SIAM J. CONTROL AND OPTIMIZATION
Vol. 30, No. 3, pp. 613-636, May 1992

006

INVESTMENT-CONSUMPTION MODELS WITH TRANSACTION FEES AND MARKOV-CHAIN PARAMETERS*

THALEIA ZARIPHOPOULOU†

Abstract. This paper considers an infinite horizon investment-consumption model in which a single agent consumes and distributes his wealth in two assets, a bond and a stock. The problem of maximization of the total utility from consumption is treated. State (amount allocated in assets) and control (consumption, rates of trading) constraints are present. It is shown that the value function is the unique viscosity solution of a system of variational inequalities with gradient constraints.

Key words. viscosity solutions, state constraints, variational inequalities, singular controls

AMS(MOS) subject classification. 90A35

Introduction. In this paper we examine a general investment and consumption decision problem for a single agent. The investor consumes at a nonnegative rate and he distributes his current wealth between two assets continuously in time. One asset is a bond, i.e., a riskless security with instantaneous rate of return r. The other asset is a stock, whose rate of return z_t is a continuous time Markov chain. In our version of the model the investor *cannot borrow* money to finance his investment in bond and he *cannot short-sell* the stock. In other words, the amount of money allocated in bond and stock must stay nonnegative.

When the investor makes a transaction, he pays transaction fees which are assumed to be proportional to the amount transacted. More specifically, let x_t and y_t be the investor's holdings in the riskless and the risky security prior to a transaction at time t. If the investor increases (or decreases) the amount invested in the risky asset to y_t+h_t (or y_t-h_t), the holding of the riskless asset decreases (increases) to $x_t-h_t-\lambda h_t$ (or $x_t+h_t-\mu h_t$). The numbers λ and μ are assumed to be nonnegative and one of them must always be positive. The control objective is to maximize, in an infinite horizon, the expected discounted utility which comes only from consumption. Due to the presence of the transaction fees, this is a singular control problem.

The paper is organized as follows. Section 1 is devoted to the description of the model and its history; the two main theorems are also stated here. Section 2 contains preliminaries about the value function. In § 3 we approximate the problem by using absolutely continuous controls. Finally, in § 4 we show that the value function is the unique constrained viscosity solution of a system of Variational Inequalities with gradient constraints.

1. We consider a market with two assets: a *bond* and a *stock*. The price P_t^0 of the bond is given by

$$dP_t^0=rP_t^0\,dt,\qquad P_0^0=p_0, \tag{1.1}$$

where $r>0$. The price P_t of the stock satisfies

$$dP_t=z(t)P_t\,dt,\qquad P_0=p. \tag{1.2}$$

* Received by the editors November 20, 1989; accepted for publication (in revised form) December 21, 1990.

† Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester , Massachusetts 01609. This work was partially supported by National Science Foundation Research planning grant 530177.

The rate of return z is a *finite state continuous time Markov chain*, defined on some underlying probability space (Ω, F, P) with jumping rate $q_{zz'}$ from state z to state z'. The state space is denoted by Z. The associated generator $\mathscr{L}$ of the Markov chain has the form

$$\mathscr{L}v(z) = \sum_{z' \neq z} q_{zz'}[v(z') - v(z)].$$

Let $K = \max_{z \in Z} z$. A natural assumption is $K \geqq r$. The amount of wealth x_t and y_t, invested at time t in bond and stock respectively, are the state variables and they evolve (see [17]) according to the equations

$$\begin{aligned} &dx_t = (rx_t - C_t)\,dt - (1+\lambda)\,dM_t + (1-\mu)\,dN_t, \\ (1.3) \qquad &dy_t = z(t)y_t\,dt + dM_t - dN, \\ &x_0 = x,\ y_0 = y,\ z(0) = z. \end{aligned}$$

For simplicity we assume here that all financial charges are paid from the holdings in bond. The investor cannot borrow money or short-sell the stock. The control processes are the *consumption rate* C_t and the processes M_t and N_t which represent *the cumulative purchases* and *sales of stock* respectively. The controls (C_t, M_t, N_t) are *admissible* if

(i) C_t is F_t-measurable, where $F_t = \sigma(z_s : 0 \leqq s \leqq t)$, $C_t \geqq 0$ almost everywhere for all $t \geqq 0$, and $E \int_0^\infty e^{-rs} C_s\,ds < +\infty$.

(ii) M_t, N_t are F_t-measurable, right continuous, and nondecreasing processes.

(iii) $x_t \geqq 0$, $y_t \geqq 0$ almost everywhere for all $t \geqq 0$, where x_t, y_t are the trajectories given by the state equation (1.3) using the controls (C_t, M_t, N_t). We denote by A the set of admissible controls.

The total expected discounted utility J coming from consumption is given by

$$J(x, y, z, C, M, N) = E \int_0^{+\infty} e^{-\beta t} U(C_t)\,dt$$

with $(C, M, N) \in A$ and $z(0) = z$, where the utility function $U : [0, +\infty) \to [0, +\infty)$ is assumed to have the following properties:

$$U \text{ is strictly increasing, bounded, concave, } C^1 \text{ function,}$$

and

$$U(0) = 0, \lim_{c \to 0} U'(c) = +\infty, \lim_{c \to \infty} U'(c) = 0.$$

The discount factor $\beta > 0$ weights consumption now versus consumption later, large β denoting instant gratification. Note that the controls M and N are acting implicitly through the constraint (iii).

The value function u is given by

$$u(x, y, z) = \sup_A E \int_0^{+\infty} e^{-\beta t} U(C_t)\,dt.$$

Our goal is to derive the Bellman equation associated with this singular control problem and to characterize u as its unique solution. It turns out that the Bellman equation here is a system of variational inequalities.

We now state one of the main results. (For the definition of constrained viscosity solution, see Definition 3.1.)

THEOREM. *The value function u is the unique constrained viscosity solution of*

$$\min[(1+\lambda)u_x - u_y, -(1-\mu)u_x + u_y,$$
$$\beta u - rxu_x - zyu_y - \max_{c \geqq 0}[-cu_x + U(c)] - \mathscr{L}u(z)] = 0 \quad (1.4)$$
$$\forall (x, y, z) \in (0, +\infty) \times (0, +\infty) \times Z$$

with

$$u(0, 0, z) = 0, \qquad \forall z \in Z,$$

in the class of bounded and uniformly continuous functions.

We continue with a discussion about the history of the model.

Transaction costs are an essential feature of some economic theories, and many times are incorporated in the two-asset portfolio selection model. In [3] Constantinides assumes that the transaction costs deplete only the riskless asset and that the stock price is a logarithmic Brownian motion. He shows that if an optimal policy exists, it has to be *simple*. An investment policy is defined as simple if it is characterized by two reflecting barriers $\underline{\lambda}, \bar{\lambda}$ with $\underline{\lambda} \leqq \bar{\lambda}$, such that the investor does not trade as long as the ratio y_t/x_t lies in $[\underline{\lambda}, \bar{\lambda},]$, and transacts to the closest boundary of the region of no transactions $[\underline{\lambda}, \bar{\lambda}]$ whenever this ratio lies outside this interval. He also shows that proportional transaction costs have only a second-order effect on equilibrium asset returns: the investors accommodate large transaction costs by drastically reducing the frequency and the volume of trade. Finally, he proves that the investor's expected utility of consumption is insensitive to deviations of the asset proportions from those proportions that are optimal in the absence of transaction costs. In a discrete-time version of the model, Constantinides [2], [3] proves that an optimal investment policy exists and it is simple.

In the continuous time framework, Taksar, Klass, and Assaf [16] assume that the investor does not consume but maximizes the long term expected rate of growth of wealth. In the same framework, but with more general assumptions, Fleming et al. [6] study the finite horizon problem, the average cost per unit time problem, and the growth problem and their relation.

Davis and Norman [5] relax the assumption that the transaction costs are charged only to the nonrisky asset. They consider a particular class of utility functions of the form $U(c) = c^p (0 < p < 1)$, and they prove that the optimal strategy confines the investor's portfolio to a certain wedge-shaped region in the portfolio plane.

Finally, there are several directions in which the two-asset problem with transaction costs can be extended. First, more than one risky asset can be allowed. Although this extension is straightforward, the computational requirements are enormous. Second, fixed transaction costs can be introduced. Some single-period models with fixed transaction costs are discussed in [1], [8], [11]–[14]. In multiperiod extensions of these models the optimal investment policy is complex, because the derived value function $u(x, y)$ is no longer homogeneous in x and y. Kandel and Ross [10] introduce quasifixed transaction costs. They use some aspects of fixed transaction costs and prove the homogeneity of the derived value function.

2. We examine some of the properties of the value function. Throughout the paper we assume

$$\beta > 2K + 1. \quad (2.1)$$

PROPOSITION 2.1. *For each $z \in Z$, u is jointly concave in x and y, strictly increasing in x, and increasing in y.*

Proof. Consider two points (x_1, y_1, z), (x_2, y_2, z). Let $\varepsilon > 0$ and $(C_1^\varepsilon, M_1^\varepsilon, N_1^\varepsilon)$, $(C_2^\varepsilon, M_2^\varepsilon, N_2^\varepsilon)$ be ε-optimal controls for these points respectively. Then

$$u(x_1, y_1, z) \leqq E \int_0^{+\infty} e^{-\beta t} U(C_1^\varepsilon)\, dt + \varepsilon$$

and

$$u(x_2, y_2, z) \leqq E \int_0^{+\infty} e^{-\beta t} U(C_2^\varepsilon)\, dt + \varepsilon.$$

Moreover, the policy $(\alpha C_1^\varepsilon + (1-\alpha) C_2^\varepsilon, \alpha M_1^\varepsilon + (1-\alpha) M_2^\varepsilon, \alpha N_1^\varepsilon + (1-\alpha) N_2^\varepsilon)$ is admissible for the point $(\alpha x_1 + (1-\alpha) x_2, \alpha y_1 + (1-\alpha) y_2, z)$. Therefore $u(\alpha x_1 + (1-\alpha) x_2, \alpha y_1 + (1-\alpha) y_2, z) \geqq E \int_0^{+\infty} e^{-\beta t} U(\alpha C_1^\varepsilon + (1-\alpha) C_2^\varepsilon)\, dt$. Using the concavity of U, the inequalities above and sending $\varepsilon \to 0$ we conclude.

We now show that $u(\cdot, \cdot, z)$ is increasing. Consider the points (x_1, y_1, z) and (x_2, y_2, z) with $x_1 \geqq x_2$, $y_1 \geqq y_2$. Let $\varepsilon > 0$ and $(C^\varepsilon, M^\varepsilon, N^\varepsilon)$ be an ε-optimal policy for (x_1, y_1, z). Since the policy $(C^\varepsilon, M^\varepsilon, N^\varepsilon)$ is admissible for the point (x_2, y_2, z), we have

$$u(x_1, y_1, z) \leqq u(x_2, y_2, z) + \varepsilon.$$

Sending $\varepsilon \to 0$ yields that $u(x_1, y_1, z) \leqq u(x_2, y_2, z)$.

Finally, we show that $u(\cdot, y, z)$ is strictly increasing. To this end, let us suppose that there exist two points (x_1, y, z) and (x_2, y, z) such that $x_1 < x_2$ and $u(x_1, y, z) = u(x_2, y, z)$. Then $u(x, y, z) = u(x_1, y, z)$, for all $x \in [x_1, x_2]$. Since u is concave and nondecreasing, the interval $[x_1, x_2]$ cannot be finite. Therefore there exists a point $x_0 \geqq 0$ such that $u(x, y, z) = u(x_0, y, z)$, for all $x \geqq x_0$. Let $(C^\varepsilon, M^\varepsilon, N^\varepsilon)$ be an ε-optimal policy for (x_0, y, z). Then

$$u(x_0, y, z) \leqq E \int_0^{+\infty} e^{-\beta t} U(C_t^\varepsilon)\, dt + \varepsilon.$$

However, if $x_1 > \max(x_0, (U^{-1}[\beta(E \int_0^{+\infty} e^{-\beta t} U(C_t^\varepsilon)\, dt + \varepsilon)]/r)$, the policy $(rx_1, 0, 0)$ is admissible for (x_1, y, z). Therefore

$$u(x_0, y, z) < \frac{1}{\beta} U(rx_1) = E \int_0^{+\infty} e^{-\beta t} U(rx_1)\, dt \leqq u(x_1, y, z),$$

which contradicts our assumption. □

PROPOSITION 2.2. *The value function u is uniformly continuous on $\bar{\Omega} = \{(x, y) \colon x \geqq 0, y \geqq 0\}$.*

Proof. We first show that u is continuous on $\bar{\Omega}$. The value function is continuous in Ω, because it is concave. As a matter of fact, u is Lipschitz continuous in Ω with Lipschitz constant of order $\beta^{-1} \| U \|_\infty |(x, y)|^{-1}$.

We next show that u is continuous on the boundary. We start with the point $(0, 0)$. Since $u(0, 0, z) = 0$ (this is an immediate consequence of the assumptions in the model), we argue by contradiction.

Let us assume that for some fixed $z_0 \in Z$ there exists a positive constant M such that $\overline{\lim}_{(x,y)\to(0,0)} u(x, y, z_0) = M$. Then there exists a sequence $(x_n, y_n) \to 0$ such that $u(x_n, y_n, z_0) > M/2$, for all $n \in N$. If (C^n, M^n, N^n) is an ε-optimal policy for the point (x_n, y_n, z_0) and (x_t^n, y_t^n) is the corresponding trajectory, let $w_t^n = x_t^n + (1-\mu) y_t^n$ and $w^n = x_n + (1-\mu) y_n$. Since the process M_t^n is nondecreasing we get

$$dw_t^n \leqq (r + K) w_t^n\, dt - C_t^n\, dt$$

and

$$E\int_0^t e^{-\beta s}C_s^n\,ds \leqq w^n - E[e^{\beta t}w_t^n] \leqq w^n,$$

where we used (2.1) and $w_t^n \geqq 0$, almost everywhere for all $t \geqq 0$. From Jensen's inequality we have

$$u(x_n, y_n, z_0) - \varepsilon \leqq E\int_0^{+\infty} e^{-\beta t}U(C_t^n)\,dt \leqq \frac{1}{\beta}\,U(\beta w^n).$$

Sending $n \to \infty$ and using that $U(0) = 0$, we get

$$0 < \frac{M}{2} < \varepsilon.$$

Sending $\varepsilon \to 0$ we get a contradiction.

We now show that $\lim_{(x,y)\to(x_0,0)} u(x, y, z) = u(x_0, 0, z)$, for all $z \in Z$. As a matter of fact, it will be an immediate consequence of the proof that $\lim_{y\to 0} u(x, y, z) = u(x, 0, z)$ uniformly with respect to x. Consider a point $(x_0, 0, z)$ with $x_0 > 0$ fixed and a sequence (x^n, y^n, z) such that $x^n, y^n > 0$ and $\lim_{n\to\infty}(x^n, y^n) = (x_0, 0)$. Since u is locally Lipschitz it suffices to show that $\lim_{n\to\infty}|u(x_0, y^n, z) - u(x_0, 0, z)| = 0$. Finally, since u is increasing, we only need to show that $u(x_0, y^n, z) \leqq u(x_0, 0, z) + \varepsilon$ for any $\varepsilon > 0$ and n sufficiently large.

Let (C^n, M^n, N^n) be an ε-optimal policy at (x_0, y^n, z). Then

$$u(x_0, y^n, z) \leqq E\int_0^{+\infty} e^{-\beta t}U(C_t^n)\,dt + \varepsilon.$$

Moreover, the control $(C^n, \bar{M}^n, N^n)$, where $d\bar{M}_t^n = dM_t^n + y^n\delta_0(t)$, is admissible for $(x_0 + (1+\lambda)y^n, 0, z)$. Therefore

$$E\int_0^{+\infty} e^{-\beta t}U(C_t^n)\,dt \leqq u(x_0 + (1+\lambda)y^n, 0, z).$$

Combining the last two inequalities, we conclude. Note that all the above arguments were uniform with respect to x_0.

We next show that $\lim_{(x,y)\to(0,y_0)} u(x, y, z) = u(0, y_0, z)$, $\forall z \in Z$. Moreover, it will be an immediate consequence of the proof that $\lim_{(x,y)\to(0,y_0)} u(x, y, z) = u(0, y_0, z)$, uniformly with respect to y.

Let $(0, y_0, z)$ with $y_0 > 0$ fixed. Arguing as before, we simply have to show that if $\varepsilon > 0$ and $x^n \to 0$ then $u(x^n, y_0, z) \leqq u(0, y_0, z) + \varepsilon$.

Let (C^n, M^n, N^n) be an ε-optimal policy for (x^n, y_0, z). Then

$$u(x^n, y_0, z) \leqq E\int_0^{+\infty} e^{-\beta t}U(C_t^n)\,dt + \varepsilon.$$

Moreover, the policy $(C^n, M^n, \bar{N}^n)$ is admissible for the point $(0, y_0 + (x^n/1-\mu), z)$, where $\bar{N}^n$ is given by $d\bar{N}_t^n = dN_t^n + (x^n/1-\mu)\delta_0(t)$. Therefore

$$E\int_0^{+\infty} e^{-\beta t}U(C_t^n)\,dt \leqq u\left(0, y_0 + \frac{x^n}{1-\mu}, z\right).$$

Combining the last two inequalities, we conclude.

We now show that u is uniformly continuous on $\bar{\Omega}$.

We argue by contradiction. If u is not uniformly continuous, then there exist sequences (X_n) and $(\bar{X}_n)$, $X_n, \bar{X}_n \in \bar{\Omega}$, such that, as $n \to \infty$, $|X_n - \bar{X}_n| \to 0$ and

$$|u(X_n, z_0) - u(\bar{X}_n, z_0)| \geqq \varepsilon \tag{2.2}$$

for some $\varepsilon > 0$ and $z_0 \in Z$.

In view of the first part of the proof, u is uniformly continuous on compact sets. Hence either (X_n) or $(\bar{X}_n)$, and therefore by assumption both must be unbounded. Let $X_n = (x_n, y_n)$ and $\bar{X}_n = (\bar{x}_n, \bar{y}_n)$. If $\underline{\lim}_{n\to\infty} x_n > 0$ and $\underline{\lim}_{n\to\infty} y_n > 0$, then the same holds for $(\bar{x}_n, \bar{y}_n)$. Since u is concave, locally Lipschitz with Lipschitz constant of order $|X|^{-1}$, (2.2) cannot hold.

We finally need to check what happens when either $\underline{\lim}_{n\to\infty} x_n \to 0$ or $\underline{\lim}_{n\to\infty} y_n = 0$. Here we only study the first case, since the other is similar. Without any loss of generality, we may assume that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = +\infty$, otherwise we work along an appropriate subsequence. Then $\lim_{n\to\infty} \bar{x}_n = 0$ and $\lim_{n\to\infty} \bar{y}_n = +\infty$. On the other hand,

$$\begin{aligned} |u(x_n, y_n) - u(\bar{x}_n, \bar{y}_n)| &\leqq |u(x_n, y_n) - u(x_n, \bar{y}_n)| + |u(x_n, \bar{y}_n) - u(\bar{x}_n, \bar{y}_n)| \\ &\leqq |u(x_n, y_n) - u(x_n, \bar{y}_n)| + |u(x_n, \bar{y}_n) - u(0, \bar{y}_n)| \\ &\quad + |u(0, \bar{y}_n) - u(\bar{x}_n, \bar{y}_n)|. \end{aligned}$$

Letting $n \to \infty$ above and using the fact that u is Lipschitz continuous with respect to y uniformly with respect to x (the Lipschitz constant being of order y^{-1}) and that $\lim_{x\to\infty} u(x, y) = u(0, y)$ uniformly with respect to y, we conclude. □

We now consider a similar control problem in which the controls, which represent the rates of trading, are assumed to be absolutely continuous processes. More precisely, we fix a positive constant L and we consider a market which offers a bond and a stock with prices evolving according to (1.1) and (1.2), respectively. The state variables x_t and y_t, which are the amount of money invested in bond and stock, obey the state equations

$$\begin{aligned} dx_t &= (rx_t - C_t)\, dt - (1+\lambda) m_t\, dt + (1-\mu) n_t\, dt, \\ dy_t &= z(t) y_t\, dt + m_t\, dt - n_t\, dt, \\ x_0 &= x,\ y_0 = y,\ z(0) = z. \end{aligned} \tag{2.3}$$

The controls of the investor are the *consumption rate* C_t and the *rates of trading* m_t and n_t, which are assumed to be almost everywhere bounded by L. The set of admissible controls A_L consists of controls (C, m, n) such that

(i) C_t is F_t-measurable where $F_t = \sigma(z_s : 0 \leqq s \leqq t)$, $C_t \geqq 0$ almost everywhere for all $t \geqq 0$ and $E \int_0^{+\infty} e^{-rs} C_s\, ds < +\infty$.

(ii) m_t, n_t are F_t-measurable right continuous and nonnegative processes.

(iii) $0 \leqq m_t, n_t \leqq L$ almost everywhere $t \geqq 0$.

(iv) $x_t \geqq 0$, $y_t \geqq 0$ almost everywhere $t \geqq 0$, where x_t, y_t are the solutions of (2.3) using the controls (C, m, n).

The assumption that $E \int_0^{+\infty} e^{-rs} C_s\, ds < \infty$ is redundant here. Indeed, one can easily show that it follows from (iii) and (iv).

The control objective is to maximize the expected discounted utility from consumption over the set of admissible controls. For each fixed $L > 0$, the value function is given by

$$u^L(x, y, z) = \sup_{A_L} E \int_0^{+\infty} e^{-\beta t} U(C_t)\, dt,$$

where U is the usual utility function and $\beta > 0$ is the discount factor.

PROPOSITION 2.3. *The value function u^L is jointly concave in x and y, strictly increasing in x, and increasing in y.*

Proof. The proof follows along the lines of Proposition 2.1. □

PROPOSITION 2.4. *The value function u^L is uniformly continuous on $\bar{\Omega}$ uniformly in L.*

Proof. u^L is concave and therefore locally Lipschitz in Ω. Moreover, the Lipschitz constant is independent of L, since u^L is uniformly bounded by $\|U\|_\infty/\beta$. Therefore u^L is continuous in Ω uniformly in L. Working as in Proposition 2.2, we can prove that u^L is continuous at the point $(0, 0)$ independently of L.

We now show that u^L is continuous in $\Omega_1 = \{(x, y): x > 0, y = 0\}$. We argue by contradiction. Since u^L is locally Lipschitz in Ω_1 and nondecreasing, it suffices to assume that there exist $z_0 \in Z$ and $x_0 > 0$ such that $u^L(x_0, 0, z_0) < \lim_{y_n \to 0} u^L(x_0, y_n, z_0)$. This is equivalent to assuming that there exist $\theta > 0$ and $N_0 > 0$ such that

$$u^L(x_0, 0, z_0) + \theta \leqq u^L(x_0, y_n, z_0), \qquad \forall n \geqq N_0.$$

On the other hand, the principle of dynamic programming gives

$$u^L(x_0, 0, z_0) \geqq E\left[\int_0^\tau e^{-\beta s} U(C_s)\, ds + e^{-\beta\tau} u^L(x_\tau, y_\tau, z_\tau)\right]$$

for any random time τ.

Let $C_t = 0$, $m_t = 1$ and $n_t = 0$, for all $t \geqq 0$, $t_n > 0$ and $\tau_n = t_n \wedge \tau_1$, where τ_1 is the first jump time of the process z_t. Then (2.3) gives

$$x_{\tau_n} = x_0\, e^{r\tau_n} - \frac{1+\lambda}{r}(\exp(r\tau_n) - 1)$$

and

$$y_{\tau_n} = \frac{\exp(z_0\tau_n) - 1}{z_0}.$$

Therefore

$$u^L(x_0, 0, z_0) \geqq E\left[\exp(-\beta\tau_n) u^L\left(x_0 \exp(r\tau_n) - \frac{1+\lambda}{r}(\exp(r\tau_n) - 1), \frac{\exp(z_0\tau_n) - 1}{z_0}, z_{\tau_n}\right)\right],$$

and, since u^L is nondecreasing

$$u^L(x_0, 0, z_0) \geqq \exp(-\beta t_n) u^L\left(x_0 - \frac{1+\lambda}{r}(\exp(rt_n) - 1), \frac{\exp(z_0 t_n) - 1}{z_0}, z_0\right) P(z_{\tau_n} = z_0).$$

We now choose t_n such that $(\exp(z_0 t_n) - 1)/z_0 = y_n$. Using that u^L is Lipschitz continuous in Ω (with the Lipschitz constant $k = k(x_0)$ independent of L), we obtain that

$$u^L(x_0, 0, z_0) \geqq \exp(-\beta t_n) P(z_{\tau_n} = z_0)\left[u^L(x_0, y_n, z_0) - k\frac{1+\lambda}{r}(\exp(rt_n) - 1)\right].$$

Combining the above yields

$$u^L(x_0, 0, z_0) \geqq \exp(-\beta t_n) P(z_{\tau_n} = z_0) \left[u^L(x_0, 0, z_0) + \theta - k \frac{1+\lambda}{r} (\exp(rt_n) - 1) \right].$$

We now send $y_n \to 0$. Then $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} P(z_{\tau_n} = z_0) = 1$ and $\theta \leqq 0$, which is a contradiction. Therefore u^L is continuous in Ω_1. It is also easily seen that the continuity was proved uniformly in L. Working similarly we can show that u^L is continuous in $\Omega_2 = \{(x, y): x = 0, y > 0\}$ uniformly in L. The proof that u^L is uniformly continuous on $\bar{\Omega}$ is similar to the one of Proposition 2.2 and therefore we omit it. □

3. In this section we characterize the value functions u^L and u. We show that u^L is the unique viscosity solution of the corresponding Hamilton–Jacobi equation. We also show that the limit of u^L as $L \to \infty$ coincides with u, which is a unique viscosity solution of a system of variational inequalities. We first give the definition of viscosity solution, which was introduced by Crandall and Lions [4].

We consider a nonlinear partial differential equation of the form

$$F(X, z, u(X, z), Du(X, z)) = 0 \tag{3.1}$$

where $z \in Z$, $X = (x, y)$ with $(x, y) \in \bar{\Omega}$, $Du(X, z) = (\partial u(X, z)/\partial x, \partial u(X, z)/\partial y)$ and $F: \bar{\Omega} \times Z \times \Re \times \Re^2 \to \Re$ is continuous, for each $z \in Z$.

DEFINITION 3.1. A continuous function $u: \bar{\Omega} \times Z \to R$ is a *constrained viscosity solution* of (3.1) if

(i) *u is a viscosity subsolution* of (3.1) on $\bar{\Omega}$, i.e., if for each $z \in Z$

$$F(X, z, u(X, z), r) \leqq 0, \qquad \forall X = (x, y) \in \bar{\Omega} \text{ and } r \in D^+_{(x,y)} u(X, z), \tag{3.2}$$

where

$$D^+_{(x,y)} u(X, z) = \left\{ r \in \Re^2 : \limsup_{h \to 0} \frac{u(X + h, z) - u(X, z) - r \cdot h}{|h|} \leqq 0 \right\};$$

(ii) u is a *viscosity supersolution* of (3.1) in Ω, i.e., if for each $z \in Z$

$$F(X, z, u(X, z), r) \geqq 0, \qquad \forall X = (x, y) \in \Omega \text{ and } r \in D^-_{(x,y)} u(X, z), \tag{3.3}$$

where

$$D^-_{(x,y)} u(X, z) = \left\{ r \in \Re^2 : \liminf_{h \to 0} \frac{u(X + h, z) - u(X, z) - r \cdot h}{|h|} \geqq 0 \right\}.$$

We now give an equivalent definition.

LEMMA 3.1. *The above definition is equivalent to the following*:

A continuous function $u: \bar{\Omega} \times Z \to \mathbb{R}$ is a constrained viscosity solution of (3.1) *if*

(i) *u is a viscosity subsolution of* (3.1) *on* $\bar{\Omega}$, *i.e., if for all $\phi \in C^1(\bar{\Omega})$ at any local maximum point $X_0 \in \bar{\Omega}$* of $u - \phi$ *the following holds*:

$$F(X_0, z, u(X_0, z), D\phi(X_0, z)) \leqq 0,$$

for each $z \in Z$.

(ii) *u is a viscosity supersolution of* (3.1) *in* Ω, *i.e., if for all $\phi \in C^1(\Omega)$ at any local minimum point $X_0 \in \Omega$ of $u - \phi$ the following holds*:

$$F(X_0, z, u(X_0, z), D\phi(X_0, z)) \geqq 0$$

for each $z \in Z$.

For the proof see [4]. □

We next show that u^L is the unique constrained viscosity solution of

$$\beta u^L = rxu_x^L + zyu_y^L + \max_{c\geqq 0}[-cu_x^L + U(c)] + \mathscr{L}u^L(z)$$

$$(3.4) \qquad + \max_{0\leqq m\leqq L}[-(1+\lambda)u_x^L + u_y^L]m + \max_{0\leqq n\leqq L}[(1-\mu)u_x^L - u_y^L]n,$$

$$u^L(0,0,z)=0, \qquad \forall z\in Z.$$

This fact follows along the results of Fleming, Sethi, and Soner [7] and Soner [15], appropriately modified to deal with the generator of the process.

THEOREM 3.1. (i) *u^L is a viscosity subsolution of* (3.4) *on* $\bar{\Omega}\times Z$.

(ii) *u^L is a viscosity supersolution of* (3.4) *in* $\Omega\times Z$.

Proof. We first approximate u^L by a sequence of functions $\{u^{L,N}\}$ defined by

$$u^{L,N} = \sup_{A_{L,N}} E\int_0^{+\infty} e^{-\beta t}U(C_t)\,dt,$$

where

$$A_{L,N} = \{(C,m,n)\in A_L : 0\leqq C_t\leqq N \text{ a.e. } \forall t\geqq 0\}.$$

Working exactly as in Propositions 2.3 and 2.4, we can prove that $u^{L,N}$ is concave in (x,y) and continuous on $\bar{\Omega}\times Z$. The corresponding Hamilton-Jacobi equation is

$$\beta u^{L,N} = rxu_x^{L,N} + zyu_y^{L,N} + \max_{N\geqq c\geqq 0}[-cu_x^{L,N} + U(c)] + \mathscr{L}u^{L,N}(z)$$

$$(3.5) \qquad + \max_{0\leqq m\leqq L}[-(1+\lambda)u_x^{L,N} + u_y^{L,N}]m + \max_{0\leqq n\leqq L}[(1-\mu)u_x^{L,N} - u_y^{L,N}]n.$$

We first prove that $u^{L,N}$ is a viscosity subsolution of (3.5) on $\bar{\Omega}$. We will need the following lemma.

LEMMA 3.2. *Let $v\in C_b(\bar{\mathscr{O}})$ be concave, where $\mathscr{O}$ is an open subset of $\mathscr{R}^n$. Then*

(i) $D^+v(X)\neq\emptyset, \forall X\in\mathscr{O}$

and

(ii) *if $p\in D^+v(X_0)$ and $\lambda(X-X_0)+X_0\in\bar{\mathscr{O}}$, $\forall X\in\bar{O}$ and $\lambda\in[0,1]$, then* $v(X)\leqq v(X_0)+p(X-X_0)$.

Proof. (i) Let $X_0\in\mathscr{O}$ be fixed and consider the functions $v^\varepsilon = v*\rho_\varepsilon$, where $\varepsilon>0$, ρ_ε is a standard molifier and $*$ denotes convolution. Since $v^\varepsilon\to v$ as $\varepsilon\to 0$ in $\overline{B(X_0,r)}\subset\mathscr{O}$, for some $r>0$, the functions v^ε are bounded in $B(X_0,r)$ uniformly in ε. Since the v^ε's are also concave (recall that v is concave), the v^ε's are also Lipschitz continuous in $B(X_0,r)$ and the Lipschitz constant in dependent of ε. By Taylor's theorem and concavity, we get

$$v^\varepsilon(X)\leqq v^\varepsilon(X_0)+Dv^\varepsilon(X_0)(X-X_0), \qquad \forall X\in B(X_0,r).$$

Since $|Dv^\varepsilon(X_0)|\leqq C$, along subsequences $\varepsilon_n\to 0$ we have $Dv_{\varepsilon_n}(X_0)\to p$ with $p\in\Re^N$. Letting $\varepsilon_n\to 0$ above, we get

$$v(X)\leqq v(X_0)+p(X-X_0), \qquad \forall X\in B(X_0,r),$$

which in turn yields that $p\in D^+v(X_0)$.

(ii) Let $p\in D^+v(X_0)$. Then

$$v(X)\leqq v(X_0)+p(X-X_0)+o(|X-X_0|), \qquad \forall X\in\bar{\mathscr{O}}.$$

Fix $X\in\bar{\mathscr{O}}$. Since $\lambda(X-X_0)+X_0\in\bar{\mathscr{O}}$, for all $\lambda\in[0,1]$, the concavity of v yields

$$v(\lambda(X-X_0)+X_0)\geqq\lambda v(X)+(1-\lambda)v(X_0).$$

Combining the last two inequalities, we get

$$\lambda v(X)+(1-\lambda)v(X_0)\leqq v(X_0)+\lambda p(X-X_0)+o(|\lambda(X-X_0)|).$$

Therefore

$$\lambda v(X)\leqq \lambda v(X_0)+\lambda p(X-X_0)+o(\lambda|X-X_0|).$$

Dividing first by λ and then sending $\lambda\to 0$, we conclude. □

We continue now with the proof of Theorem 3.1.

Proof. (i) In view of the definition of the constrained viscosity solution, we need to show that, if $(x_0, y_0, z)\in\bar{\Omega}\times Z$ is such that $D^+u^{L,N}(x_0, y_0, z)\neq\emptyset$, then

$$\beta u^{L,N}(x_0, y_0, z)\leqq rx_0p_z+zy_0q_z+\max_{c\geqq 0}[-cp_z+U(c)]+\max_{0\leqq m\leqq L}[-(1+\lambda)p_z+q_z]m$$
$$+\max_{0\leqq n\leqq L}[(1-\mu)p_z-q_z]n+\mathcal{L}u^{L,N}(x_0, y_0, z)$$

for every $(p_z, q_z)\in D^+u^{L,N}(x_0, y_0, z)$. To this end, assume that (x_0, y_0, z) and (p_z, q_z) are such that $(p_z, q_z)\in D^+u^{L,N}(x_0, y_0, z)$, $z\in Z$, and define $\Phi:\Re\times\Re\times Z\to\Re$ by

$$\Phi(x, y, z)=u^{L,N}(x_0, y_0, z)+p_z(x-x_0)+q_z(y-y_0).$$

Lemma 3.2 yields

$$u^{L,N}(x, y, z)\leqq\Phi(x, y, z),\text{ for all }(x, y, z)\in\bar{\Omega}\times Z.$$

On the other hand, the dynamic programming principle implies that for any stopping time $\tau>0$,

$$u^{L,N}(x_0, y_0, z_0)=\sup_{A_{L,N}}E\left[\int_0^\tau e^{-\beta s}U(C_s)\,ds+e^{-\beta\tau}u^{L,N}(x_\tau, y_\tau, z(\tau))\right].$$

Since $u^{L,N}\leqq\Phi$, the above equality yields

$$\Phi(x_0, y_0, z_0)\leqq\sup_{A_{L,N}}E\left[\int_0^\tau e^{-\beta s}U(C_s)\,ds+e^{-\beta\tau}\Phi(x_\tau, y_\tau, z(\tau))\right].$$

Let θ be a positive constant and τ_1 be the first jump time of the process $z(t)$. Using Dynkin's formula and the fact that $\Phi_x(x_0, y_0, z_0)=p_{z_0}$ and $\Phi_y(x_0, y_0, z_0)=q_{z_0}$ we obtain

$$E[e^{-\beta(\theta\wedge\tau_1)}\Phi(x_{\theta\wedge\tau_1}, y_{\theta\wedge\tau_1}, z(\theta\wedge\tau_1))-\Phi(x_0, y_0, z_0)]$$
$$=E\left[\int_0^{\theta\wedge\tau_1}e^{-\beta s}(-\beta\Phi(x_s, y_s, z_0)+rp_{z_0}x_s+z_0q_{z_0}y_s+\mathcal{L}\Phi(x_s, y_s, z_0)\right.$$
$$\left.-p_{z_0}C_s+[-(1+\lambda)p_{z_0}+q_{z_0}]m_s+[(1-\mu)p_{z_0}-q_{z_0}]n_s)\,ds\right].$$

Let $\theta=1/\ell$ and $(C^\ell, m^\ell, n^\ell)\in A_{L,N}$ be an $1/\ell^2$-optimal policy. Then, combining the above inequalities, we get

$$-\frac{1}{\ell^2}\leqq E\int_0^{(1/\ell)\wedge\tau_1}e^{-\beta s}[U(C_s^\ell)+rp_{z_0}x_s+z_0q_{z_0}y_s+\mathcal{L}\Phi(x_s, y_s, z_0)-p_{z_0}C_s^\ell$$
$$-\beta\Phi(x_s, y_s, z_0)+[-(1+\lambda)p_{z_0}+q_{z_0}]m_s^\ell+[(1-\mu)p_{z_0}-q_{z_0}]n_s^\ell]\,ds.$$

On the other hand, the state equations together with the constraint $0\leqq m_t, n_t\leqq L$ for almost every $t\geqq 0$ give

$$|x_t^\ell-x_0|\leqq(e^{rt}-1)\left[x_0+\frac{1-\mu}{r}L\right],$$

$$|y_t^\ell-y_0|\leqq(e^{Kt}-1)\left[y_0+\frac{L}{K}\right].$$

Using the above and the form of Φ, we can find a constant C_1 such that

$$-\frac{1}{\ell^2} \leqq E \int_0^{(1/\ell)\wedge\tau_1} e^{-\beta s}[rp_{z_0}x_0 + z_0 q_{z_0} y_0 + \mathscr{L}\Phi(x_0, y_0, z_0) - \beta\Phi(x_0, y_0, z_0)]\, ds$$

$$+ E \int_0^{(1/\ell)\wedge\tau_1} e^{-\beta s}[U(C_s^\ell) - p_{z_0}C_s^\ell + (-(1+\lambda)p_{z_0} + q_{z_0})m_s^\ell + ((1-\mu)p_{z_0} - q_{z_0})n_s^\ell]\, ds$$

$$+ C_1 E \int_0^{(1/\ell)\wedge\tau_1} e^{-\beta s}\left[(e^{rs}-1)\left(x_0 + \frac{1-\mu}{r} L\right) + (e^{Kt}-1)\left(y_0 + \frac{L}{K}\right)\right] ds.$$

Taking into account that the controls and the utility function are bounded and that $\Phi(x_0, y_0, z_0) \leqq \|U\|_\infty/\beta$, we can also find a constant C_2 such that

$$-\frac{1}{\ell^2} \leqq C_2 E \int_0^{(1/\ell)\wedge\tau_1} (1 - e^{-\beta s})\, ds$$

$$+ C_1 E \int_0^{(1/\ell)\wedge\tau_1} e^{-\beta s}\left[(e^{rs}-1)\left(x_0 + \frac{1-\mu}{r} L\right) + (e^{Ks}-1)\left(y_0 + \frac{L}{K}\right)\right] ds$$

$$+ E \int_0^{(1/\ell)\wedge\tau_1} [rp_{z_0}x_0 + z_0 q_{z_0} y_0 + \mathscr{L}\Phi(x_0, y_0, z_0) - \beta\Phi(x_0, y_0, z_0)]\, ds$$

$$+ E \int_0^{(1/\ell)\wedge\tau_1} U(C_s^\ell)\, ds - E \int_0^{(1/\ell)\wedge\tau_1} p_{z_0} C_s^\ell\, ds$$

$$+ E \int_0^{(1/\ell)\wedge\tau_1} (-(1+\lambda)p_{z_0} + q_{z_0})m_s^\ell\, ds + E \int_0^{(1/\ell)\wedge\tau_1} ((1-\mu)p_{z_0} - q_{z_0})n_s^\ell\, ds.$$

We now divide both sides by $E[(1/\ell) \wedge \tau_1]$ and we pass to the limit as $\ell \to \infty$. The first two terms will go to zero. Let

$$A_1^\ell = \frac{1}{E((1/\ell)\wedge\tau_1)} E \int_0^{(1/\ell)\wedge\tau_1} U(C_s^\ell)\, ds$$

$$A_2^\ell = -\frac{1}{E((1/\ell)\wedge\tau_1)} E \int_0^{(1/\ell)\wedge\tau_1} C_s^\ell\, ds$$

$$A_3^\ell = \frac{1}{E((1/\ell)\wedge\tau_1)} E \int_0^{(1/\ell)\wedge\tau_1} (-(1+\lambda)p_{z_0} + q_{z_0})m_s^\ell\, ds$$

$$A_4^\ell = \frac{1}{E((1/\ell)\wedge\tau_1)} E \int_0^{(1/\ell)\wedge\tau_1} ((1-\mu)p_{z_0} - q_{z_0})n_s^\ell\, ds.$$

Let $\Gamma_{L,N} = \{U(C), C, m, n\}$ for $0 \leqq C \leqq N, 0 \leqq m, n \leqq L$. Then $(A_1^\ell, A_2^\ell, A_3^\ell, A_4^\ell) \in \overline{co}\Gamma_N$. Since the latter is a compact set, there is an element (A_1, A_2, A_3, A_4) to which $(A_1^\ell, A_2^\ell, A_3^\ell, A_4^\ell)$ converges along a subsequence. We conclude easily that

$$\beta\Phi(x_0, y_0, z_0) \leqq rx_0 p_{z_0} + z_0 y_0 q_{z_0} + \max_{c \geqq 0}[-cp_{z_0} + U(c)] + \max_{0 \leqq m \leqq L}[-(1+\lambda)p_{z_0} + q_{z_0}]m$$

$$+ \max_{0 \leqq n \leqq L}[(1-\mu)p_{z_0} - q_{z_0}]n + \mathscr{L}\Phi(x_0, y_0, z_0).$$

Using that $\Phi = u^{L,N}$ at (x_0, y_0, z_0), we get (3.2).

We will next show that $\lim_{N\to\infty} u^{L,N} = u^L$ uniformly on compact subsets of $\bar{\Omega} \times Z$. We will need the following lemma.

LEMMA 3.3. *Let* $(C., m., n.) \in A_L$ *and* $N > 0$. *Then* $(C. \wedge N, m., n.) \in A_{L,N}$ *and*

$$\lim_{N\to\infty} E \int_0^\infty e^{-\beta s} U(C_s \wedge N)\, ds = E \int_0^\infty e^{-\beta s} U(C_s)\, ds.$$

We first prove the above claim and then we present the proof of Lemma 3.3. To this end, fix $(x_0, y_0, z_0) \in \bar{\Omega} \times Z$ and let $(C^\varepsilon_., m^\varepsilon_., n^\varepsilon_.) \in A_L$ be an ε-optimal policy. Then

$$u^L(x_0, y_0, z_0) \leqq E \int_0^{+\infty} e^{-\beta s} U(C^\varepsilon_s)\, ds + \varepsilon.$$

On the other hand, Lemma 3.3 yields that, for each $N > 0$, $(C^\varepsilon_. \wedge N, m^\varepsilon_., n^\varepsilon_.) \in A_{L,N}$ and

$$E \int_0^{+\infty} e^{-\beta s} U(C^\varepsilon_s)\, ds \leqq E \int_0^{+\infty} e^{-\beta s} U(C^\varepsilon_s \wedge N)\, ds + \varepsilon \text{ for } N \geqq N(\varepsilon).$$

Combining the last two inequalities yields

$$u^L(x_0, y_0, z_0) \leqq E \int_0^{+\infty} e^{-\beta s} U(C^\varepsilon_s \wedge N)\, ds + 2\varepsilon \leqq u^{L,N}(x_0, y_0, z_0) + 2\varepsilon \quad \text{for } N \geqq N(\varepsilon);$$

hence $u^{L,N}(x_0, y_0, z_0)$ is a nondecreasing sequence that converges to $u^L(x_0, y_0, z_0)$. Since both $u^{L,N}$ and u^L are continuous functions, Dini's theorem implies that $u^{L,N} \to u^L$ locally uniformly.

Proof of Lemma 3.3. The fact that $(C. \wedge N, m., n.) \in A_{L,N}$ follows immediately from the definitions of A_L and $A_{L,N}$.

On the other hand, since U is increasing, bounded and $U(0) = 0$,

$$0 \leqq E \int_0^\infty e^{-\beta s} U(C_s)\, ds - E \int_0^\infty e^{-\beta s} U(C_s \wedge N)\, ds \leqq \|U\|_\infty E\left[\int_{\{C_s \geqq N\}} e^{-\beta s}\, ds\right].$$

To conclude we need to show that

$$\lim_{N\to\infty} E\left[\int_{\{C_s \geqq N\}} e^{-\beta s}\, ds\right] = 0.$$

However,

$$NE\left[\int_{\{C_s \geqq N\}} e^{-\beta s}\, ds\right] \leqq E\left[\int_{\{C_s \geqq N\}} e^{-\beta s} C_s\, ds\right] \leqq E \int_0^\infty e^{-\beta s} C_s\, ds < \infty,$$

where the last inequality follows from the facts that $(C., m., n.) \in A_L$ and $\beta > r$. Hence

$$E\left[\int_{\{C_s \geqq N\}} e^{-\beta s}\, ds\right] \leqq \frac{1}{N} E \int_0^\infty e^{-\beta s} C_s\, ds.$$

Letting $N \to \infty$, we conclude. □

Finally, we show that u^L is a viscosity subsolution of (3.4) on $\bar{\Omega}$. Let $(p_{z_0}, q_{z_0}) \in D^+_{(x,y)} u^L(x_0, y_0, z_0)$. Then there exists (cf. [4]) a smooth function $\phi: \Re \times \Re \times Z \to \Re$ such that $\phi_x(x_0, y_0, z_0) = p_{z_0}$, $\phi_y(x_0, y_0, z_0) = q_{z_0}$, and $u^L - \phi$ has a strict local maximum at (x_0, y_0, z_0). Then $(u^{L,N} - \phi)(\cdot, \cdot, z_0)$ has a local maximum at (x_N, y_N, z_0) and $(x_N, y_N, z_0) \to (x_0, y_0, z_0)$. Moreover,

$$\beta u^{L,N}(x_N, y_N, z_0) \leqq r x_N p_{z_0} + z_0 y_N q_{z_0} + \max_{c \geqq 0} [-c p_{z_0} + U(c)] + \max_{0 \leqq m \leqq L} [-(1+\lambda) p_{z_0} + q_{z_0}] m$$
$$+ \max_{0 \leqq n \leqq L} [(1-\mu) p_{z_0} - q_{z_0}] n + \mathscr{L} u^{L,N}(x_N, y_N, z_0).$$

Sending $N \to \infty$, we get that u^L is a viscosity subsolution of (3.4).

(ii) We now prove that $u^{L,N}$ is a supersolution of (3.4) in Ω. Let $(x_0, y_0, z_0) \in \Omega$ and $(p_{z_0}, q_{z_0}) \in D^-_{(x,y)} u^{L,N}(x_0, y_0, z_0)$. Since, by Lemma 3.2(i), $D^+ u^{L,N}(x_0, y_0, z) \neq \varnothing$, the properties of D^+ and D^- yield that $u^{L,N}$ is differentiable in the X-direction at the point (x_0, y_0, z_0) (cf. [4]). Let Φ be defined as in (i). There exists a continuous function h with $h(0)=0$ such that

$$u^{L,N}(x, y, z_0) \geqq \Phi(x, y, z_0) + |X - X_0| h(|X - X_0|).$$

Using again the dynamic programming principle, we get

$$\begin{aligned}\Phi(x_0, y_0, z_0) \geqq \sup_{A_{L,N}} E\bigg[\int_0^\theta & e^{-\beta s} U(C_s)\, ds + e^{-\beta\theta}\Phi(x_\theta, y_\theta, z_\theta) \\ & + e^{-\beta\theta}|X_\theta - X_0| h(|X_\theta - x_0|)\bigg].\end{aligned} \tag{3.6}$$

Let $(C, m, n) \in A_{L,N}$ with $C_t = C_0$, $m_t = m_0$, $n_t = n_0$, $\forall t \geqq 0$ and $\tau = \theta \wedge \tau_1$ where $\theta = T \wedge \inf\{\tau > 0 \colon x_\tau^0 = 0\} \wedge \inf\{\tau > 0 \colon y_\tau^0 = 0\}$ and x_τ^0, y_τ^0 are the corresponding trajectories. Using Dynkin's formula and (3.6), we get

$$\begin{aligned} 0 \geqq C_1 E & \int_0^{\theta\wedge\tau_1} e^{-\beta s}\left[(e^{rs}-1)\left(x_0 + \frac{1-\mu}{r} L\right) + (e^{Ks}-1)\left(y_0 + \frac{L}{K}\right)\right] ds \\ & + C_2 E \int_0^{\theta\wedge\tau_1} (1 - e^{-\beta s})\, ds \\ & + E \int_0^{\theta\wedge\tau_1} [U(C_0) - p_{z_0} C_0 + r x_0 p_{z_0} + z_0 y_0 q_{z_0} + [-(1+\lambda) p_{z_0} + q_{z_0}] m_0 \\ & + [(1-\mu) p_{z_0} - q_{z_0}] n_0 + \mathscr{L}\Phi(x_0, y_0, z_0) - \beta\Phi(x_0, y_0, z_0)]\, ds \end{aligned}$$

for some constants C_1 and C_2. Dividing by $E[\theta \wedge \tau_1]$, sending $T \to 0$ and using that $u^{L,N}(x_0, y_0, z_0) = \Phi(x_0, y_0, z_0)$ we obtain (3.3). Finally, working similarly as in (i) we can show that u^L is a viscosity supersolution of (3.4) in Ω. □

THEOREM 3.2. *Let u and v be bounded, uniformly continuous such that u is a viscosity subsolution of* (3.4) *on $\bar{\Omega}$ and v is a viscosity supersolution of* (3.4) *in Ω. Then $u \leqq v$ on $\bar{\Omega}$.*

Proof. Let $X = (x, y) \in \bar{\Omega}$, $P = (p, q) \in R \times R$ and $H \colon \bar{\Omega} \times Z \times R^2 \to R$ be given by

$$H(X, z, P) = rxp + zyq + F(p) + \max_{0 \leqq m \leqq L} [(1+\lambda)p - q] m + \max_{0 \leqq n \leqq L} [-(1-\mu)p + q] n,$$

where $F(p) = \max_{c \geqq 0} [-cp + U(c)]$, $p > 0$.

We argue by contradiction; i.e., we assume that

$$\max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X, z) - v(X, z)] > 0. \tag{3.7}$$

Then for sufficiently small $\theta > 0$

$$\max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X, z) - v(X, z) - \theta|X|^2] > 0. \tag{3.8}$$

Indeed, if not, there would be a sequence $\theta_n \downarrow 0$ such that $\max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X, z) - v(X, z) - \theta_n |X|^2] \leqq 0$, which in turn yields $\max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X, z) - v(X, z)] \leqq 0$, contradicting (3.7).

Since the process z takes a finite number of values and u and v are bounded, we can find points $z_0 \in Z$ and $\bar{X} \in \bar{\Omega}$ such that

$$u(\bar{X}, z_0) - v(\bar{X}, z_0) - \theta|\bar{X}|^2 = \max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X, z) - v(X, z) - \theta|X|^2]. \tag{3.9}$$

In what follows we omit z_0.

Next, for $\varepsilon > 0$ we define $\psi : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ by

$$\psi(X, Y) = u(X) - v(Y) - \left| \frac{Y - X}{\varepsilon} - 4(1, 1) \right|^2 - \theta |X|^2,$$

and we claim that ψ attains its maximum at a point, say (X_0, Y_0), such that for ε small and some $\ell > 0$

$$|Y_0 - X_0| \leqq \ell \varepsilon. \tag{3.10}$$

Indeed, observe that ψ is bounded. Let (X_n, Y_n) be a maximizing sequence. Then

$$\lim_{n \to \infty} \left[u(X_n) - v(Y_n) - \left| \frac{Y_n - X_n}{\varepsilon} - 4(1, 1) \right|^2 - \theta |X_n|^2 \right] = \sup_{\bar{\Omega} \times \bar{\Omega}} \psi(X, Y) < +\infty.$$

However,

$$\sup_{\bar{\Omega} \times \bar{\Omega}} \psi(X, Y) \geqq \psi(\bar{X}, \bar{X} + 4\varepsilon(1, 1)) \geqq u(\bar{X}) - v(\bar{X}) - \theta |\bar{X}|^2 - \omega_v(k\varepsilon), \tag{3.11}$$

where ω_v is the modulus of continuity of v and $k > 0$. Using (3.8) and (3.9), we see that for ε sufficiently small

$$\sup_{\bar{\Omega} \times \bar{\Omega}} \psi(X, Y) > 0. \tag{3.12}$$

We also observe that if

$$u(X_n) - v(Y_n) - \left| \frac{Y_n - X_n}{\varepsilon} - 4(1, 1) \right|^2 \leqq 0$$

as $n \to \infty$ we contradict (3.12). Therefore

$$u(X_n) - v(Y_n) \geqq \left| \frac{Y_n - X_n}{\varepsilon} - 4(1, 1) \right|^2,$$

which implies (3.10). On the other hand, the choice of (X_n, Y_n) and (3.12) yield that the sequence (X_n), and, in view of the above observation, (Y_n) are bounded as $n \to \infty$. Hence, along subsequences, (X_n, Y_n) converge to a maximum point of ψ, which we denote by (X_0, Y_0).

Moreover, (3.10) and (3.11) give

$$\left| \frac{Y_0 - X_0}{\varepsilon} - 4(1, 1) \right|^2 \leqq \omega_n(k\varepsilon) + \omega_v(\ell\varepsilon). \tag{3.13}$$

We can choose ε small such that $\omega_n(k\varepsilon) + \omega_v(\ell\varepsilon) \leqq 1$. Then there is a vector e with $|e| \leqq 1$ such that $Y_0 = X_0 + 4\varepsilon(1, 1) + \varepsilon e$, which implies that $Y_0 \in \Omega$.

We now consider the functions

$$\phi(Y) = u(X_0) - \left| \frac{Y - X_0}{\varepsilon} - 4(1, 1) \right|^2 - \theta |X_0|^2,$$

$$\bar{\phi}(X) = v(Y_0) + \left| \frac{Y_0 - X}{\varepsilon} - 4(1, 1) \right|^2 + \theta |X|^2.$$

Since $u - \bar{\phi}$ has a maximum at $X_0 \in \bar{\Omega}$ and $v - \phi$ has a minimum at $Y_0 \in \Omega$, applying the definition of viscosity solution as in Lemma 3.1, we get

$$\beta u(X_0, z_0) \leqq H(X_0, z_0, P_\varepsilon + 2\theta X_0) + \sum_{z_0 \neq z'} q_{z_0 z'}[u(X_0, z) - u(X_0, z_0)]$$

and

$$\beta v(Y_0, z_0) \geqq H(Y_0, z_0, P_\varepsilon) + \sum_{z_0 \neq z'} q_{z_0 z'}[v(Y_0, z) - v(Y_0, z_0)],$$

where

$$P_\varepsilon = -\frac{2}{\varepsilon}\left(\frac{Y_0 - X_0}{\varepsilon} - 4(1, 1)\right).$$

Combining the above inequalities yields

$$\begin{aligned} \beta[u(X_0, z_0) - v(Y_0, z_0)] \leqq & [H(X_0, z_0, P_\varepsilon + 2\theta X_0) - H(Y_0, z_0, P_\varepsilon)] \\ & + \sum_{z_0 \neq z'} q_{z_0 z'}[u(X_0, z) - u(X_0, z_0) - v(Y_0, z) + v(Y_0, z_0)]. \end{aligned} \tag{3.14}$$

On the other hand, from the definition of H, we have that

$$\begin{aligned} & |H(X_0, z_0, P_\varepsilon + 2\theta X_0) - H(Y_0, z_0, P_\varepsilon)| \\ & \quad \leqq |H(X_0, z_0, P_\varepsilon + 2\theta X_0) - H(X_0, z_0, P_\varepsilon)| + K|X_0 - Y_0||P_\varepsilon| \end{aligned} \tag{3.15}$$

for some $K > 0$. Let $X_0 = (x_0, y_0)$ and $P_\varepsilon = (p_\varepsilon, q_\varepsilon)$. Then

$$\begin{aligned} H(x_0, y_0, z_0, & 2\theta x_0 + p_\varepsilon, 2\theta y_0 + q_\varepsilon) - H(x_0, y_0, z_0, p_\varepsilon, q_\varepsilon) \leqq 2\theta[rx_0^2 + zy_0^2] \\ & + \left[[\max_{0 \leqq m \leqq L}[(1+\lambda)(2\theta x_0 + p_\varepsilon) - (2\theta y_0 + q_\varepsilon)]m] - [(1+\lambda)p_\varepsilon - q_\varepsilon]L\right] \\ & + \left[[\max_{0 \leqq n \leqq L}[-(1-\mu)(2\theta x_0 + p_\varepsilon) + (2\theta y_0 + q_\varepsilon)]n] - [-(1-\mu)p_\varepsilon + q_\varepsilon]L\right] \\ \leqq & C\theta L|X_0| + 2\theta[rx_0^2 + zy_0^2] \leqq \theta|X_0|^2 + \theta C^2 L^2 + 2\theta[rx_0^2 + zy_0^2] \\ \leqq & \beta\theta|X_0|^2 + C^2 L^2 \theta, \end{aligned}$$

for some $C > 0$, where we used that $K \geqq r$, (2.1), and that F is a decreasing function. Using the above inequality and (3.10), (3.13), and (3.15), we get

$$\begin{aligned} & |H(X_0, z_0, P_\varepsilon + 2\theta X_0) - H(Y_0, z_0, P_\varepsilon)| \\ & \quad \leqq 2K\ell[\omega_v(\ell\varepsilon) + \omega_v(k\varepsilon)]^{1/2} + \beta\theta|X_0|^2 + C^2 L^2 \theta. \end{aligned} \tag{3.16}$$

Moreover, from (3.10) we have

$$u(X_0, z) - v(X_0, z) \leqq u(X_0, z_0) - v(Y_0, z_0) + \omega_v(k\varepsilon),$$

which combined with (3.11) gives

$$[u(X_0, z) - u(X_0, z_0)] - [v(Y_0, z) - v(Y_0, z_0)] \leqq \omega_v(k\varepsilon) + \omega_v(\ell\varepsilon).$$

Finally, using that $0 \leqq q_{zz'} \leqq 1$ and $\sum_{z' \in Z} q_{zz'} = 1$, we get

$$\mathscr{L}u(X_0, z_0) - \mathscr{L}v(Y_0, z_0) \leqq \omega_v(k\varepsilon) + \omega_v(\ell\varepsilon). \tag{3.17}$$

Using now (3.14), (3.16), and (3.17), we have

$$\begin{aligned} \beta[u(X_0, z_0) - v(Y_0, z_0) - \theta|X_0|^2] \leqq & 2K\ell[\omega_v(\ell\varepsilon) + \omega_v(k\varepsilon)]^{1/2} \\ & + C^2 L^2 \theta + \omega_v(k\varepsilon) + \omega_v(\ell\varepsilon), \end{aligned}$$

and, using the definition of (X_0, Y_0),

$$\max_{\bar{\Omega} \times \bar{\Omega}} \psi(X, Y) \leqq \frac{1}{\beta}\left[2K\ell[\omega_v(\ell\varepsilon) + \omega_v(k\varepsilon)]^{1/2} + \omega_v(k\varepsilon) + \omega_v(\ell\varepsilon)\right] + \frac{C^2 L^2 \theta}{\beta}.$$

Then, however, (3.9) and (3.11) yield

$$\max_{z\in Z}\sup_{X\in\bar\Omega}\,[u(X,z)-v(X,z)-\theta|X|^2]$$

$$\leqq \omega_v(k\varepsilon)+\frac{1}{\beta}\,[2K\ell[\omega_v(\ell\varepsilon)+\omega_v(k\varepsilon)]^{1/2}+\omega_v(k\varepsilon)+\omega_v(\ell\varepsilon)]+\frac{C^2L^2}{\beta}\,\theta,$$

which, in turn, implies that

$$u(X,z)-v(X,z)-\theta|X|^2\leqq\frac{C^2L^2}{\beta}\,\theta\qquad\forall X\in\bar\Omega\text{ and }\forall z\in Z.$$

Letting $\theta\to 0$ contradicts (3.7). □

Remark 3.1. Although it was not necessary for the above proof, but it will be used later on, we show that

$$\lim_{\theta\downarrow 0}\lim_{\varepsilon\downarrow 0}\theta|X_0(\theta,\varepsilon)|^2=0. \tag{3.18}$$

Indeed, from (3.11) we have

$$\psi(X_0,X_0)+\omega_v(k\varepsilon)\geqq u(\bar X)-v(\bar X)-\theta|\bar X|^2-\omega_v(k\varepsilon),$$

which yields

$$u(X_0,z_0)-v(X_0,z_0)-\theta|X_0|^2\geqq[u(\bar X,z_0)-v(\bar X,z_0)-\theta|\bar X|^2]-2\omega_v(k\varepsilon), \tag{3.19}$$

and, in turn $\sup_{\varepsilon>0}|X_0(\theta,\varepsilon)|<\infty$.

Therefore there exists $\hat X_0(\theta)$ such that $\lim_{\varepsilon\to 0}|X_0(\theta,\varepsilon)|^2=\hat X_0(\theta)$, otherwise we contradict (3.19). The limit here is taken along subsequences, which to simplify notation we denote the same way as the whole family. By sending $\varepsilon\downarrow 0$, (3.19) combined with (3.9) implies

$$u(\hat X_0,z_0)-v(\hat X_0,z_0)-\theta|\hat X_0|^2\geqq u(X,z)-v(X,z)-\theta|X|^2\qquad\forall X\in\bar\Omega,\forall z\in Z. \tag{3.20}$$

We now send $\theta\to 0$. If $\lim_{\theta\downarrow 0}\theta|\hat X_0|^2=\alpha\neq 0$, again along subsequences, (3.20) yields

$$\max_{z\in Z}\sup_{\bar\Omega}\,[u(X,z)-v(X,z)]-\alpha\geqq\max_{z\in Z}\sup_{\bar\Omega}\,[u(X,z)-v(X,z)].$$

Therefore $\max_{z\in Z}\sup_{\bar\Omega}[u(X,z)-v(X,z)]<0$, which contradicts (3.7).

PROPOSITION 3.1. *As* $L\to\infty$, $u^L\to w\in C(\bar\Omega)$.

Proof. Fix $(x_0,y_0)\in\bar\Omega$. The sequence $u^L(x_0,y_0)$ is increasing as $L\to\infty$, therefore $\lim_{L\to\infty}u^L(x_0,y_0)=w(x_0,y_0)$ exists. Moreover, since the functions u^L are continuous at (x_0,y_0) uniformly in L, w is continuous on $\bar\Omega$. □

4. In this section we prove that w coincides with the value function $u\in C(\bar\Omega)$. We first show that u is a constrained viscosity solution of a certain variational inequality and second that this variational inequality has a unique constrained viscosity solution.

THEOREM 4.1. *The value function u is a constrained viscosity solution of*

$$\begin{gathered}\min[(1+\lambda)u_x-u_y,-(1-\mu)u_x+u_y,\beta u-rxu_x-zyu_y-F(u_x)-\mathscr{L}u(z)]=0\\ \forall(x,y,z)\in(0,+\infty)\times(0,+\infty)\times Z\text{ with }u(0,0,z)=0,\forall z\in Z.\end{gathered} \tag{4.1}$$

Proof. We first show that u is viscosity subsolution of (4.1) on $\bar\Omega$. To this end, let (x_0,y_0,z_0) be fixed with $(x_0,y_0)\in\bar\Omega$, consider $(p_{z_0},q_{z_0})\in D^+_{x,y}u(x_0,y_0,z_0)$ and define $\Phi:\mathfrak{R}\times\mathfrak{R}\times Z\to\mathfrak{R}$ by

$$\Phi(x,y,z)=u(x_0,y_0,z)+p_z(x-x_0)+q_z(y-y_0),$$

where $(p_z,q_z)\in D^+_{(x,y)}u(x_0,y_0,z)$. Lemma 3.2 yields

$$u(x,y,z)\leqq\Phi(x,y,z). \tag{4.2}$$

We are going to show that

(4.3) $\min[(1+\lambda)p_{z_0}-q_{z_0}, -(1-\mu)p_{z_0}+q_{z_0}, \beta u - rx_0p_{z_0}-z_0y_0q_{z_0}-F(p_{z_0})-\mathcal{L}u(z_0)]\leqq 0$

where $F(p)=\max_{c\geqq 0}[-cp+U(c)]$.

If $(1+\lambda)p_{z_0}-q_{z_0}\leqq 0$ or $-(1-\mu)p_{z_0}+q_{z_0}\leqq 0$, the above inequality is obvious. So let us assume that

$$(1+\lambda)p_{z_0}-q_{z_0}>0 \text{ and } -(1-\mu)p_{z_0}+q_{z_0}>0. \tag{4.4}$$

In the following, we are first going to assume that the control C is such that $0\leqq C_t\leqq N$ for almost every $t\geqq 0$, and then we will remove the upper bound. Since the arguments are similar to the ones used in Theorem 3.1, we proceed as if there is no bound on C. Later, we will mention when we use this upper bound.

Applying the dynamic programming principle at the point (x_0, y_0, z_0) with stopping time $\theta=(1/\ell)\wedge\tau_1=\min\{1/\ell, \tau_1\}$, where τ_1 is the first jump time of z_t, we obtain

$$\begin{aligned} u(x_0,y_0,z_0)\leqq E\bigg[\int_0^\theta e^{-\beta s}U(C_s^\ell)\,ds+e^{-\beta\theta}u\bigg(e^{r\theta}\bigg(x_0-\int_0^\theta e^{-rs}C_s^\ell\,ds \\ -(1+\lambda)m_\theta^\ell+(1-\mu)n_\theta^\ell\bigg), e^{z_0\theta}(y_0+\hat m_\theta^\ell-\hat n_\theta^\ell), z_\theta\bigg)\bigg]+\frac{1}{\ell^2}, \end{aligned} \tag{4.5}$$

where (C^ℓ, M^ℓ, N^ℓ) is an $1/\ell^2$-optimal policy and

$$m_\theta^\ell=\int_0^\theta e^{-rs}\,dM_s^\ell,\ \hat m_\theta^\ell=\int_0^\theta \exp(-z_0s)\,dM_s^\ell$$

and

$$n_\theta^\ell=\int_0^\theta e^{-rs}\,dN_s^\ell,\ \hat n_\theta^\ell=\int_0^\theta \exp(-z_0s)\,dN_s^\ell.$$

Let $r\geqq z_0$ (the case $r<z_0$ is treated similarly). Then

$$m_\theta^\ell\leqq\hat m_\theta^\ell \text{ and } n_\theta^\ell\leqq\hat n_\theta^\ell. \tag{4.6}$$

Since the control (C^ℓ, M^ℓ, N^ℓ) is admissible, we also have

$$x_0-\int_0^\theta e^{-rs}C_s^\ell\,ds\geqq(1+\lambda)m_\theta^\ell-(1-\mu)n_\theta^\ell \tag{4.7}$$

and

$$y_0\geqq-\hat m_\theta^\ell+\hat n_\theta^\ell. \tag{4.8}$$

Moreover,

$$\hat m_\theta^\ell\leqq(\exp(r-z_0)\theta)\int_0^\theta e^{-rs}\,dM_s^\ell\leqq\left(\exp\left(\frac{r-z_0}{\ell}\right)\right)m_\theta^\ell \tag{4.9}$$

and, similarly,

$$\hat n_\theta^\ell\leqq\left(\exp\left(\frac{r-z_0}{\ell}\right)\right)n_\theta^\ell. \tag{4.10}$$

From (4.7)-(4.10) we get

$$m_\theta^\ell\leqq\frac{x_0+(1-\mu)y_0}{c}, \tag{4.11}$$

where $c>0$ is such that $(1+\lambda)-(1-\mu)\exp((r-z_0)/\ell)\geqq c$ for ℓ sufficiently large. Similarly,

$$n_\theta^\ell \leqq \frac{\exp((r-z_0)/\ell)x_0+(1+\lambda)y_0}{c} \leqq \frac{2x_0+(1+\lambda)y_0}{c} \tag{4.12}$$

for ℓ sufficiently large.

Moreover, since

$$m_\theta^\ell = \int_0^\theta e^{-rs}\,dM_s^\ell \geqq \exp(-r/\ell)(M_\theta^\ell - M_0^\ell),$$

using (4.11) we obtain

$$M_\theta^\ell - M_0^\ell \leqq \exp(r/\ell)\,\frac{x_0+(1-\mu)y_0}{c} \leqq k_1 \tag{4.13}$$

for some $k_1>0$, and, similarly

$$N_\theta^\ell - N_0^\ell \leqq k_2 \tag{4.14}$$

for some $k_2>0$.

Therefore there exist constants $\bar{C}$, K_1 and K_2 such that

$$\hat{m}_\theta^\ell - m_\theta^\ell \leqq E\int_0^\theta \bar{C}s\,dM_s^\ell \leqq \frac{K_1}{\ell} \tag{4.15}$$

and

$$\hat{n}_\theta^\ell - n_\theta^\ell \leqq \frac{K_2}{\ell}. \tag{4.16}$$

Using that u is a nondecreasing function, $r\geqq z_0$, (4.5), and (4.15), we have

$$\begin{aligned} u(x_0,y_0,z_0) \leqq \|U\|_\infty E(\theta) &+ E[e^{-\beta\theta}u(e^{r\theta}(x_0+\bar{X}_\theta), e^{r\theta}(y_0+\bar{Y}_\theta), z_\theta) \\ &- e^{-\beta\theta}u(x_0+\bar{X}_\theta, y_0+\bar{Y}_\theta, z_0)] \\ &+ E[e^{-\beta\theta}u(x_0+\bar{X}_\theta, y_0+\bar{Y}_\theta, z_0)], \end{aligned} \tag{4.17}$$

where

$$\bar{X}_\theta = -(1+\lambda)\hat{m}_\theta^\ell + \frac{(1+\lambda)K_1}{\ell} + (1-\mu)\hat{n}_\theta^\ell$$

and

$$\bar{Y}_\theta = \hat{m}_\theta^\ell - \hat{n}_\theta^\ell.$$

Let ω be the modulus of continuity of u. Then

$$\begin{aligned} E[e^{-\beta\theta}u(e^{r\theta}(x_0+\bar{X}_\theta), e^{r\theta}(y_0+\bar{Y}_\theta), z_\theta) - e^{-\beta\theta}u(x_0+\bar{X}_\theta, y_0+\bar{Y}_\theta, z_0)] \\ \leqq K_3[\|u\|_\infty P(z_\theta \neq z_0) + \omega(e^{r\theta}-1)] \end{aligned} \tag{4.18}$$

for some positive constant K_3.

Moreover, (4.2), (4.17), and (4.18) yield

$$\begin{aligned} u(x_0,y_0,z_0) \leqq \|U\|_\infty E(\theta) &+ K_3[\|u\|_\infty P(z_\theta\neq z_0) + \omega(e^{r\theta}-1)] \\ &+ E\,e^{-\beta\theta}[u(x_0,y_0,z_0) + p_{z_0}\bar{X}_\theta + q_{z_0}\bar{Y}_\theta]. \end{aligned}$$

Since $P[z_\theta \neq z_0] = 0(\theta)$, we get

$$E[((1+\lambda)p_{z_0} - q_{z_0})\hat{m}_\theta^\ell + (-(1-\mu)p_{z_0} + q_{z_0})\hat{n}_\theta^\ell]$$
$$\leqq K_3[\|u\|_\infty E(\theta) + \omega(e^{r\theta} - 1)] + \|U\|_\infty E(\theta) + \frac{(1+\lambda)K_1}{\ell}.$$

Finally, in view of (4.4), we can find positive constants M_1 and M_2 such that

$$E(\hat{m}_\theta^\ell) \leqq M_1\left[E(\theta) + \omega\left(\frac{1}{\ell}\right) + \frac{1}{\ell}\right]$$

and

$$E(\hat{n}_\theta^\ell) \leqq M_2\left[E(\theta) + \omega\left(\frac{1}{\ell}\right) + \frac{1}{\ell}\right].$$

From (4.2) and (4.5) we get

$$u(x_0, y_0, z_0) \leqq E\int_0^\theta e^{-\beta s} U(C_s^\ell)\, ds + E[e^{-\beta\theta}\Phi(x_\theta^\ell, y_\theta^\ell, z_\theta)] + \frac{1}{\ell^2},$$

where $x_\theta^\ell, y_\theta^\ell$ are given by (1.3) with control (C^ℓ, M^ℓ, N^ℓ). Using Dynkin's formula, we have

$$u(x_0, y_0, z_0) \leqq E\int_0^\theta e^{-\beta s} U(C_s^\ell)\, ds + u(x_0, y_0, z_0)$$
$$+ E\int_0^\theta e^{-\beta s}[-\beta\Phi(x_s^\ell, y_s^\ell, z_0) + r x_s^\ell p_{z_0} + z_0 y_s^\ell q_{z_0} - C_s^\ell p_{z_0}$$
$$+ \mathscr{L}\Phi(x_s^\ell, y_s^\ell, z_0)]\, ds + E\int_0^\theta e^{-\beta s}[(-(1+\lambda)p_{z_0} + q_{z_0})\, dM_s^\ell$$
$$+ E\int_0^\theta e^{-\beta s}((1-\mu)p_{z_0} - q_{z_0})\, dN_s^\ell] + \frac{1}{\ell^2}.$$

Since M_t and N_t are nondecreasing processes, using (4.4), we get

$$0 \leqq E\int_0^\theta e^{-\beta s} U(C_s^\ell)\, ds + E\int_0^\theta [r p_{z_0}(x_s^\ell - x_0) + z_0 q_{z_0}(y_s^\ell - y_0)]\, ds$$
$$+ E\int_0^\theta e^{-\beta s}[-\beta u(x_0, y_0, z_0) + r p_{z_0} x_0$$
$$+ z_0 q_{z_0} y_0 - p_{z_0} C_s^\ell + \mathscr{L}\Phi(x_s^\ell, y_s^\ell, z_0)]\, ds + \frac{1}{\ell^2}.$$

Let

$$A(\theta) = E\int_0^\theta [r p_{z_0}(x_s^\ell - x_0) + z_0 q_{z_0}(y_s^\ell - y_0)]\, ds$$

and

$$B(\theta) = E\int_0^\theta e^{-\beta s}[-\beta u(x_0, y_0, z_0) + r x_0 p_{z_0} + z_0 y_0 q_{z_0} - p_{z_0} C_s^\ell + \mathscr{L}\Phi(x_s^\ell, y_s^\ell, z_0)]\, ds.$$

Then

$$A(\theta)=E\int_0^\theta (rp_{z_0}[(e^{rs}-1)x_0-(1+\lambda)e^{rs}m_s^\ell+(1-\mu)\,e^{rs}n_s^\ell]$$
$$+z_0q_{z_0}[(\exp(z_0s)-1)y_0+(\exp(z_0s))\hat{m}_s^\ell-(\exp(z_0s))\hat{n}_s^\ell])\,ds.$$

Since $E\int_0^\theta e^{hs}k_s^\ell\,ds\leqq(1/\ell)\,e^{h/\ell}E(k_\theta^\ell)\leqq(K/\ell)\,e^{h/\ell}[E(\theta)+\omega(1/\ell)+1/\ell]$, where h is r or z_0 and k_s^ℓ is m_s^ℓ, $\hat{m}_s^\ell$, n_s^ℓ, or $\hat{n}_s^\ell$, we have

$$\lim_{\ell\to\infty}\frac{1}{E(\theta)}E\int_0^\theta e^{hs}k_s^\ell\,ds=0. \tag{4.19}$$

Therefore $\lim_{\ell\to\infty}A(\theta)/E(\theta)=0$.

Relations (4.7), (4.12), and $\beta>r$ give

$$\int_0^\theta e^{-\beta s}C_s^\ell\,ds\leqq x_0+\frac{(1-\mu)[2x_0+(1+\lambda)y_0]}{c}. \tag{4.20}$$

On the other hand,

$$\begin{aligned}\mathscr{L}\Phi(x_s^\ell,y_s^\ell,z_0)&=\sum_{z'\neq z}q_{z'z}[\Phi(x_s^\ell,y_s^\ell,z')-\Phi(x_s^\ell,y_s^\ell,z_0)]\\&=\sum_{z'\neq z}q_{z'z}[u(x_0,y_0,z')+p_{z'}(x_s^\ell-x_0)+q_{z'}(y_s^\ell-y_0)\\&\quad-u(x_0,y_0,z_0)-p_{z_0}(x_s^\ell-x_0)-q_{z_0}(y_s^\ell-y_0)].\end{aligned}$$

Using that Z is a finite set and (4.19), we get

$$\lim_{\ell\to\infty}\frac{E\int_0^\theta e^{-\beta s}\mathscr{L}\Phi(x_s^\ell,y_s^\ell,z_0)\,ds}{E(\theta)}=\mathscr{L}u(x_0,y_0,z_0). \tag{4.21}$$

Finally, from (4.19)-(4.21) we conclude that

$$\beta u(x_0,y_0,z_0)\leqq rx_0p_{z_0}+z_0y_0q_{z_0}+\mathscr{L}u(x_0,y_0,z_0)+F(p_{z_0}).$$

This last conclusion follows along the lines of the analogous argument in the proof of Theorem 3.1. To complete the proof, we need to remove the bound on C_t, which can be done again as in Theorem 3.1.

We finally show that u is a supersolution of (4.1) in Ω. To this end, fix $(x_0,y_0,z_0)\in\Omega\times Z$ and consider $(p_{z_0},q_{z_0})\in D^-_{(x,y)}u(x_0,y_0,z_0)$. Then there exists a smooth function $\psi:\mathfrak{R}\times\mathfrak{R}\times Z\to\mathfrak{R}$ such that $u-\psi$ has a strict minimum at (x_0,y_0,z_0), $u(x_0,y_0,z_0)=\psi(x_0,y_0,z_0)$ and $p_{z_0}=\psi_x(x_0,y_0,z_0)$, $q_{z_0}=\psi_y(x_0,y_0,z_0)$. Then

$$\psi(x_0,y_0,z_0)\geqq\sup_A E\left[\int_0^\theta e^{-\beta s}U(C_s)\,ds+e^{-\beta\theta}\psi(x_\theta,y_\theta,z_\theta)\right]. \tag{4.22}$$

In particular, if we use a control (C,M,N) such that $C_t=0$, for all $t\geqq0$, $M_0=M$, $N_0=N$ and $M_t=N_t=0$, for all $t>0$ in (4.22), we get

$$\psi(x_0,y_0,z_0)\geqq\psi(x_0-(1+\lambda)M+(1-\mu)N,\,y_0+M-N,\,z_0).$$

This yields

$$\min[(1+\lambda)\psi_x(x_0,y_0,z_0)-\psi_y(x_0,y_0,z_0),\,-(1-\mu)\psi_x(x_0,y_0,z_0)+\psi_y(y_0,y_0,z_0)]\geqq0. \tag{4.23}$$

Now if we use a constant control $(C_0, 0, 0)$ in (4.22), we obtain

$$\psi(x_0, y_0, z_0) \geqq E\left[\int_0^\theta e^{-\beta s} U(C_s)\, ds + e^{-\beta\theta}\psi(x_\theta, y_\theta, z_\theta)\right],$$

where $\theta = (1/\ell) \wedge \inf\{\tau: x_\tau^0 = 0\}$, where $\ell > 0$ and x_τ^0 is the corresponding trajectory. We proceed as in Theorem 3.1 and we obtain

$$\beta u(x_0, y_0, z_0) \geqq rx_0 p_{z_0} + z_0 y_0 q_{z_0} + F(p_{z_0}) + \mathscr{L}u(x_0, y_0, z_0). \tag{4.24}$$

Combining (4.23) and (4.24), we get

$$\begin{aligned}\min\,[(1+\lambda)p_{z_0} - q_{z_0}, -(1-\mu)p_{z_0} - q_{z_0}, \beta u - rx_0 p_{z_0} - z_0 y_0 q_{z_0}\\ - F(p_{z_0}) - \mathscr{L}u(x_0, y_0, z_0)] \geqq 0.\end{aligned}$$

□

THEOREM 4.2. *The variational inequality* (4.1) *has a unique constrained viscosity solution in the class of bounded uniformly continuous functions.*

Proof. We are going to show that if v and u are respectively a supersolution in Ω and a subsolution on $\bar{\Omega}$ of (4.1) then $v \geqq u$ on $\bar{\Omega}$. To end this, we follow the strategy of Ishii [9]. Let $\phi: \bar{\Omega} \to \Re$ be defined by $\phi(x, y) = C_1 x + C_2 y + k$, where C_1, C_2, k are positive constants satisfying

$$(1-\mu)C_1 < C_2 < (1+\lambda)C_1$$

and

$$\beta k > rC_1 + KC_2 + F(C_1).$$

Let $X = (x, y) \in \bar{\Omega}$, $P = (p, q) \in \Re \times \Re$ and $H: \bar{\Omega} \times Z \times \Re \times \Re^2 \to \Re$ given by

$$H(X, z, v, P) = \min\,[(1+\lambda)p - q, -(1-\mu)p + q, \beta v - rxp - zyq - F(p) - \mathscr{L}v].$$

An easy calculation shows that there exists a positive constant M such that

$$H(X, z, \phi, \nabla\phi) \geqq M > 0, \qquad \forall X \in \bar{\Omega}\backslash\{0, 0\}, z \in Z. \tag{4.25}$$

Let $\theta \in (0, 1)$ and define $v_\theta = \theta v + (1-\theta)\phi$ on $\bar{\Omega} \times Z$. The functions v_θ are bounded and uniformly continuous. Moreover, since the map $(v, P) \to H(X, z, v, P)$ is concave, ϕ satisfies (4.25) and v is a supersolution in Ω, we have that

$$\begin{aligned}H(X, z, v_\theta, \nabla v_\theta) &\geqq \theta H(X, z, v, \nabla v) + (1-\theta)H(X, z, \phi, \nabla\phi)\\ &\geqq M(1-\theta) > 0, \qquad \forall (X, z) \in \Omega \times Z\end{aligned} \tag{4.26}$$

holds in the viscosity sense. Therefore v_θ is a supersolution of $H(X, z, v_\theta, \nabla v_\theta) = M(1-\theta)$ in Ω. We now need the following lemma.

LEMMA 4.1. *Let $w, v: \bar{\Omega} \times Z \to \Re$ be uniformly continuous and nondecreasing with respect to x. If w is a subsolution of $H(X, z, w, \nabla w) = 0$ on $\bar{\Omega}$ and v is a supersolution of $H(X, z, v, \nabla v) = c$ in Ω for some $c > 0$, v is bounded from below and w is bounded, then $v \geqq w$ on $\bar{\Omega}$.*

Proof. Since the proof is similar to the one of Theorem 3.2, we only show the main steps. We assume that

$$\max_{z \in Z} \sup_{X \in \bar{\Omega}} [w(X, z) - v(X, z)] > 0. \tag{4.27}$$

This implies that for sufficiently small $\theta > 0$

$$\max_{z \in Z} \sup_{X \in \bar{\Omega}} [w(X, z) - v(X, z) - \theta|X|^2] > 0. \tag{4.28}$$

We can find points $z_0 \in Z$ and $\bar{X} \in \bar{\Omega}$ such that

$$w(\bar{X}, z_0) - v(\bar{X}, z_0) - \theta|\bar{X}|^2 = \max_{z \in Z} \sup_{X \in \bar{\Omega}} [w(X, z) - v(X, z) - \theta|X|^2]. \tag{4.29}$$

Next, we consider the function $\psi : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ with

$$\psi(X, Y) = w(X, z_0) - v(Y, z_0) - \left|\frac{Y - X}{\varepsilon} - 4(1, 1)\right|^2 - \theta|X|^2$$

for $\varepsilon > 0$ and we look at its maximum denoted by (X_0, Y_0). Working as in Theorem 3.2 we can show that $Y_0 \in \Omega$. Moreover, from Remark 3.1 we have that

$$\lim_{\theta \downarrow 0} \lim_{\varepsilon \downarrow 0} \theta|X_0|^2 = 0. \tag{4.30}$$

We now consider the functions

$$\phi(Y) = w(X_0) - \left|\frac{Y - X_0}{\varepsilon} - 4(1, 1)\right|^2 - \theta|X_0|^2,$$

$$\bar{\phi}(X) = v(Y_0) + \left|\frac{Y_0 - X}{\varepsilon} - 4(1, 1)\right|^2 + \theta|X|^2.$$

Since $w - \bar{\phi}$ has a maximum at X_0 and $v - \phi$ has a minimum at Y_0, from Lemma 3.1 we have

$$H(X_0, z_0, w(X_0, z_0), P_\varepsilon + 2\theta X_0) \leqq 0$$

and

$$H(Y_0, z_0, v(Y_0, z_0), P_\varepsilon) \geqq c,$$

where

$$P_\varepsilon = -\frac{2}{\varepsilon}\left(\frac{Y_0 - X_0}{\varepsilon} - 4(1, 1)\right).$$

Combining the above inequalities yields

$$H(X_0, z_0, w(X_0, z_0), P_\varepsilon + 2\theta X_0) - H(Y_0, z_0, v(Y_0, z_0), P_\varepsilon) \leqq -c. \tag{4.31}$$

Let $X_0 = (x_0, y_0)$, $Y_0 = (\bar{x}_0, \bar{y}_0)$ and $P_\varepsilon = (p_\varepsilon, q_\varepsilon)$. Using the form of H, (4.31) becomes

$$\begin{aligned} &\min[(1+\lambda)(p_\varepsilon + 2\theta x_0) - (q_\varepsilon + 2\theta y_0), -(1-\mu)(p_\varepsilon + 2\theta x_0) + (q_\varepsilon + 2\theta y_0), \\ &\quad \beta w - r x_0(p_\varepsilon + 2\theta x_0) - z_0 y_0(q_\varepsilon + 2\theta y_0) - F(p_\varepsilon + 2\theta x_0) - \mathscr{L}w(X_0, z_0)] \\ &\quad - \min[(1+\lambda)p_\varepsilon - q_\varepsilon, -(1-\mu)p_\varepsilon + q_\varepsilon, \beta v - r\bar{x}_0 p_\varepsilon - z_0 \bar{y}_0 q_\varepsilon \\ &\quad - F(p_\varepsilon) - \mathscr{L}v(Y_0, z_0)] \leqq -c. \end{aligned} \tag{4.32}$$

We now look at the following cases.

Case (i). $H(X_0, z_0, w(X_0, z_0),\ P_\varepsilon + 2\theta X_0) = (1+\lambda)(p_\varepsilon + 2\theta x_0) - (q_\varepsilon + 2\theta y_0)$. Then (4.32) yields

$$(1+\lambda)2\theta x_0 - 2\theta y_0 \leqq -c.$$

Using (4.30), we get a contradiction.

Case (ii). $H(X_0, z_0, x(X_0, z_0), P_\varepsilon + 2\theta X_0) = -(1-\mu)(p_e + 2\theta x_0) + (q_\varepsilon + 2\theta y_0)$.
In this case, (4.32) yields

$$2\theta[-(1-\mu)x_0 + y_0] \leqq -c,$$

which contradicts (4.30).

Case (iii).

$$H(X_0, z_0, w(X_0, z_0), P_\varepsilon + 2\theta X_0) = \beta w - rx_0(p_\varepsilon + 2\theta x_0) - z_0 y_0(q_\varepsilon + 2\theta y_0) - F(p_\varepsilon + 2\theta x_0) - \mathscr{L}w(X_0, z_0).$$

Then (4.32) yields

$$\beta[w(X_0, z_0) - v(Y_0, z_0)] + c \leqq r(x_0 - \bar{x}_0)p_\varepsilon + z_0(y_0 - \bar{y}_0)q_\varepsilon + F(p_\varepsilon + 2\theta x_0) - F(p_\varepsilon) + \mathscr{L}w(X_0, z_0) - \mathscr{L}v(Y_0, z_0).$$

Working similarly as in the proof of Theorem 3.2, we get

$$\beta[w(X_0, z_0) - v(Y_0, z_0)] + c \leqq \theta|X_0|^2 + 2K\ell[\omega_v(\ell\varepsilon) + \omega_v(k\varepsilon)]^{1/2} + \omega_v(k\varepsilon) + \omega_v(\ell\varepsilon).$$

Again working as in the proof of Theorem 3.2, sending first $\varepsilon \to 0$, then $\theta \to 0$, and using (4.30), we contradict (4.27). □

Acknowledgment. This work is part of the author's Ph.D. dissertation under the direction of Professor W. H. Fleming, whom I thank for his suggestions and advice.

REFERENCES

[1] M. J. BRENNAN, *The optimal number of securities in a risky asset portfolio when there are fixed costs of transacting: theory and some empirical results*, J. Financial Quantitative Anal., 10 (1975), pp. 483–496.

[2] G. M. CONSTANTINIDES, *Multiperiod consumption and investment behavior with convex transactions costs*, Management Sci., 25 (1979), pp. 1127–1137.

[3] ———, *Capital market equilibrium with transaction costs*, J. Political Econom., 94 (1986), pp. 842–862.

[4] M. G. CRANDALL AND P.-L. LIONS, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), pp. 1–42.

[5] M. H. A. DAVIS AND A. R. NORMAN, *Portfolio selection with transaction costs*, Math. Oper. Res., 15 (1990), pp. 676–713.

[6] W. H. FLEMING, S. GROSSMAN, J.-L. VILA, AND T. ZARIPHOPOULOU, *Optimal portfolio rebalancing with transaction costs*, Econometrica, submitted.

[7] W. H. FLEMING, S. SETHI, AND M. H. SONER, *An optimal stochastic planning problem with randomly fluctuating demand*, SIAM J. Control Optim., 25 (1987), pp. 1494–1502.

[8] D. GOLDSMITH, *Transaction costs and the theory of portfolio selection*, J. Finance, 31 (1976), pp. 1127–1139.

[9] H. ISHII, *A simple, direct proof of uniqueness for solutions of the Hamilton-Jacobi equations of Eikonal type*, Proc. Amer. Math. Soc., 100 (1987), pp. 247–251.

[10] S. KANDEL, AND S. A. ROSS, *Some intertemporal models of portfolio selection with transaction costs*, Working Paper 107, University of Chicago, Grad. School Bus., Center Res. Security Prices, Chicago, IL, 1983.

[11] H. E. LELAND, *On consumption and portfolio choices with transaction costs*, in Essays on Economic Behavior under Uncertainty, M. Balch, D. McFadden, and S. Wu, eds., North-Holland, Amsterdam (1974).

[12] H. LEVY, *Equilibrium in an imperfect market: A constraint on the number of securities in the portfolio*, Amer. Econom. Rev., 68 (1978), pp. 643–658.

[13] J. MAYSHAR, *Transaction costs in a model of capital market equilibrium*, J. Political Econom., 87 (1979), pp. 673–700.

[14] R. MUKHERJEE AND E. ZABEL, *Consumption and portfolio choices with transaction costs*, in Essays on Economic Behavior under Uncertainty, M. Balch, D. McFadden, and S. Wu, eds., North-Holland, Amsterdam, 1974.

[15] M. H. SONER, *Optimal control with state-space constraints*, SIAM J. Control Optim., 24 (1986), pp. 552–561.

[16] M. TAKSAR, M. J. KLASS, AND D. ASSAF, *A diffusion model for optimal portfolio selection in the presence of brokerage fees*, Dept. of Statistics, Florida State University, Tallahasee, FL, 1986.

[17] T. ZARIPHOPOULOU, *Optimal investment-consumption models with constraints*, Ph.D. thesis, Brown University, Providence, RI, 1988.