Utility Valuation of Credit Derivatives: Single and Two-Name Cases

Ronnie Sircar* Thaleia Zariphopoulou†

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Abstract

We study the effect of risk aversion on the valuation of credit derivatives. Using the technology of utility-indifference valuation in intensity-based models of default risk, we analyze resulting yield spreads for single-name defaultable bonds, and a simple representative two-name credit derivative. The impact of risk averse valuation on prices and yield spreads is expressed in terms of effective correlation.

Keywords: Credit derivatives, indifference pricing, reaction-diffusion equations.

Contents

*ORFE Department, Princeton University, E-Quad, Princeton NJ 08544; *sircar@princeton.edu*. Work partially supported by NSF grant DMS-0456195.

†Departments of Mathematics and Information, Risk and Operations Management, The University of Texas at Austin, Austin, TX 78712, *zariphop@math.utexas.edu*. Work partially supported by NSF grants DMS-0456118 and DMS-0091946.

1 Introduction

In this article, we analyze the impact of risk aversion on the valuation of defaultable bonds, and a simple multi-name credit derivative. Our approach is to work within intensity-based models, as initiated by, among others, Artzner and Delbaen [1] , Madan and Unal [25], Lando [22] and Jarrow and Turnbull [18]. However, rather than pricing using *no arbitrage* arguments, we study the utility-indifference valuation mechanism, which entails analysis of portfolio optimization problems under default risk.

A major limitation of many traditional approaches is the inability to capture and explain high premiums observed in credit derivatives markets for unlikely events, for example the spreads quoted for senior tranches of CDOs written on investment grade firms. The approach explored here, and in our related work [28], aims to explain such phenomena as a consequence of tranche holders' risk aversion, and to quantify this through the mechanism of utility-indifference valuation.

For a general introduction to *Credit Risk*, including other approaches to default, such as structural models, we refer, for example, to the books [5, 13, 23, 26].

Valuation Mechanisms

In complete financial market environments, such as in the classical Black-Scholes model, the payoffs of derivative securities can be replicated by trading strategies in the underlying securities, and their prices are naturally deduced from the value of these associated portfolios. However, once non-traded risks, such as unpredictable defaults, are considered, the possibility of replication and, therefore, risk elimination breaks down and alternative ways are needed for the quantification of risk and assignation of price. One approach is to use market derivatives data, when available, to identify which of the many feasible arbitrage-free pricing measures is consistent with market prices. In a different direction, valuation of claims involving nontradable risks can be based on optimality of decisions once this claim is incorporated in the investor's portfolio. Naturally, the risk attitude of the individual needs to be taken into account, and this is typically modeled by a concave and increasing utility function U. In a static framework, prices are determined through the *certainty equivalent*, otherwise known as the principle of equivalent utility [6, 16]. The utility-based value of the claim, written on the risk Y and yielding payoff $C(Y)$, is $\nu(C) = U^{-1}\{\mathbb{E}_{\mathbb{P}}(U(C(Y)))\}$. Note that the arbitrage free price and the certainty equivalent are very different. The first is *linear* and uses the *risk neutral* measure. The certainty equivalent price is *nonlinear* and uses the *historical* assessment of risks.

Prompted by the ever-increasing number of applications (event risk sensitive claims, insurance plans, mortgages, weather derivatives, etc.), considerable effort has been put into analyzing the utility-based valuation mechanism. Due to the prevalence of instruments dependent on non-market risks (like default), there is a great need for building new dynamic pricing rules. These rules should identify and price unhedgeable risks and, at the same time, build optimal risk monitoring policies. In this direction, a dynamic utility-based valuation theory has been developed producing the so-called *indifference prices.* The mechanism is based on finding the amount at which the buyer of the claim is indifferent, in terms of maximum expected utility, between holding or not holding the derivative. Specification of the indifference price requires understanding how investors act optimally with or without the derivative at hand. These issues are naturally addressed through stochastic optimization problems of utility maximization. We refer to [20, 21] and [8] as classical references in this area. The indifference approach was initiated for European claims by Hodges and Neuberger [17] and further extended by Davis *et al.* [10].

Credit Derivatives

As well as single-name securities, such as *credit default swaps* (CDSs), in which there is a relatively liquid market, basket, or multi-name products have generated considerable over-the-counter activity. Popular cases are *collateralized debt obligations* (CDOs) whose payoffs depend on the default events of a basket portfolio of up to 300 firms, over a five year period. As long as there are no defaults, investors in CDO tranches enjoy high yields, but, as defaults start occurring, they affect first the high-yield equity tranche, then the mezzanine tranches and, perhaps, the senior and super-senior tranches. See Davis and Lo [9] or Elizalde [15] for a concise introduction to these products.

The focus of modeling in the credit derivatives industry has been on *correlation* between default times. Partly, this is due to the adoption of the one-factor Gaussian copula model as industry standard and the practice (up till recently) of analyzing tranche prices through implied correlation. This revealed that traded prices of senior tranches could only be realised through these models with an implausibly high correlation parameter, the so-called correlation smile.

Rather than focusing on models with "enough correlation" to reproduce market observations via traditional no-arbitrage pricing, our goal is to understand the effects of risk aversion on valuation of single- and two-name credit derivatives. Questions of interest are i) how does risk aversion affect the value of portfolios that are sensitive to the potential default of a number of firms, and so to correlation between these events? and ii) does the nonlinearity of the indifference pricing mechanism enhance the impact of correlation? It seems natural that some of the prices, or spreads, seen in credit markets are due mainly to "crash-o-phobia" in a relatively illiquid market, with the effect enhanced nonlinearly in baskets. When super-senior tranches offer non-trivial spreads (albeit a few basis points) for protection against the default risk of $15-30\%$ of investment grade US firms over the next five years, they are ascribing a seemingly large probability to "the end of the world as we know it". We seek to capture this directly as an effect of risk aversion leading to effective or perceived correlation, in contrast to a mechanism of high direct correlation.

Taking the opposite angle, the method of indifference valuation should be attractive to participants in this still quite illiquid OTC market. It is a direct way for them to quantify the default risks they face in a portfolio of complex instruments, when calibration data is scarce. Unlike well developed equity and fixed-income derivatives markets, where the case for traditional arbitrage-free valuation is more compelling, the potential for utility valuation to account for high-dimensionality in a way that is consistent with investors' fears of a cascade of defaults is a case for its application here.

For applications of indifference valuation to credit risk, see also Collin-Dufresne and Hugonnier [7], Bielecki *et al.* [4, 3], and Shouda [27].

2 Indifference Valuation: Single Name

We start with single name defaultable bonds to illustrate the approach. We will work within models incorporating information from the firm's stock price S, but unlike in a traditional structural approach, default occurs at a non-predictable stopping time τ with stochastic intensity process λ, which is correlated with the firm's stock price. These are sometimes called *hybrid* models (see, for example, [24]). The process S could alternatively be taken as the price of another firm or index used to hedge the default risk. Of course, the choice of the investment opportunity set affects the ensuing indifference price.

The stock price S is taken to be a geometric Brownian motion. The intensity process is $\lambda(Y_t)$, where $\lambda(\cdot)$ is a non-negative, locally Lipschitz, smooth and bounded function, and Y is a correlated diffusion. The dynamics of S and Y are

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t^{(1)}, \qquad S_0 = S > 0, \\ dY_t &= b(Y_t)\, dt + a(Y_t)\left(\rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\right), \qquad Y_0 = y \in I\!R. \end{aligned}$$

The coefficients a and b are taken to be Lipschitz functions with sublinear growth. The processes W^1 and W^2 are independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, I\!P)$, and we denote by $\mathcal{F}_t$ the augmented σ-algebra generated by $((W_u^1, W_u^2);\ 0 \le u \le t)$. The parameter $\rho \in (-1, 1)$ measures the instantaneous correlation between shocks to the stock price S and shocks to the intensity-driving process Y. In applications, it is natural to expect that $\lambda(\cdot)$ and ρ are specified in a way such that the intensity tends to rise when the stock price falls.

There also exists a standard exponential random variable ξ, independent of the Brownian motions. The default time τ of the firm is defined by

$$\tau = \inf\left\{ t \ge 0 : \int_0^t \lambda(Y_s)\, ds = \xi \right\},$$

the first time the cumulated intensity reaches the random draw ξ.

Maximal Expected Utility Problem

Let $T < \infty$ denote our finite fixed horizon, chosen later to coincide with the expiration date of the derivatives contracts of interest. The investor's control process is π_t, representing the dollar amount held in the stock at time t, until $\tau \wedge T$. For $t < \tau \wedge T$, her wealth process X follows

$$\begin{aligned} dX_t &= \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)\, dt \\ &= (rX_t + \pi_t(\mu - r))\, dt + \sigma \pi_t\, dW_t^{(1)}. \end{aligned}$$

The control π is called admissible if it is $\mathcal{F}_t$-measurable and satisfies the integrability constraint $E\{\int_0^T \pi_s^2\, ds\} < \infty$. The set of admissible policies is denoted by $\mathcal{A}$.

If the default event occurs before T, the investor can no longer trade the firm's stock. She has to liquidate holdings in the stock and deposit in the bank account, so the effect is to reduce her investment opportunities. For simplicity, we assume she receives full pre-default market value on her stock holdings on liquidation, though one might extend to consider some loss, or jump downwards in the stock price at the default time. Therefore, given that $\tau < T$, for $\tau \leq t \leq T$, we have

$$X_t = X_\tau e^{r(t-\tau)},$$

as the bank account is the only remaining investment.

We shall work with exponential utility of discounted (to time zero) wealth. We are first interested in the optimal investment problem up to time T of the investor who does not hold any derivative security. At time zero, the maximum expected utility payoff then takes the form

$$\sup_{\pi\in\mathcal{A}} I\!\!E \left\{ -e^{-\gamma(e^{-rT}X_T)} \mathbf{1}_{\{\tau>T\}} + (-e^{-\gamma(e^{-r\tau}X_\tau)}) \mathbf{1}_{\{\tau\leq T\}} \right\}.$$

We switch to the discounted variable $X_t \mapsto e^{-rt}X_t$ and excess growth rate $\mu \mapsto \mu - r$; with a slight abuse, we use the same notation.

Next, we consider the stochastic control problem initiated at time $t \leq T$, and define the default time τ_t by

$$\tau_t = \inf \left\{ s \geq t : \int_t^s \lambda(Y_u)\, du = \xi \right\},$$

where ξ is an independent standard exponential random variable.

In the absence of the defaultable claim, the investor's value function is given by

$$M(t,x,y) = \sup_{\pi\in\mathcal{A}} I\!\!E \left\{ -e^{-\gamma X_T} \mathbf{1}_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}}) \mathbf{1}_{\{\tau_t\leq T\}} \mid X_t = x, Y_t = y \right\}. \qquad (1)$$

Proposition 1 *The value function $M : [0,T]\times I\!\!R\times I\!\!R \to I\!\!R^-$ is the unique viscosity solution in the class of functions that are concave and increasing in x, and uniformly bounded in y of the HJB equation*

$$M_t + \mathcal{L}_y M + \max_\pi \left(\frac{1}{2}\sigma^2\pi^2 M_{xx} + \pi(\rho\sigma a(y) M_{xy} + \mu M_x) \right) + \lambda(y)(-e^{-\gamma x} - M) = 0, \qquad (2)$$

with $M(T,x,y) = -e^{-\gamma x}$ and

$$\mathcal{L}_y = \frac{1}{2}a(y)^2 \frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}.$$

PROOF: The proof follows by extension of the arguments used in Theorem 4.1 of Duffie and Zariphopoulou [14] and is omitted. □

Bond Holder's Problem and Indifference Price

We now consider the same problem from the point of view of an investor who owns a defaultable bond of the firm. The bond pays \$1 on date T if the firm has survived till then. Defining $c = e^{-rT}$, we have the bond holder's value function

$$H(t,x,y) = \sup_{\pi\in\mathcal{A}} I\!\!E \left\{ -e^{-\gamma(X_T+c)} \mathbf{1}_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}}) \mathbf{1}_{\{\tau_t\leq T\}} \mid X_t = x, Y_t = y \right\}. \qquad (3)$$

As in Proposition 1 for the plain investor's value function M, we have the following HJB characterization.

Proposition 2 *The value function $H : [0,T] \times I\!R \times I\!R \to I\!R^-$ is the unique viscosity solution in the class of functions that are concave and increasing in x, and uniformly bounded in y of the HJB equation*

$$H_t + \mathcal{L}_y H + \max_{\pi} \left(\frac{1}{2}\sigma^2 \pi^2 H_{xx} + \pi(\rho\sigma a(y) H_{xy} + \mu H_x) \right) + \lambda(y)(-e^{-\gamma x} - H) = 0, \quad (4)$$

with $H(T, x, y) = -e^{-\gamma(x+c)}$.

The indifference value of the defaultable bond, from the point of view of the bond holder, is the reduction in her initial wealth level such that her maximum expected utility H is the same as the plain investor's value function M.

Definition 1 *The buyer's indifference price $p_0(T)$ (at time zero) of a defaultable bond with expiration date T is defined by*

$$M(0, x, y) = H(0, x - p_0, y). \quad (5)$$

The indifference price at times $0 < t < T$ can be defined similarly, with minor modifications to the previous calculations (in particular, with quantities discounted to time t dollars.)

2.1 Variational Results

In this section, we present some simple bounds for the value functions and the indifference price introduced above.

Proposition 3 *The value functions M and H satisfy, respectively*

$$-e^{-\gamma x} \le M(t, x, y) \le -e^{-\gamma x - \frac{\mu^2}{2\sigma^2}(T-t)}, \quad (6)$$

$$-e^{-\gamma x} + (e^{-\gamma x} - e^{-\gamma(x+c)}) I\!P\{\tau_t > T \mid Y_t = y\} \le H(t, x, y) \le -e^{-\gamma(x+c) - \frac{\mu^2}{2\sigma^2}(T-t)}. \quad (7)$$

Proof: We start with establishing (6). We first observe that the function $\tilde{M}(t, x, y) = -e^{-\gamma x}$ is a subsolution of the HJB equation (2). Moreover, $\tilde{M}(T, x, y) = M(T, x, y)$. The lower bound then follows from the comparison principle.

Similarly, testing the function

$$\tilde{M}(t, x, y) = -e^{-\gamma x - \frac{\mu^2}{2\sigma^2}(T-t)}$$

yields

$$\tilde{M}_t + \mathcal{L}_y \tilde{M} + \max_{\pi} \left(\frac{1}{2}\sigma^2\pi^2 \tilde{M}_{xx} + \pi(\rho\sigma a(y)\tilde{M}_{xy} + \mu \tilde{M}_x) \right)$$
$$+\lambda(y)\left(-e^{-\gamma x} + e^{-\gamma x - \frac{\mu^2}{2\sigma^2}(T-t)}\right) = \lambda(y) e^{-\gamma x}\left(e^{-\frac{\mu^2}{2\sigma^2}(T-t)} - 1\right) \le 0.$$

Therefore, $\tilde{M}$ is a supersolution, with $\tilde{M}(T, x, y) = M(T, x, y)$, and the upper bound follows.

Next, we establish (7). To obtain the lower bound, we follow the sub-optimal policy of investing exclusively in the default-free bank account (that is, taking $\pi \equiv 0$). Then

$$\begin{aligned} H(t,x,y) &\geq \mathbb{E}\left\{-e^{-\gamma(x+c)}\mathbf{1}_{\{\tau_t>T\}} + (-e^{-\gamma x})\mathbf{1}_{\{\tau_t\leq T\}} \mid X_t = x, Y_t = y\right\} \\ &= -e^{-\gamma(x+c)}\mathbb{P}\{\tau_t > T \mid Y_t = y\} + (-e^{-\gamma x})\mathbb{P}\{\tau_t \leq T \mid Y_t = y\} \\ &= -e^{-\gamma x} + (e^{-\gamma x} - e^{-\gamma(x+c)})\mathbb{P}\{\tau_t > T \mid Y_t = y\}, \end{aligned}$$

and the lower bound follows. The upper bound is established by testing the function

$$\tilde{H}(t,x,y) = -e^{-\gamma(x+c)-\frac{\mu^2}{2\sigma^2}(T-t)}$$

in the HJB equation (4) for H, and showing that $\tilde{H}$ is a supersolution. □

Remark 1 *The bounds given above reflect that, in the presence of default, the value functions are bounded between the solutions of two extreme cases. For example, the lower bounds correspond to a degenerate market (only the bank account available for trading in* $[0,T]$*), while the upper bounds correspond to the standard Merton case with no default risk.*

2.2 Reduction to Reaction-Diffusion Equations

The HJB equation (2) can be simplified by the familiar distortion scaling

$$M(t,x,y) = -e^{-\gamma x}u(t,y)^{1/(1-\rho^2)}, \tag{8}$$

with $u : [0,T] \times \mathbb{R} \to \mathbb{R}^+$ solving the reaction-diffusion equation

$$\begin{aligned} u_t + \widetilde{\mathcal{L}}_y u - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)u + (1-\rho^2)\lambda(y)u^{-\theta} &= 0, \\ u(T,y) &= 1, \end{aligned} \tag{9}$$

where

$$\theta = \frac{\rho^2}{1-\rho^2},$$

and

$$\widetilde{\mathcal{L}}_y = \mathcal{L}_y - \frac{\rho\mu}{\sigma}a(y)\frac{\partial}{\partial y}. \tag{10}$$

Similar equations arise in other utility problems in incomplete markets, for example, in portfolio choice with recursive utility [29], valuation of mortgage-backed securities [30] and life-insurance problems [2]. One might work first with (9) and then provide the verification results for the HJB equation (2), since the solutions of (2) and (9) are related through (8). It is worth noting, however, that the reaction-diffusion equation (9) does not belong to the class of such equations with Lipschitz reaction term. Therefore, more detailed analysis is needed for *directly* establishing existence, uniqueness and regularity results. In the context of a portfolio choice problem with stochastic differential utilities, the analysis can be found in [29]. The equation at hand is slightly more complicated than the one analyzed there, in that the reaction term has the multiplicative intensity factor. Because $\lambda(\cdot)$ is taken to be bounded and Lipschitz, an adaptation of the arguments in [29] can be used to show that the reaction-diffusion problem (9) has a unique bounded and smooth solution. Furthermore, using (8) and the bounds obtained for M in Proposition 3, we have

$$e^{-(1-\rho^2)\frac{\mu^2}{2\sigma^2}(T-t)} \leq u(t,y) \leq 1.$$

For the bond holder's value function, the transformation

$$H(t,x,y) = -e^{-\gamma(x+c)}w(t,y)^{1/(1-\rho^2)}$$

reduces to

$$\begin{aligned} w_t + \widetilde{\mathcal{L}}_y w - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) w + (1-\rho^2)e^{\gamma c}\lambda(y)w^{-\theta} &= 0, \\ w(T,y) &= 1, \end{aligned} \tag{11}$$

which is a similar reaction-diffusion equation as (9). The only difference is the coefficient $e^{\gamma c} > 1$ in front of the reaction term. Existence of a unique smooth and bounded solution follows similarly.

The following lemma gives a relationship between u and w.

Lemma 1 *Let u and w be solutions of the reaction-diffusion problems (9) and (11). Then*

$$u(t,y) \leq w(t,y) \qquad \textit{for } (t,y) \in [0,T] \times I\!R.$$

PROOF: We have $u(T,y) = w(T,y) = 1$. Moreover, because $e^{\gamma c} > 1$ and $\lambda > 0$,

$$(1-\rho^2)e^{\gamma c}\lambda(y)w^{-\theta} > (1-\rho^2)\lambda(y)w^{-\theta},$$

which yields

$$w_t + \widetilde{\mathcal{L}}_y w - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) w + (1-\rho^2)\lambda(y)w^{-\theta} < 0.$$

Therefore, w is a supersolution of (9), and the result follows. □

From this, we easily obtain the following sensible bounds on the indifference value of the defaultable bond, and the yield spread.

Proposition 4 *The indifference bond price p_0 in (5) is given by*

$$p_0(T) = e^{-rT} - \frac{1}{\gamma(1-\rho^2)}\log\left(\frac{w(0,y)}{u(0,y)}\right), \tag{12}$$

and satisfies $p_0(T) \leq e^{-rT}$. The yield spread defined by

$$\mathcal{Y}_0(T) = -\frac{1}{T}\log(p_0(T)) - r$$

is non-negative for all $T > 0$.

Remark 2 *We denote the seller's indifference price by $\tilde{p}_0(T)$. In order to construct it, we replace c by $-c$ in the definition (3) of the value function H and in the ensuing transformations. If $\tilde{w}$ is the solution of*

$$\tilde{w}_t + \widetilde{\mathcal{L}}_y \tilde{w} - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)\tilde{w} + (1-\rho^2)e^{-\gamma c}\lambda(y)\tilde{w}^{-\theta} = 0, \tag{13}$$

with $\tilde{w}(T,y) = 1$, then

$$\tilde{p}_0(T) = e^{-rT} - \frac{1}{\gamma(1-\rho^2)}\log\left(\frac{u}{\tilde{w}}\right).$$

Using comparison results, we obtain $u > \tilde{w}$ as $e^{-\gamma c} < 1$. Therefore $\tilde{p}_0(T) \leq e^{-rT}$ and the seller's yield spread is non-negative for all $T > 0$.

2.3 Connection with Relative Entropy Minimization

For completeness, we provide the connection between the HJB equations characterizing the primal optimal investment problem that we study, to the dual problem of relative entropy minimization. Let G be the bounded $\mathcal{F}_T-$measurable payoff of a credit derivative, and let $\mathcal{P}$ denote the primal problem's value (for simplicity, at time zero):

$$\mathcal{P} = \sup_{\pi \in \mathcal{A}} I\!\!E\left\{-e^{-\gamma(X_{\tau\wedge T} + G\mathbf{1}_{\{\tau > T\}})}\right\}.$$

In our problem (3), we have $G = c$.

As is well-known, under quite general conditions, we have the duality relation

$$\mathcal{P} = -e^{-\gamma x - \gamma \mathcal{D}},$$

where x is the initial wealth and $\mathcal{D}$ is the value of the dual optimization problem:

$$\mathcal{D} = \inf_{Q \in \mathbb{P}_f}\left(I\!\!E^Q\{G\} + \frac{1}{\gamma}\mathcal{H}(Q|I\!\!P)\right). \tag{14}$$

Here $\mathcal{H}(Q|I\!\!P)$ is the relative entropy between Q and $I\!\!P$, namely,

$$\mathcal{H}(Q|I\!\!P) = \begin{cases} I\!\!E\left\{\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right\}, & Q \ll I\!\!P, \\ \infty, & \text{otherwise.} \end{cases} \tag{15}$$

In (14), $\mathbb{P}_f$ denotes the set of absolutely continuous local martingale measures with finite relative entropy with respect to $I\!\!P$. We refer the reader to [11, 19] for full details.

We now derive the related HJB equation for the dual problem. This approach is taken in [3]. Under $Q \in \mathbb{P}_f$, the stock price S is a local martingale, but the intensity-driving process Y need not be. Under mild regularity conditions, the measure change from $I\!\!P$ to Q is parametrized by a pair of adapted processes, ψ_t and $\phi_t \geq 0$, with

$$I\!\!E^Q\left\{\int_0^T \psi_t^2\,dt\right\} < \infty, \quad \text{and} \quad \int_0^T \phi_t \lambda(Y_t)\,dt < \infty \text{ a.s.},$$

such that

$$\begin{aligned} dS_t &= \sigma S_t\, dW_t^{Q(1)} \\ dY_t &= \left(b(Y_t) - \frac{\rho\mu}{\sigma}a(Y_t) - \psi_t \rho' a(Y_t)\right)\, dt + a(Y_t)\, \left(\rho\, dW_t^{Q(1)} + \rho'\, dW_t^{Q(2)}\right), \end{aligned}$$

and the intensity is

$$\lambda_t^Q = \phi_t \lambda(Y_t).$$

Here, $W^{Q(1)}$ and $W^{Q(2)}$ are independent $Q-$Brownian motions, and $\rho' = \sqrt{1-\rho^2}$. The control ψ can be interpreted as a risk premium for the non-traded component of Y, while ϕ affects directly the stochastic intensity. The Radon-Nikodym derivative is given by

$$\begin{aligned} \log \frac{dQ}{d\mathbb{P}} &= -\frac{1}{2}\int_0^{\tau\wedge T} \left(\frac{\mu^2}{\sigma^2} + \psi_t^2\right)\, dt - \int_0^{\tau\wedge T} \frac{\mu}{\sigma}\, dW_t^{(1)} - \int_0^{\tau\wedge T} \psi_t\, dW_t^{(2)} \\ &\quad + \int_0^{\tau\wedge T} (1-\phi_t)\lambda(Y_t)\, dt + \log \phi_\tau \mathbf{1}_{\{\tau<T\}}. \end{aligned}$$

See, for example, [12, Appendix E]. Therefore, we have the expression

$$\mathcal{H}(Q|\mathbb{P}) = \mathbb{E}^Q\left\{\int_0^{\tau\wedge T} \left(\frac{\mu^2}{2\sigma^2} + \frac{1}{2}\psi_t^2 + (1-\phi_t)\lambda(Y_t)\right)\, dt + \log\phi_\tau \mathbf{1}_{\{\tau<T\}}\right\}.$$

In passing to the associated HJB equation, we use the fact that if ϕ is bounded and adapted to the filtration generated by the two Brownian motions, then τ retains the so-called "doubly stochastic" property under Q. This means that, conditioned on the path of Y, the distribution of τ under Q is given by

$$\mathbb{P}^Q\left\{\tau > t \mid (Y_s)_{0\le s\le t}\right\} = e^{-\int_0^t \phi_s \lambda(Y_s)\, ds}.$$

We will hereon assume G is bounded and of European-type, in the sense that $G = G(Y_T)$ (note that the payoff does not depend on the stock). In the defaultable bond case of interest, G is a constant.

We are thus led to define the stochastic optimization problem

$$J(t,y) = \inf_{\psi;\phi\ge 0} \mathbb{E}_{t,y}^Q\left\{\gamma \mathcal{E}_{t,T} G + \int_t^T \left(\frac{\mu^2}{2\sigma^2} + \frac{1}{2}\psi_s^2 + (1-\phi_s)\lambda(Y_s) + \phi_s\lambda(Y_s)\log\phi_s\right)\mathcal{E}_{t,s}\, ds\right\},$$

where

$$\mathcal{E}_{t,s} = e^{-\int_t^s \phi_u \lambda(Y_u)\, du}.$$

The associated HJB equation is

$$\begin{aligned} J_t + \widetilde{\mathcal{L}}_y J + \frac{\mu^2}{2\sigma^2} + \lambda(y) + \inf_\psi \left(\frac{1}{2}\psi^2 - \psi\rho' a(y) J_y\right) & \\ + \inf_{\phi\ge 0}\left(\phi\lambda(y)\log\phi - (J+1)\phi\lambda(y)\right) &= 0, \end{aligned}$$

with $J(T,y) = \gamma G(y)$, and $\widetilde{\mathcal{L}}$ as in (10). In turn, the optimizing ϕ is given by $\phi^* = e^J$, so we have

$$J_t + \tilde{\mathcal{L}}_y J + \frac{\mu^2}{2\sigma^2} - \frac{1}{2}(1-\rho^2)a(y)^2 J_y^2 + \lambda(y)(1-e^J) = 0.$$

Finally, setting $G = c = e^{-rT}$ and making the transformation

$$J = \gamma c - \frac{1}{(1-\rho^2)} \log w$$

recovers the reaction-diffusion equation (11).

2.4 Intensity Bounds

We next investigate the behaviour of the prices with respect to the intensity process. Specifically, we assume that, for $y \in I\!R$,

$$0 < \underline{\lambda} \le \lambda(y) \le \bar{\lambda} < \infty. \tag{16}$$

Proposition 5 *Let*

$$\bar{\alpha} = \frac{\mu^2}{2\sigma^2} + \bar{\lambda}, \quad \textit{and} \quad \underline{\alpha} = \frac{\mu^2}{2\sigma^2} + \underline{\lambda}.$$

Then, under assumption (16), the value functions M and H satisfy for $x, y \in I\!R$,

$$(-e^{-\gamma x}) \left[\left(1 - \frac{\bar{\lambda}}{\bar{\alpha}}\right) e^{-\bar{\alpha}(T-t)} + \frac{\bar{\lambda}}{\bar{\alpha}} \right] \le M(t,x,y) \le (-e^{-\gamma x}) \left[\left(1 - \frac{\underline{\lambda}}{\underline{\alpha}}\right) e^{-\underline{\alpha}(T-t)} + \frac{\underline{\lambda}}{\underline{\alpha}} \right], \tag{17}$$

and

$$\begin{aligned}(-e^{-\gamma(x+c)}) \left[\left(1 - \frac{\bar{\lambda} e^{\gamma c}}{\bar{\alpha}}\right) e^{-\bar{\alpha}(T-t)} + \frac{\bar{\lambda} e^{\gamma c}}{\bar{\alpha}} \right] &\le H(t,x,y) \\ &\le (-e^{-\gamma(x+c)}) \left[\left(1 - \frac{\underline{\lambda} e^{\gamma c}}{\underline{\alpha}}\right) e^{-\underline{\alpha}(T-t)} + \frac{\underline{\lambda} e^{\gamma c}}{\underline{\alpha}} \right].\end{aligned} \tag{18}$$

Proof: To show (17), we introduce the function

$$\bar{M}(t,x,y) = (-e^{-\gamma x}) \left[\left(1 - \frac{\bar{\lambda}}{\bar{\alpha}}\right) e^{-\bar{\alpha}(T-t)} + \frac{\bar{\lambda}}{\bar{\alpha}} \right].$$

Direct calculations show that, for $x \in I\!R, t \in [0,T]$,

$$\bar{M}(t,x,y) \ge -e^{-\gamma x},$$

and that

$$\bar{M}_t + \mathcal{L}_y \bar{M} + \max_{\pi} \left(\frac{1}{2}\sigma^2\pi^2 \bar{M}_{xx} + \pi(\rho\sigma a(y)\bar{M}_{xy} + \mu \bar{M}_x) \right) + \lambda(y)(-e^{-\gamma x} - \bar{M}) \ge 0.$$

Moreover, $\bar{M}(T,x,y) = -e^{-\gamma x}$. We easily conclude using the comparison principle. The other bounds are obtained similarly. □

Proposition 6 *The indifference price satisfies*

$$e^{-rT} - \frac{1}{\gamma} \log \left(\frac{\left(1 - \frac{\bar{\lambda} e^{\gamma c}}{\bar{\alpha}}\right) e^{-\bar{\alpha}(T-t)} + \frac{\bar{\lambda} e^{\gamma c}}{\bar{\alpha}}}{\left(1 - \frac{\underline{\lambda}}{\underline{\alpha}}\right) e^{-\underline{\alpha}(T-t)} + \frac{\underline{\lambda}}{\underline{\alpha}}} \right) \le p_0 \le e^{-rT} - \frac{1}{\gamma} \log \left(\frac{\left(1 - \frac{\underline{\lambda} e^{\gamma c}}{\underline{\alpha}}\right) e^{-\underline{\alpha}(T-t)} + \frac{\underline{\lambda} e^{\gamma c}}{\underline{\alpha}}}{\left(1 - \frac{\bar{\lambda}}{\bar{\alpha}}\right) e^{-\bar{\alpha}(T-t)} + \frac{\bar{\lambda}}{\bar{\alpha}}} \right).$$

Proof: The assertion follows from the definition of the indifference price and the inequalities (17) and (18). □

2.5 Constant Intensity Case

We study explicitly the case of constant intensity, when the default time τ is independent of the level of the firm's stock price S, and is simply an exponential random variable with parameter λ . This simplified structure will employed in the multi-name models that we analyze for CDO valuation in [28].

Proposition 7 *When λ is constant, the indifference price $p_0(T)$ (at time zero) of the defaultable bond expiring on date T is given by*

$$p_0(T) = e^{-rT} - \frac{1}{\gamma}\log\left(\frac{e^{-\alpha T} + \frac{\lambda}{\alpha}e^{\gamma c}\left(1-e^{-\alpha T}\right)}{e^{-\alpha T} + \frac{\lambda}{\alpha}\left(1-e^{-\alpha T}\right)}\right), \tag{19}$$

where

$$\alpha = \frac{\mu^2}{2\sigma^2} + \lambda.$$

PROOF: We construct the explicit solutions of the HJB equations solved by the two value functions M and H. When λ is constant, the value functions M and H do not depend on y, and the HJB equation (2) reduces to

$$M_t - \frac{\mu^2}{2\sigma^2}\frac{M_x^2}{M_{xx}} + \lambda(-e^{-\gamma x} - M) = 0, \tag{20}$$

with $M(T,x) = -e^{-\gamma x}$. Substituting $M(t,x) = -e^{-\gamma x}m(t)$, we obtain $m' - \alpha m + \lambda = 0$, with $m(T) = 1$, and α as above. The unique solution is

$$m(t) = e^{-\alpha(T-t)} + \frac{\lambda}{\alpha}\left(1 - e^{-\alpha(T-t)}\right).$$

Similarly, the defaultable bond holder's value function $H(t,x)$ satisfies the same equation as M, but with terminal condition $H(T,x) = -e^{-\gamma(x+c)}$. Substituting $H(t,x) = -e^{-\gamma(x+c)}h(t)$, we obtain $h' - \alpha h + \lambda e^{\gamma c} = 0$, with $h(T) = 1$. The unique solution is

$$h(t) = e^{-\alpha(T-t)} + \frac{\lambda e^{\gamma c}}{\alpha}\left(1 - e^{-\alpha(T-t)}\right).$$

We easily deduce that the indifference price of the defaultable bond at time zero is given by

$$p_0(T) = e^{-rT} - \frac{1}{\gamma}\log\left(\frac{h(0)}{m(0)}\right),$$

leading to formula (19). □

Remark 3 *The seller's indifference price is given by*

$$\tilde{p}_0(T) = e^{-rT} + \frac{1}{\gamma}\log\left(\frac{e^{-\alpha T} + \frac{\lambda}{\alpha}e^{-\gamma c}\left(1-e^{-\alpha T}\right)}{e^{-\alpha T} + \frac{\lambda}{\alpha}\left(1-e^{-\alpha T}\right)}\right).$$

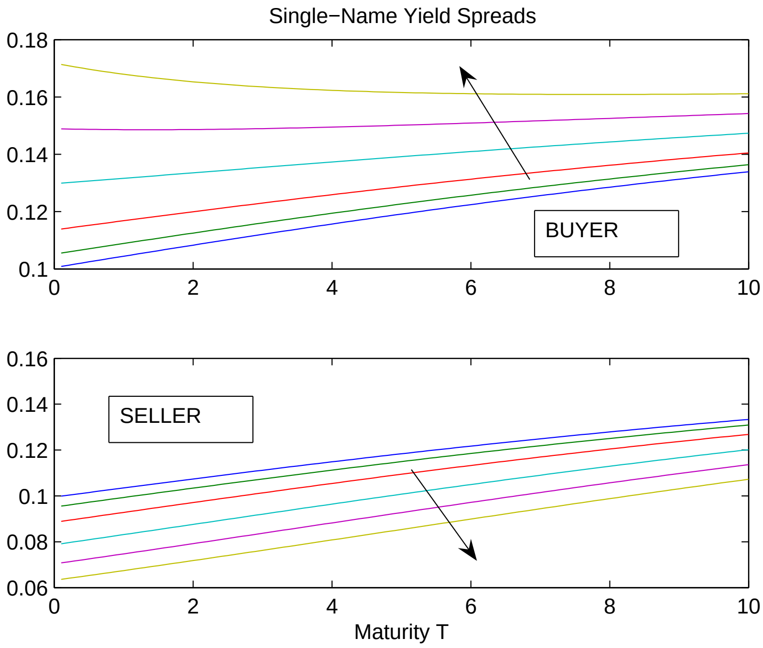

Figure 1: *Single name buyer's and seller's indifference yield spreads. The parameters are* $\lambda = 0.1$, *along with* $\mu = 0.09, r = 0.03$ *and* $\sigma = 0.15$. *The curves correspond to different risk aversion parameters* γ *and the arrows show the direction of increasing* γ *over the values* $(0.01, 0.1, 0.25, 0.5, 0.75, 1)$.

A plot of the yield spreads $\mathcal{Y}_0(T) = -\frac{1}{T}\log(p_0(T)/e^{-rT})$ for the buyer, and similarly $\tilde{\mathcal{Y}}_0(T)$ for the seller, for various risk aversion coefficients, is shown in Figure 1. Observe that both spread curves are, in general, sloping, so the spreads are not flat even though we started with a constant intensity model. While the seller's curve is upward sloping, the buyer's may become downward sloping when the risk aversion is large enough. The short term limit of the yield spread is nonzero, as we would expect in the presence of non-predictable defaults. For the buyer's yield spread, we have

$$\lim_{T\downarrow 0} \mathcal{Y}_0(T) = \frac{(e^{\gamma} - 1)}{\gamma}\lambda,$$

which is larger than λ since $\gamma > 0$. This is amplified as γ becomes larger. In other words, the buyer values the claim *as though* the intensity were larger than the historically estimated value λ. The seller, on the other hand, values short-term claims as though the intensity were lower, since

$$\lim_{T\downarrow 0} \tilde{\mathcal{Y}}_0(T) = \frac{(1 - e^{-\gamma})}{\gamma}\lambda \leq \lambda.$$

The long time limit for both buyer's and seller's spread is simply α,

$$\lim_{T\to\infty} \mathcal{Y}_0(T) = \lim_{T\to\infty} \tilde{\mathcal{Y}}_0(T) = \frac{\mu^2}{2\sigma^2} + \lambda,$$

which is always larger than λ. Both long-term yield spreads converge to the intensity plus a term proportional to the square of the Sharpe ratio of the firm's stock.

3 A Two-Name Credit Derivative

To illustrate how the nonlinearity of the utility indifference valuation mechanism affects basket or multi-name claims, we look at the case of two ($N = 2$) firms. The more realistic application to CDOs, where N might be on the order of a hundred, is studied in [28]. As with other approaches to these problems, particularly copula models, it becomes necessary to make substantial simplifications, typically involving some sort of symmetry assumption, in order to be able to handle the high-dimensional computational challenge. We will assume throughout this section that intensities are constant or, equivalently, that the default times of the firms are independent.

The firms' stock prices processes $(S^{(i)})$ follow geometric Brownian motions:

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (r + \mu_i)\, dt + \sigma_i\, dW_t^{(i)}, \quad i = 1, 2,$$

where $(W^{(i)})$ are Brownian motions with instantaneous correlation coefficient $\rho \in (-1, 1)$, and the volatilities $\sigma_i > 0$. The two firms have independent exponentially distributed default times τ_1 and τ_2, with intensities λ_1 and λ_2 respectively. We let $\tau = \min(\tau_1, \tau_2)$, and value a claim which pays \$1 if both firms survive till time T, and zero otherwise.

While both firms are alive, investors can trade the two stocks and the risk-free money market. When one defaults, its stock can no longer be traded, and if the second also subsequently defaults, the portfolio is invested entirely in the bank account. As in the single-name case, we work with discounted wealth X, and μ_i denotes the excess growth rate of the ith stock. The control processes $\pi^{(i)}$ are the dollar amount held in each stock, and the discounted wealth process X evolves according to

$$dX_t = \sum_{i=1}^{2} \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \mu_i\, dt + \sum_{i=1}^{2} \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \sigma_i\, dW_t^{(i)}.$$

When both firms are alive, the investor's objective is to maximize her expected utility from terminal wealth,

$$\sup_{\pi^{(1)}, \pi^{(2)}} I\!E \left\{ -e^{-\gamma X_T} \right\}.$$

To solve the problem, we need to recursively deal with the cases when there are no firms left, when there is one firm left and, finally, when both firms are present. Let $M^{(j)}(t, x)$ denote the value function of the investor who starts at time $t \le T$, with wealth x, when there are $j \in \{0, 1, 2\}$ firms available to invest in. In the case $j = 1$, we denote by $M_i^{(j)}(t, x)$ the sub-cases when it is the firm $i \in \{1, 2\}$ that is alive.

When there are no firms left, we have the value function

$$M^{(0)}(t, x) = -e^{-\gamma x}. \tag{21}$$

When only firm i is alive, we have the single-name value functions computed in the proof of Proposition 7, namely

$$M_i^{(1)}(t, x) = -e^{-\gamma x} v_i^{(1)}(t),$$

where

$$v_i^{(1)}(t) = e^{-\alpha_i(T-t)} + \frac{\lambda_i}{\alpha_i}\left(1 - e^{-\alpha_i(T-t)}\right), \tag{22}$$

and

$$\alpha_i = \frac{\mu_i^2}{2\sigma_i^2} + \lambda_i.$$

When both firms are alive, the value function $M^{(2)}(t,x)$ solves

$$M_t^{(2)} - \frac{1}{2}D_2\frac{(M_x^{(2)})^2}{M_{xx}^{(2)}} + \sum_{i=1}^{2}\lambda_i(M_i^{(1)} - M^{(2)}) = 0, \tag{23}$$

with $M^{(2)}(T,x) = -e^{-\gamma x}$. Here, the *diversity coefficient* D_2 is given by

$$D_2 = \mu^T A^{-1}\mu,$$

where

$$\mu = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}.$$

Evaluating D_2 yields

$$D_2 = \frac{\mu_1^2}{\sigma_1^2} + \frac{1}{(1-\rho^2)}\left(\rho\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\right)^2. \tag{24}$$

Remark 4 *In the single-name case (with constant intensity), the analog of (23) is (20). It is clear that the analog of D_2 would be $D_1 = \mu_1^2/\sigma_1^2$ (if it is firm 1, for example, in question), the square of the Sharpe ratio of the stock $S^{(1)}$. The formula (24) implies $D_2 \geq \mu_1^2/\sigma_1^2$, and, by interchanging subscripts, $D_2 \geq \mu_2^2/\sigma_2^2$. Therefore, it is natural to think of D_2 as a measure of the improved investment opportunity set offered by the diversity of having two stocks to invest in. This idea may be naturally extended to $N > 2$ dimensions (see [28]).*

Proposition 8 *The value function $M^{(2)}(t,x)$, solving (23), is given by*

$$M^{(2)}(t,x) = -e^{-\gamma x}v^{(2)}(t), \tag{25}$$

where

$$v^{(2)}(t) = e^{-\alpha_{1,2}(T-t)} + \sum_{i=1}^{2}\lambda_i\left[\left(1 - \frac{\lambda_1}{\alpha_1}\right)\frac{1}{(\alpha_{1,2}-\alpha_i)}\left(e^{-\alpha_i(T-t)} - e^{-\alpha_{1,2}(T-t)}\right) + \frac{\lambda_i}{\alpha_i\alpha_{1,2}}\left(1 - e^{-\alpha_{1,2}(T-t)}\right)\right], \tag{26}$$

and $\alpha_{1,2} = D_2 + \lambda_1 + \lambda_2$.

Proof: Inserting (25) into (23) gives the following ODE for $v^{(2)}(t)$:

$$\frac{d}{dt}v^{(2)} - \alpha_{1,2}v + \sum_{i=1}^{2}\lambda_i v_i^{(1)}(t) = 0,$$

with $v^{(2)}(T) = 1$. Using the formula (22) for $v_i^{(1)}$ and solving the ODE leads to (26). $\square$

We next consider the investment problem for the holder of the basket claim that pays \$1 if both firms survive up to time T. With $c = e^{-rT}$ as before, the value function $H^{(2)}(t, x)$ for the claim holder, starting with wealth x at time $t \le T$ when both firms are still alive, solves

$$H_t^{(2)} - \frac{1}{2} D_2 \frac{(H_x^{(2)})^2}{H_{xx}^{(2)}} + \sum_{i=1}^{2} \lambda_i (M_i^{(1)} - H^{(2)}) = 0, \tag{27}$$

with $H^{(2)}(T, x) = -e^{-\gamma(x+c)}$. Notice that in the case of this simple claim, we do not have to consider separately the case of one or no firm left because the claim pays nothing in these cases. Once one firm defaults, the bond holder's problem reduces to the previous case of no claim. In general, however, for a more complicated claim, there will be a chain of value functions $H^{(j)}$.

Working as above, we can show the following.

Proposition 9 *The value function $H^{(2)}(t, x)$, solution of (27), is given by*

$$H^{(2)}(t, x) = -e^{-\gamma(x+c)} w^{(2)}(t), \tag{28}$$

where

$$w^{(2)}(t) = e^{-\alpha_{1,2}(T-t)} + \sum_{i=1}^{2} \lambda_i e^{\gamma c} \left[\left(1 - \frac{\lambda_1}{\alpha_1}\right) \frac{1}{(\alpha_{1,2} - \alpha_i)} \left(e^{-\alpha_i (T-t)} - e^{-\alpha_{1,2}(T-t)}\right) \right. \tag{29}$$
$$\left. + \frac{\lambda_i}{\alpha_i \alpha_{1,2}} \left(1 - e^{-\alpha_{1,2}(T-t)}\right) \right].$$

Finally, the buyer's indifference price, at time zero, of the claim with maturity T is given by

$$p_0(T) = c + \frac{1}{\gamma} \log \left(\frac{v^{(2)}(0)}{w^{(2)}(0)} \right).$$

Next, we collect the analogous formulas for the seller of the claim, which are found by straightforward calculations, in the following proposition.

Proposition 10 *The value function of the seller is given by $\tilde{H}^{(2)}(t, x) = -e^{-\gamma(x-c)} \tilde{w}^{(2)}(t)$, where*

$$\tilde{w}^{(2)}(t) = e^{-\alpha_{1,2}(T-t)} + \sum_{i=1}^{2} \lambda_i e^{-\gamma c} \left[\left(1 - \frac{\lambda_1}{\alpha_1}\right) \frac{1}{(\alpha_{1,2} - \alpha_i)} \left(e^{-\alpha_i (T-t)} - e^{-\alpha_{1,2}(T-t)}\right) \right.$$
$$\left. + \frac{\lambda_i}{\alpha_i \alpha_{1,2}} \left(1 - e^{-\alpha_{1,2}(T-t)}\right) \right].$$

The seller's indifference price of the claim with maturity T at time zero is given by

$$p_0(T) = c - \frac{1}{\gamma} \log \left(\frac{v^{(2)}(0)}{\tilde{w}^{(2)}(0)} \right).$$

A plot of the yield spreads,

$$\mathcal{Y}_0(T) = -\frac{1}{T}\log(p_0(T)/e^{-rT})$$

for the buyer, and similarly for the seller, for various risk aversion coefficients, is shown in Figure 2. As in the single-name case, both spread curves are, in general, sloping, so the spreads are not flat even though we started with a constant intensity model. While the seller's curve is upward sloping, the buyer's may become downward sloping when the risk aversion is large enough. The long-term limit of both buyer's and seller's yield spread is

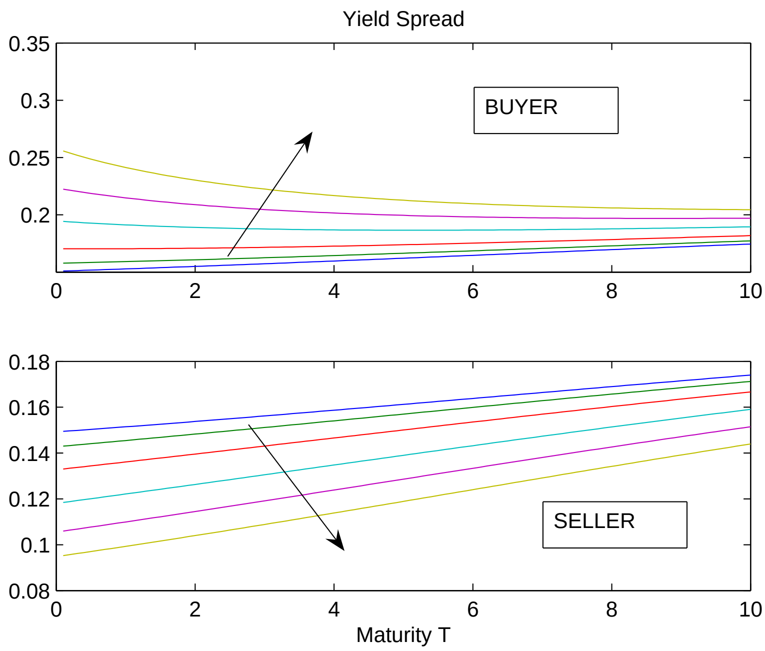

Figure 2: *Buyer's and seller's indifference yield spreads for two-name survival claim. The curves correspond to different risk aversion parameters* γ *and the arrows show the direction of increasing* γ *over the values* $(0.01, 0.1, 0.25, 0.5, 0.75, 1)$. *The other parameters are: excess growth rates* $(0.04, 0.06)$*; volatilities* $(0.2, 0.15)$*; correlation* $\rho = 0.2$*; intensities* $(0.05, 0.1)$.

simply $\alpha_{1,2}$:

$$\lim_{T\to\infty} \mathcal{Y}_0(T) = D_2 + \lambda_1 + \lambda_2,$$

which dominates the actual joint survival probability's hazard rate $\lambda_1 + \lambda_2$.

Another way to express the *correlating effecting* of utility indifference valuation is through the linear correlation coefficient. Let $p_1(T)$ and $p_2(T)$ denote the indifference prices of the (single-name) defaultable bonds for firms 1 and 2 respectively, computed as in Section 2.5. Let $p_{12}(T)$ denote the value of the two-name survival claim, as in Proposition 9. Then we define the linear correlation coefficient

$$\varrho(T) = \frac{p_{12}(T) - p_1(T)p_2(T)}{\sqrt{p_1(T)(1-p_1(T))p_2(T)(1-p_2(T))}}.$$

This is plotted for different maturities, risk aversions, and for buyer and seller in Figure 3. We

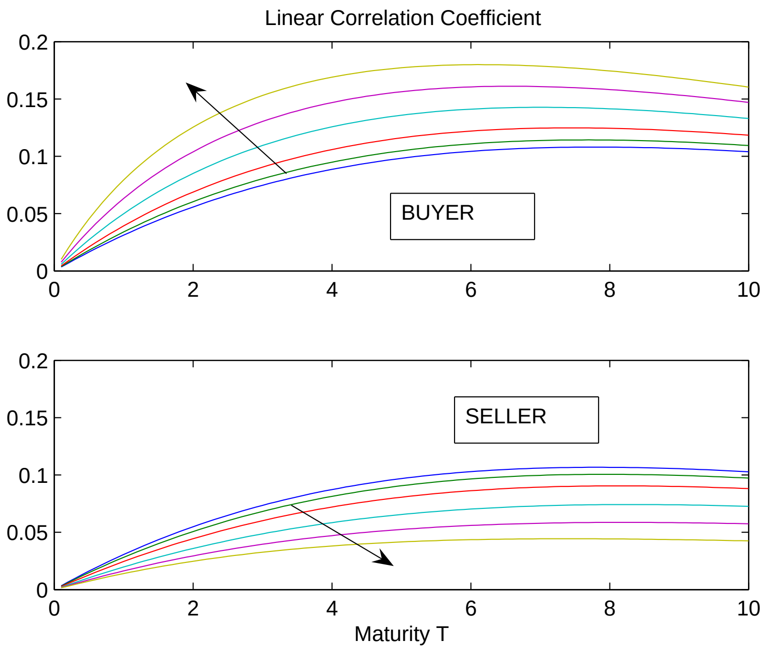

Figure 3: *Linear correlation coefficient* $\varrho(T)$ *from buyer's and seller's indifference values for single-name and two-name survival claims. The curves correspond to different risk aversion parameters* γ *and the arrows show the direction of increasing* γ *over the values* $(0.01, 0.1, 0.25, 0.5, 0.75, 1)$. *The other parameters are as in Figure 2.*

observe that the correlating effect is enhanced by the more risk averse buyer, and reduced by the risk averse seller. For both, the effect increases over short to medium maturities, before plateauing, or dropping off slightly.

4 Conclusions

The preceding analysis demonstrates that utility valuation produces non-trivial yield spreads and effective correlations within even the simplest of intensity-based models of default. They are able to incorporate equity market information (growth rates, volatilities of the non-defaulted firms) as well as investor risk aversion to provide a relative value mechanism for credit derivatives.

Here we have studied single-name defaultable bonds, whose valuation under a stochastic (diffusion) intensity process leads naturally to the study of reaction-diffusion equations. However, even with constant intensity, the yield spreads due to risk aversion are striking. The subsequent analysis of the simple two-name claim demonstrates how nonlinear pricing can be interpreted as high effective correlation. The impact on more realistic multi-name basket derivatives, such as CDOs, is investigated in detail in [28].

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