BOUNDS AND ASYMPTOTIC APPROXIMATIONS FOR UTILITY PRICES WHEN VOLATILITY IS RANDOM[*]

RONNIE SIRCAR[†] AND THALEIA ZARIPHOPOULOU[‡]

Abstract. This paper is a contribution to the valuation of derivative securities in a stochastic volatility framework, which is a central problem in financial mathematics. The derivatives to be priced are of European type with the payoff depending on both the stock and the volatility. The valuation approach uses utility-based criteria under the assumption of exponential risk preferences. This methodology yields the indifference prices as solutions to second order quasilinear PDEs. Two sets of price bounds are derived that highlight the important ingredients of the utility approach, namely, nonlinear pricing rules with dynamic certainty equivalent characteristics, and pricing measures depending on correlation and the Sharpe ratio of the traded asset. The problem is further analyzed by asymptotic methods in the limit of the volatility being a fast mean-reverting process. The analysis relates the traditional market-selected volatility risk premium approach and the preference-based valuation techniques.

Key words. financial mathematics, derivative pricing, stochastic volatility, utility indifference pricing

AMS subject classifications. 91B28, 91B70, 93E20, 34E05

DOI. 10.1137/S0363012903409253

1. Introduction. We study the utility indifference pricing mechanism for European derivative contracts in financial markets with uncertain volatility. As is well known, in such incomplete markets, there are many possible no-arbitrage pricing (or "risk-neutral") measures and typically an interval of arbitrage-free option prices. The traditional pricing methodology is that the market selects a pricing probability measure that is reflected in the prices of liquidly traded derivative contracts (for example, at-the-money call options). Indifference pricing is an alternative mechanism whereby a no-arbitrage price is selected according to investment optimality criteria of a risk-averse investor. Our analysis, using bounds and asymptotic approximations, sheds some light on the relation between the two. Specifically, Theorem 3.2 shows that the nonlinear utility pricing rule lies between a linear no-arbitrage pricing rule and an insurance-type certainty equivalent pricing rule.

Stochastic volatility models are popular because they capture the deviation of stock price data from the Black–Scholes geometric Brownian motion model in a parsimonious way. They were originally introduced in the late 1980's by Hull and White [22] and others for option pricing. Much of their success derives from their predicted option prices exhibiting the implied volatility skew that is observed in many options markets. See [17], for example, for details.

However, a market with stochastic volatility is incomplete in that volatility is a source of uncertainty that is not traded. Therefore, enforcement of *no arbitrage* does not lead to a unique derivative pricing rule. The usual way to "close" the model is

---

[†]Department of Operations Research & Financial Engineering, Princeton University, E-Quad, Princeton, NJ 08544 (sircar@princeton.edu). The research of this author was partially supported by NSF grant SES-0111499.

[‡]Departments of Mathematics and Management Science and Information Systems, The University of Texas at Austin, Austin, TX 78712 (zariphop@math.utexas.edu). The research of this author was partially supported by NSF grants DMS-0102909 and DMS-0091946.

to assume the market chooses a pricing measure which is implicit in the prices of liquidly traded options. The indifference pricing mechanism is an alternative in which the price is uniquely (and endogenously) determined at the cost of depending on the preferences of the pricer. It has been studied in various incomplete market problems, for example, when there are transaction costs [6, 11, 21] or nontraded assets [10, 31], and under exponential utilities [33].

Our analysis is presented as follows. In section 2, we describe the mechanism in the context of a standard stochastic volatility model and characterize the indifference price in terms of solutions of related Hamilton–Jacobi–Bellman (HJB) equations. We derive a quasilinear PDE (2.32) that the pricing function satisfies. In addition, a specific measure, $\mathcal{Q}$ in Definition 2.7, emerges as a natural "prior" pricing measure, and the indifference price can be characterized as a worst-case expected payoff, penalized by relative entropy with respect to this prior (section 2.1).

In section 3, we derive two sets of bounds for the indifference price by analysis of the associated HJB equations. Section 4 presents asymptotic approximations that relate the indifference price to a particular *no-arbitrage* price. Finally, section 5 concludes and lists some remaining questions about the mechanism for future investigation.

2. Indifference prices. We assume a dynamic market setting with two assets, a riskless bond B, and a stock S. The stock price is modeled as a diffusion process satisfying

$$(2.1) \qquad\qquad dS_s = \mu S_s\, ds + \sigma(Y_s, s) S_s\, dW_s^1, \qquad s \geq 0,$$

with $\mu > 0$. The volatility coefficient of the stock is driven by the stochastic factor $Y \in \mathbb{R}$ which is modeled as a correlated diffusion satisfying

$$(2.2) \qquad\qquad dY_s = b(Y_s, s)\, ds + a(Y_s, s)\, (\rho\, dW_s^1 + \rho'\, dW_s^2),$$

with $\rho \in (-1, 1)$ the correlation coefficient and $\rho' = \sqrt{1 - \rho^2}$.

The processes W^1 and W^2 are independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$, where $\mathcal{F}_s$ is the augmented σ-algebra generated by $((W_u^1, W_u^2); 0 \leq u \leq s)$. We assume that a riskless bond with maturity T is available for trading, yielding constant interest rate $r = 0$. The case $r \neq 0$ can be treated using standard discounting arguments and it is not presented herein. The derivative to be priced is of European type with payoff $g(S_T, Y_T)$ at expiration T. We make the following assumptions throughout.

ASSUMPTION 1.
1. *The volatility function $\sigma(\cdot)$ and the diffusion coefficient $a(\cdot, \cdot)$ are smooth and bounded above and below away from zero.*
2. *The drift coefficient $b(\cdot, \cdot)$ in (2.2) is Lipschitz continuous on $\mathbb{R} \times [0, T]$.*
3. *The payoff function $g(\cdot, \cdot)$ is smooth and bounded.*

Under these assumptions, (2.1) and (2.2) have a unique solution with $S_s \geq 0$ $\mathbb{P}$-a.s., $s \geq 0$ a.e. The assumption on the payoff excludes put options (whose payoff has discontinuous first derivative) and call options (which are unbounded). Handling these issues will require regularization methods (see, for example, [19]) which we do not address in this paper.

The utility-based valuation method relies on the comparison of maximal expected utilities corresponding to investment opportunities with and without the derivative. In both settings, trading takes place between the bond and the stock, and the objective

is to maximize the terminal utility of wealth. The investor starts, at time $t \geq 0$, with initial endowment x and dynamically rebalances his portfolio allocations, say, π_s^0 and π_s, representing the amounts invested at time $s \geq t$ in the bond and the stock accounts. It is assumed that no intermediate consumption or infusion of extraneous funds is allowed. The total current wealth satisfies

$$(2.3) \qquad X_s = \pi_s^0 + \pi_s, \quad t \leq s \leq T,$$

and thus solves the state controlled diffusion equation

$$(2.4) \qquad \begin{cases} dX_s = \mu \pi_s \, ds + \sigma(Y_s, s) \pi_s \, dW_s^1, & t \leq s \leq T, \\ X_t = x. \end{cases}$$

The above equation can be easily derived from (2.1) and the budget constraint (2.3) (see Merton [30]). Note that because the coefficients in (2.1) are linear in S, the latter does not appear explicitly in (2.4). Moreover, the budget constraint results in eliminating the first control variable π_s^0. The single control variable π_s is called admissible if it is $\mathcal{F}_s$-measurable and satisfies the integrability constraint $E \int_t^T \sigma(Y_s, s)^2 \pi_s^2 \, ds < +\infty$. The set of admissible policies is denoted by $\mathcal{A}$.

The next task is to introduce and analyze the three fundamental optimal investment problems via which the indifference prices for the writer and the buyer of the derivative will be constructed. Throughout the analysis, it is assumed that the individual preferences are modeled via an exponential utility function

$$(2.5) \qquad U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R},$$

with risk-aversion parameter $\gamma > 0$, and that they remain the same, independently of whether the derivative is written, bought, or not traded at all. The first is the classical Merton portfolio optimization problem, appropriately modified to accommodate stochastic volatility. Its value function is

$$(2.6) \qquad V(x, y, t) = \sup_{\mathcal{A}} E(-e^{-\gamma X_T} \mid X_t = x, \ Y_t = y),$$

where X and Y solve (2.4) and (2.2), respectively. The investor seeks to maximize his terminal expected utility.

If the derivative with payoff $g(S_T, Y_T)$ is written, the *writer's* value function is

$$(2.7) \qquad u^w(x, S, y, t) = \sup_{\mathcal{A}} E(-e^{-\gamma(X_T - g(S_T, Y_T))} \mid X_t = x, \ S_t = S, \ Y_t = y),$$

and if the derivative is bought, the *buyer's* value function is

$$(2.8) \qquad u^b(x, S, y, t) = \sup_{\mathcal{A}} E(-e^{-\gamma(X_T + g(S_T, Y_T))} \mid X_t = x, \ S_t = S, \ Y_t = y).$$

It is immediate that

$$(2.9) \qquad u^w(x, S, y, t; g) = u^b(x, S, y, t; -g).$$

The payoffs in (2.7) and (2.8) reflect, respectively, the obligation of the writer and the compensation of the buyer at expiration T.

A fundamental assumption is that both the writer and the buyer optimize over the same set of admissible strategies . Moreover, the traditional no-bankruptcy constraint $X_s \geq 0$ a.e. $t \leq s \leq T$ is not imposed herein due to the fact that exponential

utilities may allow for negative wealth levels. Imposing such constraints affects significantly the nature of the indifference prices and, in most cases, yields prices that are considerably high and not applicable. For models with transaction costs, such issues were studied by Constantinides and Zariphopoulou [6, 7].

We next review the definition of indifference prices (see Hodges and Neuberger [21]). The *writer's indifference price* of the European claim $g(S_T, Y_T)$ is defined as the amount $h^w = h^w(x, S, y, t)$, such that the writer is indifferent to the following two scenarios: optimize the utility payoff without writing the derivative and optimize the utility payoff with the *liability* $g(S_T, Y_T)$ at expiration, but with an initial *compensation* $h^w(x, S, y, t)$ at the time of inscription t. Similarly, the *indifference buyer's price* of the European claim $g(S_T, Y_T)$ is defined as the amount $h^b = h^b(x, S, y, t)$ such that the buyer is indifferent to the following two scenarios: optimize the utility payoff without buying the derivative and optimize the utility payoff with the *payoff* $g(S_T, Y_T)$ at expiration, but with the initial *cost* $h^b(x, S, y, t)$ at the time of inscription t.

DEFINITION 2.1. *The indifference prices h^w and h^b are defined by*

$$(2.10) \qquad V(x, y, t) = u^w\Big(x + h^w(x, S, y, t), S, y, t\Big),$$

$$(2.11) \qquad V(x, y, t) = u^b\Big(x - h^b(x, S, y, t), S, y, t\Big).$$

The above definition allows for derivative prices that depend on the investor's wealth, as reflected in their x-argument. Such dependence might look like an undesirable feature given the wealth-independent prices that arbitrage-free theory yields in complete markets. As the calculations below show, the choice of exponential utility leads to wealth-independent prices, at least for the case of European claims and in the absence of trading constraints. Wealth independence, however, might not hold for other choices of risk preferences and/or trading constraints. In such situations, universality may be achieved by relaxing the notion of indifference price to *reservation prices*. The latter prices are defined as wealth-independent price bounds for which the price equalities (2.10) and (2.11) hold as inequalities (see, for example, [6, 7]).

The value functions V, u^w, and u^b, whose arguments will yield the indifference prices, may be studied via their associated HJB equations. Although these equations are fully nonlinear, the convenience of the exponential utility function allows us to construct classical solutions after an initial separation of variables. It follows from a standard viscosity solution argument that these solutions coincide with the respective value functions.

To facilitate the presentation, we introduce the following operators and Hamiltonians:

$$(2.12) \qquad \mathcal{A}^{(S,y)}u = \frac{1}{2}\sigma(y, t)^2 S^2 u_{SS} + \rho\sigma(y, t)a(y, t)Su_{Sy}$$

$$+ \frac{1}{2}a(y, t)^2 u_{yy} + \mu Su_S + b(y, t)u_y,$$

$$(2.13) \qquad \mathcal{A}^{(y)}u = \frac{1}{2}a(y, t)^2 u_{yy} + b(y, t)u_y,$$

$$\mathcal{H}^{(S,y)}(u_{xx}, u_{xy}, u_{xS}, u_x) = \max_\pi \Big(\frac{1}{2}\sigma(y, t)^2\pi^2 u_{xx} + \pi(\rho\sigma(y, t)a(y, t)u_{xy}$$

$$(2.14) \qquad + \sigma(y, t)^2 Su_{xS} + \mu u_x)\Big),$$

$$(2.15) \quad \mathcal{H}^{(y)}(u_{xx}, u_{xy}, u_x) = \max_\pi \Big(\frac{1}{2}\sigma(y, t)^2\pi^2 u_{xx} + \pi(\rho\sigma(y, t)a(y, t)u_{xy} + \mu u_x)\Big).$$

Note that $\mathcal{A}^{(S,y)}$ is the infinitesimal generator of the Markov process (S, Y), and $\mathcal{A}^{(y)}$ is the infinitesimal generator of Y, which is a Markov process by itself in our stochastic volatility models.

The HJB equation associated with the value function u^w defined in (2.7) is

$$(2.16) \qquad u_t + \mathcal{A}^{(S,y)}u + \mathcal{H}^{(S,y)}(u_{xx}, u_{xy}, u_{xS}, u_x) = 0,$$
$$u(x, S, y, T) = -e^{-\gamma(x-g(S,y))},$$

on $D = \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times [0, T]$. The relevant PDE for u^b, defined in (2.8), is the same as (2.16), with the sign of g changed in the terminal condition. For V, defined in (2.6), we simply set g to zero and remove the S-derivatives from the equation (i.e., $\mathcal{A}^{(S,y)}$ and $\mathcal{H}^{(S,y)}$ are replaced by $\mathcal{A}^{(y)}$ and $\mathcal{H}^{(y)}$, respectively).

In the theorems below, we produce a closed form expression for the value function V and we provide regularity results for $u^{w,b}$. The proofs are based on the construction of a candidate solution that is actually smooth and therefore a classical solution of the HJB equation. We readily identify it also as the unique viscosity solution and, from there, by a standard argument, with the value function. The intermediate step is required because classical verification theorems require polynomial growth (in x) restrictions on the value functions, which do not hold with exponential utility.

For convenience, we introduce

$$(2.17) \qquad \mathcal{L}^{(S,y)}u = \mathcal{A}^{(S,y)}u - \rho\frac{\mu}{\sigma(y,t)}a(y,t)u_y,$$

$$(2.18) \qquad \mathcal{L}^{(y)}u = \mathcal{A}^{(y)}u - \rho\frac{\mu}{\sigma(y,t)}a(y,t)u_y,$$

$$(2.19) \quad \mathcal{M}(G_S, G_y, G) = \frac{1}{2}\sigma(y,t)^2 S^2 \frac{G_S^2}{G} + \rho\sigma(y,t)a(y,t)S\frac{G_S G_y}{G} + \frac{1}{2}\rho^2 a(y,t)^2 \frac{G_y^2}{G}.$$

THEOREM 2.2. *The value function V is given by*

$$(2.20) \qquad V(x, y, t) = -e^{-\gamma x}f(y,t)^{\frac{1}{1-\rho^2}},$$

where f solves

$$(2.21) \qquad f_t + \mathcal{L}^{(y)}f = \frac{1}{2}(1-\rho^2)\frac{\mu^2}{\sigma(y,t)^2}f$$

in $y \in \mathbb{R}$, $t < T$, with $f(y,T) = 1$ for $y \in \mathbb{R}$.

Proof. We first consider a candidate solution of the form $\tilde{V}(x, y, t) = -e^{-\gamma x}F(y,t)$. This form is suggested by the scaling properties of the exponential utility. We recall that V solves the HJB equation (2.16) for $g = 0$ which reduces to

$$(2.22) \qquad V_t + \mathcal{H}^{(y)}(V_{xx}, V_{xy}, V_x) + \mathcal{A}^{(y)}u = 0,$$

with $V(x, y, T) = -e^{-\gamma x}$. Evaluating this equation at the candidate solution $\tilde{V}$ yields that F must satisfy the quasilinear equation

$$(2.23) \qquad F_t + \mathcal{L}^{(y)}F = \frac{1}{2}\frac{\mu^2}{\sigma(y,t)^2}F + \frac{1}{2}\rho^2 a(y,t)^2\frac{F_y^2}{F},$$

with $F(y, T) = 1$. A power transformation $F = f^\delta$ for $\delta = \frac{1}{1-\rho^2}$ yields that f must solve the linear PDE (2.21), with $f(y, T) = 1$. The fact that the linear equation

(2.21) has a unique smooth and bounded solution follows from Assumption 1 on the coefficients (see [26, Theorem 2.9.10], for example). We then deduce that the candidate $\tilde{V}$ satisfies the HJB equation (2.22), with terminal value $-e^{-\gamma x}$, and it is also smooth. Therefore, it is a classical solution of (2.22). To conclude, we use uniqueness results for viscosity solutions of the HJB equation. This approach has become by now familiar in incomplete market models (see [34] for an overview).

Following the arguments of Theorems 4.1 and 4.2 in [13], we deduce that the value function V defined in (2.6) is the *unique* viscosity solution of (2.22). Uniqueness holds in the class of functions that are concave and of exponential growth in the wealth argument and are uniformly bounded in the variable y. (We note that in the model analyzed in [13], market incompleteness was generated by stochastic labor income modeled as a correlated diffusion. The two associated HJB equations have similar nonlinearities, and the technical arguments work under rather minor modifications.) We next observe that the candidate $\tilde{V}$ is smooth and therefore a viscosity solution of (2.22). Moreover, the assumptions on the model coefficients yield that it belongs to the class of viscosity solutions in which uniqueness holds. Therefore, $\tilde{V} \equiv V$ and the result follows. Note that classical verification results (for instance, [14, Theorem III.8.1]) require more stringent polynomial growth conditions on the candidate solution, a requirement that is bypassed by the viscosity arguments. $\square$

DEFINITION 2.3. *Let $\mathbb{P}$ be the historical measure. We define an equivalent measure $\widetilde{\mathbb{P}}$ by*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{\mu}{\sigma(Y_s, s)}\, dW_s^1 - \frac{1}{2}\int_0^T \frac{\mu^2}{\sigma(Y_s, s)^2}\, ds\right).$$

By Girsanov's theorem, the dynamics of (S, Y) under $\widetilde{\mathbb{P}}$ are

$$(2.24) \quad dS_s = \sigma(Y_s, s)S_s\, d\widetilde{W}_s^1,$$

$$(2.25) \quad dY_s = \left(b(Y_s, s) - \rho\frac{\mu}{\sigma(Y_s, s)}a(Y_s, s)\right)ds + a(Y_s, s)\left(\rho\, d\widetilde{W}_s^1 + \rho'd\widetilde{W}_s^2\right),$$

where $\widetilde{W}_s^1 = W_s^1 + \int_0^s \frac{\mu}{\sigma(Y_u, u)}\, du$ and $\widetilde{W}_s^2 = W_s^2$ are independent $\widetilde{\mathbb{P}}$-Brownian motions. The measure $\widetilde{\mathbb{P}}$ is often known as the minimal martingale measure [16].

From the formula (2.20) for V, the Feynman–Kac representation of the solution to (2.21), and the definition of the measure $\widetilde{\mathbb{P}}$, we obtain the following proposition.

PROPOSITION 2.4. *The solution f of (2.21) admits the probabilistic representation*

$$(2.26) \qquad f(y, t) = \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{-\int_t^T \frac{\mu^2(1-\rho^2)}{2\sigma(Y_s, s)^2}\, ds} \mid Y_t = y\right),$$

where the process Y satisfies (2.25). The value function V is then given by

$$(2.27) \qquad V(x, y, t) = -e^{-\gamma x}\left(\mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{-\int_t^T \frac{\mu^2(1-\rho^2)}{2\sigma(Y_s, s)^2}\, ds} \mid Y_t = y\right)\right)^{\frac{1}{1-\rho^2}}.$$

THEOREM 2.5. *The writer's value function u^w is given by*

$$(2.28) \qquad u^w(x, S, y, t) = -e^{-\gamma x}G(S, y, t),$$

where $G \in C^{2,2,1}(\mathbb{R}^+ \times \mathbb{R} \times [0,T])$ is the unique bounded solution of the quasilinear equation

$$(2.29) \qquad G_t + \mathcal{L}^{(S,y)}G = \frac{1}{2}\frac{\mu^2}{\sigma(y,t)^2}G + \mathcal{M}(G_S, G_y, G),$$

with $G(S,y,T) = e^{\gamma g(S,y)}$.

Proof. We look at a candidate solution of the form $\tilde{u}(x,S,y,t) = -e^{-\gamma x}G(S,y,t)$. Straightforward calculations in (2.16) imply that G must satisfy the quasilinear equation (2.29). Note that G must be positive as u is negative.

A logarithmic transformation $G = e^\phi$ gives that ϕ should solve a quasilinear equation with quadratic nonlinearity, namely,

$$(2.30) \qquad \phi_t + \mathcal{L}^{(S,y)}\phi + \frac{1}{2}a(y)^2(1-\rho^2)\phi_y^2 = \frac{\mu^2}{2\sigma(y)^2},$$

with $\phi(S,y,T) = \gamma g(S,y)$. Equation (2.30) is the familiar HJB equation of a *quadratic cost* stochastic control problem (see Fleming and Rishel [15, section VI.5]). Under Assumption 1, we obtain that ϕ is bounded, $\phi \in C^{2,2,1}(\mathbb{R}^+ \times \mathbb{R} \times [0,T])$, and that it is the unique solution in this class (see, for example, Ladyzenskaja, Solonnikov, and Uralceva [27], Fleming and Rishel [15], or Pham [32]). This in turn yields the same properties for G. Following similar arguments as in the proof of Theorem 2.2 to identify $\tilde{u}$ as the unique viscosity solution of (2.16) and therefore the value function, we conclude that $u^w = \tilde{u}$.   □

We note that even though a simple power transformation can linearize the reduced equation (2.23) in the one dimensional case, quasilinear equations of the form (2.29) cannot be linearized in higher dimensional settings unless the nonlinearity is a quadratic in ∇G, where ∇ denotes the gradient with respect to the spatial variables, and there are no cross-derivative terms. Of course under a logarithmic transformation, one may reduce (2.29) to (2.30).

Before we construct the indifference prices, we introduce some convenient notation.

LEMMA 2.1. *Let*

$$(2.31) \qquad L(y,t) = \frac{1}{\rho'}a(y,t)\frac{f_y(y,t)}{f(y,t)},$$

with f given in (2.26). Then under Assumption 1, L is smooth, and bounded.

Proof. From (2.21), f is positive, smooth, and bounded for fixed $t < T$ under Assumption 1. To establish that $f_y(y,t)$ is also smooth and bounded, it suffices to differentiate (2.21) with respect to y and use the relevant probabilistic representation of f_y.   □

THEOREM 2.6. *(i) The writer's indifference price h^w is the unique $C^{2,2,1}(\mathbb{R}^+ \times \mathbb{R} \times [0,T])$ bounded solution of the pricing equation*

$$(2.32) \qquad h_t^w + \mathcal{L}^{(S,y)}h^w + \rho'a(y,t)L(y,t)h_y^w + \frac{1}{2}\gamma(1-\rho^2)a(y,t)^2(h_y^w)^2 = 0$$

with $h^w(S,y,T) = g(S,y)$.

(ii) The buyer's indifference price satisfies

$$(2.33) \qquad h^w(S,y,t;g) = -h^b(S,y,t;-g)$$

and solves

$$(2.34) \qquad h_t^b + \mathcal{L}^{(S,y)} h^b + \rho' a(y,t) L(y,t) h_y^b - \frac{1}{2}\gamma(1-\rho^2) a(y,t)^2 (h_y^b)^2 = 0$$

with $h^b(S,y,T) = g(S,y)$.

Proof. We discuss only the arguments for h^w. To this end, we first observe that the pricing equality (2.10) together with the representations (2.20) and (2.28) of the value functions V and u^w yield that h^w is given by

$$(2.35) \qquad h^w(S,y,t) = \frac{1}{\gamma} \ln \frac{G(S,y,t)}{f(y,t)^{1/(1-\rho^2)}},$$

and as such is independent of x. Direct substitution shows that h^w solves the claimed quasilinear equation (2.32). It also satisfies the terminal condition $h^w(S,y,T) = g(S,y)$. In (2.32), the coefficient of the additional h_y^w term is smooth and bounded by Lemma 2.1. The uniqueness and regularity results for h^w follow from an appropriate adaptation of Theorem 4.1 in Pham [32], which utilizes that (2.32), like (2.30) for ϕ, is the HJB equation of a quadratic cost control problem. The parity property (2.33) follows from the definition of the indifference prices and the properties of the value functions u^w and u^b. $\quad\square$

The indifference price equation (2.32) indicates that a new measure, defined below, emerges from the utility-based valuation.

DEFINITION 2.7. *We define $\mathcal{Q}$ by*

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \frac{\mu}{\sigma(Y_s,s)}\, dW_s^1 + \int_0^T L(Y_s,s)\, dW_s^2\right.$$
$$\left. -\frac{1}{2}\int_0^T \left(\frac{\mu^2}{\sigma(Y_s,s)^2} + L(Y_s,s)^2\right) ds\right).$$

In the language of stochastic volatility models, the function $-L$ is a particular market price of volatility risk (and $\mathcal{Q}$ a particular equivalent martingale measure), as described in section 4.1.1. In fact, $\mathcal{Q}$ is the minimal relative entropy martingale measure, as we discuss in section 2.1.

It is worth observing that the price equation (2.32) does not have a zeroth order term or drift terms in S. It does not have a nonlinear term involving h_S either. This is an immediate consequence of the assumptions of zero interest rate and that the stock is tradeable. We also observe that the only place where the risk-aversion coefficient γ appears is in front of the non-linear term that directly reflects market incompleteness. The latter has also a fixed sign with respect to γ which, in turn, yields the following intuitive result.

THEOREM 2.8. *The writer's (resp., buyer's) indifference price is nondecreasing (resp., nonincreasing) with respect to the risk-aversion parameter γ. As $\gamma \to 0$, the writer's and buyer's indifference prices satisfy*

$$(2.36) \qquad \lim_{\gamma\downarrow 0} h^{w,b}(S,y,t) = \mathbb{E}_{\mathcal{Q}}(g(S_T,Y_T)\mid S_t = S, Y_t = y),$$

where the measure $\mathcal{Q}$ is defined above.

Proof. We denote by h^{γ_1} and h^{γ_2} the writer's indifference prices corresponding to risk-aversion coefficients γ_1 and γ_2 with $\gamma_1 < \gamma_2$. Straightforward calculations show

that h^{γ_1} is a subsolution of the indifference pricing equation that h^{γ_2} satisfies and that $h^{\gamma_1}(S, y, T) = h^{\gamma_2}(S, y, T)$. Using classical comparison results for (2.32), we conclude.

We observe that as $\gamma \to 0$, the indifference pricing equation (2.32) formally converges to the linear equation

$$(2.37) \qquad h_t^0 + \mathcal{L}^{(S,y)} h^0 + \rho' a(y, t) L(y, t) h_y^0 = 0$$

with terminal condition $h^0(S, y, T) = g(S, y)$. Under Assumption 1, the latter has a unique smooth (and thus viscosity) solution given by the Feynman–Kac formula

$$(2.38) \qquad h^0(S, y, t) = \mathbb{E}_{\mathcal{Q}}(g(S_T, Y_T) \mid S_t = S, Y_t = y).$$

The stability properties of viscosity solutions (see Lions [29, Proposition I.3]) yield that $\{h^\gamma(S, y, t)\}$ converges, along subsequences, to the viscosity solution of (2.37) and, by uniqueness, we conclude.

The analogous result for the buyer's price follows from similar calculations.   □

Expressions such as (2.38) have also been obtained in other utility-based pricing approaches [9, 25, 24]. It is well known that the indifference price (buyer's or writer's) of $\alpha > 0$ derivative contracts with bounded payoff g, written $h(\alpha; \gamma)$ as a function of the quantity and risk-aversion parameter, satisfies

$$h(\alpha; \gamma) = \alpha h(1; \alpha \gamma),$$

as is clear in the current context from the PDEs (2.32) and (2.34) by replacing g by αg in the terminal condition and making the change of variable $h = \alpha h'$. Therefore, taking the limit of zero risk-aversion is analogous to taking the limit of the price per unit $h(\alpha; \gamma)/\alpha$ as α goes to zero (with $\gamma > 0$ fixed). This is how the "fair" price is defined in [9], and the measure $\mathcal{Q}$ that arises in its characterization is labeled the neutral pricing measure in [24]. In the next section, we also point out that $\mathcal{Q}$ is in fact the minimal relative entropy measure.

2.1. Interpretation via relative entropy penalization. One way to interpret the utility-based valuation mechanism is in terms of relative entropy penalization. This is a specific example of the well-known connection between exponential utility and entropy as discussed, for example, in [12, 33].

Recall that $\mathcal{Q}$ denotes the probability measure under which the dynamics of (S, Y) are

$$(2.39) \qquad dS_s = \sigma(Y_s, s) S_s \, dW_s^{\mathcal{Q}(1)},$$

$$dY_s = \left(b(Y_s, s) - \rho \frac{\mu}{\sigma(Y_s, s)} a(Y_s, s) + \rho' a(Y_s, s) L(Y_s, s) \right) ds$$

$$\qquad + a(Y_s, s) \left(\rho \, dW_s^{\mathcal{Q}(1)} + \rho' dW_s^{\mathcal{Q}(2)} \right),$$

where $W_s^{\mathcal{Q}(1)} = W_s^1 + \int_0^s \frac{\mu}{\sigma(Y_u, u)} \, du$ and $W_s^{\mathcal{Q}(2)} = W_s^2 - \int_0^s L(Y_u, u) \, du$ are independent $\mathcal{Q}$-Brownian motions. Note that $\mathcal{Q}$ is already a "risk-neutral" martingale measure because S is a $\mathcal{Q}$-martingale.

Then, let $\mathbb{P}^{(\lambda)}$ be any *equivalent local martingale measure*, which, in this context, is parameterized by an adapted process, λ, say, with $\int_0^T \lambda_s^2 \, ds < \infty$ a.s., such that

$$\frac{d\mathbb{P}^{(\lambda)}}{d\mathcal{Q}} = \exp \left(-\int_0^T \lambda_s \, dW_s^{\mathcal{Q}(2)} - \frac{1}{2} \int_0^T \lambda_s^2 \, ds \right).$$

That is, defining $W_s^{\lambda(2)} = W_s^{\mathcal{Q}(2)} + \int_0^s \lambda_u\, du$, $(W^{\mathcal{Q}(1)}, W^{\lambda(2)})$ are independent Brownian motions under $\mathbb{P}^{(\lambda)}$, and the dynamics of (S, Y) are described by (2.39) and

$$dY_s = \left(b(Y_s, s) - \rho \frac{\mu}{\sigma(Y_s, s)} a(Y_s, s) + \rho' a(Y_s, s)(L(Y_s, s) - \lambda_s) \right) ds$$
$$+ a(Y_s, s)\, (\rho\, dW_s^{\mathcal{Q}(1)} + \rho'\, dW_s^{\lambda(2)}).$$

We can think of $\mathcal{Q}$ as a *prior* risk-neutral measure, and we define the relative entropy $H_t(\mathbb{P}^{(\lambda)} \mid \mathcal{Q})$ between the conditional laws on the process $\{(S_s, Y_s), t \le s \le T\}$ starting at the same point (S, y) at time t as follows. First, let (ξ_s) denote the Radon–Nikodym process

$$\xi_s = \mathbb{E}_{\mathcal{Q}} \left(\frac{d\mathbb{P}^{(\lambda)}}{d\mathcal{Q}} \mid \mathcal{F}_s \right).$$

We then define

$$H_t(\mathbb{P}^{(\lambda)} \mid \mathcal{Q}) = \mathbb{E}_{\mathbb{P}^{(\lambda)}} \left(\ln(\xi_T / \xi_t) \mid \mathcal{F}_t \right).$$

By direct calculation, this is a quadratic penalization on the "additional" volatility risk premium λ:

$$(2.40) \qquad H_t(\mathbb{P}^{(\lambda)} \mid \mathcal{Q}) = \frac{1}{2} \mathbb{E}_{\mathbb{P}^{(\lambda)}} \left(\int_t^T \lambda_s^2\, ds \mid \mathcal{F}_t \right).$$

We denote by $\mathcal{M}_f$ the set of λ with

$$\mathbb{E}_{\mathbb{P}^{(\lambda)}} \left(\int_0^T \lambda_s^2\, ds \right) < \infty,$$

which guarantees finiteness of the relative entropies $H_t(\mathbb{P}^{(\lambda)} \mid \mathcal{Q})$.

Therefore, we can interpret the writer's indifference pricing mechanism as choosing a measure which tries to maximize the derivative's expected payout but is constrained from deviating too far from the prior in terms of relative entropy:

$$(2.41) \qquad h^w = \sup_{\lambda \in \mathcal{M}_f} \left[\mathbb{E}_{\mathbb{P}^{(\lambda)}} \left(g(S_T, Y_T) \mid \mathcal{F}_t \right) - \frac{1}{\gamma} H_t(\mathbb{P}^{(\lambda)} \mid \mathcal{Q}) \right].$$

This is because the HJB equation associated with this stochastic control problem is (2.32), as follows from using the formula (2.40). Notice that the upshot of the utility mechanism is to identify the prior $\mathcal{Q}$ which does not depend on the risk-aversion γ or the claim g being priced. The relative entropy arises naturally as the dual of the exponential utility, and γ^{-1} weights the penalty term.

In [12], the writer's indifference price (at time $t = 0$) is characterized as

$$(2.42) \qquad h^w = \sup_{Q \in P_f} \left[\mathbb{E}_Q(g(S_T, Y_T)) - \frac{1}{\gamma} H_0(Q \mid \mathbb{P}) \right] - \sup_{Q \in P_f} \left[-\frac{1}{\gamma} H_0(Q \mid \mathbb{P}) \right]$$

(their equation (5.6)) under quite general conditions. The set P_f consists of measures Q that are absolutely continuous with respect to $\mathbb{P}$, such that the wealth process X is a $(Q, \mathcal{F})$-local martingale, and $H_0(Q \mid \mathbb{P}) < \infty$. In (2.42), the indifference price

is given as the difference between the solutions of two optimization problems, with $\mathbb{P}$ as the prior measure. Our expression (2.41) gives the indifference price under our diffusion stochastic volatility models as the solution of a single optimization problem with prior (risk-neutral) measure $\mathcal{Q}$ (which arises from the solution of the Merton problem).

In fact, it is easy to show from the associated HJB equations that $\mathcal{Q}$ is the minimal relative entropy martingale measure (see Fritelli [20]) solution of

$$\sup_{Q \in P_f} \left(-H_0(Q \mid \mathbb{P}) \right)$$

and that

$$H_0(\mathcal{Q} \mid \mathbb{P}) = -\frac{1}{1-\rho^2} \ln f \mid_{t=0},$$

$$\sup_{Q \in P_f} \left[\mathbb{E}_Q(g(S_T, Y_T)) - \frac{1}{\gamma} H_0(Q \mid \mathbb{P}) \right] = \frac{1}{\gamma} \ln G \mid_{t=0},$$

which connects (at time $t = 0$) our expression (2.35) with (2.42), which is taken from [12].

Relative entropy minimization has been extensively used for calibration from market data, with additional constraints that some benchmark derivative contracts are priced exactly [2, 4]. The prior measure $\mathcal{Q}$ will also emerge later in section 3.1 in obtaining bounds for the indifference prices.

3. Indifference price spreads. The indifference pricing equations (2.32) and (2.34) do not in general have explicit solutions. Given that risk aversion is taken into account in the utility-based valuation, one would naturally expect to recover indifference prices in terms of the so-called certainty equivalent pricing rule. This is in fact the classical risk-based pricing device used in the de facto incomplete insurance market (see, for example, Bowers et al. [3]). However, as simple calculations show, indifference prices do not correspond to straightforward generalizations of certainty equivalents given that they result from an interplay between dynamic optimization among investment opportunities and risk monitoring. We note that classical arbitrage-free financial markets use a risk-neutral measure and incomplete insurance markets use the historical one.

In what follows we aim at addressing some of the above issues by looking at price bounds. We derive bounds on the indifference prices that involve characteristics of linear and nonlinear prices, namely, expected payoffs, certainty equivalents, and related pricing measures.

PROPOSITION 3.1. *Let $\widetilde{\mathbb{P}}$ be the minimal martingale measure described in Definition 2.3, and define*

$$R(y, t) = \frac{1}{2\gamma} \left(\frac{\mu}{\sigma(y, t)} \right)^2,$$

with $\mu/\sigma(y, t)$ being the (time-varying) Sharpe ratio of the traded stock, and

$$\zeta(S, y, t) = \frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}_{\widetilde{\mathbb{P}}} \left(e^{-\gamma(1-\rho^2) \int_t^T R(Y_s, s) \, ds} \mid S_t = S, Y_t = y \right).$$

(i) *The writer's indifference price h^w satisfies*

(3.1) $h^w(S,y,t) \le \frac{1}{\gamma} \ln \mathbb{E}_{\widetilde{\mathbb{P}}} \left(e^{\gamma \left(g(S_T, Y_T) - \int_t^T R(Y_s, s)\, ds \right)} \mid S_t = S, Y_t = y \right) - \zeta(S,y,t),$

(3.2) $h^w(S,y,t) \ge \mathbb{E}_{\widetilde{P}} \left(g(S_T, Y_T) - \int_t^T R(Y_s, s)\, ds \mid S_t = S, Y_t = y \right) - \zeta(S,y,t).$

(ii) *The buyer's indifference price h^b satisfies*

$h^b(S,y,t) \le \mathbb{E}_{\widetilde{\mathbb{P}}} \left(g(S_T, Y_T) + \int_t^T R(Y_s, s)\, ds \mid S_t = S, Y_t = y \right) + \zeta(S,y,t),$

$h^b(S,y,t) \ge -\frac{1}{\gamma} \ln \mathbb{E}_{\widetilde{\mathbb{P}}} \left(e^{-\gamma \left(g(S_T, Y_T) + \int_t^T R(Y_s, s)\, ds \right)} \mid S_t = S, Y_t = y \right) + \zeta(S,y,t).$

Proof. We first present the arguments for the derivation of the lower bound (3.2). We recall that

(3.3) $$h(S,y,t) = \frac{1}{\gamma} \ln \frac{G(S,y,t)}{V(y,t)} = \frac{1}{\gamma} \ln \frac{G(S,y,t)}{f(y,t)^{1/(1-\rho^2)}}$$

and that G solves (2.29). Because $\rho^2 \le 1$ and $G > 0$, we easily conclude that G is a supersolution of

(3.4)
$$\bar{G}_t + \mathcal{L}^{(S,y)} \bar{G} = \frac{1}{2} \frac{\mu^2}{\sigma(y,t)^2} \bar{G} + \frac{1}{2} a^2(y,t) \frac{\bar{G}_y^2}{\bar{G}} + \rho \sigma(y,t) a(y,t) S \frac{\bar{G}_S \bar{G}_y}{\bar{G}} + \frac{1}{2} \sigma^2(y,t) S^2 \frac{\bar{G}_S^2}{\bar{G}}$$

and, thus,

(3.5) $$G(S,y,t) \ge \bar{G}(S,y,t).$$

The solution $\bar{G}$ to (3.4) can be derived via an exponential transformation $\bar{G} = e^{\bar{\phi}}$, with $\bar{\phi}$ solving

(3.6) $$\bar{\phi}_t + \mathcal{L}^{(S,y)} \bar{\phi} = \frac{\mu^2}{2\sigma^2(y,t)}, \qquad \bar{\phi}(S,y,T) = \gamma g(S,y).$$

Under the boundedness assumptions on the coefficients and payoff, (3.6) has a unique classical solution, and $\bar{\phi}$ has the probabilistic representation

(3.7) $$\bar{\phi}(S,y,t) = \gamma \mathbb{E}_{\widetilde{P}} \left(g(S_T, Y_T) - \int_t^T R(Y_s, s)\, ds \mid S_t = S, Y_t = y \right).$$

Combining $G \ge e^{\bar{\phi}}$ with (3.3) and (2.26) gives the lower bound (3.2) for the writer's indifference price h^w.

Following similar arguments, we can derive the upper bound for the indifference price. In fact, from (2.19), $\mathcal{M}(G_S, G_y, G) \ge 0$, so (2.29) yields

$$G_t + \mathcal{L}^{(S,y)} G \ge \frac{\mu^2}{2\sigma^2(y,t)} G.$$

Therefore, G is a subsolution to

(3.8) $\hat{G}_t + \mathcal{L}^{(S,y)}\hat{G} = \dfrac{\mu^2}{2\sigma^2(y,t)}\hat{G}, \qquad \hat{G}(S,y,T) = e^{\gamma g(S,y)}.$

This in turn yields $G(S,y,t) \leq \hat{G}(S,y,t)$, and, in view of the probabilistic representation of the solution of (3.8), gives

$$G(S,y,t) \leq \mathbb{E}_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(g(S_T,Y_T) - \int_t^T R(Y_s,s)\,ds\right)} \mid S_t = S, Y_t = y \right).$$

The upper bound (3.1) follows easily. Part(ii) can be derived from the parity formula (2.33).   □

3.1. Alternative bounds. We continue with the derivation of alternative reservation prices. We stress that the bounds derived below have rather natural and desirable properties. The lower bound is given by an *arbitrage-free–type* price of the payoff g. This price corresponds to the limiting case $\gamma \to 0$ described in Theorem 2.8. The upper bound is given in terms of a *certainty-equivalent–type* price of the payoff g. It corresponds to what the writer would charge under a pricing device based entirely on static certainty equivalent valuation without taking into account dynamic rebalancing and optimal investments. It is important to observe that all bounds are expressed in terms of the same measure $\mathcal{Q}$.

THEOREM 3.2. *Let $\mathcal{Q}$ be the measure introduced in Definition 2.7.*

(i) *The writer's indifference price satisfies*

$$\mathbb{E}_{\mathcal{Q}}\left(g(S_T,Y_T) \mid S_t = S, Y_t = y \right) \leq h^w(S,y,t)$$

(3.9)
$$\leq \frac{1}{\gamma} \ln \mathbb{E}_{\mathcal{Q}}\left(e^{\gamma g(S_T,Y_T)} \mid S_t = S, Y_t = y \right).$$

(ii) *The buyer's indifference price satisfies*

$$-\frac{1}{\gamma} \ln \mathbb{E}_{\mathcal{Q}}\left(e^{-\gamma g(S_T,Y_T)} \mid S_t = S, Y_t = y \right) \leq h^b(S,y,t)$$

$$\leq \mathbb{E}_{\mathcal{Q}}\left(g(S_T,Y_T) \mid S_t = S, Y_t = y \right).$$

Proof. We first construct appropriate sub- and supersolutions of (2.29). In particular, we look for sub- and supersolutions of the separable form $M(y,t)N(S,y,t)$. Inserting the above function in (2.29) yields

$$\begin{cases} N\left(M_t + \mathcal{L}^{(y)}M - \dfrac{1}{2}\rho^2 a^2(y,t)\dfrac{M_y^2}{M} - \dfrac{\mu^2}{2\sigma^2(y,t)}M \right) \\ + M\left(N_t + \mathcal{L}^{(S,y)}N + (1-\rho^2)a^2(y,t)\dfrac{M_y}{M}N_y - \mathcal{M}(N_S, N_y, N) \right) = 0, \end{cases}$$

where $\mathcal{M}$ was defined in (2.19), and with $M(y,T)N(S,y,T) = e^{\gamma g(S,y)}$. Next, we choose M and N to solve, respectively,

(3.10) $M_t + \mathcal{L}^{(y)}M = \dfrac{\mu^2}{2\sigma^2(y,t)}M + \dfrac{1}{2}\rho^2 a^2(y,t)\dfrac{M_y^2}{M},$

with $M(y, T) = 1$, and

$$(3.11) \qquad N_t + \mathcal{L}^{(S,y)} N + (1 - \rho^2) a^2(y, t) \frac{M_y}{M} N_y = \mathcal{M}(N_S, N_y, N),$$

with $N(S, y, T) = e^{\gamma g(S,y)}$. We observe that M and F solve the same equations, (3.10) and (2.23), and satisfy the same terminal condition. By uniqueness, we deduce that $M \equiv F$ or, equivalently,

$$(3.12) \qquad M(y, t) = f(y, t)^{1/(1-\rho^2)},$$

with f solving (2.21). Therefore, we can write the solution of (2.29) as $G(S, y, t) = f(y, t)^{1/(1-\rho^2)} N(S, y, t)$, with N solving (3.11). It is worth observing that N solves the quasilinear equation (3.11) that is similar to (2.29) but with two modifications; namely, (3.11) does not have a potential term, and, also, its first derivative coefficient contains the extra term $(1 - \rho^2) a^2(y, t) \frac{M_y(y,t)}{M(y,t)}$.

Because of (3.12), we have

$$(3.13) \qquad \frac{M_y(y, t)}{M(y, t)} = \frac{1}{1 - \rho^2} \frac{f_y(y, t)}{f(y, t)} \ ,$$

which is smooth and bounded as observed in Lemma 2.1. Therefore, (3.11) can be written as

$$(3.14) \qquad \begin{cases} N_t + \mathcal{L}^{(S,y)} N + \rho' a(y, t) L(y, t) N_y = \mathcal{M}(N_S, N_y, N), \\ N(S, y, T) = e^{\gamma g(S,y)}, \end{cases}$$

where $L(y, t)$ was introduced in (2.31). Observing that $\mathcal{M}(N_S, N_y, N) \geq 0$ and that $\rho^2 \leq 1$, by arguments similar to the ones used in the derivation of the lower and upper bounds, (3.2) and (3.1), respectively, we readily deduce that

$$(3.15) \qquad \underline{N}(S, y, t) \leq N(S, y, t) \leq \bar{N}(S, y, t),$$

where $\underline{N}$ and $\bar{N}$ solve, respectively,

$$(3.16) \qquad \bar{N}_t + \mathcal{L}^{(S,y)} \bar{N} + \rho' a(y, t) L(y, t) \bar{N}_y = 0,$$

with $\bar{N}(S, y, T) = e^{\gamma g(S,y)}$, and

$$(3.17) \quad \underline{N}_t + \mathcal{L}^{(S,y)} \underline{N} + \rho' a(y, t) L(y, t) \underline{N}_y = \frac{1}{2} a^2(y, t) \frac{\underline{N}_y^2}{\underline{N}} + \rho a(y, t) \sigma(y, t) S \frac{\underline{N}_S \underline{N}_y}{\underline{N}}$$
$$+ \frac{1}{2} \sigma^2(y, t) S^2 \underline{N}_{SS},$$

with $\underline{N}(S, y, T) = e^{\gamma g(S,y)}$. The linear equation (3.16) has a unique classical solution under Assumption 1, and the Feynman–Kac formula yields

$$(3.18) \qquad \bar{N}(S, y, t) = \mathbb{E}_{\mathcal{Q}} \left(e^{\gamma g(S_T, Y_T)} \mid S_t = S, Y_t = y \right),$$

with $\mathcal{Q}$ given in Definition 2.7. Using an exponential transformation $\underline{N}(S, y, t) = e^{k(S,y,t)}$ gives that k must solve

$$k_t + \mathcal{L}^{(S,y)} k + \rho' a(y, t) L(y, t) k_y = 0,$$

with $k(S, y, T) = \gamma g(S, y)$, which has a unique classical solution under our assumptions. The same then follows for (3.17). The probabilistic representation for k is

$$k(S, y, t) = \mathbb{E}_{\mathcal{Q}}\Big(\gamma g(S_T, Y_T) \mid S_t = S, Y_t = y\Big).$$

Using (3.12) and (3.3), we deduce

$$\frac{1}{\gamma} \ln \underline{N}(S, y, t) \leq h^w(S, y, t) \leq \frac{1}{\gamma} \ln \bar{N}(S, y, t),$$

which yields the desired upper and lower bounds (3.9). □

From these bounds, we easily obtain the following bounds on the *price spread*.

PROPOSITION 3.3. *The price spread $h^w - h^b$ is bounded by*

$$0 \leq h^w - h^b \leq \frac{1}{\gamma} \ln \Big[\mathbb{E}_{\mathcal{Q}}\Big(e^{-\gamma g(S_T, Y_T)} \mid S_t = S, Y_t = y\Big) \mathbb{E}_{\mathcal{Q}}\Big(e^{\gamma g(S_T, Y_T)} \mid S_t = S, Y_t = y\Big)\Big].$$

4. Fast mean-reverting stochastic volatility. We now study the indifference price using asymptotic approximations. In this section, we assume the European contract is a claim on S_T only and not on Y_T. That is, $g = g(S_T)$, as is usually the case.

4.1. Stochastic volatility framework. For clarity of exposition, we take the volatility-driving process (Y_t) to be an Ornstein–Uhlenbeck (OU) process, namely,

$$dY_t = \alpha(m - Y_t)\, dt + \beta\,(\rho\, dW_t^1 + \rho' dW_t^2),$$

where α is the rate of mean-reversion, m the long-run mean, and β the volatility of the volatility factor Y, which we shall call the "v-vol." In terms of the previous notation, we have $b(y) = \alpha(m - y)$ and $a(y) = \beta$. The process admits a unique invariant distribution, $\mathcal{N}(m, \nu^2)$, where $\nu^2 = \beta^2/(2\alpha)$. We also define at this stage the density of this distribution $\Phi(y)$ and the average $\langle \cdot \rangle$ with respect to this density:

$$\langle \chi \rangle = \int \chi \Phi.$$

In particular, we denote by $\bar{\sigma}$ the long-run (root-mean-square) volatility

(4.1) $$\bar{\sigma} = \sqrt{\langle \sigma^2 \rangle}.$$

The utility-based pricing equation (2.32) describes the indifference price h as the solution of a quasilinear differential equation depending on the risk-aversion parameter γ, the level of the volatility driving process y, and the parameters of this model α, β, m, ρ as well as the function $\sigma(\cdot)$. Given a fully specified model and parameters estimated from data, we could compute the utility price by numerically discretizing (2.32). Another approach is to use asymptotic approximations that are exact in some limit in which the problem simplifies. These can be used in certain parameter ranges which may be valid in some markets. They also give a deeper analytical understanding of the pricing mechanism and its relationship to modeling assumptions.

The limit we focus on here is fast mean-reversion of the volatility process, meaning that α is large. We write

$$\alpha = 1/\varepsilon, \qquad 0 < \varepsilon << 1,$$

and we are interested in the limit $\varepsilon \downarrow 0$ with the variance of the invariant distribution ν^2 fixed. This implies the scaling $\beta = \sqrt{2}\nu/\sqrt{\varepsilon}$. The choice of scaling is natural, because it allows us to pick up the effects of both mean-reversion and v-vol in the correction (first order) term of the approximation (4.10) below.

Evidence of a rapidly mean-reverting volatility factor in the S&P 500 is presented in the empirical study [18] of high-frequency data. Another recent empirical study [1] has found evidence of a fast volatility scale in exchange rate dynamics. Chernov et al. [5] propose and give evidence from data for two-factor stochastic volatility models in which one factor mean-reverts on a short time-scale. For present purposes, the method of this section can be regarded as yielding an approximation whose validity will depend on specific market conditions, in particular over time horizons when other slower factors can be considered effectively constant. The extension to incorporate slower scales using mixed singular and regular perturbation techniques is the subject of future investigation.

This method was previously used for *no-arbitrage* derivative pricing and hedging problems in Fouque, Papanicolaou, and Sircar [17]. The arguments were extended for stochastic control problems in [23]. We summarize the main findings from the former for *no-arbitrage* pricing and hedging European claims in order to compare the analogous results for the indifference pricing mechanism.

4.1.1. No-arbitrage pricing of European claims. Let $P(S, y, t)$ be the pricing function for a European claim with payoff $g(S_T)$. By *no-arbitrage* arguments, this price is given by

$$(4.2) \qquad P(S, y, t) = \mathbb{E}^{(\lambda_m)}\{g(S_T) \mid S_t = S, Y_t = y\},$$

where the expectation is taken with respect to the equivalent martingale measure $\mathbb{P}^{(\lambda_m)}$, and λ_m is the market price of volatility risk. We assume that $\lambda_m = \lambda_m(Y_t)$ in that it is a bounded function of Y_t only and therefore that (S, Y) is also a Markov process under $\mathbb{P}^{(\lambda_m)}$, which justifies the notation in (4.2). This premium is implicit in the prices of liquidly traded options or the market-set implied volatility skew. Under the measure $\mathbb{P}^{(\lambda_m)}$, the dynamics of (S, Y) can be written

$$dS_t = \sigma(Y_t)S_t \, dW_t^\star,$$

$$dY_t = \left[\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho\frac{\mu}{\sigma(Y_t)} - \rho'\lambda_m(Y_t)\right)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\rho \, dW_t^\star + \rho' dZ_t^\star\right),$$

where $(W_t^\star)$ and $(Z_t^\star)$ are independent $\mathbb{P}^{(\lambda_m)}$-Brownian motions.

We make the following assumption.

ASSUMPTION 2. *The market price of volatility risk function $\lambda_m(\cdot)$ is smooth and bounded.*

The analysis of [17, Chapter 5] leads to the following approximation for P in the limit of fast mean-reversion:

$$(4.3) \qquad P(S, y, t) \approx P^{(0)}(S, t) + \widetilde{P^{(1)}}(S, t),$$

where $P^{(0)}(S, t)$ is the Black–Scholes pricing function for the claim using the long-run average volatility parameter $\bar{\sigma}$ which is related to the original stochastic volatility model through (4.1). In other words, $P^{(0)}(S, t)$ solves the Black–Scholes PDE problem

$$(4.4) \qquad \mathcal{L}_{BS}(\bar{\sigma})P^{(0)} = 0; \qquad t < T,$$
$$P^{(0)}(S, T) = g(S),$$

where

$$(4.5) \qquad \mathcal{L}_{BS}(\bar{\sigma}) = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 S^2 \frac{\partial^2}{\partial S^2},$$

the Black–Scholes differential operator at volatility level $\bar{\sigma}$. Under Assumptions 1 and 2, $P^{(0)}$ is smooth and bounded with bounded derivatives.

The correction term $\widetilde{P^{(1)}}$ accounting for stochastic volatility effects is given by

$$(4.6) \qquad \widetilde{P^{(1)}}(S,t) = -(T-t)\left(V_2 S^2 P_{SS}^{(0)} + V_3 S^3 P_{SSS}^{(0)}\right)$$

for some constants V_2 and V_3 related to the original parameters and the functions σ and λ_m by formulas given below. In the case of smooth payoff g, we have the following convergence result, shown in [17]. We use the order notation

$$f(\varepsilon) = \mathcal{O}(g(\varepsilon)), \qquad \text{as } \varepsilon \downarrow 0 \qquad \Rightarrow \qquad \lim_{\varepsilon \downarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} = c,$$

for some constant c independent of ε, and $f(\varepsilon) = o(g(\varepsilon))$ if $c = 0$.

PROPOSITION 4.1. *Under Assumptions 1 and 2, for a fixed point* (S, y, t),

$$(4.7) \qquad |P(S,y,t) - (P^{(0)}(S,t) + \widetilde{P^{(1)}}(S,t))| = \mathcal{O}(\varepsilon).$$

For the case of call and put options when the payoff is only $\mathcal{C}^0$, the following convergence result is proved in [19] using a regularization technique.

PROPOSITION 4.2. *Under Assumptions 1 and 2, for a fixed point* (S, y, t),

$$|P(S,y,t) - (P^{(0)}(S,t) + \widetilde{P^{(1)}}(S,t))| = o(\varepsilon |\log \varepsilon|^{1+p})$$

for any $p > 0$.

In Theorem 4.3 below, we prove the analogue of (4.7) for smooth and bounded g and the indifference pricing mechanism.

We note the following points about the approximation (4.3).

- To this level, namely, zeroth plus first order of approximation, the price is insensitive to the present level of the stochastic volatility process $\sigma(Y_t)$.
- The group parameters V_2 and V_3 are related to the original model as follows:

$$(4.8) \qquad V_2 = \frac{\nu}{\sqrt{2\alpha}}\left(2\rho\langle\sigma\psi_1'\rangle - \left\langle\left(\frac{\mu\rho}{\sigma} + \rho'\lambda_m\right)\psi_1'\right\rangle\right),$$

$$(4.9) \qquad V_3 = \frac{\rho\nu}{\sqrt{2\alpha}}\langle\sigma\psi_1'\rangle,$$

  where $\nu^2 = \beta^2/(2\alpha)$ and $\psi_1(y)$ is a solution of the Poisson equation (4.23) below. In practice, these relations are not used and V_2 and V_3 are estimated directly from the market-implied volatility skew and then can be used for pricing American and exotic claims to the same order of approximation.
- The formulas (4.8) and (4.9) show that V_2 and V_3 are of order $1/\sqrt{\alpha}$ and so are small under the assumption of fast mean-reversion. Moreover, V_3 is zero when $\rho = 0$ and in the case of nonzero correlation, the third-derivative term describes the leverage effect. In the case of equities, ρ is typically negative and returns distributions are asymmetric with a fatter left tail. The second-derivative term contains effects due to the extra kurtosis of stochastic volatility models over geometric Brownian motion, and the market price of volatility risk λ_m.

- Finally, the approximation is *robust* in the sense that it does not depend on specification of σ or λ_m within the class of functions described in Assumptions 1 and 2.

4.2. Approximation of indifference prices and interpretation. The main result of this section is the following theorem.

THEOREM 4.3. *Let h denote either the writer's or the buyer's indifference price, defined in (2.10) and (2.11). Under Assumption 1, for a fixed point (S, y, t),*

$$(4.10) \qquad |h(S, y, t) - (P^{(0)}(S, t) - (T - t)(V_3 S^3 P^{(0)}_{SSS} + V_2^{(0)} S^2 P^{(0)}_{SS}))| = \mathcal{O}(\varepsilon).$$

Here, V_3 is defined in (4.9) and $V_2^{(0)}$ denotes V_2 defined in (4.8), with $\lambda_m = 0$.

Comparing with the *no-arbitrage* fast mean-reverting stochastic volatility approximation (4.6), we see from (4.8) that to this order of approximation, *the (writer's or buyer's) utility indifference price is exactly the no-arbitrage price in which there is zero risk premium from the second Brownian motion ($\lambda_m(\cdot) \equiv 0$). In particular, the risk-aversion coefficient γ does not appear in these first two terms of the approximation.*

The intuition for this is best understood from the relative entropy formulation of the indifference pricing mechanism discussed in section 2.1. Given a prior risk-neutral measure $\mathcal{Q}$ under which the volatility is *fast mean-reverting*, the utility pricer does not use his freedom to deviate from this belief up to the level of accuracy we have computed. In particular, he would have to choose a very large volatility risk premium λ in (2.41) for the asymptotics to lead to a different approximation in the first two terms, and this is penalized heavily by the entropy.

4.3. Expansions for indifference prices. To produce an asymptotic expansion for the indifference price, one may either analyze the indifference price equation (2.32) directly or, alternatively, approximate the value functions V and u involved in the pricing mechanism and, subsequently, approximate h via (2.10). We choose to proceed in the latter way because it also gives, as an intermediate output, useful results for the optimization problems that are interconnected with the utility-based prices.

We recall that the value functions u and V of the writer and the plain investor, respectively, can be written $u(x, S, y, t) = -e^{-\gamma x} G(S, y, t)$, with G solving (2.29), and $V(x, y, t) = -e^{-\gamma x} F(y, t)$, with F solving (2.23), and that the indifference price is given by

$$(4.11) \qquad h(S, y, t) = \frac{1}{\gamma} \ln \frac{G(S, y, t)}{F(y, t)}.$$

It is convenient to work with the logarithmic transformations of G and F. To this end, we set

$$(4.12) \qquad G = e^{\phi}, \qquad F = e^{\psi},$$

where $\phi(S, y, t)$ solves (2.30) and $\psi(y, t)$ solves

$$(4.13) \qquad \psi_t + \mathcal{L}^{(y)}\psi + \frac{1}{2}a(y)^2(1 - \rho^2)\psi_y^2 - \frac{\mu^2}{2\sigma(y)^2} = 0,$$

with $\psi(y, T) = 0$. In the latter case, we could as easily work with f, which solves the linear equation (2.21), but since we will need the expansion for ϕ, it is simpler to

construct that and then obtain the expansion for ψ by setting $g \equiv 0$ and removing the S dependence.

The first two terms of the asymptotic expansions of ϕ and ψ in powers of $\sqrt{\varepsilon}$ are summarized in the following proposition, which is proven in the following sections.

PROPOSITION 4.4. *Under Assumption 1, for a fixed point (S, y, t) the following hold.*

(i) *The first two terms of the expansion for ϕ, solution of (2.30), lead to the approximation*

$$(4.14) \qquad |\phi(S, y, t) - (\phi^{(0)}(S, t) + \widetilde{\phi^{(1)}}(S, t))| = \mathcal{O}(\varepsilon),$$

where

$$(4.15) \quad \phi^{(0)}(S, t) = \gamma P^{(0)}(S, t) - \frac{\mu^2}{2\sigma_\star^2}(T - t),$$

$$(4.16) \quad \widetilde{\phi^{(1)}}(S, t) = -\gamma(T - t)\left[V_3 S^3 P_{SSS}^{(0)}(S, t) + V_2^{(0)} S^2 P_{SS}^{(0)}(S, t) + \frac{\mu^3 C_4}{\gamma}\right].$$

Here, V_3 is defined in (4.9), $V_2^{(0)}$ denotes V_2 defined in (4.8), with $\lambda_m = 0$, C_4 is a market constant, $C_4 = \frac{\rho\nu}{\sqrt{2\alpha}}\langle\frac{\psi_2'}{\sigma}\rangle$, with ψ_2 defined below in (4.24), and

$$(4.17) \qquad \bar{\sigma}^2 = \langle\sigma^2\rangle, \qquad \frac{1}{\sigma_\star^2} = \left\langle\frac{1}{\sigma^2}\right\rangle.$$

(ii) *The first two terms of the expansion for ψ, solution of (4.13), lead to the approximation*

$$(4.18) \qquad |\psi(S, y, t) - (\psi^{(0)}(S, t) + \widetilde{\psi^{(1)}}(S, t))| = \mathcal{O}(\varepsilon),$$

where

$$\psi^{(0)}(S, t) = -\frac{\mu^2}{2\sigma_\star^2}(T - t), \qquad \widetilde{\psi^{(1)}}(S, t) = -(T - t)\mu^3 C_4.$$

Before we give the proof of Proposition 4.4, we introduce some convenient notation.

4.4. Operator notation. We write (2.30) in the compact form

$$(4.19) \qquad \mathcal{L}^\varepsilon \phi + \frac{\nu^2}{\varepsilon}(1 - \rho^2)\phi_y^2 = \frac{\mu^2}{2\sigma(y)^2},$$

where we define

$$(4.20) \qquad \begin{aligned} \mathcal{L}^\varepsilon &= \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{L}_0 &= \nu^2\frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}, \\ \mathcal{L}_1 &= \sqrt{2}\rho\nu\left(\sigma(y)S\frac{\partial^2}{\partial S\partial y} - \frac{\mu}{\sigma(y)}\frac{\partial}{\partial y}\right), \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}\sigma(y)^2 S^2\frac{\partial^2}{\partial S^2}. \end{aligned}$$

We notice that

1. $\mathcal{L}_0$ is the infinitesimal generator of the OU process with unit rate of mean-reversion;
2. $\mathcal{L}_1$ takes derivatives in y and so kills any function that does not depend on y; and
3. $\mathcal{L}_2 = \mathcal{L}_{BS}(\sigma(y))$, where $\mathcal{L}_{BS}(\cdot)$ is the Black–Scholes differential operator as a function of the volatility level defined in (4.5).

4.5. Proof of Proposition 4.4. We first describe an expansion of the form

$$\phi(S, y, t) = \phi^{(0)}(S, y, t) + \sqrt{\varepsilon}\phi^{(1)}(S, y, t) + \varepsilon\phi^{(2)}(S, y, t)$$
$$(4.21) \qquad\qquad\qquad + \varepsilon^{3/2}\phi^{(2)}(S, y, t) - Z^\varepsilon(S, y, t),$$

and write an equation for the *error term* Z^ε.

We define $\phi^{(0)}$ and $\sqrt{\varepsilon}\phi^{(1)} = \widetilde{\phi^{(1)}}$ by (4.15) and (4.16), respectively. In particular, $\phi^{(0)}$ and $\phi^{(1)}$ do not depend on y, and, by the smoothness assumptions on g, they are also smooth.

We define $\phi^{(2)}$ by

$$(4.22) \qquad\qquad \phi^{(2)}(S, y, t) = -\frac{1}{2}\psi_1(y)S^2\phi^{(0)}_{SS} + \frac{\mu^2}{2}\psi_2(y),$$

where ψ_1 and ψ_2 are solutions of the Poisson equations

$$(4.23) \qquad\qquad\qquad \mathcal{L}_0\psi_1 = \sigma(y)^2 - \bar{\sigma}^2,$$

$$(4.24) \qquad\qquad\qquad \mathcal{L}_0\psi_2 = \frac{1}{\sigma(y)^2} - \frac{1}{\sigma_\star^2}.$$

The average volatilities $\bar{\sigma}$ and $\sigma_\star$ are defined in (4.17) and are finite as σ is bounded. The expressions (4.15), (4.16), and (4.22) are motivated by a formal expansion. Finally, we define $\phi^{(3)}(S, y, t)$ by

$$(4.25) \quad \phi^{(3)}(S, y, t) = \frac{\rho\nu}{\sqrt{2}}\left[\psi_3(y)(S^3\phi^{(0)}_{SSS} + 2S^2\phi^{(0)}_{SS}) - \mu\psi_4(y)S^2\phi^{(0)}_{SS} - \mu\psi_5(y)\right],$$

where ψ_3, ψ_4, and ψ_5 are solutions of the Poisson equations

$$(4.26) \qquad\qquad\qquad \mathcal{L}_0\psi_3 = \sigma(y)\psi_1'(y) - \langle\sigma\psi_1'\rangle,$$

$$(4.27) \qquad\qquad\qquad \mathcal{L}_0\psi_4 = \frac{\psi_1'(y)}{\sigma(y)} - \left\langle\frac{\psi_1'}{\sigma}\right\rangle,$$

$$(4.28) \qquad\qquad\qquad \mathcal{L}_0\psi_5 = \frac{\psi_2'(y)}{\sigma(y)} - \left\langle\frac{\psi_2'}{\sigma}\right\rangle.$$

Here, $\phi^{(3)}$ has been chosen to be a solution of

$$\mathcal{L}_0\phi^{(3)} + \mathcal{L}_1\phi^{(2)} + \mathcal{L}_2\phi^{(1)} = 0,$$

which is a Poisson equation (in y). Its solvability condition (see [17, section 5.2.2]) is

$$(4.29) \qquad\qquad\qquad \langle\mathcal{L}_2\phi^{(1)} + \mathcal{L}_1\phi^{(2)}\rangle = 0,$$

which is satisfied because that is how $\phi^{(1)}$ and $\phi^{(2)}$ were chosen.

LEMMA 4.1. *The functions $\phi^{(2)}$ and $\phi^{(3)}$ are smooth as functions of S and t and can be chosen to be at most logarithmically growing in y at infinity.*

Proof. As shown in [19, Appendix C], the boundedness assumption on σ implies that we can choose ψ_1 and ψ_2 to be at most logarithmically growing at infinity:

$$|\psi_{1,2}(y)| \leq c(1 + \ln(1 + |y|))$$

for some constant c. In particular, the derivatives $\psi'_{1,2}$ are bounded, and the right-hand sides of (4.26)–(4.28) are therefore bounded. Then we can also choose ψ_3, ψ_4, and ψ_5 to be at most logarithmically growing at infinity. Since $\phi^{(0)}(S, t)$ and $\phi^{(1)}(S, t)$ are smooth, the result follows from the explicit expressions (4.22) and (4.25). □

We assume hereon that $\phi^{(2)}$ and $\phi^{(3)}$ *are* chosen to be at most logarithmically growing in y at infinity.

Inserting the expansion (4.21) into the PDE (2.30), we obtain the following equation and terminal condition for Z^ε:

$$(4.30) \qquad \mathcal{L}^\varepsilon Z^\varepsilon + 2\nu^2(1 - \rho^2)(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)})Z_y^\varepsilon - \frac{\nu^2}{\varepsilon}(1 - \rho^2)(Z_y^\varepsilon)^2 = \varepsilon J,$$

$$(4.31) \qquad Z^\varepsilon(S, y, T) = \varepsilon K(S, y).$$

In this calculation, we have used the fact that $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$, and $\phi^{(3)}$ satisfy

$$\mathcal{L}_0\phi^{(0)} + \nu^2(1 - \rho^2)(\phi_y^{(0)})^2 = 0, \qquad \mathcal{L}_0\phi^{(1)} + \mathcal{L}_1\phi^{(0)} = 0,$$

$$(4.32) \qquad \mathcal{L}_0\phi^{(2)} + \mathcal{L}_1\phi^{(1)} + \mathcal{L}_2\phi^{(0)} = \frac{\mu^2}{2\sigma(y)^2},$$

as well as the solvability condition (4.29) for (4.32).

In (4.30) and (4.31), the source term J and terminal data K are given by

$$(4.33) \qquad J = \mathcal{L}_1\phi^{(3)} + \mathcal{L}_2\phi^{(2)} + \sqrt{\varepsilon}\mathcal{L}_2\phi^{(3)} + \nu^2(1 - \rho^2)\left(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)}\right)^2,$$

$$(4.34) \qquad K = \phi^{(2)}(S, y, T) + \sqrt{\varepsilon}\phi^{(3)}(S, y, T).$$

From Lemma 4.1, it follows that J and K are smooth and at most logarithmically growing in y at infinity. Notice that Z^ε is the unique classical solution of a quasilinear parabolic PDE problem. Defining

$$(4.35) \qquad \theta(S, y, t) = 2\nu^2(1 - \rho^2)\left(\phi_y^{(2)} + \sqrt{\varepsilon}\phi_y^{(3)}\right),$$

we can write (4.30) as

$$(4.36) \qquad \hat{\mathcal{L}}^\varepsilon Z^\varepsilon = \varepsilon J + \frac{\nu^2}{\varepsilon}(1 - \rho^2)(Z_y^\varepsilon)^2,$$

where $\hat{\mathcal{L}}^\varepsilon = \mathcal{L}^\varepsilon + \theta(S, y, t)\frac{\partial}{\partial y}$ is a linear parabolic operator. Notice that θ is bounded.

LEMMA 4.2. *Let $\bar{Z}^\varepsilon(S, y, t)$ be the unique classical solution of the linear PDE problem*

$$(4.37) \qquad \hat{\mathcal{L}}^\varepsilon \bar{Z}^\varepsilon = \varepsilon J,$$
$$\bar{Z}^\varepsilon(S, y, T) = \varepsilon K(S, y).$$

Then $Z^\varepsilon \leq \bar{Z}^\varepsilon$.

Proof. The result follows from the nonnegativity of the nonlinear term in (4.36) and classical comparison results for linear parabolic equations. ☐

LEMMA 4.3. *Define* $\underline{Z}^\varepsilon(S, y, t)$ *by* $\underline{Z}^\varepsilon = -\ln \bar{q}^\varepsilon$, *where* $\bar{q}^\varepsilon(S, y, t)$ *is the unique classical solution of the linear PDE problem*

$$(4.38) \qquad \hat{\mathcal{L}}^\varepsilon \bar{q}^\varepsilon + \varepsilon J \bar{q}^\varepsilon = 0,$$
$$\bar{q}^\varepsilon(S, y, T) = e^{-\varepsilon K(S,y)}.$$

Then $Z^\varepsilon \geq \underline{Z}^\varepsilon$.

Proof. We first observe that $q^\varepsilon = e^{-Z^\varepsilon}$ satisfies

$$(4.39) \qquad \hat{\mathcal{L}}^\varepsilon q^\varepsilon + \varepsilon J q^\varepsilon = \frac{1}{2q^\varepsilon} \left(\frac{\rho \nu \sqrt{2}}{\sqrt{\varepsilon}} q_y^\varepsilon + \sigma(y) S q_S^\varepsilon \right)^2,$$

with terminal condition

$$q^\varepsilon(S, y, T) = e^{-\varepsilon K(S,y)}.$$

By the nonnegativity of the nonlinear term on the right-hand side of (4.39) and classical comparison results for linear parabolic equations, the result follows. ☐

We next need to show that the upper and lower bounds go to zero with ε.

LEMMA 4.4. *At fixed* (S, y, t),

$$\bar{Z}(S, y, t) = \mathcal{O}(\varepsilon).$$

Proof. We can write the probabilistic representation of (4.37),

$$(4.40) \qquad \bar{Z}^\varepsilon(S, y, t) = \mathbb{E}^\star \left(\varepsilon K(\hat{S}_T, \hat{Y}_T) - \varepsilon \int_t^T J(\hat{S}_u, \hat{Y}_u, u) \, du \mid \hat{S}_t = S, \hat{Y}_t = y \right),$$

in terms of the processes $(\hat{S}, \hat{Y})$ defined by

$$d\hat{S} = \sigma(\hat{Y}) \hat{S} \left(\rho \, d\hat{B}^1 + \rho' d\hat{B}^2 \right),$$
$$(4.41) \qquad d\hat{Y} = \frac{1}{\varepsilon} \left((m - \hat{Y}) - \sqrt{2\varepsilon} \frac{\rho \nu \mu}{\sigma(\hat{Y})} + \varepsilon \theta(\hat{S}, \hat{Y}, t) \right) dt + \frac{\nu \sqrt{2}}{\sqrt{\varepsilon}} \, d\hat{B}^1,$$

on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}}^\star)$ where $\hat{B}^1$ and $\hat{B}^2$ are independent Brownian motions and where $\mathbb{E}^\star$ denotes expectation with respect to $\hat{\mathbb{P}}^\star$. This is because the infinitesimal generator of $(\hat{S}, \hat{Y})$ is $\hat{\mathcal{L}}^\varepsilon$.

We recall that $J(S, y, t)$ and $K(S, y)$ are at most logarithmically growing as functions of y and are bounded as functions of (S, t). Then the expectation (4.40) can be bounded by a combination of terms of the form $\varepsilon \mathbb{E}^\star \{\chi(\hat{Y}_u) \mid \hat{S}_t = S, \hat{Y}_t = y\}$, with $t < u \leq T$, for some logarithmically growing functions χ. Using Lemma 4.6 below, for ε sufficiently small, the terms $\mathbb{E}^\star \{\chi(\hat{Y}_u) \mid \hat{S}_t = S, \hat{Y}_t = y\}$ are bounded independent of ε. The result follows from (4.40). ☐

LEMMA 4.5. *At fixed* (S, y, t),

$$\underline{Z}(S, y, t) = \mathcal{O}(\varepsilon).$$

   Proof. The solution of (4.38) has the probabilistic representation, via the Feynman–Kac formula

$$\bar{q}^{\varepsilon}(S, y, t) = \mathbb{E}^{\star}\left\{\exp\left(-\varepsilon K(\hat{S}_T, \hat{Y}_T) + \varepsilon \int_t^T J(\hat{S}_u, \hat{Y}_u, u)\, du\right) \mid \hat{S}_t = S, \hat{Y}_t = y\right\},$$

where $(\hat{S}, \hat{Y})$ were defined in the previous lemma. From the properties of J and K, this can be bounded above by

$$\mathbb{E}^{\star}\left\{\exp\left(\varepsilon\chi(\hat{Y}_T, T) + \varepsilon \int_t^T \chi(\hat{Y}_u, u)\, du\right) \mid \hat{S}_t = S, \hat{Y}_t = y\right\}$$

for some functions χ at most logarithmically growing in $\hat{Y}$. From the ε-independent exponential moments of $\hat{Y}$ given in Lemma 4.6 below, it follows that

$$\bar{q}^{\varepsilon}(S, y, t) = 1 + \mathcal{O}(\varepsilon)$$

for fixed (S, y, t). The result follows from $\underline{Z}^{\varepsilon} = -\ln \bar{q}^{\varepsilon}$.   □

   LEMMA 4.6. *For $\hat{Y}$ defined in (4.41), there exist some $\varepsilon_0 > 0$ and a constant $C(t, T, v, y)$, independent of ε, such that for any $t \le s \le T$ and $0 < \varepsilon < \varepsilon_0$,*

$$\mathbb{E}^{\star}\{e^{v\hat{Y}_s} \mid \hat{S}_t = S, \hat{Y}_t = y\} < C(t, T, v, y).$$

   Proof. The proof is a modification of [8, Proposition 2]. We first rewrite (4.41) as

$$d\hat{Y}_t = \left(\frac{1}{\varepsilon}(m - \hat{Y}_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(\hat{S}, \hat{Y}, t)\right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\, d\hat{B}_t^1,$$

with

$$\Lambda(S, y, t) = \frac{\rho\mu}{\sigma(y)} - \sqrt{\varepsilon}\frac{\theta(S, y, t)}{\nu\sqrt{2}},$$

where θ was defined in (4.35). Since θ is bounded, for any $\varepsilon_0 > 0$ and $0 < \varepsilon < \varepsilon_0$, Λ is bounded independent of ε.

   Next, using Girsanov's theorem, we define an equivalent measure $\hat{\mathbb{P}}$ under which $\hat{Y}$ is a standard OU process. Introducing $\hat{W}_t^1 = \hat{B}_t^1 - \int_0^t \Lambda(\hat{S}_u, \hat{Y}_u, u)\, du$, we define $\hat{\mathbb{P}}$ by

$$\frac{d\hat{\mathbb{P}}^{\star}}{d\hat{\mathbb{P}}} = M_T^{(\Lambda)},$$

where

$$M_s^{(\Lambda)} = e^{-\int_t^s \Lambda(\hat{S}_u, \hat{Y}_u, u)\, d\hat{W}_u^1 - \frac{1}{2}\int_t^s \Lambda(\hat{S}_u, \hat{Y}_u, u)^2\, du}.$$

Then $\hat{W}^1$ is a $\hat{\mathbb{P}}$-Brownian motion and $M^{(\Lambda)}$ is a $(\hat{\mathbb{P}}, (\hat{\mathcal{F}}_t))$-martingale.

   Now we have

(4.42) $$\mathbb{E}^{\star}\{e^{v\hat{Y}_s} \mid \hat{S}_t = S, \hat{Y}_t = y\} = \mathbb{E}\{e^{v\hat{Y}_s} M_s^{(\Lambda)} \mid \hat{S}_t = S, \hat{Y}_t = y\},$$

where $\mathbb{E}$ denotes the expectation under $\hat{\mathbb{P}}$. We rewrite (4.42) as

$$\mathbb{E}\{e^{v\hat{Y}_s}M_s^{(\Lambda)} \mid \hat{S}_t = S, \hat{Y}_t = y\} = \mathbb{E}\left\{e^{v\hat{Y}_s}e^{\frac{1}{2}\int_t^s \Lambda(\hat{S}_u,\hat{Y}_u,u)^2 du}\sqrt{M_s^{(2\Lambda)}} \mid \hat{S}_t = S, \hat{Y}_t = y\right\},$$

and, using the Cauchy–Schwarz inequality, we deduce that

$$(4.43) \quad \mathbb{E}^{\star}\{e^{v\hat{Y}_s} \mid \hat{S}_t = S, \hat{Y}_t = y\} \leq \sqrt{\mathbb{E}\left\{e^{2v\hat{Y}_s}e^{\int_t^s \Lambda(\hat{S}_u,\hat{Y}_u,u)^2 du} \mid \hat{S}_t = S, \hat{Y}_t = y\right\}},$$

since $M^{(2\Lambda)}$ is a martingale with expected value equal to one. Therefore,

$$(4.44) \quad \mathbb{E}^{\star}\{e^{v\hat{Y}_s} \mid \hat{S}_t = S, \hat{Y}_t = y\} \leq e^{\frac{1}{2}(s-t)||\Lambda||_{\infty}^2}\sqrt{\mathbb{E}\left\{e^{2v\hat{Y}_s} \mid \hat{S}_t = S, \hat{Y}_t = y\right\}}.$$

Under $\hat{\mathbb{P}}$, the dynamics of $\hat{Y}$ are given by

$$d\hat{Y}_t = \frac{1}{\varepsilon}(m - \hat{Y}_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\,d\hat{W}_t^1.$$

That is, it is an autonomous standard OU process. From [8, Lemma 2], there exists a constant $c(v,y)$ such that for any $t \leq s \leq T$ and $\varepsilon > 0$ we have

$$\mathbb{E}\{e^{v\hat{Y}_s} \mid \hat{S}_t = S, \hat{Y}_t = y\} \leq c(v,y).$$

The result follows by setting

$$C(t,T,v,y) = e^{\frac{1}{2}(T-t)||\lambda||_{\infty}^2}\sqrt{c(2v,y)}. \qquad \square$$

Combining these results, we have

$$\mathcal{O}(\varepsilon) = -\ln\bar{q}^{\varepsilon}(S,y,t) \leq Z^{\varepsilon}(S,y,t) \leq \bar{Z}^{\varepsilon}(S,y,t) = \mathcal{O}(\varepsilon)$$

for fixed (S,y,t). It follows that

$$Z^{\varepsilon}(S,y,t) \to 0 \qquad \text{as } \varepsilon \downarrow 0,$$

and part (i) of Proposition 4.4 follows from (4.21).

The proof of convergence of the approximation (4.18) for the solution ψ of (4.13) is a simpler version of the preceding, setting $g \equiv 0$ and thereby removing the S-dependences.

4.6. Proof of Theorem 4.3. From (4.11) and (4.12), we have that

$$h(S,y,t) = \frac{1}{\gamma}(\phi(S,y,t) - \psi(S,y,t)).$$

Using (4.14) and (4.18) and the triangle inequality trivially leads to (4.10), completing the proof.

5. Conclusions. The practical implications of the preceding analysis are pricing spreads in section 3 that can be tuned by the pricer's risk aversion; robust asymptotic approximations in section 4 that do not require precise specification of a stochastic volatility model; and analytical results for the problem of optimizing a portfolio consisting of dynamic positions in a liquid underlying asset combined with a static position in a derivative security. Such a problem is important in incomplete markets when investors would like to use derivatives to "trade volatility" indirectly but cannot rebalance frequently because of high transaction costs. Our results herein are applied to this problem in work in preparation.

Many questions about the utility pricing mechanism remain for future investigation: the implications for hedging derivative risk; the effect on implied volatilities, especially the term-structure, or variation with time-to-maturity; the multidimensional problem, meaning both the case of multifactor stochastic volatility models (where perhaps one factor is slow and another is fast mean-reverting), and the case of options on a basket of stocks. Mathematically, the main challenge is in extending the present analysis to the important cases of unbounded and nonsmooth payoffs.

<div align="center">REFERENCES</div>

[1] S. ALIZADEH, M. BRANDT, AND F. DIEBOLD, *Range-based estimation of stochastic volatility models*, J. Finance, 57 (2002), pp. 1047–1092.

[2] M. AVELLANEDA, *The minimum-entropy algorithm and related methods for calibrating asset-pricing models*, Doc. Math., extra volume III (1998), pp. 545–563.

[3] N.L. BOWERS, H.U. GERBER, J.C. HICKMAN, AND D.A. JONES, *Actuarial Mathematics*, 2nd ed., Society of Actuaries, Schaumburg, IL, 1997.

[4] R. CARMONA AND L. XU, *Calibrating Arbitrage-Free Stochastic Volatility Models by Relative Entropy Method*, Technical report, CEOR, Princeton University, Princeton, NJ, 1997.

[5] M. CHERNOV, R. GALLANT, E. GHYSELS, AND G. TAUCHEN, *Alternative models for stock price dynamics*, J. Econometrics, 116 (2003), pp. 225–257.

[6] G.M. CONSTANTINIDES AND T. ZARIPHOPOULOU, *Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences*, Finance Stoch., 3 (1999), pp. 345–369.

[7] G.M. CONSTANTINIDES AND T. ZARIPHOPOULOU, *Bounds on derivative prices in an intertemporal setting with proportional transaction costs and multiple securities*, Math. Finance, 11 (2001), pp. 331–346.

[8] P. COTTON, J.-P. FOUQUE, G. PAPANICOLAOU, AND K.R. SIRCAR, *Stochastic volatility corrections for interest rate derivatives*, Math. Finance, 14 (2004), pp. 173–200.

[9] M.H.A. DAVIS, *Option pricing in incomplete markets*, in Mathematics of Derivative Securities, M.A.H. Dempster and S.R. Pliska, eds., Cambridge University Press, Cambridge, UK, 1997, pp. 216–226.

[10] M.H.A. DAVIS, *Optimal Hedging with Basis Risk*, Technical report, Vienna University of Technology, Vienna, Austria, 2000.

[11] M.H.A. DAVIS, V. G. PANAS, AND T. ZARIPHOPOULOU, *European option pricing with transaction costs*, SIAM J. Control Optim., 31 (1993), pp. 470–493.

[12] F. DELBAEN, P. GRANDITS, T. RHEINLANDER, D. SAMPERI, M. SCHWEIZER, AND C. STRIKER, *Exponential hedging and entropic penalties*, Math. Finance, 12 (2002), pp. 99–123.

[13] D. DUFFIE AND T. ZARIPHOPOULOU, *Optimal investment with undiversifiable income risk*, Math. Finance, 3 (1993), pp. 135–148.

[14] W.H. FLEMING AND H.M. SONER, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, 1993.

[15] W.H. FLEMING AND R.W. RISHEL, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.

[16] H. FOELLMER AND M. SCHWEIZER, *Hedging of contingent claims under incomplete information*, in Applied Stochastic Analysis, M.H.A. Davis and R.J. Elliott, eds., Gordon and Breach, London, 1990.

[17] J.-P. FOUQUE, G. PAPANICOLAOU, AND K.R. SIRCAR, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, UK, 2000.

[18] J.-P. Fouque, G. Papanicolaou, K.R. Sircar, and K. Solna, *Short time-scale in S&P 500 volatility*, J. Computat. Finance, 6 (2003), pp. 1–23.

[19] J.-P. Fouque, G. Papanicolaou, K.R. Sircar, and K. Solna, *Singular perturbations in option pricing*, SIAM J. Appl. Math., 63 (2003), pp. 1648–1665.

[20] M. Fritelli, *The minimal entropy martingale measure and the valuation problem in incomplete markets*, Math. Finance, 10 (2000), pp. 39–52.

[21] S. Hodges and A. Neuberger, *Optimal replication of contingent claims under transaction costs*, Rev. Futures Markets, 8 (1989), pp. 222–239.

[22] J. Hull and A. White, *The pricing of options on assets with stochastic volatilities*, J. Finance, 42 (1987), pp. 281–300.

[23] M. Jonsson and K.R. Sircar, *Partial hedging in a stochastic volatility environment*, Math. Finance, 12 (2002), pp. 375–409.

[24] J. Kallsen, *Utility-based derivative pricing in incomplete markets*, in Mathematical Finance Finance Bachelier Congress 2000, Springer-Verlag, New York, 2000, pp. 313–338.

[25] I. Karatzas and S. Kou, *On the pricing of contingent claims under constraints*, Ann. Appl. Probab., 6 (1996), pp. 321–369.

[26] N.V. Krylov, *Controlled Diffusion Processes*, Springer-Verlag, New York, 1980.

[27] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Uralceva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monogr. 23, AMS, Providence, RI, 1968.

[28] P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part 1: The dynamic programming principle and applications*, Comm. Partial Differential Equations, 8 (1983), pp. 1101–74.

[29] P.-L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part 2: Viscosity solutions and uniqueness*, Comm. Partial Differential Equations, 8 (1983), pp. 1229–76.

[30] R. C. Merton, *Lifetime portfolio selection under uncertainty: The continuous-time case*, Rev. Econom. Statist., 51 (1969), pp. 247–257.

[31] M. Musiela and T. Zariphopoulou, *An example of indifference prices under exponential preferences*, Finance Stoch., 8 (2004), pp. 229–239.

[32] H. Pham, *Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints*, Appl. Math. Optim., 46 (2002), pp. 55–78.

[33] R. Rouge and N. ElKaroui, *Pricing via utility maximization and entropy*, Math. Finance, 10 (2000), pp. 259–276.

[34] T. Zariphopoulou, *Stochastic control methods in asset pricing*, in Handbook of Stochastic Analysis and Applications, D. Kannan and V. Lakshmikanathan, eds., Marcel Dekker, New York, 2001.