Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model

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Abstract

We introduce a new class of dynamic utilities that are generated forward in time. We discuss the associated value functions, optimal investments and indifference prices and we compare them with their traditional counterparts, implied by backward dynamic utilties.

1 Introduction

This paper is a contribution to integrated portfolio management in incomplete markets. Incompleteness stems from a correlated stochastic factor affecting the dynamics of the traded risky security (stock). The investor trades between a riskless bond and the stock, and may incorporate in his portfolio derivatives and liabilities. The optimal investment problem is embedded into a partial equilibrium one that can be solved by the so called utility-based pricing approach. The optimal portfolios can be, in turn, constructed as the sum of the policy of the plain investment problem and the indifference hedging strategy of the associated claim.

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In a variety of applications, the investment horizon and the maturities of the claims do not coincide. This misalignment might cause price discrepancies, if the current optimal expected utility is not correctly specified. The focus herein is in exploring which classes of utilities preclude such pathological situations.

In the traditional framework of expected utility from terminal wealth, the correct dynamic utility is easily identified, namely, it is given by the implied value function. Such a utility is, then, called self-generating in that it is indistinguishable from the value function it produces. This is an intuitively clear consequence of the Dynamic Programming Principle. There are, however, two important underlying ingredients. Firstly, the risk preferences are a priori specified at a future time, say T, and, secondly, the utility, denoted by $U_t^B(x;T)$, is generated at previous times $(0 \leq t \leq T)$. Herein, T denotes the end of the investment horizon and x represents the wealth argument. Due to the backward in time generation, T is called the backward normalization point and $U_t^B(x;T)$ the backward dynamic utility.

Albeit their popularity, the traditional backward dynamic utilities considerably constraint the set of claims that can be priced to the ones that expire before the normalization point T. Moreover, even in the absence of payoffs and liabilities, utilities of terminal wealth do not seem to capture very accurately changes in the risk attitude as the market environment evolves. In many aspects, in the familiar utility framework utilities "move" backwards in time while market shocks are revealed forward in time.

Motivated by such considerations, the authors recently introduced the notion of forward dynamic utilities (see Musiela and Zariphopoulou (2005a) and (2005b)) in a simple multi-period incomplete binomial model. These utilities, like their backward counterparts, are created via an expected criterion but, in contrast, they evolve forward in time. Specifically, they are determined today, say at s, and they are generated for future times, via a self-generating criterion. In other words, the forward dynamic utility, $U_t^F(x;s)$, is normalized at present time and not at the end of the generic investment horizon.

In this paper, we extend the notion of forward dynamic utilities in a diffusion model with a correlated stochastic factor. For simplicity, we assume that the utility data, at both backward and forward normalization points, are taken to be of exponential type with constant risk aversion. This assumption can be relaxed without losing the fundamental properties of the dynamic utilities. We, also, concentrate our analysis to European-type liabilities so that closed form variational results can be obtained.

The two classes of dynamic utilities, as well as the emerging prices and investment strategies, have similarities but, also, striking differences. As mentioned above, both utilities are self-generating and, therefore, price discrepancies are precluded in the associated backward and forward indifference pricing systems. A consequence of self-generation is that an investor endowed with backward and forward utilities receives the same dynamic utility across different investment horizons. It is worth noting that while the backward dynamic utility is unique, the forward one might not be.

The associated indifference prices have very distinct characteristics. Backward indifference prices depend on the backward normalization point, (and, thus, implicitly on the trading horizon) even if the claim matures before T. However, forward indifference prices are not affected by the choice of the forward normalization point. For the class of European claims examined herein, backward prices are represented as nonlinear expectations associated with the minimal relative entropy measure. On the other hand, forward prices are also represented as nonlinear expectation but with respect to the minimal martingale measure. The two prices do not coincide unless the market is complete. This is a direct consequence of the fact that the internal market incompleteness - coming form the stochastic factor - is processed by the backward and forward dynamic utilities in a very distinct manner.

The portfolio strategies related to the backward and forward utilities have also very different characteristics. The optimal backward investments consist of the myopic portfolio, the backward indifference deltas and the excess risky demand. The latter policy reflects, in contrast to the myopic portfolio, the incremental changes in the optimal behavior due to the movement of the stochastic factor. The forward optimal investments have the same structure as their backward counterparts but do not include the excess risky demand.

The paper is organized as follows. In Section 2, we introduce the investment model and its dynamic utilities. In Section 3, we provide some auxiliary technical results related to the minimal martingale and minimal entropy measures. In Sections 4 and 5, respectively, we construct the backward and forward dynamic utilities, the associated prices and the optimal investments. We conclude in Section 6, where we provide a comparative study for integrated portfolio problems under the two classes of dynamic risk preferences.

2 The model and its dynamic utilities

Two securities are available for trading, a riskless bond and a risky stock whose price solves

$$dS_s = \mu\left(Y_s\right) S_s ds + \sigma\left(Y_s\right) S_s dW_s^1 \tag{1}$$

for $s \geq 0$ and $S_0 = S > 0$. The bond offers zero interest rate. The case of (deterministic) non-zero interest rate may be handled by straightforward scaling arguments and is not discussed.

The process Y, to be referred to as the stochastic factor, is assumed to satisfy

$$dY_s = b\left(Y_s\right) ds + a\left(Y_s\right) dW_s \tag{2}$$

for $s \geq 0$ and $Y_0 = y \in \mathcal{R}$.

The processes W^1 and W are standard Brownian motions defined on a probability space $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_s\right), \mathbb{P}\right)$ with $\mathcal{F}_s$ being the augmented σ-algebra. We assume that the correlation coefficient $\rho \in (-1, 1)$ and, thus, we may write

$$dW_s = \rho dW_s^1 + \sqrt{1-\rho^2} dW_s^{1,\perp} \tag{3}$$

with $W^{1,\perp}$ being a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$ orthogonal to W^1. For simplicity, we assume that the dynamics in (1) and (2) are autonomous. We denote the stock's *Sharpe ratio* process by

$$\lambda_s = \lambda(Y_s) = \frac{\mu(Y_s)}{\sigma(Y_s)}. \tag{4}$$

The following assumption will be standing throughout.

Assumption 1: The market coefficients μ, σ, a and b are assumed to be $C^2(\mathcal{R})$ functions that satisfy, $|f(y)| \leq C(1 + |y|)$ for $f = \mu, \sigma, a$ and b, and are such that (1) and (2) have a unique strong solution satisfying $S_s > 0$ a.e. for $s \geq 0$. There also exists $\varepsilon > 0$ such that $\sigma(y) > \varepsilon$, for $y \in \mathcal{R}$.

Next, we consider an arbitrary trading horizon $[0, T]$, and an investor who starts, at time $t_0 \in [0, T]$, with initial wealth $x \in \mathcal{R}$ and trades between the two securities. His/her current wealth X_s, $t_0 \leq s \leq T$, satisfies the budget constraint $X_s = \pi_s^0 + \pi_s$ where π_s^0 and π_s are self-financing strategies representing the amounts invested in the bond and the stock accounts. Direct calculations, in the absence of intermediate consumption, yield the evolution of the wealth process

$$dX_s = \mu(Y_s)\pi_s ds + \sigma(Y_s)\pi_s dW_s \tag{5}$$

with $X_{t_0} = x \in \mathcal{R}$. The set $\mathcal{A}$ of admissible strategies is defined as $\mathcal{A} = \left\{\pi : \pi \text{ is } \mathcal{F}_s - \text{measurable, self-financing and } E_{\mathbb{P}}\left(\int_0^T \sigma^2(Y_s)\pi_s^2 ds\right) < \infty\right\}$. Further constraints might be binding due to the specific application and/or the form of the involved utility payoffs. In order, however, to keep the exposition simple and to concentrate on the new notions and insights, we choose to abstract from such constraints. We denote by $\mathcal{D}$ the generic spatial solvency domain for (x, y).

We start with an informal motivational discussion for the upcoming notions of backward and forward dynamic utilities. In the traditional economic model of expected utility from terminal wealth, a utility datum is assigned at a given time, representing the end of the investment horizon. We denote the utility datum by $u(x)$ and the time at which it is assigned by T. At $t_0 \geq 0$, the investor starts trading between the available securities till T. At intermediate times $t \in [t_0, T]$, the associated value function $v : \mathcal{D} \times [0, T] \to \mathcal{R}$ is defined as the maximal expected (conditional on $\mathcal{F}_t$) utility that the agent achieves from investment. For the model at hand, v takes the form

$$v(x, y, t) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(u(X_T) | X_t = x, Y_t = y\right), \qquad t \in [t_0, T] \tag{6}$$

with the wealth and stochastic factor processes X, Y solving (5) and (2).

The scope is to specify v and to construct the optimal control policies. The duality approach can be applied to general market models and provides characterization results for the value function, but limited results for the optimal portfolios. The latter can be constructed via variational methods for certain classes of diffusion models. To date, while there is a rich body of work for the value function, very little is understood about how investors adjust their portfolios in terms of their risk preferences, trading horizon and the market environment.

i) Integrated models of portfolio choice

In a more realistic setting, the investor might be interested in incorporating in his portfolio derivative securities, liabilities, proceeds from additional assets, labor income etc. Given that such situations arise frequently in practice, it is important to develop an approach that accommodates integrated investment problems and yields quantitative and qualitative results for the optimal portfolios. This is the aim of the study below.

To simplify the presentation, we assume, for the moment, that the investor faces a liability at T, represented by a random variable $C_T \in \mathcal{F}_T$. We recall that $\mathcal{F}_T$ is generated by both the traded stock and the stochastic factor and that the investor uses only self-financing strategies.

In a complete market set-up (e.g. when the processes S and Y are perfectly correlated) the optimal strategy for this generalized portfolio choice model is as follows: at initiation t_0, the investor splits the initial wealth, say x, into the amounts $E_{\mathbb{Q}}\left(C_T|\,\mathcal{F}_{t_0}\right)$ and $\tilde{x} = x - E_{\mathbb{Q}}\left(C_T|\,\mathcal{F}_{t_0}\right)$, with $E_{\mathbb{Q}}\left(C_T|\,\mathcal{F}_{t_0}\right)$ being the arbitrage-free price of C_T. The residual amount $\tilde{x}$ is used for investment as if there was no liability. The dynamic optimal strategy is, then, the sum of the optimal portfolio, corresponding to initial endowment $\tilde{x}$ and the hedging strategy, denoted by $\delta_s\left(C_T\right)$, of a European-type contingent claim written on the traded stock, maturing at T and yielding C_T. Using $*$ to denote optimal policies, we may write

$$\pi_s^{x,*} = \pi_s^{\tilde{x},*} + \delta_s\left(C_T\right) \qquad \text{with} \quad \tilde{x} = x - E_{\mathbb{Q}}\left(C_T|\,\mathcal{F}_{t_0}\right) \tag{7}$$

This can be established either by variational methods or duality. This remarkable additive structure, arising in the highly nonlinear utility setting, is a direct consequence of the ability to replicate the liability. Note that for a fixed choice of wealth units, the second portfolio component is not affected by the risk preferences.

When the market is incomplete, similar argumentation can be developed by formulating the problem as a partial equilibrium one and, in turn, using results from the utility-based valuation approach. The liability may be, then, viewed as a derivative security and the optimal portfolio choice problem is embedded to an indifference valuation one. Using payoff decomposition results (see, for example, Musiela and Zariphopoulou (2001 and 2004a), Stoikov and Zariphopoulou (2004) and Monoyios (2006)), we associate to the liability an indifference hedging strategy, say $\Delta_s\left(C_T\right)$, that is the incomplete market counterpart of its arbitrage-free replicating portfolio. Denoting the relevant indifference price by $\nu_t\left(C_T\right)$, we obtain an analogous to (7) decomposition of the optimal investment strategy in the stock account, namely,

$$\pi_s^{x,*} = \pi_s^{\tilde{x},*} + \Delta_s\left(C_T\right) \qquad \text{with} \quad \tilde{x} = x - \nu_{t_0}\left(C_T\right) \tag{8}$$

Because the indifference valuation approach incorporates the investor's risk preferences, the choice of utility will influence - in contrast to the complete market case - both components of the emerging optimal investment strategy. Note, however, that due to the dynamic nature of the problem, utility effects evolve both with time and market information. In order to correctly quantify these effects, it is imperative to be able to specify the *dynamic value* of our investment strategies across horizons, maturities and units. As the analysis below indicates, the cornerstone of this endeavor is the specification of a dynamic utility structure that yields consistent valuation results and investment behavior across optimally chosen self-financing strategies.

Before we introduce the dynamic utilities, we first recall the auxiliary concept of *indifference value*. To preserve simplicity, we consider the aforeintroduced single liability C_T. To calculate its indifference price, $\nu_t(C_T)$, for $t \in [0,T]$, we look at the investor's modified utility,

$$v^{C_T}(x,y,t) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(u\left(X_T - C_T\right) \middle| X_t = x, Y_t = y\right) \tag{9}$$

and, subsequently, impose the equilibrium condition

$$v\left(x - \nu_t\left(C_T\right), y, t\right) = v^{C_T}(x,y,t). \tag{10}$$

The optimal policy is given by (8) and can be retrieved in closed form for special cases. For example, when the utility is exponential, $u(x) = -e^{-\gamma x}$ with $\gamma > 0$, the stock's Sharpe ratio is constant and $C_T = G(Y_T)$, for some bounded function G, variational arguments yield the optimal investment representation

$$\pi_s^{x,*} = \pi_s^{\tilde{x},*} + \rho \frac{a(Y_s)}{\sigma(Y_s)} \left(\left. \frac{\partial g(y,t)}{\partial y} \right|_{y=Y_s, t=s} \right)$$

with $\tilde{x} = x - g(y,t_0)$, and $g : \mathcal{R} \times [0,T] \to \mathcal{R}$ solving a quasilinear pde, of quadratic gradient nonlinearities., with terminal condition $g(y,T) = G(y)$.

ii) Liabilities and payoffs of shorter maturities

Consider a liability to be paid *before* the fixed horizon T, say at $T_0 < T$. There are two ways to proceed. The first alternative is to work with portfolio choice in the initial investment horizon, $[t_0,T]$. In this case, the utility (9) becomes

$$v^{C_{T_0}}(x,y,t) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(u\left(X_T - C_{T_0}\right) \middle| X_t = x, Y_t = y\right), \tag{11}$$

where we took into consideration that the riskless interest rate is zero. The indifference value is, then, calculated by the pricing condition (10), for $t \in [t_0,T_0]$. However, such arguments might not be easily implemented, if at all, as it is the case of liabilities and payoffs of random maturity and/or various exotic characteristics.

The second alternative is to derive the indifference value by considering the investment opportunities, with and without the liability, up to the claim's maturity T_0. For this, we first need to correctly specify the value functions, denoted, respectively, by $\bar{v}$ and $\bar{v}^{C_{T_0}}$, that correspond to optimality of investments in the *shorter* investment horizon, $[t_0, T_0]$. Working along the lines their long-horizon counterparts, v and v^{C_T}, were defined, let us, hypothetically, assume that we are given a utility datum for the point T_0. We denote this datum by $\bar{u}(x, y, T_0)$. We will, henceforth, use the '$-$' notation for all quantities, i.e. utilities, investments and indifference values, associated with the shorter horizon. For $t \in [t_0, T_0]$,

$$\bar{v}(x, y, t) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(\bar{u}(X_{T_0}, Y_{T_0}, T_0) \middle| X_t = x, Y_t = y \right)$$

and

$$\bar{v}^{C_{T_0}}(x, y, t) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(\bar{u}(X_{T_0} - C_{T_0}, Y_{T_0}, T_0) \middle| X_t = x, Y_t = y \right).$$

The associated indifference value, $\bar{\nu}_t(C_{T_0})$, will be, then, given by

$$\bar{v}(x - \bar{\nu}_t(C_{T_0}), y, t) = \bar{v}^{C_{T_0}}(x, y, t).$$

Clearly, in order to have a well specified valuation system, we must have, for *all* $C_{T_0} \in \mathcal{F}_{T_0}$ and $t \in [t_0, T_0]$,

$$\nu_t(C_{T_0}) = \bar{\nu}_t(C_{T_0}),$$

which strongly suggests that the utility datum $\bar{u}(x, y, T_0)$ cannot be exogenously assigned in an arbitrary manner.

Such issues, related to the correct specification and alignment of intermediate utilities, and their value functions, with the claims' possibly different and/or random maturities, were first discussed in Davis and Zariphopoulou (1995) in the context of utility-based valuation of American claims in markets with transaction costs. Recall that when early exercise is allowed, the first alternative computational step, (cf. (11)), cannot be implemented because T_0 is not a priori known. For the same class of early exercise claims, but when incompleteness comes exclusively from a non-traded asset (which does not affect the dynamics of the stock), and the claim is written on both the traded and nontraded assets, further analysis on the specification of preferences across exercise times, was provided in Kallsen and Kuehn (2004), Oberman and Zariphopoulou (2003) and Musiela and Zariphopoulou (2004b). In the latter papers, the related intermediate utilities, and valuation condition took, respectively, the forms

$$\bar{u}(x, S, t) = v(x, S, t) \quad \text{and} \quad \bar{v}(x, S, t) = \bar{u}(x, S, t),$$
$$\bar{v}^{C_\tau}(x, S, z, t) = \sup_{\mathcal{A} \times \mathcal{T}} E_{\mathbb{P}} \left(v(X_\tau - C(S_\tau, Z_\tau), S_\tau, \tau) \middle| X_t = x, S_t = S, Z_t = z \right),$$

where $\mathcal{T}$ is the set of stopping times in $[t_0, T]$. The processes S and Z represent the traded and nontraded assets and X the wealth process. The early exercise indifference price of C_τ is, then, given by

$$\bar{v}(x, S, t) = \bar{v}^{C_\tau}(x + \bar{\nu}_t(C(S_\tau, Z_\tau)), S, z, t).$$

While the above calculations might look pedantic when a single exogenous cash flow (liability or payoff) is incorporated, the arguments get much more involved when a family of claims is considered and arbitrary, or stochastic, maturities are allowed. Naturally, the related difficulties disappear when the market is complete. However, when perfect replication is not viable and a utility-based approach is implemented for valuation, discrepancies leading to arbitrage might arise if we fail to properly incorporate in our model dynamic risk preferences that process and price the market incompleteness in a consistent manner. This issue was exposed by the authors in Musiela and Zariphopoulou (2005a) and (2005b), who initiated the construction of indifference pricing systems based on the so-called *backward* and *forward* dynamic exponential utilities. In these papers, indifference valuation of arbitrary claims and specification of integrated optimal policies were studied in an incomplete binomial case. Even though this model set-up was rather simple, it offered a starting point in exploring the effects of the evolution of risk preferences to prices and investments. What follows is, to a great extent, a generalization of the theory developed therein.

iii) Utility measurement across investment times

Let us now see how a dynamic utility can be introduced and incorporated in the stochastic factor model we are interested in. We recall that the standing assumptions are: i) the trading horizon $[0,T]$ is preassigned, ii) a utility datum is given for T and iii) T dominates the maturities of all claims and liabilities in consideration.

We next assume that instead of having the single *static* measurement of utility, u, at expiration, the investor is endowed with a dynamic utility, $u_t\left(x,y;T\right)$, $t\in[0,T]$. Being vague, for the moment, we view this utility as a functional, at each intermediate time t, of her current wealth and the level of the stochastic factor. Obviously, we must have

$$u_T\left(x,y;T\right)=u\left(x\right),$$

in which case, we say that $u_t\left(x,y;T\right)$ is *normalized* at T. As a consequence, we will refer to T as the *normalization* point. For reasons that will be apparent in the sequel, we choose to carry T in our notation.

If a maximal expected criterion is involved, the associated value function, denoted with a slight abuse of notation by v_t, will naturally take the form

$$v_t\left(x,y,\bar{T}\right)=\sup_{\mathcal{A}}E_{\mathbb{P}}\left(\left.u_T\left(X_{\bar{T}},Y_{\bar{T}};T\right)\right|X_t=x,Y_t=y\right),$$

in an arbitrary sub-horizon $\left[t,\bar{T}\right]\in[t_0,T]$ and with X, Y solving (5) and (2).

Let us now see how the generic liability $C_{T_0}\in\mathcal{F}_{T_0}$ would be valued under such a utility structure. For $t\in[t_0,T_0]$, the relevant maximal expected dynamic utility will be

$$v_t^{C_{T_0}}\left(x,y,T_0\right)=\sup_{\mathcal{A}}E_{\mathbb{P}}\left(\left.u_{T_0}\left(X_{T_0}-C_{T_0},Y_{T_0};T\right)\right|X_t=x,Y_t=y\right).$$

Respectively, the indifference value, $\nu_t(C_{T_0};T)$, must satisfy, for $t \in [t_0, T_0]$,

$$v_t(x - \nu_t(C_{T_0};T), y, T_0) = v_t^{C_{T_0}}(x, y, T_0).$$

Observe that because u_t is normalized at T, the associated value functions will depend on the normalization point. The latter will also affect the indifference price $\nu_t(C_{T_0};T)$, even though the claim matures at an earlier time.

So far, the above formulation seems convenient, and flexible enough, for the valuation of claims with arbitrary maturities, as long as these maturities are shorter than the time at which risk preferences are normalized. However, as the next two examples show, it is wrong to assume that a dynamic utility can be introduced in an ad hoc way.

In both examples, it is assumed that the terminal utility datum is of exponential type, and independent of the level of the stochastic factor,

$$u_T(x, y; T) = -e^{-\gamma x} \tag{12}$$

with $(x, y) \in \mathcal{D}$ and γ being a given positive constant. It is also assumed that there is a single claim to be priced. Its payoff is taken to be of the form $C_{T_0} = G(Y_{T_0})$, for some bounded function $G : \mathcal{R} \rightarrow \mathcal{R}^+$. Albeit the fact that in the model considered herein, such a payoff is, to a certain extent, artificial, we, nevertheless, choose to work with it because explicit formulae can be obtained and the exposition is, thus, considerably facilitated.

Example 1: Consider a dynamic utility of the form

$$u_t(x, y; T) = \begin{cases} -e^{-\gamma x} & \bar{T} < t \leq T \\ -e^{-\bar{\gamma} x} & 0 < t \leq \bar{T}, \end{cases}$$

with $\bar{T} > T_0$, γ as in (12) and $\bar{\gamma} \neq \gamma$.

Let us now see how C_{T_0} will be valued under the above choice of dynamic utility. If the investor chooses to trade in the *original* horizon $[t_0, T]$, the associated intermediate utilities are

$$\begin{aligned} v_t^{0,C_{T_0}}(x, y, T) &= \sup_{\mathcal{A}} E_{\mathbb{P}}\left(u_T(X_T - C_{T_0}, Y_T) \middle| X_t = x, Y_t = y \right) \\ &= \sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma\left(X_T - C_{T_0}\right)} \middle| X_t = x, Y_t = y \right). \end{aligned}$$

Obviously, the discontinuity, with regards to the risk aversion coefficient of u_t will not alter the above value functions. Following the results of Sircar and Zariphopoulou (2005) yields

$$\nu_t(C_{T_0}) = \frac{1}{\gamma(1-\rho^2)} \ln E_{\mathbb{Q}^{me}}\left(e^{\gamma\left(1-\rho^2\right)G\left(Y_{T_0}\right)} \middle| Y_t = y \right), \tag{13}$$

with $\mathbb{Q}^{me}$ being the minimal relative entropy martingale measure (see next section for the relevant technical arguments).

If, however, the investor chooses to trade solely in the *shorter* horizon $\left[t_0, \bar{T}\right]$, analogous argumentation yields

$$\bar{v}_t^{0,C_{T_0}}\left(x, y, \bar{T}\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(\left. -e^{-\bar{\gamma}\left(X_{\bar{T}} - G\left(Y_{T_0}\right)\right)}\right| X_t = x, Y_t = y\right),$$

where we used the " $-$ " notation to denote the shorter horizon choice. The associated indifference price is

$$\bar{\nu}_t\left(C_{T_0}\right) = \frac{1}{\bar{\gamma}\left(1-\rho^2\right)} \ln E_{\mathbb{Q}^{me}}\left(\left. e^{\bar{\gamma}\left(1-\rho^2\right)G\left(Y_{T_0}\right)}\right| Y_t = y\right),$$

and we easily deduce that, in general,

$$\nu_t\left(C_{T_0}\right) \neq \bar{\nu}_t\left(C_{T_0}\right),$$

an obviously wrong result.

Note that even if we naively allow $\gamma = \bar{\gamma}$ price discrepancies will still emerge.

Example 2: Consider the dynamic utility

$$u_t\left(x, y; T\right) = -e^{-\gamma x - F(y,t;T)}$$

with γ as in (12) and

$$F\left(y, t; T\right) = E_{\mathbb{P}}\left(\left.\int_t^T \frac{1}{2}\lambda^2\left(Y_s\right) ds\right| Y_t = y\right),$$

where $\mathbb{P}$ is the historical measure. If the agent chooses to invest in the longer horizon, $[t_0, T]$, the indifference value remains the same as in (13). However, if he chooses to invest exclusively till the liability is met, we have, for $t \in [t_0, T_0]$,

$$\bar{v}_t^{0,C_{T_0}}\left(x, y, T_0\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(\left. u_{T_0}\left(X_{T_0} - G\left(Y_{T_0}\right); T\right)\right| X_t = x, Y_t = y\right)$$

$$= \sup_{\mathcal{A}} E_{\mathbb{P}}\left(\left. -e^{-\gamma\left(X_{T_0} - G\left(Y_{T_0}\right)\right) - F\left(Y_{T_0}, T_0; T\right)}\right| X_t = x, Y_t = y\right).$$

Setting

$$Z_{T_0} = \frac{1}{\gamma} F\left(Y_{T_0}, T_0; T\right)$$

we deduce that, in the absence of the liability, the current utility is

$$\bar{v}_t^0\left(x, y, T_0\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(\left. -e^{-\gamma\left(X_{T_0} + Z_{T_0}\right)}\right| X_t = x, Y_t = y\right).$$

Note that, by definition, $Z_{T_0} \in \mathcal{F}_{T_0}$. Therefore, we may interpret $\bar{v}_t^0$ as a buyer's value function for the claim Z_{T_0}, in a traditional (non-dynamic) exponential utility setting of constant risk aversion γ and investment horizon $[0, T_0]$. Then,

$$\bar{v}_t^0\left(x,y,T_0\right)=-e^{-\gamma\left(x+\bar{\mu}_t\left(Z_{T_0}\right)\right)-\tilde{H}\left(y,t;T_0\right)}$$

with $\tilde{H}\left(y,t,T_0\right)$ being the aggregate entropy function (see equation **(24)** in next Section) and

$$\bar{\mu}_t\left(Z_{T_0}\right)=-\frac{1}{\gamma\left(1-\rho^2\right)}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma\left(1-\rho^2\right)Z_{T_0}}\Big|\,Y_t=y\right).$$

Proceeding similarly, we deduce,

$$\bar{v}_t^{C_{T_0}}\left(x,y,T_0\right)=\sup_{\mathcal{A}}E_{\mathbb{P}}\left(-e^{-\gamma\left(X_{T_0}+\left(Z_{T_0}-G\left(Y_{T_0}\right)\right)\right)}\Big|\,X_t=x,Y_t=y\right)$$

$$=-e^{-\gamma\left(x+\bar{\mu}_t\left(Z_{T_0}-G\left(Y_{T_0}\right)\right)\right)-\tilde{H}\left(y,t;T_0\right)}$$

with

$$\bar{\mu}_t\left(Z_{T_0}-G\left(Y_{T_0}\right)\right)=-\frac{1}{\gamma\left(1-\rho^2\right)}\ln E_{\mathbb{Q}^{me}}\left(e^{-\gamma\left(1-\rho^2\right)\left(Z_{T_0}-G\left(Y_{T_0}\right)\right)}\Big|\,Y_t=y\right).$$

Applying the definition of the indifference value, we deduce that, with regards to the shorter horizon,

$$\bar{\nu}_t\left(G\left(Y_{T_0}\right)\right)=\bar{\mu}_t\left(Z_{T_0}\right)-\bar{\mu}_t\left(Z_{T_0}-G\left(Y_{T_0}\right)\right)$$

$$=\frac{1}{\gamma\left(1-\rho^2\right)}\ln\frac{E_{\mathbb{Q}^{me}}\left(e^{-\gamma\left(1-\rho^2\right)\left(Z_{T_0}-G\left(Y_{T_0}\right)\right)}\Big|\,Y_t=y\right)}{E_{\mathbb{Q}^{me}}\left(e^{-\gamma\left(1-\rho^2\right)\left(Z_{T_0}\right)}\Big|\,Y_t=y\right)}$$

which, in general, does *not* coincide with $\nu_t\left(G\left(Y_{T_0}\right)\right)$ given in (13).

iv) Backward and forward dynamic exponential utilities

The above examples expose that an ad hoc choice of dynamic utility might lead to price discrepancies. This, in view of the structural form of the optimal policy for the integrated model (cf. (8)) would, in turn, yield wrongly specified investment policies. It is thus important to investigate which classes of dynamic utilities preclude such pathological situations.

For the simple examples above, the correct choice of the dynamic utility is essentially obvious, namely,

$$u_t\left(x,y;T\right)=\begin{cases}u\left(x\right) & \text{for}\quad t=T\\ \\ v\left(x,y,t\right) & \text{for}\quad t\in\left[t_0,T\right],\end{cases}$$

with v as in (6).

This simple observation indicates the following: first, observe that if u_t is the candidate dynamic utility, then, in all trading sub-horizons, say $\left[t,\tilde{t}\right]$, the associated dynamic value function v_t will be

$$v_t\left(x,y;T\right)=\sup_{\mathcal{A}}E_{\mathbb{P}}\left(\left.u_{\tilde{t}}\left(X_{\tilde{t}},Y_{\tilde{t}};T\right)\right|X_t=x,Y_t=y\right).$$

Discrepancies in prices will be, then, precluded if at all intermediate times the dynamic utility coincides with the dynamic value function it generates,

$$u_t(x, y; T) = v_t(x, y; T).$$

We then say that the dynamic utility is *self-generating.*

Building on this concept, we are led to two classes of dynamic utilities, the *backward* and *forward ones.* Their definitions are given below. Because the applications herein are concentrated on exponential preferences, we work with such utility data. Throughout, we take the risk aversion coefficient to be a positive constant γ.

While the backward dynamic utility is essentially the traditional value function, the concept of forward utility is, to the best of our knowledge, new. As mentioned earlier, it was recently introduced by the authors in an incomplete binomial setting (see Musiela and Zariphopoulou (2005a) and (2005b)) and it is herein extended to the diffusion case.

We continue with the definition of the backward dynamic utility. This utility takes the name *backward* because it is first specified at the normalization point T and is then generated at previous times.

Definition 1 *Let $T > 0$. An $\mathcal{F}_t$-measurable stochastic process $U_t^B(x; T)$ is called a backward dynamic utility (BDU), normalized at T, if for all $t, \bar{T}$ it satisfies the stochastic optimality criterion*

$$U_t^B(x; T) = \begin{cases} -e^{-\gamma x}, & t = T \\ \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_{\bar{T}}^B(X_{\bar{T}}; T) \middle| \mathcal{F}_t \right), & 0 \le t \le \bar{T} \le T \end{cases} \tag{14}$$

with X given by (5) and $X_t = x \in \mathcal{R}$.

The above equation provides the constitutive law for the backward dynamic utility. Note that even though this dynamic utility coincides with the familiar value function, its notion was created from a very different point of view and scope. Under mild regularity assumptions on the coefficients of the state processes, it is easy to deduce that the above problem has a solution that is unique. There is ample literature on the value function and, thus, on the backward dynamic utility (see, for example, Kramkov and Schachermayer (1999), Rouge and El Karoui (2000), Delbaen et al. (2002) and Kabanov and Stricker (2002)).

The fact that U_t^B is self-generating, is immediate. Indeed, in an arbitrary sub-horizon $[t, \bar{T}]$, the associated value function V_t^B, given by

$$V_t^B\left(x, \bar{T}; T\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_{\bar{T}}^B(X_{\bar{T}}; T) \middle| \mathcal{F}_t \right),$$

coincides with its associated dynamic utility,

$$U_t^B(x; T) = V_t^B\left(x, \bar{T}; T\right)$$

by Definition 1.

A consequence of self-generation is that the investor receives the same dynamic utility across different investment horizons. This is seen by the fact that, for $\bar{T} \leq \bar{T}'$, self-generation yields

$$V_t^B\left(x, \bar{T}; T\right) = U_t^B\left(x; T\right)$$

and

$$V_t^B\left(x, \bar{T}'; T\right) = U_t^B\left(x; T\right)$$

and the horizon invariance,

$$V_t^B\left(x, \bar{T}; T\right) = V_t^B\left(x, \bar{T}'; T\right)$$

follows.

In most of the existing utility models, the dynamic utility - or, equivalently, its associated value function - is generated backwards in time. The form of the utility might be more complex, as it is the case of recursive utilities where dynamic risk preferences are generated by an aggregator. Nevertheless, the features of utility prespecification at a future fixed point in time and generation at previous times are still prevailing.

One might argue that an ad hoc specification of utility at a future time is, to a certain extent, non intuitive, given that our risk attitude might change with the way the market environment enfolds from one time period to the next. Note that changes in the investment opportunities and losses/gains are revealed forward in time while the traditional value function appears to process this information backwards in time. Such issues have been considered in prospect theory where, however, utility normalization at a given future point is still present.

From the valuation perspective, working with utilities normalized in the future severely constraints the class of claims that can be priced. Indeed, their maturities must be always dominated by the time at which the backward utility is normalized. This precludes opportunities related to claims arriving at a later time and maturing beyond the normalization point.

In order to be able to accommodate claims of arbitrary maturities, one might propose to work in an infinite horizon framework and to employ either discounted at optimal growth utility functionals or utilities allowing for intermediate consumption. The perpetual nature of these problems, however, might not be appropriate for a variety of applications in which the agent faces defaults, constraints due to reporting periods and other "real-time" issues.

Motivated by these considerations, the authors recently introduced the concept of *forward dynamic utilities.* Their main characteristic is that they are determined at present time and, as their name indicates, are generated, via their constitutive equation, forward in time.

Definition 2 *Let $s \geq 0$. An $\mathcal{F}_t$-measurable stochastic process $U_t^F\left(x; s\right)$ is called a forward dynamic exponential utility (FDU), normalized at s, if, for all t, T,* *with $s \leq t \leq T$, it satisfies the stochastic optimization criterion*

$$U_t^F(x;s) = \begin{cases} -e^{-\gamma x}, & t = s \\ \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^F(X_T;s) \middle| \mathcal{F}_t \right), & t \geq s. \end{cases} \tag{15}$$

Observe that by construction, there is *no constraint on the length of the trading horizon.*

Like its backward dynamic counterpart, the forward dynamic utility is self-generating and makes the investor indifferent across distinct investment horizons. Indeed, self-generation, i.e.,

$$U_t^F(x;s) = V_t^F(x,T;s)$$

with

$$V_t^F(x,T;s) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^F(X_T;s) \middle| \mathcal{F}_t \right)$$

is an immediate consequence the above definition. For the horizon invariance, it is enough to observe that in different sub-horizons, say $[t,T]$ and $\left[t,\bar{T}\right]$,

$$V_t^F(x,T;s) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^F(X_T;s) \middle| \mathcal{F}_t \right),$$

and

$$V_t^F\left(x,\bar{T};s\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_{\bar{T}}^F(X_{\bar{T}};s) \middle| \mathcal{F}_t \right),$$

and, therefore,

$$V_t^F(x,T;s) = V_t^F\left(x,\bar{T};s\right).$$

We stress that, in contrast to their backward dynamic counterparts, forward dynamic utilities might *not* be unique. In general, the problem of existence and uniqueness is an open one. This issue is discussed in Section 5. Determining a natural class of forward utilities in which uniqueness is established is a challenging and, in our view, interesting question.

3 Auxiliary technical results

In the upcoming sections, two equivalent martingale measures will be used, namely, the minimal martingale and the minimal entropy ones. They are denoted, respectively, by $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$ and are defined as the minimizers of the entropic functionals

$$\mathcal{H}^0\left(\mathbb{Q}^{mm}\middle|\,\mathbb{P}\right) = \min_{Q\in\mathcal{Q}_e} E_{\mathbb{P}}\left(-\ln\frac{dQ}{d\mathbb{P}}\right)$$

and

$$\mathcal{H}\left(\mathbb{Q}^{me}\middle|\,\mathbb{P}\right) = \min_{Q\in\mathcal{Q}_e} E_{\mathbb{P}}\left(\frac{dQ}{d\mathbb{P}}\ln\frac{dQ}{d\mathbb{P}}\right),$$

where $\mathcal{Q}_e$ stands for the set of equivalent martingale measures. There is ample literature on these measures and on their role in valuation and optimal portfolio choice in the traditional framework of exponential utility; see, respectively, Foellmer and Schweizer (1991), Schweizer (1995) and (1999), Bellini and Frittelli (2002) and Frittelli (2000), Rouge and El Karoui (2000), Arai (2001), Delbaen et al. (2002), Kabanov and Stricker (2002)).

For arbitrary $T > 0$, the restrictions of $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$ on the σ-algebra $\mathcal{F}_T = \sigma\left\{\left(W_u^1, W_u\right) : 0 \leq u \leq T\right\}$, can be explicitly constructed as it is discussed next. We remark that, with a slight abuse of notation, the restrictions of the two measures are denoted as their original counterparts.

The density of the *minimal martingale measure* is given by

$$\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}} = \exp\left(-\int_0^T \lambda_s dW_s^1 - \int_0^T \frac{1}{2}\lambda_s^2 ds\right) \tag{16}$$

with λ being the Sharpe ratio process (4).

Calculating the density of the *minimal relative entropy measure* is more involved and we refer the reader to Rheinlander (2003) (see, also, Grandits and Rheinlander (2002)) for a concise treatment. For the diffusion case considered herein, the density can be found through variational arguments and is represented by

$$\frac{d\mathbb{Q}^{me}}{d\mathbb{P}} = \exp\left(-\int_0^T \lambda_s dW_s^1 - \int_0^T \hat{\lambda}_s dW_s^{1,\perp} - \int_0^T \frac{1}{2}\left(\lambda_s^2 + \hat{\lambda}_s^2\right) ds\right) \tag{17}$$

with $W^{1,\perp}$ as in (3). The process $\hat{\lambda}$ is given by

$$\hat{\lambda}_s = \hat{\lambda}\left(Y_s, s; T\right) \tag{18}$$

with Y solving (2) and $\hat{\lambda} : \mathcal{R} \times [0,T] \rightarrow \mathcal{R}^+$ defined as

$$\hat{\lambda}\left(y, t; T\right) = -\frac{1}{\sqrt{1-\rho^2}} a\left(y\right) \frac{f_y\left(y,t;T\right)}{f\left(y,t;T\right)}, \tag{19}$$

where $f : \mathcal{R} \times [0,T] \rightarrow \mathcal{R}^+$ is the unique $C^{1,2}\left(\mathcal{R} \times [0,T]\right)$ solution of the terminal value problem

$$\begin{cases} f_t + \frac{1}{2}a^2\left(y\right) f_{yy} + \left(b\left(y\right) - \rho\lambda\left(y\right) a\left(y\right)\right) f_y = \frac{1}{2}\left(1-\rho^2\right)\lambda^2\left(y\right) f \\ f(y,T) = 1. \end{cases} \tag{20}$$

The proof can be found in Benth and Karslen (2005) (see, also, Stoikov and Zariphopoulou (2004) and Monoyios (2006)).

The dependence of f and $\hat{\lambda}$ on the end of the horizon, T, is highlighted due to the role that it will play in the upcoming dynamic utilities.

It easily follows that the aggregate, relative to the historical measure, entropies of $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$ are, respectively,

$$\mathcal{H}\left(\mathbb{Q}^{mm}\right|\mathbb{P}) = E_{\mathbb{P}}\left(\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}^{mm}}{d\mathbb{P}}\right) = E_{\mathbb{Q}^{mm}}\left(\int_0^T \frac{1}{2}\lambda_s^2 ds\right)$$

and

$$\mathcal{H}\left(\mathbb{Q}^{me}\right|\mathbb{P}) = E_{\mathbb{P}}\left(\frac{d\mathbb{Q}^{me}}{d\mathbb{P}}\ln\frac{d\mathbb{Q}^{me}}{d\mathbb{P}}\right) = E_{\mathbb{Q}^{me}}\left(\int_0^T \frac{1}{2}\left(\lambda_s^2+\hat{\lambda}_s^2\right) ds\right).$$

When the market becomes complete, the two measures, $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$, coincide with the unique risk neutral measure. In general, they differ and their respective relative entropies are related in a nonlinear manner. This was explored in Stoikov and Zariphopoulou (2005, Corollary 3.1), where it was shown that

$$\mathcal{H}\left(\mathbb{Q}^{me}\right|\mathbb{P}) = \mathcal{E}_{\mathbb{Q}^{mm}}\left(\int_0^T \frac{1}{2}\lambda_s^2 ds\left|\mathcal{F}_0\right.\right). \tag{21}$$

The conditional nonlinear expectation $\mathcal{E}_Q$ of a generic random variable $Z\in\mathcal{F}_T$ and measure Q on $(\Omega,\mathcal{F}_T)$ is defined, for $t\in[0,T]$ and $\gamma\in\mathcal{R}^+$, by

$$\mathcal{E}_Q\left(Z\right|\mathcal{F}_t;\gamma) = -\frac{1}{\gamma\left(1-\rho^2\right)}\ln E_Q\left(e^{-\gamma\left(1-\rho^2\right)Z}\left|\mathcal{F}_t\right.\right), \qquad \gamma\in\mathcal{R}^+. \tag{22}$$

The aggregate entropy $\mathcal{H}\left(\mathbb{Q}^{me}\right|\mathbb{P})$ is then the nonlinear expectation of the random variable $Z_T=\int_0^T \frac{1}{2}\lambda_s^2 ds$, for $Q=\mathbb{Q}^{me}$ and $\gamma=1$.

Next we introduce two quantities that will facilitate our analysis. Namely, for $0\le t\le\tilde{T}\le T$, we define the aggregate relative entropy process

$$H\left(t,\tilde{T}\right) = E_{\mathbb{Q}^{me}}\left(\int_t^{\tilde{T}} \frac{1}{2}\left(\lambda^2\left(Y_s\right)+\hat{\lambda}\left(Y_s,s;T\right)^2\right)ds\right|\mathcal{F}_t\right) \tag{23}$$

and the function $\tilde{H}:\mathcal{R}\times\left[0,\tilde{T}\right]\to\mathcal{R}^+$,

$$\tilde{H}\left(y,t;\tilde{T}\right) = E_{\mathbb{Q}^{me}}\left(\int_t^{\tilde{T}} \frac{1}{2}\left(\lambda^2\left(Y_s\right)+\hat{\lambda}\left(Y_s,s;T\right)^2\right)ds\right|Y_t=y\right), \tag{24}$$

for λ, $\hat{\lambda}$ defined in (4) and (18). We, also, introduce the linear operators

$$\mathcal{L}^Y = \frac{1}{2}a^2\left(y\right)\frac{\partial^2}{\partial y^2}+b\left(y\right)\frac{\partial}{\partial y}, \tag{25}$$

$$\mathcal{L}^{Y,mm} = \frac{1}{2}a^2\left(y\right)\frac{\partial^2}{\partial y^2}+\left(b\left(y\right)-\rho\lambda\left(y\right)a\left(y\right)\right)\frac{\partial}{\partial y} \tag{26}$$

and

$$\mathcal{L}^{Y,me} = \frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2} + (b(y) - \rho\lambda(y)a(y))\frac{\partial}{\partial y} \tag{27}$$
$$+a^2(y)\frac{f_y(y,t;T)}{f(y,t;T)}\frac{\partial}{\partial y}$$

with f solving (20).

The following results follow directly from the definition of $\tilde{H}$, and (19) and (20).

Lemma 3 *For $\tilde{T} \leq T$, the function $\tilde{H} : \mathcal{R}\times[0,\tilde{T}] \to \mathcal{R}^+$, solves the quasilinear equation*

$$\tilde{H}_t + \mathcal{L}^{Y,mm}\tilde{H} - \frac{1}{2}(1-\rho^2)a(y)^2\,\tilde{H}_y^2 + \frac{\lambda^2(y)}{2} = 0,$$

or, equivalently, the linear equation

$$\tilde{H}_t + \mathcal{L}^{Y,me}\tilde{H} + \frac{\lambda^2(y) + \hat{\lambda}(Y_s,s;T)^2}{2} = 0$$

with $\tilde{H}\left(y,\tilde{T};T\right) = 0$, and $\mathcal{L}^{Y,mm}$, $\mathcal{L}^{Y,me}$ as in (26) and (27).

4 Investment and valuation under backward dynamic exponential utilities

In this section, we provide an analytic representation of the backward dynamic exponential utility (cf. Definition 1) and construct the agent's optimal investment in an integrated portfolio choice problem. We recall that the investment horizon is fixed, the utility is normalized at its end and that no liabilities, or cash flows, are allowed beyond the normalization point. For convenience, we occasionally rewrite some of the quantities introduced in earlier sections.

Proposition 4 *Let $\mathbb{Q}^{me}$ be the minimal relative entropy martingale measure and $H(t,T)$ the aggregate relative entropy process (cf. (23)),*

$$H\left(t,\tilde{T}\right) = E_{\mathbb{Q}^{me}}\left(\int_t^{\tilde{T}}\frac{1}{2}\left(\lambda^2(Y_s) + \hat{\lambda}(Y_s,s;T)^2\right)ds\,\middle|\,\mathcal{F}_t\right)$$

with λ and $\hat{\lambda}$ as in (4) and (18). Then, for $x \in \mathcal{R}$, $t \in [0,T]$, the process $U_t^B \in \mathcal{F}_t$, given by

$$U_t^B(x;T) = -e^{-\gamma x - H(t,T)} \tag{28}$$

is the backward dynamic exponential utility.

The proof is, essentially, a direct consequence of the Dynamic Programming Principle and the results of Rouge and El Karoui (2000). For the specific technical arguments, related to the stochastic factor model we examine herein, we refer the reader to Stoikov and Zariphopoulou (2004). We easily deduce the following result.

Corollary 5 *The backward dynamic utility is given by*

$$U_t^B(x;T) = u(x, Y_t, t; T)$$

with $u : \mathcal{R} \times \mathcal{R}^+ \times [0,T] \to \mathcal{R}^-$ *defined as*

$$u(x,y,t;T) = -e^{-\gamma x - \tilde{H}(y,t;T)}$$

with

$$\tilde{H}\left(y,t;\tilde{T}\right) = E_{\mathbb{Q}^{me}}\left(\int_t^{\tilde{T}} \frac{1}{2}\left(\lambda^2(Y_s) + \hat{\lambda}(Y_s,s;T)^2\right) ds \middle| Y_t = y\right).$$

i) Backward indifference values

Next, we revisit the classical definition of indifference values but in the framework of backward dynamic utility. This framework allows for a concise valuation of claims and liabilities of arbitrary maturities, provided that these maturities occur before the normalization point. Due to self-generation, the notion of dynamic value function becomes redundant. Herein we concentrate on the indifference treatment of a liability, or, equivalently, on the optimal portfolio choice of the writer of a claim, yielding payoff equal to the liability at hand.

Definition 6 *Let* T *be the backward normalization point and consider a claim* $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$*, written at* $t_0 \geq 0$ *and maturing at* $\bar{T} \leq T$*. For* $t \in \left[t_0, \bar{T}\right]$*, the backward indifference value process (BIV)* $\nu_t^B(C_{\bar{T}};T)$ *is defined as the amount that satisfies the pricing condition*

$$U_t^B\left(x - \nu_t^B(C_{\bar{T}};T);T\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_{\bar{T}}^B(X_{\bar{T}} - C_{\bar{T}};T) \middle| \mathcal{F}_t\right), \tag{29}$$

for all $x \in \mathcal{R}$*,* $X_t = x$*.*

We note that the backward indifference value coincides with the classical one, but it is constructed from a quite different point of view and scope. The focus herein is not on rederiving previously known quantities but, rather, in exploring how the backward indifference values are affected by the normalization point and the changes in the market environment, as well, as how they differ from their forward dynamic counterparts.

We address these questions for the class of bounded European claims and liabilities, for which we can deduce closed form variational expressions.

Proposition 7 *Let* T *be the backward normalization point and consider a European claim written at* $t_0 \geq 0$ *and maturing at* $\bar{T} \leq T$*, yielding payoff* $C_{\bar{T}} = C(S_{\bar{T}}, Y_{\bar{T}})$*. For* $t \in \left[t_0, \bar{T}\right]$*, its backward indifference value process* $\nu_t^B(C_{\bar{T}};T)$ *is given by*

$$\nu_t^B(C_{\bar{T}};T) = p^B(S_t, Y_t, t)$$

where S and Y solve (1) and (2), and $p^F : \mathcal{R}^+ \times \mathcal{R} \times [0,\bar{T}] \to \mathcal{R}$ satisfies

$$\begin{cases} p_t^B + \mathcal{L}^{(S,Y),me} p^B + \frac{1}{2}\gamma\left(1-\rho^2\right) a^2\left(y\right)\left(p_y^B\right)^2 = 0 \\ \\ p^B\left(S,y,\bar{T}\right) = C\left(S,y\right). \end{cases} \tag{30}$$

Herein,

$$\mathcal{L}^{(S,Y),me} = \frac{1}{2}\sigma^2\left(y\right)S^2\frac{\partial^2}{\partial S^2} + \rho\sigma\left(y\right)Sa\left(y\right)\frac{\partial^2}{\partial S\partial y} + \frac{1}{2}a^2\left(y\right)\frac{\partial^2}{\partial y^2} \tag{31}$$

$$+\left(b\left(y\right) - \rho\lambda\left(y\right)a\left(y\right) + a^2\left(y\right)\frac{f_y\left(y,t;T\right)}{f\left(y,t;T\right)}\right)\frac{\partial}{\partial y},$$

and f solves (cf. (20))

$$\begin{cases} f_t + \frac{1}{2}a^2\left(y\right)f_{yy} + \left(b\left(y\right) - \rho\lambda\left(y\right)a\left(y\right)\right)f_y = \frac{1}{2}\left(1-\rho^2\right)\lambda^2\left(y\right)f \\ \\ f(y,T) = 1. \end{cases}$$

Proof. For convenience, we recall the entropic quantities

$$H\left(t,t^{'}\right) = E_{\mathbb{Q}^{me}}\left(\int_t^{t^{'}}\frac{1}{2}\left(\lambda^2\left(Y_s\right) + \hat{\lambda}\left(Y_s,s;T\right)^2\right)ds\middle|\mathcal{F}_t\right)$$

and

$$\tilde{H}\left(y,t;t^{'}\right) = E_{\mathbb{Q}^{me}}\left(\int_t^{t^{'}}\frac{1}{2}\left(\lambda\left(Y_u\right)^2 + \hat{\lambda}\left(Y_u,u;T\right)^2\right)du\middle|Y_t = y\right)$$

for $0 \le t \le t^{'} \le \bar{T} \le T$. We first calculate the right hand side of (29), which, in view of Proposition 6, becomes

$$\sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}} - C_{\bar{T}}) - H\left(\bar{T};T\right)}\middle|\mathcal{F}_t\right)$$

$$= \sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}} - G_{\bar{T}})}\middle|\mathcal{F}_t\right)$$

with

$$G_{\bar{T}} = C\left(S_{\bar{T}},Y_{\bar{T}}\right) - \frac{1}{\gamma}H\left(\bar{T};T\right).$$

One may, then, view this problem as a traditional indifference valuation one in which the trading horizon is $\left[t,\bar{T}\right]$ and the utility is the exponential function at $\bar{T}$. For the stochastic factor model we consider herein, we obtain (see Sircar and Zariphopoulou (2005) and Grasselli and Hurd (2004))

$$\sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}} - G_{\bar{T}})}\middle|\mathcal{F}_t\right) = -e^{-\gamma(x - h(S_t,Y_t,t)) - H\left(t;\bar{T}\right)}$$

with $h : \mathcal{R}^+ \times \mathcal{R} \times [0, \bar{T}] \to \mathcal{R}$ solving

$$\begin{cases} h_t + \mathcal{L}^{(S,Y),me} h + \frac{1}{2}\gamma\left(1-\rho^2\right) a^2(y)\, h_y^2 = 0 \\ \\ h\left(S, y, \bar{T}\right) = C(S, y) - \frac{1}{\gamma}\tilde{H}\left(y, \bar{T}; T\right). \end{cases}$$

Next, introduce the function $p^B : \mathcal{R}^+ \times \mathcal{R} \times [0, \bar{T}] \to \mathcal{R}$

$$p^B(S, y, t) = h(S, y, t) + \frac{1}{\gamma}\left(\tilde{H}(y, t; T) - \tilde{H}\left(y, t; \bar{T}\right)\right).$$

Using the equation satisfied by h, we deduce that p^B solves (30). On the other hand, Corollary 5 and the above equalities yield

$$\sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{\bar{T}} - G_{\bar{T}})}\middle| X_t = x, S_t = S, Y_t = y\right)$$

$$= -e^{-\gamma\left(x - p^B(S,y,t)\right) - \left(\tilde{H}\left(y,t;\bar{T}\right) + \tilde{H}\left(y,\bar{T};T\right)\right)} = -e^{-\gamma\left(x - p^B(S,y,t)\right) - \tilde{H}(y,t;T)}$$

and the assertion follows from Definition 6 and Proposition 5. ■

ii) Optimal portfolios under backward dynamic utility

Next, we construct the optimal portfolio strategies in the integrated portfolio problem. We start with the agent's optimal behavior in the absence of the liability/payoff. We concentrate our attention to optimal behavior in a shorter horizon. For simplicity, its end is taken to coincide with $\bar{T}$, the point at which the liability is met.

Proposition 8 *Let T be the backward normalization point and $[t, \bar{T}] \in [t, T]$ be the trading horizon of an investor endowed with the backward exponential dynamic utility U^B. The processes, $\pi_s^{B,*}$ and $\pi_s^{B,0,*}$, representing the optimal investments in the risky and riskless asset, are given, for $s \in [t, \bar{T}]$, by*

$$\pi_s^{B,*} = \pi^{B,*}\left(X_s^{B,*}, Y_s, s\right) = \frac{\mu(Y_s)}{\gamma\sigma^2(Y_s)} - \rho\frac{a(Y_s)}{\sigma(Y_s)}\tilde{H}_y(Y_s, s; T) \tag{32}$$

and

$$\pi_s^{B,0,*} = \pi^{B,0,*}(X_s^*, Y_s, s) = X_s^{B,*} - \pi_s^{B,*}.$$

Herein, $X_s^{B,*}$ solves (5) with $\pi_s^{B,*}$ being used, and $\tilde{H} : \mathcal{R} \times [0, T] \to \mathcal{R}^+$ satisfies

$$\tilde{H}_t + \mathcal{L}^{Y,me}\tilde{H} + \frac{\lambda^2(y) + \hat{\lambda}(y, t; T)^2}{2} = 0$$

with terminal condition

$$\tilde{H}\left(y, \bar{T}; T\right) = E_{\mathbb{Q}^{me}}\left(\int_{\bar{T}}^{T} \frac{1}{2}\left(\lambda(Y_s)^2 + \hat{\lambda}(Y_s, s; T)^2\right) ds \middle| Y_{\bar{T}} = y\right). \tag{33}$$

Given the diffusion nature of the model, the form of the utility data and the regularity assumptions on the market coefficients, optimality follows from classical verification results (see, among others, Duffie and Zariphopoulou (1993), Zariphopoulou (2002), Pham (2002), Touzi (2002)).

Due to the stochasticity of the investment opportunity set, the optimal investment strategy in the stock account consists of two components, namely, the *myopic* portfolio and the so-called *excess risky demand,* given, respectively, by $\frac{\mu(Y_s)}{\gamma\sigma^2(Y_s)}$ and $-\rho\frac{a(Y_s)}{\sigma(Y_s)}\tilde{H}_y(Y_s, s; T)$. The myopic component is what the investor would follow if the coefficients of the risky security remained constant across trading periods. The excess risky demand is the required investment that emerges from the local in time changes in the Sharpe ratio (see, among others, Kim and Omberg (1996), Liu (1999), Campbell and Viceira (1999), Chacko and Viceira (1999), Wachter (2002) and Campell et al. (2004)).

Note that even though the trading horizon $[t, \bar{T}]$ is shorter than the original one, $[t, T]$, the optimal policies *depend on the longer horizon* because the dynamic risk preferences are normalized at T and not at $\bar{T}$.

Remark: The reader familiar with the representation of indifference prices, might try to interpret the excess risky demand as the indifference hedging strategy of an appropriately chosen claim. Such questions were studied in Stoikov and Zariphopoulou (2004) where the relevant claim was identified and priced.

We continue with the optimal strategies in the presence of a European-type liability $C_{\bar{T}}$, which, we recall, is taken to be bounded.

Proposition 9 *Let T be the backward normalization point and consider an investor endowed with the backward dynamic exponential utility U^B and facing a liability $C_{\bar{T}} = C(S_{\bar{T}}, Y_{\bar{T}})$. The processes, $\pi_s^{B,*}$ and $\pi_s^{B,0,*}$, representing the optimal investments in the risky and riskless asset, are given, for $s \in [t, \bar{T}]$, by*

$$\pi_s^{B,*} = \pi^{B,*}\left(X_s^{B,*}, S_s, Y_s, s\right) = \frac{\mu(Y_s)}{\gamma\sigma^2(Y_s)} - \rho\frac{a(Y_s)}{\sigma(Y_s)}\tilde{H}_y(Y_s, s; T) \tag{34}$$

$$+S_s p_S^B(S_s, Y_s, s) + \rho\frac{a(Y_s)}{\sigma(Y_s)}p_y^B(S_s, Y_s, s)$$

and

$$\pi_s^{B,0,*} = \pi^{B,0,*}(X_s^*, S_s, Y_s, s) = X_s^{B,*} - \pi_s^{B,*}.$$

Herein, $X_s^{B,*}$ solves (5) with $\pi_s^{B,*}$ being used, $\tilde{H}$ as in Proposition 8 and p^B solves (30).

Proof. In the presence of the liability, we observe

$$\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_{\bar{T}}^B(X_{\bar{T}} - C_{\bar{T}}; T)\middle|\, \mathcal{F}_t\right) = u^C(x, S_t, Y_t, t),$$

where $u^C : \mathcal{R} \times \mathcal{R}^+ \times \mathcal{R} \times \left[0, \bar{T}\right] \to \mathcal{R}^-$ solves the Hamilton-Jacobi-Bellman equation

$$u_t^C + \max_{\pi}\left(\frac{1}{2}\sigma^2(y)\pi^2 u_{xx}^C + \pi\left(\sigma^2(y) S u_{xS}^C + \rho a(y)\sigma(y) u_{xy}^C + \mu(y) u_x^C\right)\right)$$

$$+\mathcal{L}^{(S,Y)}u^C = 0,$$

with

$$u^C\left(x,S,y,\bar{T}\right) = -e^{-\gamma(x-C(S,y))-\tilde{H}\left(y,\bar{T};T\right)},$$

and

$$\mathcal{L}^{(S,Y)} = \frac{1}{2}\sigma^2(y)\,S^2\frac{\partial^2}{\partial S^2} + \rho\sigma(y)\,Sa(y)\,\frac{\partial^2}{\partial S\partial y} + \frac{1}{2}a^2(y)\,\frac{\partial^2}{\partial y^2} \tag{35}$$

$$+\mu(y)\,\frac{\partial}{\partial S} + b(y)\,\frac{\partial}{\partial y}.$$

Verification results yield that the optimal policy $\pi_s^{B,*}$ is given in the feedback form

$$\pi_s^{B,*} = \pi^{B,*}\left(X_s^{B,*}, S_s, Y_s, s\right)$$

with

$$\pi^{B,*}(x,S,y,t) = -\frac{\sigma^2(y)\,Su_{xS}^C + \rho a(y)\sigma(y)\,u_{xy}^C + \mu(y)\,u_x^C}{\sigma^2(y)\,u_{xx}^C}.$$

On the other hand, from Proposition 8,

$$u^C(x,S,y,t) = -e^{-\gamma\left(x-p^B(S,y,t)\right)-\tilde{H}(y,t:T)}.$$

Combining the above and the feedback form of $\pi^{B,*}(x,S,y,t)$, we conclude. ■

5 Investment and valuation under forward dynamic exponential utilities

We now revert our attention to portfolio choice and pricing under the newly introduced class of forward dynamic utilities. We start with the analytic construction of such a utility. As mentioned in Section 1, general existence and uniqueness results for forward dynamic utilities are lacking. As a matter of fact, an alternative solution to (15) is Example 3.

Proposition 10 *Let $s \geq 0$ be the forward normalization point. Define, for $t \geq s$, the process*

$$h(s,t) = \int_s^t \frac{1}{2}\lambda(Y_u)^2\,du \tag{36}$$

with λ being the Sharpe ratio (4). Then, the process $U_t^F(x;s)$, given, for $x \in \mathcal{R}$ and $t \geq s$, by

$$U_t^F(x;s) = -e^{-\gamma x + h(s,t)} \tag{37}$$

is a forward dynamic exponential utility, normalized at s.

Proof. The fact that $U_t^F(x;s)$ is $\mathcal{F}_t$-measurable and normalized at s is immediate. It remains to show (15), namely, that for arbitrary $T \geq t$,

$$-e^{-\gamma x + h(s,t)} = \sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma X_T + h(s,T)}\middle|\,\mathcal{F}_t\right).$$

Using (36), the above reduces to

$$-e^{-\gamma x} = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma X_T + h(t,T)} \Big| \mathcal{F}_t \right). \tag{38}$$

Next, we introduce the function $u : \mathcal{R} \times \mathcal{R} \times [0,T] \to \mathcal{R}^{-}$,

$$u(x,y,t) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma X_T + \int_t^T \frac{1}{2}\lambda^2(Y_s)ds} \Big| X_t = x, Y_t = y \right).$$

Classical arguments imply that u solves the Hamilton-Jacobi-Bellman equation

$$u_t + \mathcal{L}^Y u + \frac{\lambda^2(y)}{2} u$$

$$+ \max_{\pi} \left(\frac{1}{2}\sigma^2(y)\pi^2 u_{xx} + \pi\left(\rho a(y)\sigma(y) u_{xy} + \mu(y) u_x\right) \right) = 0$$

with

$$u(x,y,T) = -e^{-\gamma x}$$

and $\mathcal{L}^Y$ as in (15). We deduce (see, for example, Duffie and Zariphopoulou (1993) and Pham (2002)) that the above equation has a unique solution in the class of functions that are concave and increasing in x, and are uniformly bounded in y. We then see that the function $\check{u}(x,y,t) = -e^{-\gamma x}$ is such a solution and, by uniqueness, it coincides with u. The rest of the proof follows easily. ∎

We next present an alternative forward dynamic utility.

Example 3: Consider, for $x \in \mathcal{R}$ and $t \geq s$, the process

$$U_t^F(x;s) = -e^{-\gamma x - Z(s,t)}$$

with

$$Z(s,t) = \int_s^t \frac{1}{2}\lambda_s^2 ds + \int_s^t \lambda_s dW_s^1. \tag{39}$$

Observe that, for $X_t = x$, the forward stochastic criterion (cf. (15)),

$$-e^{-\gamma x - Z(s,t)} = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma X_T - Z(s,T)} \Big| \mathcal{F}_t \right)$$

will hold if we establish

$$-e^{-\gamma x} = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma X_T - Z(t,T)} \Big| \mathcal{F}_t \right)$$

or, equivalently,

$$-e^{-\gamma x} = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma X_T - Z(t,T)} \Big| X_t = x, Y_t = y, Z_t = 0 \right)$$

with Z as in (39). Defining $v : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \times [0,T] \to \mathcal{R}^-$ by

$$v(x,y,z,t) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(-e^{-\gamma X_T - Z(t,T)} \middle| X_t = x, Y_t = y, Z_t = z \right),$$

we see that it solves the Hamilton-Jacobi-Bellman equation

$$v_t + \max_{\pi} \left(\frac{1}{2}\sigma^2(y)\pi^2 u_{xx} + \pi\left(\lambda(y)\sigma(y)u_{xz} + \rho a(y)\sigma(y)u_{xy} + \mu(y)u_x\right) \right)$$

$$+\frac{1}{2}\lambda^2(y)v_{zz} + \rho\lambda(y)a(y)v_{zy} + \frac{1}{2}a^2(y)v_{yy} + b(y)v_y + \frac{1}{2}\lambda^2(y)v_z$$

with

$$v(x,y,z,T) = -e^{-\gamma x - z}.$$

Substituting above the function $\hat{v}(x,y,z,t) = -e^{-\gamma x - z}$, and after some calculations, yields

$$-\frac{\left(\lambda(y)\sigma(y)\hat{v}_{xz} + \mu(y)\hat{v}_x\right)^2}{2\sigma^2(y)\hat{v}_{xx}} + \frac{1}{2}\lambda^2(y)\left(\hat{v}_{zz} + \hat{v}_z\right) = 0.$$

We easily conclude that $\hat{v} \equiv v$ and the assertion follows.

i) Forward indifference values

We next introduce the concept of forward indifference value. Like its backward counterpart, it is defined as the amount that generates the same level of dynamic utility with and without incorporating the liability. Note, also, that in the Definition below, it is only the forward dynamic utility that enters, eliminating the need to incorporate in the definition the forward dynamic value function. This allows for a concise treatment of payoffs and liabilities of arbitrary maturities. Finally, we remark, that the nomenclature 'forward' does not refer to the terminology used in derivative valuation pertinent to wealth expressed in forward units. Rather, it refers to the forward in time manner that the dynamic utility evolves.

While the concept of forward indifference value appears to be a straightforward extension of the backward one, it is important to observe that the *maturities of the claims in consideration need not be bounded by any prespecified horizon.* This is one of the striking differences between the classes of claims that can be priced by the two distinct dynamic utilities we consider herein. Finally, we remark, that the nomenclature 'forward' does not refer to the terminology used in derivative valuation pertinent to wealth expressed in forward units. Rather, it refers to the forward in time manner that the dynamic utility evolves.

Definition 11 *Let $s \geq 0$ be the forward normalization point and consider a claim $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$, written at $t_0 \geq s$ and maturing at T. For $t \in [t_0, T]$, the* *forward indifference value process (FIP)* $\nu_t^F(C_{\bar{T}};s)$ *is defined as the amount that satisfies the pricing condition*

$$U_t^F\left(x-\nu_t^F(C_T;s);s\right)=\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^F(X_T-C_T;s)\middle|\mathcal{F}_t\right) \tag{40}$$

for all $x\in\mathcal{R},\ X_t=x$.

We continue with the valuation of a bounded European-type liability and we examine how its forward indifference value is affected by the choice of the normalization point. We show that even though both forward dynamic utilities, entering in (40) above, depend on the normalization point, the emerging forward price does *not*. This is another important difference between the backward and the forward indifference values.

Proposition 12 *Let* $s\geq 0$ *be the forward normalization point and consider a European claim written at* $t_0\geq s$ *and maturing at* T *yielding payoff* $C_T=C(S_T,Y_T)$. *For* $t\in[t_0,T]$, *its forward indifference value* $\nu_t^F(C_T;s)$ *is given by*

$$\nu_t^F(C_T;s)=p^F(S_t,Y_t,t)$$

where S *and* Y *solve (1) and (2), and* $p^F:\mathcal{R}^+\times\mathcal{R}\times[0,T]\to\mathcal{R}$ *satisfies*

$$\begin{cases} p_t^F+\mathcal{L}^{(S,Y),mm}p^F+\frac{1}{2}\gamma\left(1-\rho^2\right)a^2(y)\left(p_y^F\right)^2=0 \\ p^F(S,y,T)=C(S,y), \end{cases} \tag{41}$$

with

$$\mathcal{L}^{(S,Y),mm}=\frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2}+\rho\sigma(y)Sa(y)\frac{\partial^2}{\partial S\partial y}+\frac{1}{2}a^2(y)\frac{\partial^2}{\partial y^2}$$
$$+\left(b(y)-\rho\lambda(y)a(y)\right)\frac{\partial}{\partial y}.$$

Proof. We first note that

$$\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^F(X_T-C_T;s)\middle|\mathcal{F}_t\right)$$
$$=\sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_T-C_T)+\int_s^T\frac{1}{2}\lambda^2(Y_u)du}\middle|\mathcal{F}_t\right)$$
$$=e^{\int_s^t\frac{1}{2}\lambda^2(Y_u)du}\sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_T-C_T)+\int_t^T\frac{1}{2}\lambda^2(Y_u)du}\middle|\mathcal{F}_t\right)$$

where we used Proposition 10 and the measurability of the process h (cf. (36)).

Define $u^C:\mathcal{R}\times\mathcal{R}^+\times\mathcal{R}\times[0,T]\to\mathcal{R}^-$,

$$u^C(x,S,y,t)=\sup_{\mathcal{A}} E_{\mathbb{P}}\left(-e^{-\gamma(X_T-C_T)+\int_t^T\frac{1}{2}\lambda^2(Y_u)du}\middle|X_t=x,S_t=S,Y_t=y\right)$$

and observe that it solves the Hamilton-Jacobi-Bellman equation

$$u_t^C + \max_\pi \left(\frac{1}{2}\sigma^2(y)\,\pi^2 u_{yy}^C + \pi\left(\sigma^2(y)\,S u_{xS}^C + \rho a(y)\sigma(y)\,u_{xy}^C + \mu(y)\,u_x^C\right)\right)$$

$$+\mathcal{L}^{(S,Y)}u^C + \frac{\lambda^2(y)}{2}u^C = 0$$

with

$$u^C(x,S,y,T) = -e^{-\gamma(x-C(S,y))}$$

and $\mathcal{L}^{(S,Y)}$ as in (35). Using the transformation

$$u^C(x,S,y,t) = -e^{-\gamma\left(x-p^F(S,y,t)\right)}$$

we deduce, after tedious but straightforward calculations, that the coefficient $p^F(S,y,t)$ solves (41). We, then, easily, see that

$$\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U_T^F(X_T - C_T; s)\middle|\,\mathcal{F}_t\right)$$

$$= -e^{\int_s^t \frac{1}{2}\lambda^2(Y_u)du}u^C(x,S_t,Y_t,t),$$

and, using Proposition 11 and Definition 12 we conclude. ■

ii) Optimal portfolios under forward dynamic utilities

We continue with the optimal investment policies under the forward dynamic risk preferences.

Proposition 13 *Let $s \geq 0$ be the forward normalization point and $[t,T]$ the trading horizon, with $s \leq t$. The processes, $\pi^{F,*}$ and $\pi^{F,0,*}$, representing the optimal investments in the risky and riskless asset, are given, respectively, for $u \in [t,T]$, by*

$$\pi_u^{F,*} = \pi^{F,*}\left(X_u^{F,*}, Y_u, u\right) = \frac{\mu(Y_u)}{\gamma\sigma^2(Y_u)} \qquad (42)$$

and

$$\pi_u^{F,0,*} = \pi^{F,0,*}(X_u^*, Y_u, u) = X_u^{F,*} - \pi_u^{F,*},$$

with $X_u^{F,*}$ solving (5) with $\pi_u^{F,*}$ being used.

Two important facts emerge. Firstly, both optimal investment policies $\pi^{F,*}$ and $\pi^{F,0,*}$ are *independent of the spot normalization point.* Secondly, the investment in the risky asset *consists entirely of the myopic component.* Indeed, the excess hedging demand, which emerges due to the presence of the stochastic factor, has vanished. The investor has processed the stochasticity of the market environment into her preferences, that are dynamically updated, following, forward in time, the market movements.

Proposition 14 *Let $s \geq 0$ be the forward normalization point and consider an investor endowed with the forward exponential dynamic utility U^F and facing a liability $C_T = C(S_T, Y_T)$. The processes, $\pi_s^{F,*}$ and $\pi_s^{F,0,*}$, representing the optimal investments in the risky and riskless asset in the integrated portfolio choice problem, are given, for $u \in [t, T]$, by*

$$\pi_u^{F,*} = \pi^{F,*}\left(X_u^{F,*}, S_u, Y_u, u\right) = \frac{\mu(Y_u)}{\gamma\sigma^2(Y_u)}$$

$$+S_u p_S^F(S_u, Y_u, u) + \rho\frac{a(Y_u)}{\sigma(Y_u)} p_y^F(S_u, Y_u, u) \qquad (43)$$

and

$$\pi_u^{F,0,*} = \pi^{F,0,*}(X_u^*, S_u, Y_u, u) = X_u^{F,*} - \pi_u^{F,*}.$$

Herein, $X_s^{F,*}$ solves (5) with $\pi_s^{F,*}$ being used, and p^F satisfies (41).

6 Concluding remarks: Forward versus backward utilities and their associated indifference prices

In the previous two Sections, we analyzed the investment and pricing problems of investors endowed with backward *(BDU)* and forward *(FDU)* dynamic exponential utilities. These utilities have similarities but, also, striking differences. These features are, in turn, inherited to the associated optimal policies, indifference prices and risk monitoring strategies. Below, we provide a discussion on these issues.

We first observe that the backward and forward utilities are produced via a conditional expected criterion. They are both self-generating, in that they coincide with their implied value functions. Moreover, in the absence of exogenous cash flows, investors endowed with such utilities are indifferent to the investment horizons.

Backward and forward dynamic utilities are constructed in entirely different ways. Backward utilities are first specified at a given future time, T, and, they are, subsequently, generated at previous to T times. Forward utilities are defined at present, s, and are, in turn, generated forward in time. The times T and s, at which the backward and forward utility data are determined, are the backward and forward normalization points. We recall, from equations (28) and (37), that the *BDU* and *FDU* processes, U_t^B and U_t^F, are $\mathcal{F}_t-$ adapted and given, respectively, by

$$U_t^B(x; T) = -e^{-\gamma x - H(t,T)}$$

and

$$U_t^F(x; s) = -e^{-\gamma x - h(s,t)},$$

with

$$H\left(t,T\right)=E_{\mathbb{Q}^{me}}\left(\left.\int_t^T\frac{1}{2}\left(\lambda_u^2+\hat{\lambda}_u^2\right)du\right|\mathcal{F}_t\right)$$

and

$$h(s,t)=\int_s^t\frac{1}{2}\lambda_u^2du.$$

Herein, λ and $\hat{\lambda}$ are given in (4) and (18), and $\mathbb{Q}^{me}$ is the minimal relative entropy measure.

Both *BDU* and *FDU* have an exponential, affine in wealth, structure. However, the backward utility compiles changes in the market environment in an aggregate manner, while the forward utility does so in a much finer way. This is seen by the nature of the processes $H\left(t,T\right)$ and $h\left(s,t\right)$. It is worth observing that

$$H\left(t,T\right)\neq E_{\mathbb{Q}^{me}}\left(\left.h\left(t,T\right)\right|\mathcal{F}_t\right)$$

and that U_t^F is not affected by $\hat{\lambda}$ (cf. (18) and (19)) that represent the 'orthogonal' component of the market price of risk.

Backward and forward utilities generate different optimal investment strategies (see, respectively, Propositions 9 and 14). Under backward dynamic preferences, the investor invests in the risky asset an amount equal to the sum of the myopic portfolio and the excess risky demand. The former investment strategy depends on the risk aversion coefficient γ, but not on the backward normalization point T. The excess risky demand, however, is not affected by γ but depends on the choice of the normalization point, even if investment takes place in a shorter horizons.

Under forward preferences, the investor invests in the risk asset solely the myopic portfolio. The myopic strategy does not depend on the forward normalization point or the investment horizon.

As a consequence of the above differences, the emerging backward *(BIV)* and forward *(FIV)* indifference values, $\nu_t^B\left(C_{\bar{T}};T\right)$ and $\nu_t^F\left(C_{\bar{T}};s\right),\ 0\leq s\leq t\leq\bar{T}\leq T$, have very distinct characteristics. Concentrating on the class of bounded European claims, we see that $\nu_t^B\left(C_{\bar{T}};T\right)$ and $\nu_t^F\left(C_{\bar{T}};s\right)$ are constructed via solutions of similar quasilinear pdes. While the nonlinearities. in the pricing pdes are of the same type, the associated linear operators, $\mathcal{L}^{(S,y),me}$ and $\mathcal{L}^{(S,y),mm}$ differ (see, respectively, (30) and (41)). The former, appearing in the *BIV* equation, corresponds to the minimal relative entropy measure while the latter, appearing in the *FIV* equation*,* to the minimal martingale measure. Denoting the solutions of these pdes as nonlinear expectations, we may formally represent - with a slight abuse of notation - the two indifference values as

$$\nu_t^B\left(C_{\bar{T}};T\right)=\mathcal{E}_{\mathbb{Q}^{me}}\left(C_{\bar{T}};T\right)$$

and

$$\nu_t^F\left(C_{\bar{T}};s\right)=\mathcal{E}_{\mathbb{Q}^{mm}}\left(C_{\bar{T}};s\right).$$

The FIV is independent on the forward normalization point. The BIV depends, however, on the backward normalization point, even if the claim matures in a shorter horizon.

As the investor becomes risk neutral, $\gamma \rightarrow 0$, we obtain

$$\lim_{\gamma \rightarrow 0} \nu_t^B \left(C_{\bar{T}}; T\right) = E_{\mathbb{Q}^{me}} \left(C_{\bar{T}} \left| \mathcal{F}_t \right.\right)$$

and

$$\lim_{\gamma \rightarrow 0} \nu_t^F \left(C_{\bar{T}}; s\right) = E_{\mathbb{Q}^{mm}} \left(C_{\bar{T}} \left| \mathcal{F}_t \right.\right).$$

However, as the investor becomes infinitely risk averse, $\gamma \rightarrow \infty$, both BIV and FIV converge to the same limit given by the super replication value,

$$\lim_{\gamma \rightarrow \infty} \nu_t^B \left(C_{\bar{T}}; T\right) = \lim_{\gamma \rightarrow \infty} \nu_t^F \left(C_{\bar{T}}; s\right) = \left\| C_{\bar{T}} \right\|_{\mathcal{L}^\infty \{ . | \mathcal{F}_t \}}.$$

In the presence of the liability and under backward dynamic utility, the investment in the risky asset consists of the myopic portfolio, the excess risky demand and the backward indifference risk monitoring strategies (see Proposition 10). With the exception of the myopic portfolio, all other three portfolio components depend on the normalization point T. When, however, the investor uses forward dynamic utility, his optimal integrated policy does not include the excess risky demand. The entire policy is independent of the forward normalization point s, and depends exclusively on the maturity of the claim and the changes in the market environment.

When the market becomes complete, the backward and forward pricing measures, $\mathbb{Q}^{me}$ and $\mathbb{Q}^{mm}$, coincide with the unique risk neutral measure, $\mathbb{Q}^*$, and *(BIV)* and *(FIV)* reduce to the arbitrage free price.

In general, the *backward and forward indifference values do not coincide.* The underlying reason is that they are defined via the backward and forward dynamic utilities that process the internal model incompleteness, generated by the stochastic factor Y, in a very different manner. Characterizing the market environments as well as the claims for which the two prices coincide is an open question.

7 Bibliography

[1] Arai, T: The relation between the minimal martingale measure and the minimal entropy martingale measure, *Asia-Pacific Financial Markets*, 8(2), 167-177, (2001).

[2] Bellini, F. and M. Frittelli: On the existence of the minmax martingale measures, *Mathematical Finance,* 12, 1-21.

[3] Benth, F. E. and K. H. Karlsen: A PDE representation of the density of the minimal martingale measure in stochastic volatility markets, *Stochastics and Stochastics reports,* 77, 109-137 (2005).

[4] Campbell, J.Y. and Viceira, L.M.: Consumption and portfolio decisions when expected returns are time-verying, *Quarterly Journal of Economics*, 114, 433-495 (1999).

[5] Campell, J. Y., Cocco, J., Gomes, F., Maenhout, P. J. and Viceira, L. M: Stock market mean reversion and the optimal equity allocation of a long-lived investor, *European Finance Review*, in press (2004).

[6] Chacko, G. and L. Viceira.: Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets, NBER Working papers, No 7377 (1999).

[7] Davis, M.H.A. and T. Zariphopoulou: American options and transaction fees, *Mathematical Finance*, *IMA Volumes in Mathematics and its Application*, Springer-Verlag (1995).

[8] Delbaen, F., Grandits, P., Rheinlander, T., Samperi, D., Schweizer, M., Stricker, C.: Exponential hedging and entropic penalties, *Mathematical Finance*, 12, 99-123 (2002).

[9] Duffie, D. and T. Zariphopoulou: Optimal investment with undiversifiable income risk, *Mathematical Finance,* 3(2),135-148 (1993).

[10] Foellmer, H. and Schweizer, M.: Hedging of contingent claims under incomplete information, In *Applied Stochastic Analysis, Stochastic Monographs,* Vol. 5, M.H.A. Davis and R. J. Elliott eds., Gordon and Breach, 389-414 (1991).

[11] Frittelli, M.: The minimal entropy martingale measure and the valuation problem in incomplete markets, *Mathematical Finance*, 10, 39-52 (2000).

[12] Grandits, P. and Rheinlander, T.: On the minimal entropy measure, *Annals of Probability*, 30 (3), 1003-1038 (2002).

[13] Grasselli, M. R. and Hurd, T.: Indifference pricing and hedging in stochastic volatility models, preprint (2004).

[14] Kabanov, Y. and Stricker, C.: On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper "Exponential hedging and entropic penalties", *Mathematical Finance*, 12 (2): 125-134 (2002).

[15] Kim, T.S. and Omberg, E.: Dynamic nonmyopic portfolio behavior, *Review of Financial Studies*, 9, 141-161 (1996).

[16] Kramkov, D. and Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investments in incomplete markets, *Annals of Applied Probability*, 9, 904-950 (1999).

[17] Liu, J.: Portfolio selection in stochastic environments, working paper, Stanford University (1999).

[18] Kallsen, J. and C. Kuehn: Pricing derivatives of American and Game type in incomplete markets, *Finance and Stochastics,* 8(2), 261-284 (2004).

[19] Monoyios, M.: Characterization of optimal dual measures through distortion, *Decisions in Economics and Finance,* in print (2006).

[20] Musiela, M. and T. Zariphopoulou: Indifference prices and related measures, Technical report, The University of Texas at Austin (2001).

[21] Musiela, M. and Zariphopoulou, T.: An example of indifference prices under exponential preferences, *Finance and Stochastics*, 8, 229-239 (2004a).

[22] Musiela, M. and Zariphopoulou, T.: Indifference prices of early exercise claims. G. Yin and Q. Zhang eds. *Proceedings of AMS-IMS-SIAM Joint*

Summer Conference on Mathematical Finance, Volume 351 of Contemporary Mathematics, AMS, 259-272 (2004b).

[23] Musiela, M. and Zariphopoulou, T.: The backward and forward dynamic utilities and their associated pricing systems: The case study of the binomial model, *Indifference Pricing*, R. Carmona Ed., Princeton University Press, in print (2005a).

[24] Musiela, M. and T. Zariphopoulou: Forward dynamic utilities and indifference valuation in incomplete binomial models, preprint (2005b).

[25] Oberman, A. and Zariphopoulou, T.: Pricing early exercise contracts in incomplete markets, *Computational Management Science*, 1, 75-107 (2003).

[26] Pham, H.: Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints, *Applied Mathematics and Optimization*, 46, 55-78 (2002).

[27] Rheinlander, T.: An entropy approach to the Stein/Stein model with correlation, *Finance and Stochastics,* 9, 399-413 (2003).

[28] Rouge, R. and El Karoui, N.: Pricing via utility maximization and entropy, *Mathematical Finance*, 10, 259-276 (2000).

[29] Schweizer, M.: On the minimal martingale measure and the Foellmer-Schweizer decomposition, *Stochastic Processes and Applications*, 13, 573-599 (1995).

[30] Schweizer, M.: A minimality property of the minimal martingale measure, *Statistics and Probability Letters*, 42, 27-31 (1999).

[31] Sircar, R. and Zariphopoulou, T.: Bounds and asymptotic approximations for utility prices when volatility is random, SIAM Journal on Control and Optimization, *SIAM Journal on Control and Optimization,* 43(4), 1328-1353 (2005).

[32] Stoikov, S. and Zariphopoulou, T.: Optimal investments in the presence of unhedgeable risks and under CARA preferences, *IMA Volume in Mathematics and its Application,* in print (2004).

[33] Touzi, N.: Stochastic control problems, viscosity solutions and application to Finance, Technical report, Special research semester on Financial Markets, Pisa (2002).

[34] Wachter, J.: Portfolio consumption decisions under mean-reverting returns: an exact solution for complete markets, *Journal of Financial and Quantitative Analysis*, 37 (1), 63-91 (2002).

[35] Zariphopoulou, T.: Stochastic control methods in asset pricing, *Handbook of stochastic analysis and applications,* D. Kannan and V. Lakshmikanthan eds., Statistics, Textbooks, Monographs, 163, Dekker, 679-753 (2002).