

Credit Derivatives and Risk Aversion

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Abstract

We discuss the valuation of credit derivatives in extreme regimes such as when the time-to-maturity is short, or when payoff is contingent upon a large number of defaults, as with senior tranches of collateralized debt obligations. In these cases, risk aversion may play an important role, especially when there is little liquidity, and utility indifference valuation may apply. Specifically, we analyze how short-term yield spreads from defaultable bonds in a structural model may be raised due to investor risk aversion.

1 Introduction

The recent turbulence in the credit markets, largely due to overly optimistic valuations of complex credit derivatives by major financial institutions, highlights the need for an alternative pricing mechanism in which risk aversion is explicitly incorporated, especially in such an arena where liquidity is sporadic and has tended to dry up. A number of observations suggest that utility-based valuation may capture better than the traditional risk-neutral (expectation) valuation some common market phenomena:

- Short-term yield spreads from *single-name* credit derivative prices decay slowly and seem to approach a non-zero limit, suggesting significant anticipation (or phobia) of credit shocks over short horizons.
- Among *multi-name* products, the premia paid for senior CDO tranches have often been on the order of a dozen or so basis points (for CDX tranches, for example), ascribing quite a

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large return for providing protection against the risk of default of 15 – 30% investment grade companies over a few years.

Moreover, prevailing basement-level prices of credit-associated products in the absence of confidence suggest that risk averse quantification might currently be better used for securities where hitherto there had been better liquidity.

It is clear, also, that rating agencies, perhaps willingly neglectful, have severely underestimated the combined risk of basket credit derivatives, especially those backed by subprime mortgages. In a front page article about the recent losses of over \$8 billion by Merrill Lynch, the Wall Street Journal (on 25 October, 2007) reported: “More than 70% of the securities issued by each CDO bore triple-A credit ratings. ... But by mid-2006, few bond insurers were willing to write protection on CDOs that were ultimately backed by subprime mortgages ... Merrill put large amounts of AAA-rated CDOs onto its own balance sheet, thinking they were low-risk assets because of their top credit ratings. Many of those assets dived in value this summer.”

In this article, we focus on the first bullet point above to address whether utility valuation can improve structural models to better reproduce observed short-term yield spreads. While practitioners have long since migrated to intensity-based models where the arrival of default risk inherently comes as a surprise, hence leading to non-zero spreads in the limit of zero maturity, there has been interest in the past in adapting economically preferable structural models towards the same effect. Some examples include the introduction of jumps, [3, 11, 21], stochastic interest rates [13], imperfect information on the firm’s asset value [4], uncertainty in the default threshold [9] and fast mean-reverting stochastic volatility [6]. In related work, utility-based valuation has been applied within the framework of intensity-based models for both single-name derivatives [1, 18, 20] and, in addressing the second bullet point, for multi-name products [19].

In a complete market setting, the payoffs of any financial claims can be replicated by trading the underlying securities, and their prices are equal to the value of the associated hedging portfolios. However, in market environments with credit risks, the risks associated with defaults may not be completely eliminated. For instance, if the default of a firm is triggered by the firm’s asset value falling below a certain level, then perfect replication for defaultable securities issued by the firm requires that the firm’s asset value be liquidly traded. While the firm’s stock is tradable, its asset value is not, and hence the market completeness assumption breaks down. The buyer or seller of the firm’s defaultable securities takes on some unhedgeable risk that needs to be quantified in order to value the security. In the Black-Cox [2] structural model, the stock price is taken as proxy for the firm’s asset value (see [10] for a survey), but we will focus on the effect of the incomplete information provided by only being able to trade the firm’s stock, which is imperfectly correlated with its asset value.

We will apply the technology of utility-indifference valuation for defaultable bonds in a structural model of Black-Cox-type. The valuation mechanism incorporates the bond holder’s (or seller’s) risk aversion, and accounts for investment opportunities in the firm’s stock to optimally hedge default risk. These features have a significant impact on the bond prices and yield spreads

(see Figures 1 and 2).

2 Indifference Valuation for Defaultable Bonds

We consider the valuation of a defaultable bond in a structural model with diffusion dynamics. The firm's creditors hold a bond promising payment of \$1 on expiration date T , unless the firm defaults. In the Merton model [15], default occurs if the firm's asset value on date T is below a pre-specified debt level D . In the Black and Cox generalization [2], the firm defaults the first time the underlying asset hits the lower boundary

$$\tilde{D}(t) = De^{-\beta(T-t)}, \quad t \in [0, T],$$

which represents the threshold at which bond safety covenants cause a default. In other words, the bond becomes worthless if the value of the asset *ever* falls below \tilde{D} before expiration date T .

Let Y_t be the firm's asset value at time t , which we take to be observable. Then, the firm's default is signaled by Y_t hitting the level $\tilde{D}(t)$. The firm's stock price (S_t) follows a geometric Brownian motion, and the firm's asset value is taken to be a correlated diffusion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^1, \tag{2.1}$$

$$dY_t = \nu Y_t dt + \eta Y_t (\rho dW_t^1 + \rho' dW_t^2). \tag{2.2}$$

The processes W^1 and W^2 are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, where $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the augmented filtration generated by these two processes. The instantaneous correlation coefficient $\rho \in (-1, 1)$ measures how closely changes in stock prices follow changes in asset values and we define $\rho' := \sqrt{1 - \rho^2}$. It is easy to accommodate firms that pay continuous dividends, but for simplicity, we do not pursue this here.

2.1 Maximal Expected Utility Problem

We assume that the holder of the defaultable bond dynamically invests in a riskless bank account which pays interest at constant rate r , and the company stock. Note that the firm's asset value Y is not market-traded. The holder can partially hedge against his position by trading in the company stock S , but not the firm's asset value Y . The investor's trading horizon $T < \infty$ is chosen to coincide with the expiration date of the derivative contracts of interest. Fixing the current time $t \in [0, T)$, a trading strategy $\{\theta_u; t \leq u \leq T\}$ is the cash amount invested in the market index S , and it is deemed admissible if it is self-financing, non-anticipating and satisfies the integrability condition $\mathbb{E}\{\int_t^T \theta_u^2 du\} < \infty$. The set of admissible strategies over the period $[t, T]$ is denoted by $\Theta_{t,T}$. The employee's aggregate current wealth X then evolves according to

$$dX_s = [\theta_s(\mu - r) + rX_s] ds + \theta_s \sigma dW_s^1, \quad X_t = x. \tag{2.3}$$

Considering the problem initiated at time $t \in [0, T]$, we define the default time τ_t by

$$\tau_t := \inf\{u \geq t : Y_u \leq \tilde{D}(u)\}.$$

If the default event occurs prior to T , the investor can no longer trade the firm's stock. He has to liquidate holdings in the stock and deposit in the bank account, reducing his investment opportunities. (Throughout, we are neglecting other potential investment opportunities, but a more complex model might include these; in multi-name problems, such as valuation of CDOs, this is particularly important: see [19]). For simplicity, we also assume that he receives full pre-default market value on his stock holdings on liquidation, though one might extend to consider some loss at the default time. Therefore, given $\tau_t < T$, for $t \in (\tau_t, T]$, the investor's wealth grows at rate r :

$$X_t = X_{\tau_t} e^{r(t-\tau_t)}.$$

The investor measures utility (at time T) via the exponential utility function $U : \mathbb{R} \mapsto \mathbb{R}_-$ defined by

$$U(x) = -e^{-\gamma x}, \quad x \in \mathbb{R},$$

where $\gamma > 0$ is the coefficient of absolute risk aversion. The indifference pricing mechanism is based on the comparison of maximal expected utilities from investments with and without the credit derivative. We first look at the optimal investment problem of an investor who dynamically invests in the firm's stock as well as the bank account, and does not hold any derivative. In the absence of the defaultable bond, the investor's value function is given by

$$M(t, x, y) = \sup_{\Theta_{t,T}} \mathbb{E} \left\{ -e^{-\gamma X_T} \mathbf{1}_{\{\tau_t > T\}} + (-e^{-\gamma X_{\tau_t} e^{r(T-\tau_t)}}) \mathbf{1}_{\{\tau_t \leq T\}} \mid X_t = x, Y_t = y \right\}, \quad (2.4)$$

which is defined in the domain $\mathcal{I} = \{(t, x, y) : t \in [0, T], x \in \mathbb{R}, y \in [\tilde{D}(t), +\infty)\}$.

Proposition 2.1 *The value function $M : \mathcal{I} \mapsto \mathbb{R}_-$ is the unique viscosity solution in the class of function that are concave and increasing in x , and uniformly bounded in y of the HJB equation*

$$M_t + \mathcal{L}_y M + rxM_x + \max_{\theta} \left(\frac{1}{2} \sigma^2 \theta^2 M_{xx} + \theta (\rho \sigma \eta M_{xy} + (\mu - r) M_x) \right) = 0, \quad (2.5)$$

where the operator \mathcal{L}_y is defined as

$$\mathcal{L}_y = \frac{1}{2} \eta^2 y^2 \frac{\partial^2}{\partial y^2} + \nu y \frac{\partial}{\partial y}.$$

The boundary conditions are given by

$$M(T, x, y) = -e^{-\gamma x}, \quad M(t, x, D e^{-\beta(T-t)}) = -e^{-\gamma x e^{r(T-t)}}.$$

Proof. The proof follows the arguments in theorem 4.1 of [5], and is omitted. ■

Intuitively, if the firm's current asset value y is very high, then default is highly unlikely, so the investor is likely to be able to invest in the firm's stock S till time T . Indeed, as $y \rightarrow +\infty$, we have $\tau_t \rightarrow +\infty$, and $\mathbf{1}_{\{\tau_t > T\}} = 1$ a.s. Hence, in the limit, the value function becomes that of the standard (default-free) Merton investment problem (see [14]) which has a closed-form solution. Formally,

$$\begin{aligned} \lim_{y \rightarrow +\infty} M(t, x, y) &= \sup_{\Theta_{t,T}} \mathbb{E} \left\{ -e^{-\gamma X_T} \mid X_t = x \right\} \\ &= -e^{-\gamma x e^{r(T-t)}} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \end{aligned} \quad (2.6)$$

2.2 Bond Holder's Problem

We now consider the maximal expected utility problem from the perspective of the holder of a defaultable bond of the firm. Recall that the bond pays \$1 on date T if the firm has survived till then. Hence, the bond holder's value function is given by

$$V(t, x, y) = \sup_{\Theta_{t,T}} \mathbb{E} \left\{ -e^{-\gamma(X_T+1)} \mathbf{1}_{\{\tau_t > T\}} + (-e^{-\gamma X_{\tau_t} e^{r(T-\tau_t)}}) \mathbf{1}_{\{\tau_t \leq T\}} \mid X_t = x, Y_t = y \right\}. \quad (2.7)$$

We have a HJB characterization similar to that in Proposition 2.1.

Proposition 2.2 *The valuation function $V : \mathcal{I} \mapsto \mathbb{R}_-$ is the unique viscosity solution in the class of function that are concave and increasing in x , and uniformly bounded in y of the HJB equation*

$$V_t + \mathcal{L}_y V + rxV_x + \max_{\theta} \left(\frac{1}{2} \sigma^2 \theta^2 V_{xx} + \theta (\rho \sigma \eta V_{xy} + (\mu - r) V_x) \right) = 0, \quad (2.8)$$

with terminal and boundary conditions

$$V(T, x, y) = -e^{-\gamma(x+1)}, \quad V(t, x, D e^{-\beta(T-t)}) = -e^{-\gamma x e^{r(T-t)}}.$$

If the firm's current asset value y is far away from the default level, then it is very likely that the firm will survive through time T , and the investor will collect \$1 at maturity. In other words, as $y \rightarrow +\infty$, the value function (formally) becomes

$$\begin{aligned} \lim_{y \rightarrow +\infty} V(t, x, y) &= \sup_{\Theta_{t,T}} \mathbb{E} \left\{ -e^{-\gamma(X_T+1)} \mid X_t = x \right\} \\ &= -e^{-\gamma(1+x e^{r(T-t)})} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \end{aligned} \quad (2.9)$$

2.3 Indifference Price for the Defaultable Bond

The buyer's indifference price for a derivative is defined as the amount of money that he is willing to pay for holding the derivative. We can derive the bond holder's indifference price via the value functions M and V . Without loss of generality, we compute this price at time zero.

Definition 2.3 *The buyer's indifference price $p_{0,T}(y)$ defaultable bond with expiration date T is defined by*

$$M(0, x, y) = V(0, x - p_{0,T}(y)), \quad (2.10)$$

where M and V are given in (2.4) and (2.7).

It is well-known that the indifference price under exponential utility does not depend on the investor's initial wealth x . This can also be seen from Proposition 3.1 below. When there is no default risk, then the value functions M and V are given by (2.6) and (2.9). From the above definition, we have the indifference price for the default-free bond is e^{-rT} , which is just the present value of the \$1 to be collected at time T , and is independent of the holder's risk aversion and the firm's asset value.

2.4 Solutions for the HJB Equations

The HJB equation (2.5) can be simplified by the familiar distortion scaling

$$M(t, x, y) = -e^{-\gamma x e^{r(T-t)}} u(t, y)^{\frac{1}{1-\rho^2}}. \quad (2.11)$$

The non-negative function u is defined over the domain $\mathcal{J} = \{(t, y) : t \in [0, T], y \in [\tilde{D}(t), +\infty)\}$. It solves the *linear* (Feynman-Kac) differential equation

$$\begin{aligned} u_t + \tilde{\mathcal{L}}_y u - (1 - \rho^2) \frac{(\mu - r)^2}{2\sigma^2} u &= 0, \\ u(T, y) &= 1, \\ u(t, D e^{-\beta(T-t)}) &= 1, \end{aligned} \quad (2.12)$$

where

$$\tilde{\mathcal{L}}_y = \mathcal{L}_y - \rho \frac{(\mu - r)}{\sigma} \eta y \frac{\partial}{\partial y}.$$

This is the infinitesimal generator of Y under the minimal entropy martingale measure, \tilde{P} (defined later in (2.18)).

For the bond holder's value function, the transformation

$$V(t, x, y) = -e^{-\gamma(x e^{r(T-t)} + 1)} w(t, y)^{\frac{1}{1-\rho^2}} \quad (2.13)$$

reduces the HJB equation (2.8) to the linear PDE problem

$$\begin{aligned} w_t + \tilde{\mathcal{L}}_y w - (1 - \rho^2) \frac{(\mu - r)^2}{2\sigma^2} w &= 0, \\ w(T, y) &= 1, \\ w(t, De^{-\beta(T-t)}) &= e^{\gamma(1-\rho^2)}, \end{aligned} \quad (2.14)$$

which differs from (2.12) only by a boundary condition. By classical comparison results (see, for instance, [17]), we have

$$u(t, y) \leq w(t, y), \quad \text{for } (t, y) \in \mathcal{J}. \quad (2.15)$$

Furthermore, u and w admit the Feynman-Kac representations

$$u(t, y) = \tilde{\mathbb{E}} \left\{ e^{-(1-\rho^2)\frac{(\mu-r)^2}{2\sigma^2}(\tau_t \wedge T-t)} \mid Y_t = y \right\}, \quad (2.16)$$

$$w(t, y) = \tilde{\mathbb{E}} \left\{ e^{-(1-\rho^2)\frac{(\mu-r)^2}{2\sigma^2}(T-t)} \mathbf{1}_{\{\tau_t > T\}} + e^{\gamma(1-\rho^2)} e^{-(1-\rho^2)\frac{(\mu-r)^2}{2\sigma^2}(\tau_t-t)} \mathbf{1}_{\{\tau_t \leq T\}} \mid Y_t = y \right\}, \quad (2.17)$$

where the expectations are taken under the measure $\tilde{\mathbb{P}}$ defined by

$$\tilde{\mathbb{P}}(A) = \mathbb{E} \left\{ \exp \left(-\frac{\mu - r}{\sigma} W_T^1 - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2} T \right) \mathbf{1}_A \right\}, \quad A \in \mathcal{F}_T. \quad (2.18)$$

Hence, under $\tilde{\mathbb{P}}$, the firm's stock price is a martingale, and the dynamics of Y are

$$dY_t = \left(\nu - \rho \frac{(\mu - r)}{\sigma} \eta \right) Y_t dt + \eta Y_t d\tilde{W}_t, \quad Y_0 = y,$$

where \tilde{W} is a $\tilde{\mathbb{P}}$ -Brownian motion. The measure $\tilde{\mathbb{P}}$ is the equivalent martingale measure that has the minimal entropy relative to \mathbb{P} (see [8]). This measure arises frequently in indifference pricing theory.

The representations (2.16) and (2.17) are useful in deriving closed-form expressions for the functions $u(t, y)$ and $w(t, y)$. First, we notice that

$$u(t, y) = e^{-(1-\rho^2)\frac{(\mu-r)^2}{2\sigma^2}(T-t)} \tilde{\mathbb{P}} \{ \tau_t > T \mid Y_t = y \} + \tilde{\mathbb{E}} \left\{ e^{-(1-\rho^2)\frac{(\mu-r)^2}{2\sigma^2}(\tau_t-t)} \mathbf{1}_{\{\tau_t \leq T\}} \mid Y_t = y \right\}.$$

Under the measure $\tilde{\mathbb{P}}$, the default time τ_t is given by

$$\tau_t = \inf \left\{ u \geq t : \left(\nu - \rho \frac{(\mu - r)}{\sigma} \eta - \frac{\eta^2}{2} - \beta \right) (u - t) + \eta(\tilde{W}_u - \tilde{W}_t) \leq \log(D/Y_t) - \beta(T - t) \right\}.$$

Then, we explicitly compute the representations using the distribution of τ_t , which is well-known; see for example [12]. Standard yet tedious calculations yield the following expression for $u(t, y)$:

$$u(t, y) = e^{-\alpha(T-t)} \left[\Phi \left(\frac{-b + \psi(T-t)}{\sqrt{T-t}} \right) - e^{2\psi b} \Phi \left(\frac{b + \psi(T-t)}{\sqrt{T-t}} \right) \right] \\ + e^{b(\psi-c)} \left[\Phi \left(\frac{b-c(T-t)}{\sqrt{T-t}} \right) + e^{2bc} \Phi \left(\frac{b+c(T-t)}{\sqrt{T-t}} \right) \right].$$

Here $\Phi(\cdot)$ is the standard normal cumulative distribution function, and

$$\alpha = (1-\rho^2) \frac{(\mu-r)^2}{2\sigma^2}, \quad b = \frac{\log(D/y) - \beta(T-t)}{\eta}, \quad \psi = \frac{\nu - \beta}{\eta} - \rho \left(\frac{\mu-r}{\sigma} \right) - \frac{\eta}{2}, \quad c = \sqrt{\psi^2 + 2\alpha}.$$

A similar formula can be obtained for $w(t, y)$:

$$w(t, y) = e^{-\alpha(T-t)} \left[\Phi \left(\frac{-b + \psi(T-t)}{\sqrt{T-t}} \right) - e^{2\psi b} \Phi \left(\frac{b + \psi(T-t)}{\sqrt{T-t}} \right) \right] \\ + e^{\gamma(1-\rho^2)} e^{b(\psi-c)} \left[\Phi \left(\frac{b-c(T-t)}{\sqrt{T-t}} \right) + e^{2bc} \Phi \left(\frac{b+c(T-t)}{\sqrt{T-t}} \right) \right]. \quad (2.19)$$

3 The Yield Spread

Using (2.11) and (2.13), we can express the indifference price and the yield spread (at time zero), which can be computed using the explicit formulas for $u(0, y)$ and $w(0, y)$ above.

Proposition 3.1 *The indifference price $p_{0,T}(y)$ defined in (2.10) is given by*

$$p_{0,T}(y) = e^{-rT} \left(1 - \frac{1}{\gamma(1-\rho^2)} \log \frac{w(0, y)}{u(0, y)} \right). \quad (3.1)$$

It satisfies $p_{0,T}(y) \leq e^{-rT}$ for every $y \geq De^{-\beta T}$. The yield spread, defined by

$$\mathcal{Y}_{0,T}(y) = -\frac{1}{T} \log(p_{0,T}(y)) - r, \quad (3.2)$$

is non-negative for all $y \geq De^{-\beta T}$ and $T > 0$.

Proof. The fact that $p_{0,T} \leq e^{-rT}$ follows from the inequality $u \leq w$. To show that $\mathcal{Y}_{0,T}$ is well-defined, we need to establish that $p_{0,T} \geq 0$. For this, consider $w^* := e^{-\gamma(1-\rho^2)} w$, and observe that it satisfies the same PDE as u , as well as the same condition on the boundary $\{y = De^{-\beta(T-t)}\}$ and terminal condition $w^*(T, y) = e^{-\gamma(1-\rho^2)} \leq 1$. Therefore $w^* \leq u$ which gives $w \leq e^{\gamma(1-\rho^2)} u$, and the assertion follows. ■

3.1 The Seller's Price and Yield Spread

We can construct the bond seller's value function by replacing $+1$ by -1 in the definition (2.7) of V , and the corresponding transformation (2.13). If we denote the seller's indifference price by $\tilde{p}_{0,T}(y)$, then

$$\tilde{p}_{0,T}(y) = e^{-rT} \left(1 - \frac{1}{\gamma(1-\rho^2)} \log \frac{u(0,y)}{\tilde{w}(0,y)} \right),$$

where \tilde{w} solves

$$\begin{aligned} \tilde{w}_t + \tilde{\mathcal{L}}_y \tilde{w} - (1-\rho^2) \frac{(\mu-r)^2}{2\sigma^2} \tilde{w} &= 0, \\ \tilde{w}(T,y) &= 1, \\ \tilde{w}(t, De^{-\beta(T-t)}) &= e^{-\gamma(1-\rho^2)}. \end{aligned} \tag{3.3}$$

The comparison principle yields

$$u(t,y) \geq \tilde{w}(t,y), \quad \text{for } (t,y) \in \mathcal{J}. \tag{3.4}$$

Therefore, $\tilde{p}_{0,T}(y) \leq e^{-rT}$, and the seller's yield spread, denoted by $\tilde{\mathcal{Y}}_{0,T}(y)$, is also non-negative for all $y \geq De^{-\beta T}$ and $T > 0$, as follows from a similar calculation to that in the proof of Proposition 3.1. We obtain a closed-form expression for \tilde{w} by replacing $e^{\gamma(1-\rho^2)}$ by $e^{-\gamma(1-\rho^2)}$ in (2.19) for w .

3.2 The Term-Structure of the Yield Spread

The yield spread term-structure is a natural way to compare zero-coupon defaultable bonds with different maturities. The plots of the buyer's and seller's yield spreads for various risk aversion coefficients and Sharpe ratios of the firm's stock are shown, respectively, in Figures 1 and 2. While risk aversion induces the bond buyer to demand a higher yield spread, it reduces the spread offered by the seller. On the other hand, a higher Sharpe ratio of the firm's stock, given by $(\mu-r)/\sigma$, entices the investor to invest in the firm's stock, resulting in a higher opportunity cost for holding or selling the defaultable bond. Consequently, both the buyer's and seller's yield spreads increase with the Sharpe ratio.

It can be observed from the formulas for u and w that the yield spread depends on the ratio between the default level and the current asset value, D/y , rather than their absolute levels. As seen in Figure 3, when the firm's asset value gets closer to the default level, not only does the yield spread increase, but the yield curve also exhibits a hump. The peak of the curve moves leftward, corresponding to shorter maturities, as the default-to-asset ratio increases. In these figures, we have taken $\beta = 0$: the curves with $\beta > 0$ are qualitatively the same.

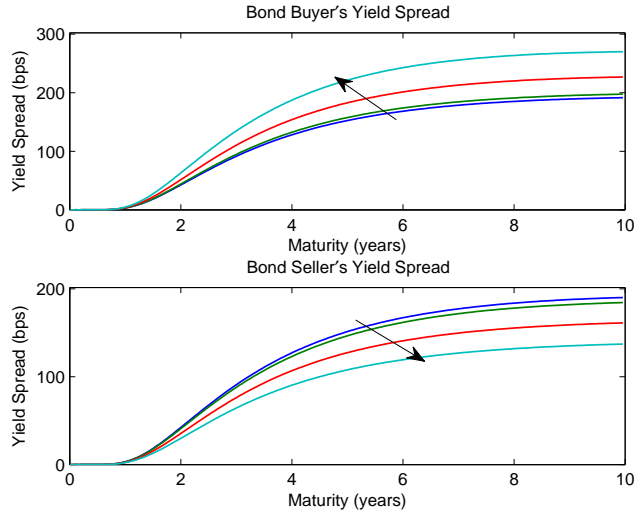


Figure 1: *The defaultable bond buyer's and seller's yield spreads. The parameters are $\nu = 8\%$, $\eta = 20\%$, $r = 3\%$, $\mu = 9\%$, $\sigma = 20\%$, $\rho = 50\%$, $\beta = 0$, along with relative default level $D/y=0.5$. The curves correspond to different risk-aversion parameters γ and the arrows show the direction of increasing γ over the values (0.01, 0.1, 0.5, 1).*

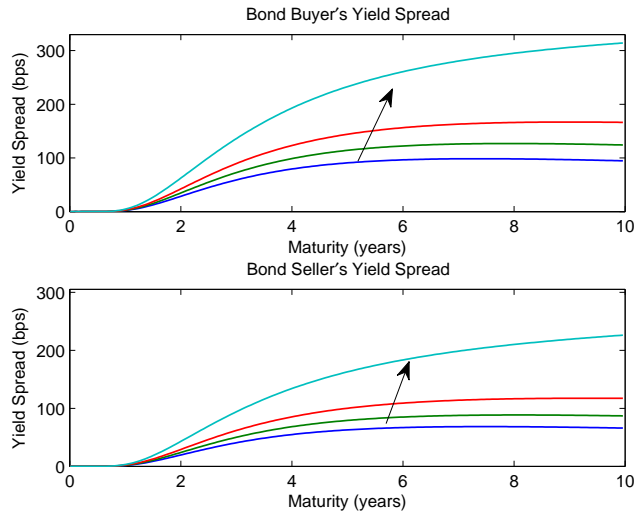


Figure 2: *The defaultable bond buyer's and seller's yield spreads. The parameters are $\nu = 8\%$, $\eta = 20\%$, $r = 3\%$, $\gamma = 0.5$, $\rho = 50\%$, $\beta = 0$, with relative default level $D/y=0.5$. The curves correspond to different sharp ratio parameters $(\mu - r)/\sigma$ and the arrows show the direction of increasing $(\mu - r)/\sigma$ over the values (0, 0.1, 0.2, 0.4).*

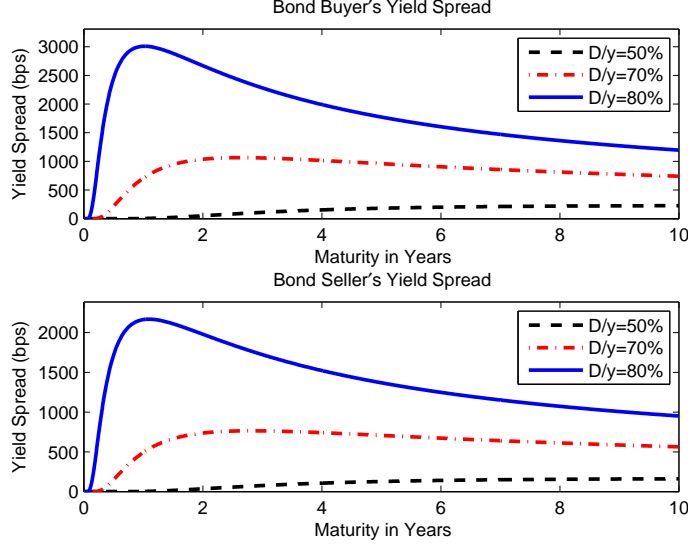


Figure 3: *The defaultable bond buyer's and seller's yield spreads for different default-to-asset ratios (D/y). The parameters are $\nu = 8\%$, $\eta = 20\%$, $r = 3\%$, $\gamma = 0.5$, $\mu = 9\%$, $\sigma = 20\%$, $\rho = 50\%$, $\beta = 0$.*

3.3 Comparison with the Black-Cox Model

We compare our utility-based valuation with the complete markets Black-Cox price. In the Black-Cox setup, the firm's asset value is assumed tradable and evolves according to the following diffusion process under the risk-neutral measure \mathbb{Q} :

$$dY_t = rY_t dt + \eta Y_t dW_t^{\mathbb{Q}}, \quad (3.5)$$

where $W^{\mathbb{Q}}$ is a \mathbb{Q} -Brownian motion. The firm defaults as soon as the asset value Y hits the boundary \tilde{D} . In view of (3.5), the default time is then given by

$$\tau = \inf\{t \geq 0 : (r - \eta^2/2 - \beta)t + \eta W_t^{\mathbb{Q}} = \log(D/y) - \beta T\}.$$

The price of the defaultable bond (at time zero) with maturity T is

$$\begin{aligned} c_{0,T}(y) &= \mathbb{E}^{\mathbb{Q}} \{ e^{-rT} \mathbf{1}_{\{\tau > T\}} \} \\ &= e^{-rT} \mathbb{Q}\{\tau > T\}, \end{aligned}$$

which can be explicitly expressed as

$$c_{0,T}(y) = e^{-rT} \left[\Phi \left(\frac{-b + \phi T}{\sqrt{T}} \right) - e^{2\phi b} \Phi \left(\frac{b + \phi T}{\sqrt{T}} \right) \right],$$

with

$$\phi = \frac{r}{\eta} - \frac{\eta}{2} - \frac{\beta}{\eta}.$$

Of course the defaultable bond price no longer depends on the holder's risk aversion parameter γ , the firm's stock price S , nor the drift of the firm's asset value ν .

Figure 4 shows buyer's and seller's yield spread curves for two different values of ν , and low and moderate risk aversion levels (left and right graphs, respectively). From the bond holder's

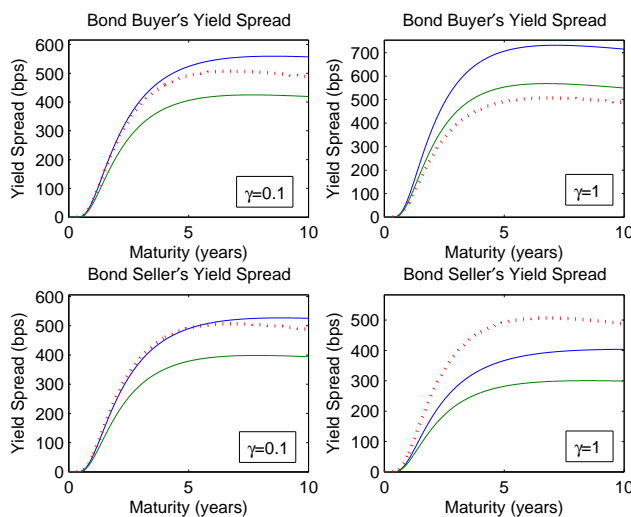


Figure 4: *The defaultable bond buyer's and seller's yield spreads. In every graph, the dotted curve represents the Black-Cox yield curve, and the top and bottom solid curves correspond to the yields from our model with ν being 7% and 9% respectively. Other common parameters are $\eta = 25\%$, $r = 3\%$, $\mu = 9\%$, $\sigma = 20\%$, $\rho = 50\%$, $\beta = 0$, and $D/y=50\%$.*

and seller's perspectives, since defaults are less likely if the firm's asset value has a higher growth rate, the yield spread decreases with respect to ν . Most strikingly, in the top-right graph, with moderate risk aversion, the utility buyer's valuation enhances short term yield spreads compared to the standard Black-Cox valuation. This effect is reversed in the seller's curves (bottom-right). We observe therefore that the risk averse buyer is willing to pay a lower price for short term defaultable bonds, so demanding a higher yield. We highlight this effect in Figure 5 for a more highly distressed firm, and plotted against log maturities.

4 Conclusions

Utility valuation offers an alternative risk aversion based explanation for significant short term yield spreads observed in single-name credit spreads. As in other approaches which modify the

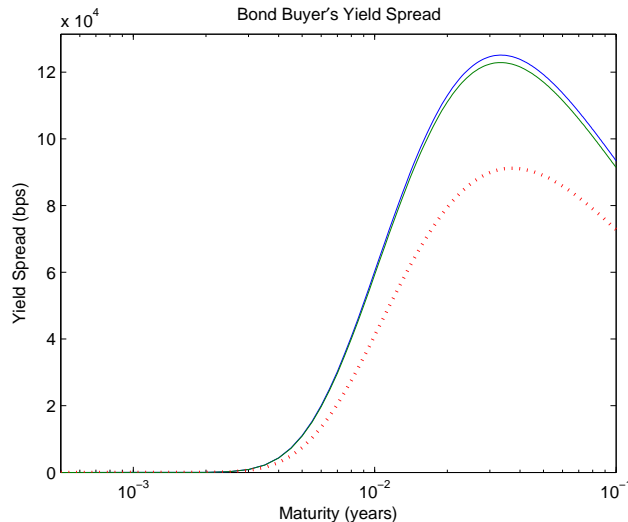


Figure 5: The defaultable bond buyer’s yield spreads, with maturity plotted on a log-scale. The dotted curve represents the Black-Cox yield curve. Here $\nu = 7\%$ and 9% in the other two curves. Other common parameters are as in Figure 4, except $D/y = 95\%$.

standard structural approach for default risk, the major challenge is to extend to complex multi-name credit derivatives. This may be done if we assume independence between default times and “effectively correlate” them through utility valuation: see [7] for small correlation expansions around the independent case with risk-neutral valuation. Another possibility is to assume a large degree of homogeneity between the names (see for example [19] with indifference pricing of CDOs under intensity models), or to adapt a homogeneous group structure to reduce dimension as in [16].

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