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Maturity-independent risk measures
(an abridged version)

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1 Introduction

Despite all the recent work in the area of risk measurement, there is still a number of theoretical, as well as practical, questions left unanswered. The one we focus on in the present paper deals with the problem one faces when the maturity (horizon, expiration date, etc.) associated with a particular risky position is not fixed. We take the view that the mechanism used to measure the risk content of a certain random variable should not depend on any a priori choice of the measurement horizon. This is, for example, the case in complete financial markets. Indeed, consider for simplicity the Samuelson (Black-Scholes) market model with zero interest rate and the procedure one would follow to price a contingent claim therein. The fundamental theorem of asset pricing tells us to simply compute the expectation of the discounted claim under the unique martingale measure. There is no explicit mention of the maturity date of the contingent claim in this algorithm, or, for that matter, any other prespecified horizon. Letting the claim's payoff stay unexercised for any amount of time after its expiry would not change its arbitrage-free price in any way. It is exactly this property that, in our opinion, has not received sufficient attention in the literature. As one of the fundamental properties clearly exhibited under market completeness, it should be shared by any workable risk measurement and pricing procedure in arbitrary incomplete markets.

Incorporation of the maturity-independence property described above into the existing framework of risk measurement has been guided by the principle of minimal impact: we strove to keep new axioms as similar as possible to the existing ones for convex risk measures, and to implement only minimally needed changes. This led us to the realization that it is the domain of the risk measure that inadvertently dictates the use of a specific time horizon. If we replace it by a more general domain, the maturity-independence would follow. Thus, our axioms are identical to the axioms of a replication-invariant convex risk measure, except for the choice of the domain which is *not* a subspace of a function space on $\mathcal{F}_T$, for some fixed time horizon T.

In addition to the novel axiom pertinent to maturity independence, a link to the notion of forward performance processes, recently proposed by M. Musiela and the first author (see [5–9]) is established. Indeed, focusing on the exponential case, it is shown that every forward performance process can be used to create an example of a maturity-independent risk measure. On one hand, this connection provides a useful and simple tool for (a non-trivial task of) constructing maturity-independent risk measures. On the other hand, we hope that it would give a firm decision-theoretic foundation to the theory of forward performances.

2 Maturity-independent risk measures

2.1 The need for maturity independence

We pose and address the following question: "*Is there a class of risk measures that are not constructed in reference to a specific time instance and can, thus, be used to measure the risk content of claims of all (arbitrary) maturities?*" Equivalently, we wish to avoid the case when two versions of the same risk measure (differing only in the choice of the maturity date) give different risk values to the same contingent claim. Before we proceed to formal definitions, let us recall some of the fundamental properties of the arbitrage-free pricing ("Black-Scholes") functional ρ_{BS} in the context of a complete financial market. For a "regular-enough" contingent claim f, the value $\rho_{BS}(f)$ is defined as the capital needed at inscription to replicate it perfectly. The functional ρ_{BS} satisfies the axioms of convex risk measures and is replication-invariant. Moreover, it is per se unaffected by the expiration date of the generic claim f. When markets are incomplete, a much more interesting set of phenomena occurs, as there is no canonical ("Black-Scholes") pricing mechanism. We shall see that, interestingly, some traditional and widely used risk measures are *not* maturity-independent. In other words, under these measures, indifference prices of the same contingent claim, but calculated in terms of two distinct maturities will, in general, *differ*.

Let $\mathbb{L}$ denote the set of all bounded random variables with finite maturities, i.e., $\mathbb{L} = \cup_{t\geq 0}\mathbb{L}^\infty(\mathcal{F}_t)$. The set $\mathbb{L}$ serves as a natural domain for the class of risk measures we propose in the sequel. Note that $\mathbb{L}$ contains all $\mathcal{F}_t$-measurable bounded contingent claims, for all times $t \geq 0$, but it avoids the (potentially pathological) cases of random variables in $\mathbb{L}^\infty(\mathcal{F}_\tau)$, where τ is a finite, but possibly unbounded, stopping time. Here, and in the sequel, S denotes the stock-price process modeled, generally, as a càdlàg semimartingale satisfying a no-arbitrage condition; $\mathcal{A}$ is an appropriate admissibility class of trading strategies.

Definition 2.1. A functional $\rho : \mathbb{L} \to \mathbb{R}$ is called a **maturity-independent convex risk measure** if it has the following properties for all $f, g \in \mathbb{L}$, and $\lambda \in [0, 1]$:

(1) $\rho(f) \leq 0, \ \forall\, f \geq 0,$ (anti-positivity)

(2) $\rho(\lambda f + (1-\lambda)g) \leq \lambda\rho(f) + (1-\lambda)\rho(g),$ (convexity)

(3) $\rho(f - m) = \rho(f) + m, \ \forall\, m \in \mathbb{R},$ and (cash-translativity)

(4) for *all* $t \geq 0$, and $\pi \in \mathcal{A}$, $\rho(f + \int_0^t \pi_s\, dS_s) = \rho(f)$. (replication and maturity independence)

We note that the properties which differentiate the maturity-independent risk measures from the existing notions are the *choice of the domain* $\mathbb{L}$ and the validity of axiom (4) for *all* maturities $t \geq 0$.

The simplest example of a maturity-independent convex risk measure is the super-hedging price function $\hat{\rho} : \mathbb{L} \to \mathbb{R}$ given by

$$\hat{\rho}(f) = \inf\left\{m \in \mathbb{R} : \exists\, \pi \in \mathcal{A},\ m + \textstyle\int_0^\infty \pi_s\, dS_s \geq f,\ \text{a.s.}\right\}.$$

It is easy to see that it satisfies all axioms in Definition 2.1. As in the maturity-dependent case, $\hat{\rho}$ has the extremal property $\hat{\rho}(f) \geq \rho(f)$, for any $f \in \mathbb{L}$ and any maturity-independent risk measure ρ.

2.1.1 Risk measures lacking maturity independence

It is tempting to assume that a maturity-independent risk measure ρ can always be constructed by identifying a maturity date t associated with a contingent claim f, and setting $\rho(f) = \rho(f; t)$, for some replication-invariant risk measure $\rho(\cdot; t)$. As shown in the following example, this construction will not always be possible even if we restrict our attention to the well-explored class of entropic risk measures.

We present a simple two-period example in which entropic risk measurement gives different results for the same, time-1-measurable contingent claim f, when considered at time 1 and time 2. The market structure is described by the simple tree in Figure 1, where the (physical) probability of each of the branches leaving the initial node is $\frac{1}{3}$, and the conditional probabilities of the two contingencies (leading to S_4 and S_5) after node S_3 are equal to $\frac{1}{3}$ and $\frac{2}{3}$, respectively. One can implement the described situation on a 4-element probability space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $\mathbb{P}[\omega_1] = \mathbb{P}[\omega_2] = 1/3$, $\mathbb{P}[\omega_3] = 1/9$ and $\mathbb{P}[\omega_4] = 2/9$. There are two financial instruments: a riskless bond $S^0 \equiv 1$, and a stock $S = S^1$ whose price is denoted by $S_0, \ldots, S_5$ for various nodes of the information tree, such that the following relations hold:

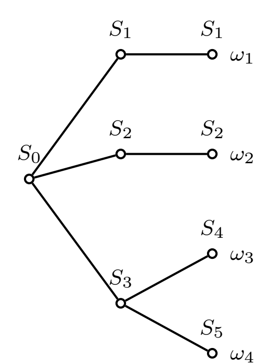

Figure 1. The market tree

$$S_0 = S_2,\ S_2 = \tfrac{1}{2}\big(S_1 + S_3\big),\ S_1 \neq S_3,\ S_3 = \tfrac{1}{2}\big(S_4 + S_5\big),\ S_4 \neq S_5.$$

This implies, in particular, that the market is arbitrage-free and, due to its incompleteness, the set of equivalent martingale measures is larger that just a singleton. Next, we consider a family $\{f_a\}_{a>0}$ of contingent claims defined by

$$f_a(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2, \\ a, & \omega = \omega_3, \omega_4. \end{cases}$$

One can show (employing a dual representation of entropic risk measures) that there exists $a \in \mathbb{R}$ such that $\rho(f_a; 1) \neq \rho(f_a; 2)$ where $\rho(f_a; t)$, $t = 1, 2$, is the value of the entropic risk measure of the contingent claim f_a, seen as a time-t random variable (note that f_a is $\mathcal{F}_1$-measurable, for all a). Consequently, the entropic risk measure does not give rise to a maturity-independent risk measure in this, simple, financial market.

3 Forward Entropic Risk Measures (FERM)

We introduce a family of risk measures closely related to indifference prices. The novelty of the approach is that the underlying risk preference functionals are not tied down to a specific maturity, as it has been the case in the standard expected utility formulation. Rather, they can be seen as specified at initiation and subsequently "generated" across all times. This approach was proposed by the first author and M. Musiela (see [5–9]) and is briefly reviewed below.

3.1 Forward exponential performances

The notion of a forward performance process has arisen from the search for ways to measure the performance of investment strategies across *all* times in $[0, \infty)$. In order to produce a nontrivial such object, we look for a random field $U = U_t(\omega, x)$ defined for *all* times $t \geq 0$ and parametrized by a wealth argument $x \in \mathbb{R}$ such that the mapping $x \mapsto U_t(\omega, x)$ admits the classical properties of utility functions.

On an arbitrary trading horizon, say $[s, t]$, $0 \leq s < t < \infty$, the investor whose preferences are described by the random field U seeks to maximize the expected investment performance:

$$V_s(x) = \operatorname*{esssup}_{\pi \in \mathcal{A}} \mathbb{E}[U_t(X_t^{x,\pi})|\mathcal{F}_s], \ 0 \leq s \leq t. \tag{3.1}$$

Herein, $X^{x,\pi}$ denotes the investor's wealth process, $x \in \mathbb{R}$ the investor's initial wealth at time s, and π a generic investment strategy belonging to the appropriate admissibility class $\mathcal{A}$.

It has been argued in [9] that the class of performance random fields with the additional property

$$V_t(x) = U_t(x), \text{ a.s. } \forall\, t \in [0, \infty), \ x \in \mathbb{R}, \tag{3.2}$$

possesses several desirable properties and gives rise to an analytically tractable theory.

Definition 3.1. A random field U satisfying (3.2), where V is defined by (3.1), is called **self-generating**.

Remark 3.2. We remind the reader that a classical example of a self-generating performance random field (albeit *only on the finite horizon* $[0, T]$) is the traditional value function, defined as

$$U_t(x) = \operatorname*{esssup}_{\pi \in \mathcal{A}} \mathbb{E}[U_T(X_T^{x,\pi})|\mathcal{F}_t], \ t \in [0, T], \, x \in \mathbb{R},$$

where T is a prespecified maturity beyond which no investment activity is measured, and $U_T(\cdot, \cdot) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a classical (state-dependent) utility function. When the horizon is infinite, such a construction will not produce any results. Indeed, there is no appropriate time for the final datum to be given.

In the traditional framework, the datum (terminal utility) is assigned at some fixed future time T. Alternatively, in the case of an infinite time horizon, it is more natural to think of the datum $u_0 : \mathbb{R} \to \mathbb{R}$ as being assigned at time $t = 0$, and a self-generating performance random field U_t chosen so that $U_0(x) = u_0(x)$.

Because of this interpretation, the self-generating performance random fields may also be referred to as **forward performances**.

While traditional performance random fields on finite horizons are straightforward to construct and characterize, producing a forward performance random field on $[0, \infty)$ from a given initial datum u_0 is considerably more difficult. Several examples of such a construction, all based on the exponential initial datum, are given below.

Definition 3.3. A performance random field U is called a **forward exponential performance** if it is self-generating, and there exists a constant $\gamma > 0$, such that $U_0(x) = -e^{-\gamma x},\ x \in \mathbb{R}$.

The construction presented below can be found in [8], and the reader is urged to consult that reference for details of the financial model, setting and notation in the following theorem:

Theorem 3.4 (Theorem 4 in [8]). *Let $(W)_{t\in[0,\infty)}$ be a one-dimensional Brownian motion, and let $(Y_t)_{t\in[0,\infty)}$ and $(Z_t)_{t\in[0,\infty)}$ be two continuous processes solving*

$$dY_t = Y_t\delta_t(\lambda_t dt + dW_t) \text{ and } dZ_t = Z_t\phi_t dW_t, \tag{3.3}$$

with $Y_0 = 1/\gamma > 0,\ Z_0 = 1$, for a fixed, but arbitrary k-dimensional coefficient processes $(\delta_t)_{t\in[0,\infty)}$ and $(\phi_t)_{t\in[0,\infty)}$ which are assumed to be $\mathbb{F}$-adapted, and with δ satisfying $\sigma_t\sigma_t^+\delta_t = \delta_t$, for all $t \geq 0$, a.s. Assume that δ and ϕ are regular enough for the integrals in (3.3) to be well defined, and that, when restricted to any finite interval $[0, t]$, the process Z is a positive martingale, and Y is uniformly bounded from above and away from zero.

Let the process $(A_t)_{t\in[0,\infty)}$ be defined as

$$A_t = \int_0^t \left\|\sigma_s\sigma_s^+(\lambda_s + \phi_s) - \delta_s\right\|^2 ds. \tag{3.4}$$

Then, the random field U, given by

$$U_t(\omega, x) = -Z_t(\omega)\exp\left(-\frac{x}{Y_t(\omega)} + \frac{A_t(\omega)}{2}\right), \tag{3.5}$$

is a forward exponential performance. In particular, for $0 \leq s \leq t$ and $\xi \in \mathbb{L}^\infty(\mathcal{F}_s)$, we have

$$U_s(\xi) = \operatorname*{esssup}_{\pi\in\mathcal{A}} \mathbb{E}\left[U_t\left(\xi + \int_s^t \pi_u dS_u\right)\Big|\,\mathcal{F}_s\right],\ \text{a.s.} \tag{3.6}$$

Remark 3.5. In (3.5) above, one can give a natural financial interpretation to the processes Y (which normalizes the wealth argument) and Z (which appears as a multiplicative factor). One might think of Y as a benchmark (or a numéraire) in relation to which we wish to measure the performance of our investment strategies. The values of the process Z, on the other hand, can be thought of as Radon-Nikodym derivatives of the investor's subjective probability measure with respect to the measure $\mathbb{P}$.

3.2 Forward entropic risk measures

We are now ready to introduce the *forward entropic risk measures* (FERM). We start with an auxiliary object, denoted by $\rho(C; t)$.

Definition 3.6. Let U be the forward exponential performance defined in (3.5), and let $t \geq 0$ be arbitrary, but fixed. For a contingent claim written at time $s = 0$ and yielding a payoff $C \in \mathbb{L}^\infty(\mathcal{F}_t)$, we define $\rho(C; t) \in \mathbb{R}$ as the unique solution of

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[U_t\Big(x + \int_0^t \pi_s dS_s\Big)\Big] = \sup_{\pi\in\mathcal{A}} \mathbb{E}\Big[U_t\Big(x + \rho(C; t) + C + \int_0^t \pi_s dS_s\Big)\Big],\ \forall\, x \in \mathbb{R}. \tag{3.7}$$

The mapping $\rho(\cdot; t) : \mathbb{L}^\infty(\mathcal{F}_t) \to \mathbb{R}$ is called the **t-normalized forward entropic measure**.

One can readily check that equation (3.7) indeed admits a unique solution (independent of the initial wealth x), so that the t-normalized forward entropic measures are well defined. The reader can convince him-/herself of the validity of the following result:

Proposition 3.7. *The t-normalized forward entropic risk measures are replication-invariant convex risk measures on $\mathbb{L}^\infty(\mathcal{F}_t)$, for each $t \geq 0$.*

The *fundamental* property in which forward entropic risk measures differ from a generic replication-invariant risk measure is the following:

Proposition 3.8. *For $0 \leq s < t < \infty$, and $C^{(s)} \in \mathbb{L}^\infty(\mathcal{F}_s)$, consider the s- and t-normalized forward entropic measures $\rho(C^{(s)}; s)$ and $\rho(C^{(s)}; t)$ applied to the contingent claim $C^{(s)}$. Then,*

$$\rho(C^{(s)}; s) = \rho(C^{(s)}; t). \tag{3.8}$$

More generally, for $C^{(r)} \in \mathbb{L}^\infty(\mathcal{F}_r)$, where $0 \leq r < s < t < \infty$, we have $\rho(C^{(r)}; s) = \rho(C^{(r)}; t)$.

We are now ready to define the forward entropic risk measures:

Definition 3.9. For $C \in \mathbb{L}$, define the **earliest maturity** $t_C \in [0, \infty)$ **of** C as $t_C = \inf\{t \geq 0 : C \in \mathcal{F}_t\}$. The **forward entropic risk measure** $\rho : \mathbb{L} \to \mathbb{R}$ is defined as

$$\rho(C) = \rho(C; t_C), \tag{3.9}$$

where $\rho(C; t_C)$ is the value of the t_C-normalized forward entropic risk measure, defined in (3.7), applied to the contingent claim C.

The focal point of the present section is the following theorem:

Theorem 3.10. *The mapping $\rho : \mathbb{L} \to \mathbb{R}$ is a maturity-independent risk measure.*

Finally, we provide (without proof) an explicit representation of the forward entropic risk measures.

Theorem 3.11. *Let Y, Z, A and $U_t(\cdot)$ be as in Theorem 3.4. For $C \in \mathbb{L}$, its forward entropic risk measure is given by*

$$\rho(C) = \inf_{\pi \in \mathcal{A}} \left(\frac{1}{\gamma} \ln \mathbb{E}[-U_t(C + \textstyle\int_0^t \pi_s \, dS_s)] \right), \quad \textit{for any } t \geq t_C. \tag{3.10}$$

4 Examples

This example uses the notation of and is set in the financial framework described in [8], where the stock-price process satisfies

$$dS_t = S_t(\mu_t \, dt + \sigma_t \, dW_t^1),$$

on an augmented filtration generated by a 2-dimensional Brownian motion (W^1, W^2). The adapted processes μ and $\sigma > 0$ are assumed to satisfy the needed regularity conditions, but are not necessarily deterministic. With $\lambda_t^1 = \mu_t/\sigma_t$, the processes Z, Y, A from Theorem 3.4 can be written as

$$dY_t = Y_t \delta_t(\lambda_t^1 dt + dW_t^1), \ Y_0 = 1/\gamma > 0, \quad dZ_t = Z_t \phi_t dW_t^1, \ Z_0 = 1, \tag{4.1}$$

and

$$A_t = \textstyle\int_0^t (\lambda_s^1 + \phi_s - \delta_s)^2 ds, \ A_0 = 0, \tag{4.2}$$

subject to a choice of two processes ϕ and δ, under the regularity conditions stated in Theorem 3.4.

a) $\phi \equiv \delta \equiv 0$. In this case, $Z_t \equiv 1$, $Y_t \equiv 1/\gamma$, $A_t \equiv \int_0^t (\lambda_s^1)^2\, ds$ and the random field U of (3.5) becomes

$$U_t(x) = -\exp(-\gamma x + \frac{A_t}{2}).$$

Using the indifference-pricing equation (3.7) and the self-generation property (3.6) of U_t, we deduce that for $C \in \mathbb{L}$, the value $\rho(C)$ satisfies

$$-\exp(\gamma\rho(C)) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\gamma(C + \int_0^t \pi_s\, dS_s) + \tfrac{A_t}{2}\right)\right], \quad \text{for any } t \geq t_C.$$

On the other hand, the classical (exponential) indifference price, $\nu(C - \frac{A_t}{2\gamma}; t)$, of the contingent claim $C - \frac{A_t}{2\gamma}$, maturing at time t, satisfies

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}[-\exp(-\gamma(\nu(C - \tfrac{A_t}{2\gamma}; t) + \int_0^t \pi_s\, dS_s))] = \sup_{\pi\in\mathcal{A}} \mathbb{E}[-\exp(-\gamma(C - \tfrac{A_t}{2\gamma} + \int_0^t \pi_s\, dS_s))].$$

With $H_t = \ln \sup_{\pi\in\mathcal{A}} \mathbb{E}[-\exp(-\gamma \int_0^t \pi_s\, dS_u)]$ (which will be recognized by the reader familiar with exponential utility maximization as the aggregate relative entropy), we obtain

$$\rho(C) = -\nu(C - \tfrac{A_t}{2\gamma}; t) - \tfrac{1}{\gamma} H_t, \text{ for any } t \geq t_C. \tag{4.3}$$

b) $\delta \equiv 0$. Then $Y_t \equiv 1/\gamma$, $A_t \equiv \int_0^t (\lambda_s^1 + \phi_s)^2\, ds$, and the random field U of (3.5) takes the form

$$U_t(x) = -Z_t \exp(-\gamma x + \frac{A_t}{2}).$$

The risk measure $\rho(C)$ can be represented as in (4.3) above, with one important difference. Specifically, the (physical) probability measure $\mathbb{P}$ has to be replaced by the probability $\tilde{\mathbb{P}}$ whose Radon-Nikodym derivative w.r.t. $\mathbb{P}$ is given by Z_t on $\mathcal{F}_t$, for any $t \geq 0$.

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