pricing scheme applied backwards in time.

i) *Valuation functional* At the beginning of each time step, say $(t, t+1)$, $t = 0, 1 ... T-1$, the price $\nu_t(C_T)$ is computed via a price functional, denoted by $\mathcal{E}_{\mathbb{Q}}^{(t,\,t+1)}$, applied to the end of the period payoff $\nu_{t+1}(C_T)$. Its pricing role is in many aspects similar to the one of its single period counterpart, analyzed in detail in the previous section.

Highlighting the main properties of $\mathcal{E}_{\mathbb{Q}}^{(t,\,t+1)}(.)$, we first observe that it is *nonlinear* and *time dependent*. Nonlinearity arises due to the way unhedgeable risks are priced. Risk preferences are injected and, the end of the period payoff $\nu_{t+1}(C_T)$ is altered to the *preference adjusted dynamic* payoff

$$\tilde{\nu}_{t+1}(C_T) = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma \nu_{t+1}(C_T)} \left| \mathcal{F}_t \vee \mathcal{F}^S_{t+1} \right. \right).$$

This step amounts to the specification, isolation and valuation of the unhedgeable risks and is inherently of nonlinear nature. Once this sub-step is executed, the remaining risks are hedgeable and, thus, priced by arbitrage free arguments yielding the price as

$$\nu_t(C_T) = E_{\mathbb{Q}}(\tilde{\nu}_{t+1}(C_T)).$$

It is worth noticing that $\mathcal{E}_{\mathbb{Q}}^{(t,\,t+1)}(.)$ is naturally affected by risk preferences only at its first step, the one that considers the unhedgeable risks. It is interesting to notice its independence on the specific payoff and its universality with respect to the pricing measure $\mathbb{Q}$.

ii) *Pricing measure* Throughout the scheme iterations, the martingale measure $\mathbb{Q}$, defined in (22), is being used. As it was pointed out, its essential role is to preserve the conditional distribution of unhedgeable risks, given the hedgeable ones, at its historical value. In order to gain some insights about its selection criterion, let us first look at all martingale measures, Q. For any such measure, the joint distributions of $(\xi_1, \xi_2, ...\xi_T, \eta_1, \eta_2, \ldots \eta_T)$ can be then computed via their conditional counterparts, namely,

$$Q(\xi_1, \ldots \xi_T, \eta_1, \ldots \eta_T) = \prod_{s=0}^{T-1} Q(\xi_{s+1}, \eta_{s+1} \left| \mathcal{F}_s \right.).$$

However, we note that

$$Q(\xi_{t+1}, \eta_{t+1} \left| \mathcal{F}_t \right.) = Q(\xi_{t+1} \left| \mathcal{F}_t \right.) Q(\eta_{t+1} \left| \mathcal{F}_t \vee \mathcal{F}^S_{t+1} \right.)$$

with the term $Q(\xi_{t+1} \left| \mathcal{F}_t \right.)$ depending exclusively on ξ^d_{t+1} and ξ^u_{t+1}. Therefore, the choice of pricing measure amounts only to the specification of the last term, $Q(\eta_{t+1} \left| \mathcal{F}_t \vee \mathcal{F}^S_{t+1} \right.)$. This conditional distribution essentially reflects the statistical vision of the unhedgeable risks that the pricing mechanism must carry through. But given their nature, these risks, need to be *identical*, when viewed, both by the real and the candidate pricing measure. This intuitive and natural condition is expressed in (22).

We conclude this section by adding an interesting remark. As it was mentioned in the introduction, indifference prices have been produced as solutions to reduced optimization problems with criteria involving relative entropy terms. The presence of entropy is anticipated due to the use of exponential preferences. One might wonder how the minimal relative entropy measure is related to the pricing measure $\mathbb{Q}$. It is interesting to realize that, even though their specification was motivated and implemented by entirely different criteria, scope and techniques, the two measures actually coincide.

Proposition 7 *A martingale measure Q satisfies (22) if and only if it has the minimal relative to $\mathbb{P}$ entropy.*

Proof In the single period case, the claim follows by straightforward calculations. For the multiperiod case, we first observe that the relative to $\mathbb{P}(\xi_1,...\xi_T,\eta_1,...\eta_T)$ entropy of $Q(\xi_1,...\xi_T,\eta_1,...\eta_T)$ satisfies

$$H(Q\,|\mathbb{P}) = E_Q\left(\log\frac{\prod_{t=0}^{T-1} Q(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}{\prod_{t=0}^{T-1}\mathbb{P}(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}\right) \tag{34}$$

$$= \sum_{t=0}^{T-1} E_Q\left(\log\frac{Q(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}{\mathbb{P}(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}\right).$$

Looking at the elementary events $A_t = \{\omega : \xi_t(\omega) = \xi_t^u\}$ and $B_t = \{\omega : \eta_t(\omega) = \eta_t^u\}$, and manipulating expressions involving conditional expectations with respect to martingale measures, we deduce that the minimization problem is essentially reduced to the single period case. Straightforward, albeit tedious calculations then yield that the martingale measure $\mathbb{Q}$ satisfying

$$\frac{\mathbb{Q}(A_tB_t\,|\mathcal{F}_{t-1})}{\mathbb{Q}(A_t\,|\mathcal{F}_{t-1})} = \frac{\mathbb{P}(A_tB_t\,|\mathcal{F}_{t-1})}{\mathbb{P}(A_t\,|\mathcal{F}_{t-1})},\ \frac{\mathbb{Q}(A_t^cB_t^c\,|\mathcal{F}_{t-1})}{\mathbb{Q}(A_t^c\,|\mathcal{F}_{t-1})} = \frac{\mathbb{P}(A_t^cB_t^c\,|\mathcal{F}_{t-1})}{\mathbb{P}(A_t^c\,|\mathcal{F}_{t-1})}$$

is indeed the claimed minimizer. In fact, we may show, that

$$E_Q\left(\log\frac{Q(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}{\mathbb{P}(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}\right) \geq E_{\mathbb{Q}}\left(\log\frac{\mathbb{Q}(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}{\mathbb{P}(\xi_{t+1},\eta_{t+1}\,|\mathcal{F}_t)}\right)$$

and the result follows. □

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