Investment performance measurement under asymptotically linear local risk tolerance[*]

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Abstract

Using forward optimality criteria, we analyze a portfolio choice problem when the local risk tolerance is time-dependent and asymptotically linear in wealth. This class corresponds to a dynamic extension of the traditional (static) risk tolerances associated with the power, logarithmic and exponential utilities. We provide explicit solutions for the optimal investment strategies and wealth processes in an incomplete non-Markovian market with asset prices modelled as Ito processes. The methodology allows for measuring the investment performance in terms of a benchmark and alternative market views.

Key words: Forward performance process, portfolio management, investment performance, local risk tolerance, fast-diffusion equation, benchmark, market views, incomplete markets.

1   Introduction

This paper is a contribution to optimal portfolio management using the forward performance approach. This approach, developed by the first author and M. Musiela (see, Musiela and Zariphopoulou [2003, 2007b]), is based on the martingale properties of the so-called forward performance process which combines the investor's preferences with market related inputs. In many aspects, it is similar to the traditional maximal expected utility methodology where the martingality of the solution (value function) is a consequence of

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the dynamic programming principle. It differs, however, in that the forward performance process is defined endogenously to the market environment and for all times. A direct consequence of these properties is that the forward solution follows the market movements "path-by-path" and, moreover, can be constructed without references to a specific trading horizon.

Constructing the forward performance process and the associated with it optimal portfolio strategies poses many difficulties due to the fact that the implicit stochastic optimization problem is posed "forward" in time. A class of such processes was recently constructed in Musiela and Zariphopoulou [2006b, 2007b] using the compilation of differential and stochastic input. The inputs are given, respectively, by the solution of a fully nonlinear pde and a triple of stochastic processes representing a benchmark, alternative market views and (random) time-rescaling. The optimal policies are given as a linear combination of the investor's optimal wealth and the time-rescaled risk tolerance processes. An important result is that these two processes solve an autonomous system of stochastic differential equations.

Pivotal role in the above analysis plays the local risk tolerance function. It is constructed from the investor's initial risk preferences and the solution of an equation of fast-diffusion type. It is, then, used to solve the aforementioned system and, in turn, to explicitly specify the optimal investment processes in a feedback form. We note that such optimal policies come as a surprise given the non-Markovian nature of the market model.

Motivated by the emerging modeling importance of the local risk tolerance, we concentrate herein on a specific class of such functions. The family we consider corresponds to a dynamic generalization of the popular utilities used in academic works of portfolio management, namely, the power, logarithmic and exponential ones. However, in contrast to the power and logarithmic cases, the risk tolerances we consider are globally defined (i.e. for positive and negative wealth levels).

The paper is organized as follows. In section 2, we introduce the model and review the definition of forward performance process and the main results of Musiela and Zariphopoulou [2007b]. In section 3, we focus on a two-parameter family of risk tolerance functions and construct the related forward performance process. In section 4, we provide an explicit construction of the associated optimal allocations and wealth processes. We conclude in section 5 where we concentrate on special limiting choices of the two risk tolerance parameters.

2 The model and its investment performance measurement

The market environment consists of one riskless and k risky securities. The risky securities are stocks and their prices are modelled as positive and

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continuous Ito processes. Namely, for $i = 1, ..., k$, the price S^i of the ith risky asset solves

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^d \sigma_t^{ji} dW_t^j \right) \tag{2.1}$$

with $S_0^i > 0$. The process $W = \left(W^1, ..., W^d \right)$ is a standard $d-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, it is assumed that the underlying filtration, $\mathcal{F}_t$, coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma \left(W_s : 0 \le s \le t \right)$.

The coefficients μ_i and σ_i, $i = 1, ..., k$, follow $\mathcal{F}_t$-adapted processes with values in $\mathcal{R}$ and $\mathcal{R}^d$, respectively. For brevity, we use σ_t to denote the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji} \right)$, whose i^{th} column represents the volatility σ_t^i of the ith risky asset. We may, then, alternatively write (2.1) as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t \right).$$

The riskless asset, the savings account, has the price process B satisfying

$$dB_t = r_t B_t dt$$

with $B_0 = 1$, and for a nonnegative, $\mathcal{F}_t$-adapted interest rate process r_t. The market coefficients, μ, σ and r are taken to be bounded.

It is postulated that there exists an $\mathcal{F}_t$-adapted process λ, known as the market price of risk, taking values in $\mathcal{R}^d$ and such that the equality

$$\mu_t^i - r_t = \sum_{j=1}^d \sigma_t^{ji} \lambda_t^j = \sigma_t^i \cdot \lambda_t$$

is satisfied for $t \ge 0$, for all $i = 1, ..., k$. Using vector and matrix notation, the above becomes

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t \tag{2.2}$$

where σ^T stands for the transpose matrix of σ, and $\mathbf{1}$ denotes the d-dimensional vector with every component equal to one. It is assumed that, for all $t \ge 0$, $E_{\mathbb{P}} \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds < \infty$, where σ^+ denotes the Moore-Penrose pseudoinverse of the volatility matrix (see Penrose [1955]). Recall that the matrix σ^+ exists and is unique even if the market fails to be complete.

Starting at $t = 0$ with an initial endowment $x \in \mathcal{R}$, the investor invests at all future times $t > 0$ in the riskless and risky assets. The present value of the amounts invested are denoted, respectively, by π_t^0 and π_t^i, $i = 1, ..., k$.

The present value of her aggregate investment is, then, given by $X_t = \sum_{i=0}^k \pi_t^i$. We will refer to X as the discounted wealth. The investment strategies $\left(\pi_t^0, \pi_t^1, ..., \pi_t^k \right)$ will play the role of control processes and are taken

to satisfy the standard assumption of being self-financing, i.e. for $s \geq 0$,

$$X_s = x + \sum_{i=1}^{k} \int_0^s \pi_u^i \left(\mu_u^i - r_u \right) du + \sum_{i=1}^{k} \int_0^s \pi_u^i \sigma_u^i \cdot dW_u. \qquad (2.3)$$

Writing the above in differential form, yields the evolution of the discounted wealth,

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot \left(\lambda_t dt + dW_t \right) = \sigma_t \pi_t \cdot \left(\lambda_t dt + dW_t \right), \qquad (2.4)$$

where the (column) vector, $\pi_t = \left(\pi_t^i ; i = 1, ..., k \right)$.

The set of admissible strategies, $\mathcal{A}$, consists of all self-financing $\mathcal{F}_t-$adapted processes π_t such that $E_{\mathbb{P}} \int_0^s |\sigma_t \pi_t|^2 dt < \infty$, for $s > 0$. It is also assumed, in order to preclude arbitrage opportunities, that, for each $s > 0$, the associated wealth process, X_t, $0 \leq t \leq s$, is a $\mathbb{Q}|_{\mathcal{F}_s}$-supermartingale for some equivalent martingale measure $\mathbb{Q}|_{\mathcal{F}_s} \sim \mathbb{P}|_{\mathcal{F}_s}$.

We continue with the definition of the forward performance process. We refer the reader to Musiela and Zariphopoulou [2007a,b] (see, also, Musiela and Zariphopoulou [2003]) for a detailed analysis on the motivation and modeling considerations that led to the development of the forward performance concept.

Definition 2.1. *An $\mathcal{F}_t$-adapted process $U_t(x)$ is a forward performance if:*

i) for each $t \geq 0$ and as a function of $x \in \mathcal{R}$, $U_t(x)$ is concave and increasing,

ii) for each $t \geq 0$ and each self-financing strategy, $\pi \in \mathcal{A}$,

$$E_{\mathbb{P}} \left[U_t \left(X_t^{\pi} \right) \right]^+ < \infty,$$

iii) for each self-financing strategy, $\pi \in \mathcal{A}$,

$$E_{\mathbb{P}} \left[U_s \left(X_s^{\pi} \right) | \mathcal{F}_t \right] \leq U_t \left(X_t^{\pi} \right), \quad s \geq t,$$

iv) there exists a self-financing strategy, $\pi^* \in \mathcal{A}$, for which

$$E_{\mathbb{P}} [U_s (X_s^{\pi^*}) | \mathcal{F}_t] = U_t (X_t^{\pi^*}), \quad s \geq t,$$

and

v) it satisfies the initial datum $U_0(x) = u_0(x)$, $x \in \mathcal{R}$ where $u_0 : \mathcal{R} \to \mathcal{R}$ is a concave and increasing function of wealth.

Related to our work is the recent paper Choulli et al. [2007] in which the authors considered random horizon choices, aiming at alleviating the dependence of the value function on a fixed (and deterministic) horizon. Their model is more general than ours, in terms of the assumptions on the price

processes. However, the focus is on horizon effects and not on additional features affecting the form of the forward solution like, for example, numeraire choice, tracking a benchmark and alternative market views. Horizon issues were also considered in Henderson and Hobson [2007a,b] who proposed the so-called horizon-unbiased utilities in the context of lognormal diffusion models and constructed a deterministic class of solutions. While preparing this work, the authors came across the preprint Berrier et al. [2007] where a special case of forward processes is considered in a model similar to ours (see Corollary 2.1 below).

We mention that forward formulations of optimal control problems have been proposed and analyzed in the past. For deterministic models, we refer the reader, among others, to Seinfeld and Lapidus [1968] and Chapter 1 in Larson [1968] (see, also, Vrr [1977]). In stochastic settings, forward optimality has been studied, primarily under Markovian assumptions, in Kurtz [1984] via the associated controlled martingale problems and the construction of the Nisio semigroup (see, Nisio [1981]).

Next, we review the results of Musiela and Zariphopoulou [2007b]. They consist of three parts, namely, the representation of a family of forward performance processes, the specification of the associated optimal investment strategies and wealth processes and the construction of an autonomous system of stochastic differential equations that the optimal wealth and risk tolerance processes solve.

Theorem 2.1. *Let the processes Y and Z solve*

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \tag{2.5}$$

and

$$dZ_t = Z_t \phi_t \cdot dW_t \tag{2.6}$$

with $Y_0 = Z_0 = 1$, δ and ϕ being $\mathcal{F}_t$-adapted and bounded with δ such that $\sigma\sigma^+\delta = \delta$ and $E_{\mathbb{P}} \int_0^t |\sigma_s\sigma_s^+\phi_s|^2 ds < \infty$. Define the process

$$A_t = \int_0^t |\sigma_s\sigma_s^+ (\lambda_s + \phi_s) - \delta_s|^2 ds, \quad t \geq 0 \tag{2.7}$$

where λ as in (2.2).

Let $u : \mathcal{R} \times (0, \infty) \to \mathcal{R}$ be a concave and increasing function of the spatial argument with $u : \mathcal{C}^{3,1} (\mathcal{R} \times (0, \infty))$ satisfying the differential constraint

$$u_t u_{xx} = \frac{1}{2} u_x^2 \tag{2.8}$$

and the initial datum

$$u(x, 0) = u_0(x) \tag{2.9}$$

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with $u_0 : \mathcal{R} \to \mathcal{R}$ be in $\mathcal{C}^3(\mathcal{R})$. Then, the process $U_t(x)$ defined by

$$U_t(x) = u\left(\frac{x}{Y_t}, A_t\right) Z_t, \quad t \geq 0 \tag{2.10}$$

is a forward performance.

The process Y, which normalizes the wealth argument, may be thought as a benchmark (or numeraire) with regards to which the investment performance is measured. The process Z refers to changes in the historical probability measure and accommodates alternative views on anticipated market movements. We will refer to Y and Z as the *benchmark* and *market view processes* respectively.

Corollary 2.1. *In the special case $\delta_t = \phi_t = 0$, $t \geq 0$, the forward performance process deduces to*

$$U_t(x) = u\left(x, \int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds\right). \tag{2.11}$$

If, in addition, the market parameters are constant, the forward solution is given by the deterministic function

$$U_t(x) = u\left(x, |\sigma \sigma^+ \lambda|^2 t\right). \tag{2.12}$$

Forward solutions of form (2.11) (resp. (2.12)) are the ones considered in Berrier et al. [2007] (resp. Henderson and Hobson [2007a,b]).

We continue with the optimal investment strategies and the wealth they generate. It is worth mentioning that despite the dimensionality and incompleteness of the model, as well as the allowed path-dependence of the coefficients, the optimal control policies are given in an explicit *feedback* form. To our knowledge this is one of the very few such examples.

For convenience and generality, we work in the benchmarked configuration, namely, we consider the processes

$$\tilde{\pi}_t^* \equiv \frac{1}{Y_t} \pi_t^* \quad \text{and} \quad \tilde{X}_t^* \equiv \frac{X_t^*}{Y_t} \tag{2.13}$$

denoting, respectively, the *benchmarked optimal portfolio* and *benchmarked optimal wealth*.

A quantity that will play an important role in the analysis herein is the *local risk tolerance* $r : \mathcal{R} \times [0, \infty) \to \mathcal{R}^+$, defined as

$$r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)} \tag{2.14}$$

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with u as in (2.10). For its initial value, we will be using the notation

$$r_0\left(x\right) = r\left(x, 0\right) = -\frac{u'\left(x\right)}{u''\left(x\right)}. \tag{2.15}$$

The following assumption will be standing throughout.

Assumption 1: There exist constants K_1 and K_2 such that, for all $t \geq 0$ and $x, \bar{x} \in \mathcal{R}$,

$$r^2\left(x, t\right) \leq K_1\left(1 + x^2\right) \quad \text{and} \quad \left|r\left(x, t\right) - r\left(\bar{x}, t\right)\right| \leq K_2\left|x - \bar{x}\right|. \tag{2.16}$$

Next, we introduce the *risk tolerance process* (at benchmarked optimal wealth)

$$\tilde{R}_t^* = r\left(\tilde{X}_t^*, A_t\right) \tag{2.17}$$

with r as in (2.24) and A being the time-rescaling process defined in (2.7).

Theorem 2.2. *The optimal benchmarked portfolio $\tilde{\pi}_t^*$, $t > 0$, is given by*

$$\tilde{\pi}_t^* = \Pi_t^*\left(\tilde{X}_t^*\right)$$

with

$$\Pi_t^*\left(x\right) = x\sigma^+\delta_t + r\left(x, A_t\right)\sigma_t^+\left(\lambda_t + \phi_t - \delta_t\right) \tag{2.18}$$

where A as in (2.7) and $\tilde{X}_t^*$, $t > 0$, solving

$$d\tilde{X}_t^* = \left(\sigma_t \tilde{\pi}_t^* - \tilde{X}_t^* \delta_t\right) \cdot \left(\left(\lambda_t - \delta_t\right) dt + dW_t\right), \tag{2.19}$$

with $\tilde{\pi}^*$ being used.

Equivalently,

$$\tilde{\pi}_t^* = m_t \tilde{X}_t^* + n_t \tilde{R}_t^* \tag{2.20}$$

with $\tilde{R}_t^*$ as in (2.17) and the portfolio weights given by

$$m_t = \sigma_t^+ \delta_t \quad \text{and} \quad n_t = \sigma_t^+\left(\lambda_t + \phi_t - \delta_t\right). \tag{2.21}$$

An important consequence of the above theorem is that, under *any* choice of risk preferences, the optimal investment strategy is represented as a *linear combination* of two funds, namely,

$$\tilde{\pi}_t^{*,X} = m_t \tilde{X}_t^* \quad \text{and} \quad \tilde{\pi}_t^{*,R} = n_t \tilde{R}_t^*. \tag{2.22}$$

The portfolio $\tilde{\pi}_t^{*,X}$ depends functionally only on current wealth and not the risk tolerance. The situation, however, is reversed for the second investment strategy, $\tilde{\pi}_t^{*,R}$. Observe that the portfolio weights m_t, n_t, $t > 0$ are affected exclusively by the market. They may take the value zero in which case the relevant optimal allocation vanishes. Such cases are discussed at the end of this section.

Next, we present the autonomous system of stochastic differential equations that the processes $\tilde{X}_t^*$ and $\tilde{R}_t^*$, $t > 0$ solve. Solving this system and using the linear representation result of (2.20) enable us to explicitly construct the optimal allocation vector $\tilde{\pi}_t^*$.

Proposition 2.1. *Let r be the local risk tolerance function, introduced in (2.14), and A the time-rescaling process given in (2.7). Then, for $t > 0$, the processes $\tilde{X}_t^*$ and $\tilde{R}_t^*$, $t > 0$, representing the (benchmarked) optimal wealth and risk tolerance, solve the system*

$$\begin{cases} d\tilde{X}_t^* = \tilde{R}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t)\, dt + dW_t) \\[2mm] \qquad d\tilde{R}_t^* = r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* \end{cases} \tag{2.23}$$

with $\tilde{X}_0^* = x$, $\tilde{R}_0^* = r_0(x)$ and n_t, $t > 0$ as in (2.21).

From (2.23) we see that the solution $\left(\tilde{X}^*, \tilde{R}^*\right)$ is fully specified once the model is chosen and the local risk tolerance function is known. Recall that r is constructed from the function u (cf. (2.14)), obtained from the nonlinear equation (2.8) and the initial datum (2.9). The form of the above system, however, motivates us to question whether one should first model the differential input u and, then, specify r (cf. (2.14)) or go the opposite direction. Herein, we follow the second approach, namely, we first choose a family of risk tolerances and, in turn, recover the associated differential input. A fundamental result used for this construction is that r satisfies an autonomous differential equation. This rather interesting property was shown in Musiela and Zariphopoulou [2006b].

Proposition 2.2. *If u satisfies (2.8), the associated local risk tolerance function r, defined in (2.14), satisfies*

$$r_t + \frac{1}{2} r^2 r_{xx} = 0. \tag{2.24}$$

It is easy to see how the differential input, u, is recovered once the local risk tolerance is known. Indeed, choosing the initial condition $r_0(x) = r(x, 0)$ and using (2.15) yields (modulo two constants) the initial datum (2.9). In turn, equation (2.24), together with the initial condition r_0, will give the values $r(x, t)$, for $t > 0$. The function $u(x, t)$, $t > 0$, can be, then, retrieved from (2.14) by successive integration, provided certain (time-dependent) quantities are correctly specified. Related arguments are found in the proof of Proposition 3.2.

The reader with expertise in nonlinear partial differential equations will find the form of (2.24) familiar. In fact, it is a nonlinear heat equation, frequently called equation of fast-diffusion type. There is a vast literature on this equation and we refer the reader, among others, to the book of Vasquez [2006]. Observe, however, that classical results might not be applicable since the equation is "ill-posed", a fact that adds various difficulties to the construction of well-defined and stable solutions.

We finish this section mentioning that there is alternative way to construct u from r which could, perhaps, provide more intuition for the evolution of the differential input. Namely, observe that (2.8) and (2.14) yield the transport equation

$$u_t + \frac{1}{2} r(x,t) u_x = 0. \tag{2.25}$$

Such first-order equations can be solved by the method of characteristics. In (2.25) these curves have slope equal to one half the local risk tolerance. The input u is, then, readily constructed through the initial condition u_0, computed from (2.15), and its propagation along the characteristic curves.

3 Asymptotically linear local risk tolerance functions

We now focus on a specific class of risk tolerance functions. To provide some motivation for our choice, let us recall that the utilities most frequently appearing in academic papers of portfolio management are the *power, logarithmic* and *exponential*[1]. In the generic problem of maximizing the expected utility of terminal wealth, these utilities are assigned at the end of the trading horizon, say $[0, T]$, and given, respectively, by

$$u^p(x;T) = \frac{1}{\gamma} x^\gamma, \quad x \geq 0, \ \gamma < 1, \ \gamma \neq 0, \tag{3.1}$$

$$u^l(x;T) = \log x, \quad x > 0, \tag{3.2}$$

and

$$u^e(x;T) = -e^{-\kappa x}, \quad x \in \mathcal{R}, \ \kappa > 0. \tag{3.3}$$

The associated risk tolerances (with a slight abuse of notation, we denote them by r but keep the argument T to emphasize their dependence on the horizon choice) are, naturally, time independent and given by

$$r^p(x;T) = \frac{1}{1-\gamma} x, \ x \geq 0 \quad \text{and} \quad r^l(x;T) = x, \ x > 0, \tag{3.4}$$

and

$$r^e(x;T) = \frac{1}{\kappa}, \quad x \in \mathcal{R}. \tag{3.5}$$

Notice that in the traditional setting [2] risk preferences are chosen exclusively at the single time instant, T. In the forward framework, however, they are set at initial time, $t = 0$, and then specified for *all future* times $t > 0$.

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[1]The quadratic utility deserves special attention due to its saturation properties and will be studied separately.

[2]We remind the reader that there is no intermediate consumption and, thus, no risk preferences are allocated to incoming consumption streams.

For the family of forward performance processes we consider herein, the specification of the future values of r comes from the differential constraint (2.24).

Next, we introduce a rich family of solutions which, from one hand, are appropriate for the new framework and, on the other, resemble a dynamic extension of their traditional counterparts (3.4) and (3.5).

Proposition 3.1. *Let $\alpha, \beta > 0$ and $r_0 : \mathcal{R} \to \mathcal{R}^+$ be given by*

$$r_0(x) = \sqrt{\alpha x^2 + \beta}.$$

Then, the function $r : \mathcal{R} \times [0, \infty) \to \mathcal{R}^+$,

$$r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \tag{3.6}$$

solves (2.24).

It is easy to verify that for *fixed* $t = T$, $r^p(x; T)$, $r^l(x; T)$ and $r^e(x; T)$ are *limiting* cases of (3.6) in their respective spatial domains. Indeed,

$$r^p(x; T) = \lim_{\beta \to 0} r(x, T; \alpha, \beta), \ x \geq 0, \quad \text{and} \quad \gamma = \frac{\sqrt{\alpha} - 1}{\sqrt{\alpha}}, \ \alpha \neq 1, \tag{3.7}$$

$$r^l(x; T) = \lim_{\beta \to 0} r(x, T; \alpha, \beta), \ x > 0 \quad \text{and} \quad \alpha = 1, \tag{3.8}$$

and

$$r^e(x; T) = \lim_{\alpha \to 0} r(x, T; \alpha, \beta) \quad \text{and} \quad \beta^2 = \kappa^{-1}. \tag{3.9}$$

It is immediate that the family $r(x, t; \alpha, \beta)$ satisfies Assumption 1. Moreover, it is *globally* defined and remains strictly positive at all positive times,

$$r(x, t; \alpha, \beta) > 0, \quad x \in \mathcal{R} \quad \text{and} \quad t > 0.$$

It has a global minimum at the origin, $(0, 0)$, at which it degenerates, $r(0, 0; \alpha, \beta) = 0$. The top panel of Figure 1 provides its graph for $\alpha = 4$ and $\beta = 0.1$.

The family (3.6) will be called *asymptotically linear* due to its limiting behavior

$$\lim_{x \to \pm\infty} \frac{r(x, t; \alpha, \beta)}{|x|} = \sqrt{\alpha}, \quad t \geq 0. \tag{3.10}$$

Remark: The above class can be readily generalized to the three-parameter family

$$r(x, t; x_0, \alpha, \beta) = \sqrt{\alpha(x - x_0)^2 + \beta e^{-\alpha t}}, \quad t > 0.$$

Since the arguments developed in the sequel can be easily extended, for the above case, we choose $x_0 = 0$.

The rest of the paper is dedicated to the construction of the forward performance process, the optimal investment allocations and the optimal wealth when the local risk tolerance is given by (3.6). The first step is to identify the differential input that is associated with (3.6), i.e. for an increasing and concave function $u(x, t; \alpha, \beta)$ satisfying

$$-\frac{u_x(x, t; \alpha, \beta)}{u_{xx}(x, t; \alpha, \beta)} = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \quad x \in \mathcal{R} \text{ and } t \geq 0.$$

It is easy to verify that the construction is invariant under affine transformations, namely, if $u(x, t; \alpha, \beta)$ satisfies the above then, for M, N constants,

$$\bar{u}(x, t; \alpha, \beta) = Mu(x, t; \alpha, \beta) + N \tag{3.11}$$

satisfies it as well. To preserve the desired monotonicity of u we need to choose $M > 0$.

As it will be clear from the proof of the next Proposition, the form of u depends on the range of the parameter α. Specifically, one needs to look at the cases $\alpha = 1$ and $\alpha \neq 1$, separately.

Proposition 3.2. *Let r be given by (3.6) with $\alpha, \beta > 0$. The following statements hold.*

i) If $\alpha \neq 1$, the associated differential input is given, for $x \in \mathcal{R}$ and $t \geq 0$, by

$$u(x, t; \alpha, \beta)$$

$$= M \frac{(\sqrt{\alpha})^{1+\frac{1}{\sqrt{\alpha}}}}{\alpha - 1} e^{\frac{1-\sqrt{\alpha}}{2}t} \frac{\left(\frac{\beta}{\sqrt{\alpha}} e^{-\alpha t} + (1 + \sqrt{\alpha}) x \left(\sqrt{\alpha} x + \sqrt{\alpha x^2 + \beta e^{-\alpha t}}\right)\right)}{\left(\sqrt{\alpha} x + \sqrt{\alpha x^2 + \beta e^{-\alpha t}}\right)^{1+\frac{1}{\sqrt{\alpha}}}} + N.$$

$$\tag{3.12}$$

ii) If, $\alpha = 1$, then, for $x \in \mathcal{R}$ and $t \geq 0$,

$$u(x, t; 1, \beta)$$

$$= \frac{M}{2} \left(\log\left(x + \sqrt{x^2 + \beta e^{-t}}\right) - \frac{e^t}{\beta} x \left(x - \sqrt{x^2 + \beta e^{-t}}\right) - \frac{t}{2}\right) + N.$$

$$\tag{3.13}$$

Proof. Rewriting (2.14) as $(\log u_x(x, t; \alpha, \beta))_x = -r(x, t; \alpha, \beta)^{-1}$ and integrating yields

$$u_x(x, t; \alpha, \beta) = m(t) \left(x + \sqrt{x^2 + \frac{\beta}{\alpha} e^{-\alpha t}}\right)^{-\frac{1}{\sqrt{\alpha}}} \tag{3.14}$$

for some function $m : [0, \infty) \to \mathcal{R}^+$. In turn,

$$u_{xx}(x, t; \alpha, \beta) = -m(t) \frac{\left(x + \sqrt{x^2 + \frac{\beta}{\alpha} e^{-\alpha t}}\right)^{-\frac{1}{\sqrt{\alpha}}}}{\sqrt{\alpha x^2 + \beta e^{-\alpha t}}}.$$

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From equation (2.8) we, then, deduce that

$$u_t(x,t;\alpha,\beta) = -\frac{1}{2}m(t)\left(x + \sqrt{x^2 + \frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\frac{1}{\sqrt{\alpha}}}\sqrt{\alpha x^2 + \beta e^{-\alpha t}}.$$

Integrating yields, for $\alpha = 1$,

$$u(x,t;1,\beta) = -\frac{1}{2\beta}m(t)\left(e^t x^2 - e^t x\sqrt{x^2 + \beta e^{-t}} - \beta \log\left(x + \sqrt{x^2 + \beta e^{-t}}\right)\right) + n(t)$$

while, for $\alpha \neq 1$,

$$u(x,t;\alpha,\beta) = m(t)\frac{\sqrt{\alpha}}{\alpha - 1}\left(x + \sqrt{x^2 + \frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\left(1 + \frac{1}{\sqrt{\alpha}}\right)} \times$$

$$\times \left(\frac{\beta}{\alpha}e^{-\alpha t} + (1 + \sqrt{\alpha})x\left(x + \sqrt{x^2 + \frac{\beta}{\alpha}e^{-\alpha t}}\right)\right) + n(t).$$

We analyze only the latter case. Differentiating the above gives

$$u_t(x,t) = n'(t) + \left(x + \sqrt{x^2 + \frac{\beta}{\alpha}e^{-\alpha t}}\right)^{-\left(1 + \frac{1}{\sqrt{\alpha}}\right)} \times$$

$$\times \left(\beta e^{-\alpha t}\left(\frac{m'(t)}{\sqrt{\alpha}(\alpha - 1)} - \frac{m(t)}{2(\sqrt{\alpha} + 1)}\right) + m'(t)\frac{\sqrt{\alpha}}{\sqrt{\alpha} - 1}x\left(x + \sqrt{x^2 + \frac{\beta}{\alpha}e^{-\alpha t}}\right)\right).$$

Reconciling the above two expressions for $u_t(x,t)$ yields

$$m'(t) = -\frac{\sqrt{\alpha} - 1}{2}m(t) \quad \text{and} \quad n'(t) = 0.$$

Thus, $m(t) = Me^{-\frac{\sqrt{\alpha}-1}{2}}$ and $n(t) = N$, and (3.12) follows. $\square$

The initial value u_0, derived from (3.12) and (3.13) for $t = 0$, will be needed for special cases presented in the sequel. For convenience we write it below, namely, for $x \in \mathcal{R}$, $\alpha > 0$ ($\alpha \neq 1$),

$$u_0(x;\alpha,\beta) = M\frac{(\sqrt{\alpha})^{1+\frac{1}{\sqrt{\alpha}}}}{\alpha - 1}\frac{\left(\frac{\beta}{\sqrt{\alpha}} + (1 + \sqrt{\alpha})x\left(\sqrt{\alpha}x + \sqrt{\alpha x^2 + \beta}\right)\right)}{\left(\sqrt{\alpha}x + \sqrt{\alpha x^2 + \beta}\right)^{1+\frac{1}{\sqrt{\alpha}}}} + N \tag{3.15}$$

while for $\alpha = 1$,

$$u_0(x,1,\beta) = \frac{M}{2}\left(\log\left(x + \sqrt{x^2 + \beta}\right) - \frac{x}{\beta}\left(x - \sqrt{x^2 + \beta}\right)\right) + N. \tag{3.16}$$

Once the differential input is specified, the construction of the forward performance process is an immediate application of Theorem 2.1.

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Proposition 3.3. *Let the local risk tolerance and* (Y, Z, A) *be as in (3.6) and (2.5), (2.6) and (2.7). Then, for* $x \in \mathcal{R}$ *and* $t \geq 0$, *the process*

$$U_t(x; \alpha, \beta) = u\left(\frac{x}{Y_t}, A_t; \alpha, \beta\right) Z_t, \qquad (3.17)$$

with $u(x, t; \alpha, \beta)$ *given in Proposition 3.2, is a forward performance.*

Remark: It is important to notice that in the classical case, the power and logarithmic utilities u^l and u^p (cf. (3.1) and (3.2)) are not everywhere defined. This restraints the applicability of such preferences especially when we introduce derivatives and liabilities. Observe, however, that their time-dependent forward counterparts, (3.12) and (3.13), are spatially globally defined. For this reason, the above process $U_t(x; \alpha, \beta)$ is also globally defined. The situation changes, however, when the time dependence disappears which occurs when $\beta \to 0$ and/or $\alpha \to 0$. These cases deserve special attention and are discussed separately (see Section 5).

In the second panel of Figure 1, we provide the graph of the function $u(x, t; \alpha, \beta)$ (cf. (3.12)), for $\alpha = 4$ and $\beta = 0.1$. We, also, provide the cross-sections $u(x, t_0; \alpha, \beta)$ and $u(x_0, t; \alpha, \beta)$. The first panel of Figure 2 shows, for fixed time t_0, the monotonicity and concavity of $u(x, t_0; \alpha, \beta)$ while the second panel shows the monotonicity of $u(x_0, t; \alpha, \beta)$ in terms of time.

4   At the optimum

We provide explicit solutions for the optimal investment policies, the associated wealth and the optimal investment performance. The key ingredients used in the construction of these processes are the autonomous system that the optimal wealth and risk tolerance processes satisfy (cf. (2.23)) together with the specific form of the local risk tolerance function (cf. (3.6)). We remind the reader that the results are stated in the benchmarked configuration.

Theorem 4.1. *The processes* $\tilde{X}_t^*$ *and* $\tilde{R}_t^*$, $t > 0$, *representing the optimal (benchmarked) wealth and risk tolerance solve the system of linear stochastic differential equations*

$$\begin{cases} d\tilde{X}_t^* = \tilde{R}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t) \, dt + dW_t) \\[2mm] d\tilde{R}_t^* = \alpha \tilde{X}_t^* \sigma_t n_t \cdot ((\lambda_t - \delta_t) \, dt + dW_t) \end{cases} \qquad (4.1)$$

with $\tilde{X}_0^* = x$ *and* $\tilde{R}_0^* = r(x, 0) = \sqrt{\alpha x^2 + \beta}$.

In turn,

$$\tilde{X}_t^* = e^{-\frac{\alpha}{2} \int_0^t |\sigma_s n_s|^2 ds} \left(x \cosh\left(\sqrt{\alpha} k_t\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha} k_t\right) \right) \qquad (4.2)$$

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and

$$\tilde{R}_t^* = e^{-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds} \left(\sqrt{\alpha}x \sinh\left(\sqrt{\alpha}k_t\right) + \sqrt{\alpha x^2 + \beta} \cosh\left(\sqrt{\alpha}k_t\right) \right), \quad (4.3)$$

where n_t, $t > 0$, as in (2.21) and

$$k_t = \int_0^t \sigma_s \sigma_s^+ \left(\lambda_s + \phi_s - \delta_s\right) \cdot \left(\left(\lambda_s - \delta_s\right) ds + dW_s\right). \quad (4.4)$$

The vector of optimal asset allocations is given by

$$\tilde{\pi}_t^* = m_t \tilde{X}_t^* + n_t \tilde{R}_t^* \quad (4.5)$$

with $\tilde{X}_t^*$, $\tilde{R}_t^*$ as above and m_t as in (2.21).

Proof. The coefficients in (4.1) follow from Theorem 2.2 (see (2.18) and (2.19)) and (3.6). The admissibility conditions for the optimal policy follow from the boundedness assumption on the market coefficients. Indeed, one can easily see that the integrability condition $E_{\mathbb{P}}\int_0^s |\pi_t^*|^2 dt < \infty$ holds for $0 \le t \le s$ and that the wealth process X_t^*, $0 \le t \le s$, is a $\mathbb{Q}|_{\mathcal{F}_s}$-martingale where

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_s} = \exp\{-\int_0^s \lambda_t \cdot dW_t - \frac{1}{2}\int_0^s |\lambda_t|^2 ds\}.$$

The arguments in the benchmarked configuration follow easily as well.

Adding and subtracting the equations in (4.1) yields

$$d\left(\sqrt{\alpha}\tilde{X}_t^* + \tilde{R}_t^*\right) = \sqrt{\alpha}\left(\sqrt{\alpha}\tilde{X}_t^* + \tilde{R}_t^*\right)\sigma_t n_t \cdot \left((\lambda_t - \delta_t)\, dt + dW_t\right)$$

and

$$d\left(\sqrt{\alpha}\tilde{X}_t^* - \tilde{R}_t^*\right) = -\sqrt{\alpha}\left(\sqrt{\alpha}\tilde{X}_t^* - \tilde{R}_t^*\right)\sigma_t n_t \cdot \left((\lambda_t - \delta_t)\, dt + dW_t\right)$$

and we easily conclude. $\qquad\square$

For completeness, we provide the optimal allocations π^* and wealth X^* in the original (non-benchmarked) formulation. Recall (see (2.13) and (2.5)) that, for $t > 0$,

$$X_t^* = Y_t \tilde{X}_t^* \quad \text{and} \quad \pi_t^* = m_t X_t^* + n_t Y_t r\left(\frac{X_t^*}{Y_t}, A_t\right).$$

Proposition 4.1. *Let $x \in \mathcal{R}$ be the investor's initial endowment. Then, the optimal allocation vector and associated optimal wealth are given, respectively, by*

$$\pi_t^* = e^{\zeta_t} m_t \left(x \cosh\left(\sqrt{\alpha}k_t\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha}k_t\right) \right)$$
$$+ e^{\zeta_t} n_t \left(\sqrt{\alpha}x \cosh\left(\sqrt{\alpha}k_t\right) + \sqrt{\alpha x^2 + \beta} \sinh\left(\sqrt{\alpha}k_t\right) \right), \quad t > 0, \tag{4.6}$$

and

$$X_t^* = e^{\zeta_t} \left(x \cosh\left(\sqrt{\alpha}k_t\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha}k_t\right) \right), \quad t \geq 0 \tag{4.7}$$

where

$$\zeta_t = \int_0^t \left(\delta_s \cdot \lambda_s - \frac{1}{2}|\delta_s|^2 - \frac{\alpha}{2}|\sigma_s n_s|^2 \right) ds + \int_0^t \delta_s \cdot dW_s \tag{4.8}$$

and m_t, n_t and k_t as in (2.21) and (4.4).

Next we look at the extreme cases $m_t = n_t = 0$, $t > 0$ leading, respectively, to $\tilde{\pi}_t^{*,X} = 0$ and $\tilde{\pi}_t^{*,R} = 0$. It is easy to check that they reduce to $\delta_t = 0$ and $\lambda_t + \phi_t - \delta_t = 0$, $t \geq 0$.

i) Absence of benchmark: $\delta_t = 0$. Then (2.5) yields $Y_t = Y_0 = 1$, $t \geq 0$.

Then, the first portfolio component vanishes, $\pi_t^{*,X} = 0$, while the second simplifies to

$$\pi_t^{*,R} = e^{-\frac{\alpha}{2}\int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s)|^2 ds} \sigma_t^+ (\lambda_t + \phi_t) \times$$
$$\times \left(\sqrt{\alpha}x \cosh\left(\sqrt{\alpha}k_t'\right) + \sqrt{\alpha x^2 + \beta} \sinh\left(\sqrt{\alpha}k_t'\right) \right)$$

with

$$k_t' = \int_0^t \sigma_s \sigma_s^+ (\lambda_s + \phi_s) \cdot (\lambda_s ds + dW_s). \tag{4.9}$$

The optimal wealth is given by

$$X_t^* = e^{-\frac{\alpha}{2}\int_0^t |\sigma_s \sigma_s^+ (\lambda_s + \phi_s)|^2 ds} \left(x \cosh\left(\sqrt{\alpha}k_t'\right) + \sqrt{x^2 + \frac{\beta}{\alpha}} \sinh\left(\sqrt{\alpha}k_t'\right) \right).$$

The (sub)case $\lambda_t + \phi_t = 0$ deserves special attention since $\pi_t^{*,R}$ also vanishes. Moreover, $A_t = 0$, $t \geq 0$, which leads to the performance process

$$U_t (x, t; \alpha, \beta) = u_0 (x; \alpha, \beta) Z_t \tag{4.10}$$

with u_0 as in (3.15) or (3.16). Moreover,

$$\tilde{\pi}_t^{*,X} = \pi_t^{*,X} = 0 \quad \text{and} \quad \tilde{\pi}_t^{*,R} = \pi_t^{*,R} = 0, \quad t \geq 0$$

and, in turn,

$$\tilde{X}_t^* = X_t^* = x, \quad t \geq 0.$$

At the optimum,

$$U_t^* (x; \alpha, \beta) = U_t (x; \alpha, \beta) = u_0 (x; \alpha, \beta) Z_t.$$

The above results show that for the above choice of coefficients ($\lambda_t + \phi_t = 0$ and $\delta_t = 0$, $t \geq 0$), it is optimal for the investor to invest *zero wealth into each risky asset,* a result that comes as a surprise given the non-zero returns. Notice that such a solution seems to capture quite accurately the strategy of a *derivatives' trader* for whom the underlying objective is to hedge as opposed to the asset manager whose objective is to invest. Naturally, under this strategy, the forward performance process is not affected by the time evolution of u. This a direct consequence of the fact that the time-rescaling process A degenerates.

ii) Tracking the benchmark: $\lambda_t + \phi_t - \delta_t = 0$, $t \geq 0$.

In this case, the portfolio $\tilde{\pi}^{*,R}$ vanishes and, thus, any dependence on the risk tolerance dissipates. The investor invests the fraction m_t of his (benchmarked) wealth to the risky assets and puts the rest in the riskless bond. We have $A_t = 0$, $t \geq 0$ and, thus, the performance process is given by (4.10). Moreover,

$$\tilde{\pi}_t^{*,X} = m_t \tilde{X}_t^* \quad \text{and} \quad \tilde{\pi}_t^{*,R} = 0, \quad t > 0.$$

The absolute wealth tracks the benchmark while the (benchmarked) risk tolerance process remains unchanged,

$$X_t^* = x Y_t \quad \text{and} \quad \tilde{R}_t^* = \tilde{R}_0^* = \sqrt{\alpha x^2 + \beta}.$$

At the optimum,

$$U_t^* (x; \alpha, \beta) = u_0 \left(\frac{X_t^*}{Y_t}; \alpha, \beta \right) Z_t = u_0 (x; \alpha, \beta) Z_t.$$

Remark: The above result shows that the investor allocates in the riskless asset the amount $\tilde{\pi}_t^{*,0} = p_t X_t^*$ with $p_t = 1 - m_t \cdot \mathbf{1}$. Notice that depending on the level of the *weight process* p_t, $t \geq 0$, which is determined only by the market parameters, the investor allocates *arbitrarily* small or large proportions of the wealth in the riskless asset. In the extreme case, $p_t = 0$, $t \geq 0$, the investor allocates zero wealth in the riskless asset while in the other such case, namely when $p_t = 1$, $t \geq 0$, the optimal allocation consists of putting all wealth in the riskless asset.

5 Special cases: CARA and CRRA forward performance processes

We now look at the behavior of the solutions when the parameters α and β vanish. Recalling equalities (3.7), (3.8) and (3.9), we anticipate that the limiting risk tolerance and differential input must resemble their classical power, logarithmic and exponential analogues. While passing to the limit in (3.6), and (3.12) and (3.13) is not difficult from the technical point of view, the emerging limits have some noteworthy properties. To simplify the notation, we skip throughout the parameter notation and use, instead, the superscripts e, p and l in a self-evident way.

i) The case $\alpha = 0$.

Passing to the limit in (3.6) and (3.12) yields, for $t \geq 0$,

$$\lim_{\alpha \to 0} r(x, t; \alpha, \beta) = \sqrt{\beta}, \quad x \in \mathcal{R} \tag{5.1}$$

and

$$u^e(x, t) = \lim_{\alpha \to 0} u(x, t; \alpha, \beta) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \quad x \in \mathcal{R}, \tag{5.2}$$

where we chose, for simplicity, $M = (\sqrt{\alpha})^{\frac{1}{\sqrt{\alpha}}} (\sqrt{\beta})^{\frac{1-\sqrt{\alpha}}{\sqrt{\alpha}}}$ and $N = 0$ [3]; Figure 3 demonstrates this convergence.

One, easily, sees that the limiting local risk tolerance (5.1) leads to an exponential forward performance process. This class of solutions was extensively analyzed in Musiela and Zariphopoulou [2006b, 2007a] and we refer the reader therein for detailed arguments.

Proposition 5.1. *For $\alpha = 0$, $\beta > 0$, $t \geq 0$, $x \in \mathcal{R}$, and (Y, Z, A) as in (2.5), (2.6), and (2.7), the process*

$$U_t^e(x) = -\exp\left(-\frac{1}{\sqrt{\beta}} \frac{x}{Y_t} + \frac{A_t}{2}\right) Z_t$$

is a forward performance. Moreover, the optimal (benchmarked) investment strategy and the associated wealth are given by the processes

$$\tilde{\pi}_t^{*, e} = \left(x + \sqrt{\beta} k_t\right) m_t + \sqrt{\beta} n_t \quad and \quad \tilde{X}_t^{*, e} = x + \sqrt{\beta} k_t \tag{5.3}$$

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[3] For the second limit, we use in (3.12) that for $\beta > 0$, $x \in \mathcal{R}$,

$$\lim_{\alpha \to 0} \left(\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\alpha}{\beta} x^2 + 1}\right)^{-\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}}} = e^{-\frac{x}{\sqrt{\beta}}}.$$

with n_t, k_t as in (2.21) and (4.4).

At the optimum,

$$U_t^e\left(X_t^*\right) = -\exp\left(-\frac{x}{\sqrt{\beta}} - k_t + \frac{1}{2}\int_0^t |\sigma_s n_s|^2 ds\right) Z_t.$$

Remark: It is interesting to observe that due to the presence of the benchmark the optimal investment policy *depends* on the current wealth. This is in contrast to the known results which yield wealth independent policies, a fact that is frequently used against the use of exponential preferences in models of investment and (indifference) valuation.

Next, we write the solutions when both the benchmark and the market view process are absent.

Corollary 5.1. *Let* $\delta_t = \phi_t = 0$, $t \geq 0$. *Then,*

$$U_t^e\left(x\right) = -\exp\left(-\frac{1}{\sqrt{\beta}}x + \frac{1}{2}\int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds\right). \tag{5.4}$$

Moreover,

$$X_t^{*,e} = x + \sqrt{\beta}\int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s) \quad and \quad \pi_t^{*,e} = \sqrt{\beta}\sigma_t^+ \lambda_t.$$

ii) The case $\beta = 0$.

Passing to the limit in (3.6) yields, for $t \geq 0$,

$$\lim_{\beta \to 0} r\left(x, t; \alpha, \beta\right) = \sqrt{\alpha}|x|, \quad x \in \mathcal{R}. \tag{5.5}$$

In turn, for $\alpha > 1$ $(\alpha < 1)$, (3.12) gives

$$u^p\left(x, t\right) = \lim_{\beta \to 0} u\left(x, t; \alpha, \beta\right) = \begin{cases} \dfrac{1}{\gamma}x^\gamma e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t} & \text{for } x \geq 0 \ (x > 0) \\ -\infty & \text{for } x < 0 \ (x \leq 0) \end{cases} \tag{5.6}$$

with

$$\gamma = \frac{\sqrt{\alpha} - 1}{\sqrt{\alpha}}, \quad \alpha > 0, \tag{5.7}$$

and where we chose the constants $M = 2^{\frac{1}{\sqrt{\alpha}}}$ and $N = 0$.

For $\alpha = 1$, (3.13) yields

$$u^l\left(x, t\right) = \lim_{\beta \to 0} u\left(x, t; 1, \beta\right) = \begin{cases} \log x - \dfrac{1}{2}t & \text{for } x > 0 \\ -\infty & \text{for } x \leq 0 \end{cases} \tag{5.8}$$

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for the choice $M = 2$ and $N = -\left(\frac{1}{2} + \log 2\right)$.

The limiting behavior of the differential inputs $u(x, t; \alpha, \beta)$ and $u(x, t; 1, \beta)$ when $\beta \to 0$ is shown in Figures 4 and 5.

We see that while the local risk tolerance in (5.5) is well defined for *all* $x \in \mathcal{R}$, the associated differential inputs u^p and u^l explode for non positive wealth levels. This impedes us from having globally defined forward performance processes. A well-defined problem may be formulated if we a priori constrain the set of admissible policies to strategies which generate nonnegative wealth. A modification of the proofs of Theorems 2.1 and 2.2 yields the following results.

Proposition 5.2. *Let the local risk tolerance be given by*

$$r(x, t; \alpha, 0) = \sqrt{\alpha} x,$$

for $x \geq 0$ when $\alpha > 1$ and $x > 0$ when $\alpha < 1$ ($\alpha \neq 0$). Let, also, (Y, Z, A) be as in (2.5), (2.6) and (2.7). Then, for $\alpha > 1$ ($\alpha < 1$), the process

$$U_t^p(x) = \frac{1}{\gamma}\left(\frac{x}{Y_t}\right)^{\gamma} e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}A_t} Z_t, \quad x \geq 0 \ (x > 0), \tag{5.9}$$

is a forward performance. Moreover, the optimal investment strategy and associated wealth processes are given by

$$\tilde{\pi}_t^{*,p} = x\left(m_t + \sqrt{\alpha} n_t\right) \exp\left(-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds + \sqrt{\alpha} k_t\right)$$

and

$$\tilde{X}_t^{*,p} = x \exp\left(-\frac{\alpha}{2}\int_0^t |\sigma_s n_s|^2 ds + \sqrt{\alpha} k_t\right),$$

with n_t, k_t as in (2.21) and (4.4).

At the optimum,

$$U_t^p\left(X_t^{*,p}\right)$$

$$= \frac{1}{\gamma} x^{\gamma} \exp\left(-\frac{\alpha - 1}{2}\int_0^t |\sigma_s n_s|^2 ds + (\sqrt{\alpha} - 1)k_t\right) Z_t, \quad \text{for } x \geq 0 \ (x > 0).$$

Similar results can be obtained for the logarithmic case.

Proposition 5.3. *Let the local risk tolerance be given by*

$$r(x, t; 1, 0) = x, \quad x > 0.$$

Then, the process

$$U_t^l(x) = \left(\log \frac{x}{Y_t} - \frac{A_t}{2}\right) Z_t, \quad x > 0$$

is a forward performance. Moreover, the optimal investment strategy and associated wealth processes are given by

$$\tilde{\pi}_t^{*,l} = x(m_t + n_t)\exp(-\frac{1}{2}\int_0^t |\sigma_s n_s|^2 ds + k_t)$$

and

$$\tilde{X}_t^{*,l} = x\exp\left(-\frac{1}{2}\int_0^t |\sigma_s n_s|^2 ds + k_t\right).$$

At the optimum,

$$U_t^l\left(X_t^{*,l}\right) = \left(\log x - \int_0^t |\sigma_s n_s|^2 ds + k_t\right) Z_t$$

with n_t, k_t as in (2.21) and (4.4).

In analogy to Corollary 5.1, we look at the case of no benchmark and no alternative market views.

Corollary 5.2. *Let $\delta_t = \phi_t = 0$, $t \geq 0$ and $\beta = 0$. Then, for $\alpha > 1$ ($\alpha < 1$),*

$$U_t^p(x) = \frac{1}{\gamma}x^\gamma \exp\left(-\frac{\gamma}{2(1-\gamma)}\int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds\right), \quad x \geq 0 \ (x > 0). \quad (5.10)$$

Moreover,

$$\pi_t^{*,p} = \sqrt{\alpha}x\sigma_t^+ \lambda_t \exp\left(-\frac{\alpha}{2}\int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \sqrt{\alpha}\int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right)$$

and

$$X_t^{*,p} = x\exp\left(-\frac{\alpha}{2}\int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \sqrt{\alpha}\int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right).$$

Corollary 5.3. *Let $\delta_t = \phi_t = 0$, $t \geq 0$ and $\beta = 0$. Then, for $\alpha = 1$,*

$$U_t^l(x) = \left(\log x - \frac{1}{2}\int_0^t |\sigma_s \sigma^+ \lambda_s|^2 ds\right), \quad x > 0. \quad (5.11)$$

Moreover,

$$\pi_t^{*,l} = x\sigma_t^+ \lambda_t \exp\left(-\frac{1}{2}\int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right)$$

and

$$X_t^{*,l} = x\exp\left(-\frac{1}{2}\int_0^t |\sigma_s \sigma_s^+ \lambda_s|^2 ds + \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot (\lambda_s ds + dW_s)\right).$$

When the market coefficients are constants, the forward processes $U_t^e(x)$, $U_t^p(x)$ and $U_t^l(x)$ in (5.4), (5.10) and (5.11) reduce to deterministic functions. These special cases can be found in Henderson and Hobson [2007a,b].

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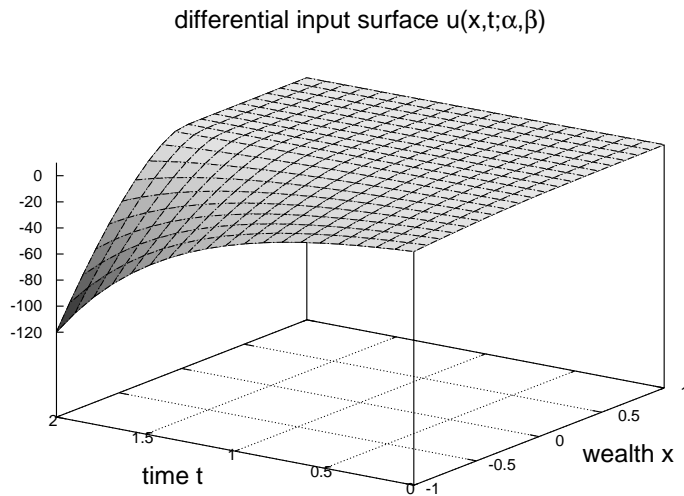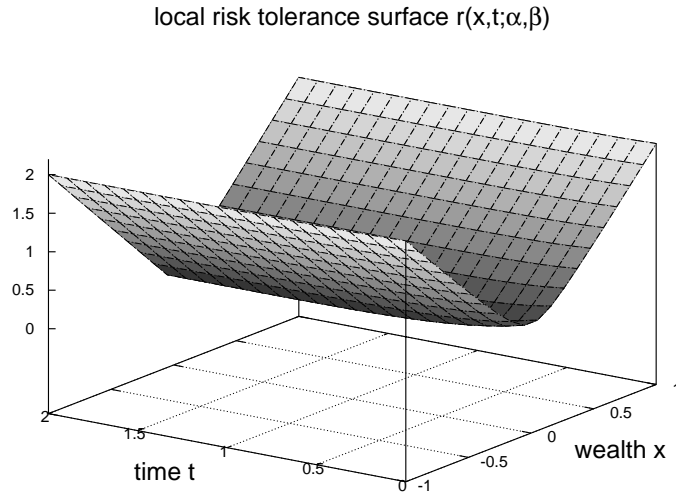
local risk tolerance surface r(x,t;α,β)

differential input surface u(x,t;α,β)

Figure 1: **The risk tolerance and differential input surfaces.** For parameters $\alpha = 4$ and $\beta = 0.1$, this figure presents the local risk tolerance surface $r(x, t; \alpha, \beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}$ (first panel) and the differential input surface $u(x, t; \alpha, \beta)$ given in (3.12), for $M = 1$ and $N = 0$ (second panel).
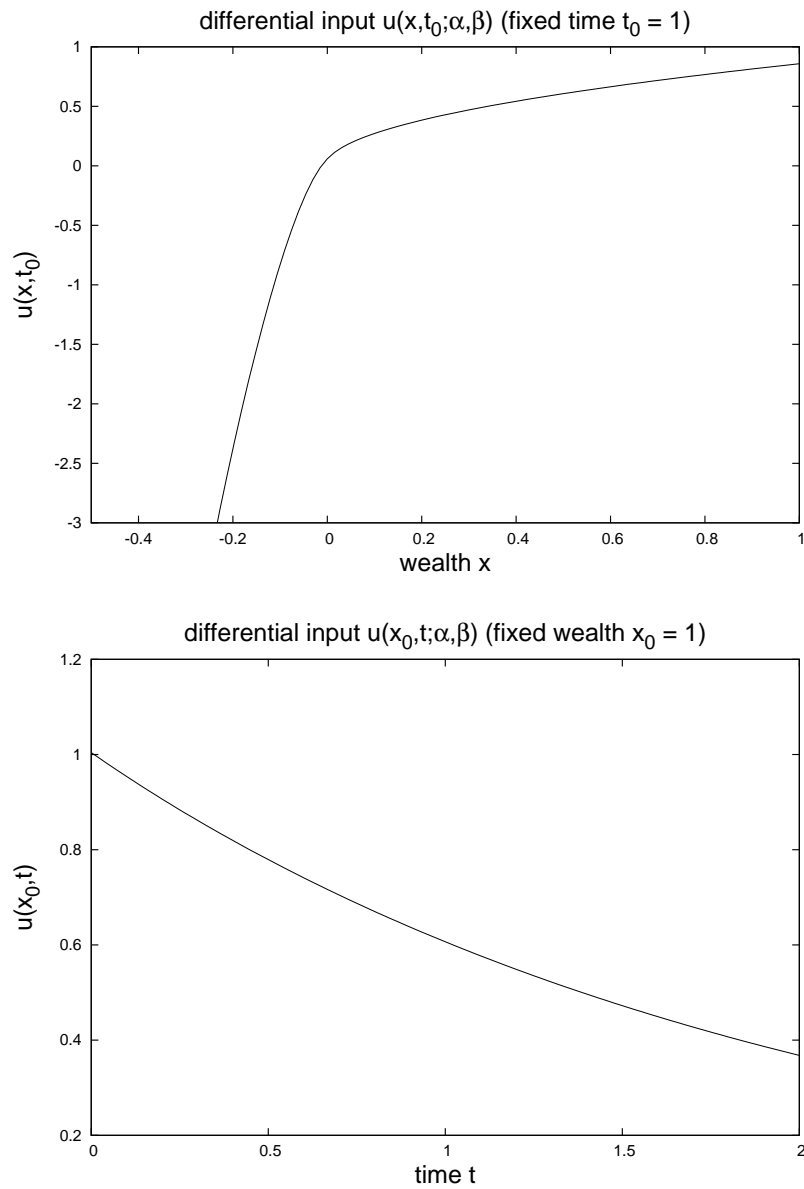
Figure 2: **Cross sections of the differential input.** For parameters $\alpha = 4$ and $\beta = 0.1$, this figure presents the cross sections of the differential input surface $u(x, t; \alpha, \beta)$ given in (3.12), for $M = 1$ and $N = 0$. The first panel corresponds to $u(x, t_0; \alpha, \beta)$, with $t_0 = 1$. The second panel corresponds to $u(x_0, t; \alpha, \beta)$, with $x_0 = 1$.
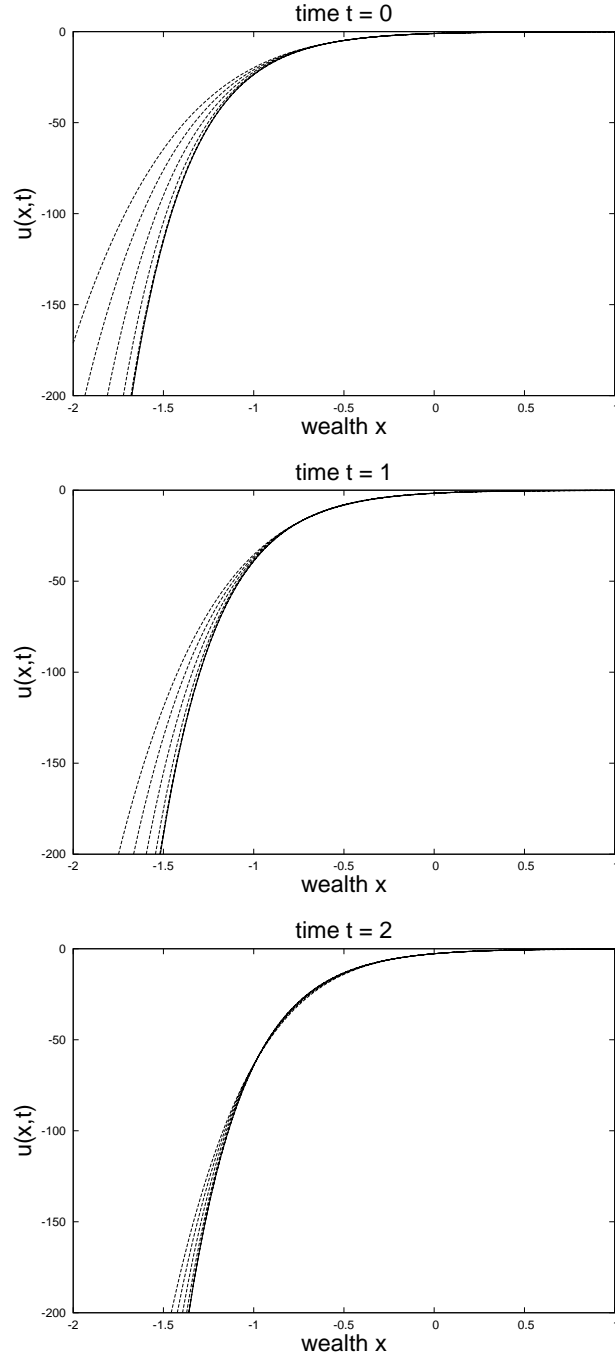
Figure 3: **Convergence to the exponential case.** We choose $\beta = 0.1$. For times $t = 0, 1, 2$, the three panels demonstrate the convergence, as $\alpha \to 0$, of the differential input $u(x, t; \alpha, \beta)$, given in (3.12), for $M = (\sqrt{\alpha})^{\frac{1}{\sqrt{\alpha}}} (\sqrt{\beta})^{\frac{1-\sqrt{\alpha}}{\sqrt{\alpha}}}$ and $N = 0$. The curve of solid line corresponds to the exponential differential input $u^e(x, t) = \lim_{\alpha \to 0} u(x, t; \alpha, \beta) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{1}{2}t}$. The curves of dotted lines correspond to $u(x, t; \alpha, \beta)$ for $\alpha = 1 \times 10^{-1}, 6 \times 10^{-2}, 3 \times 10^{-2}, 1 \times 10^{-2}, 1 \times 10^{-3}$ and 1×10^{-4}, respectively.
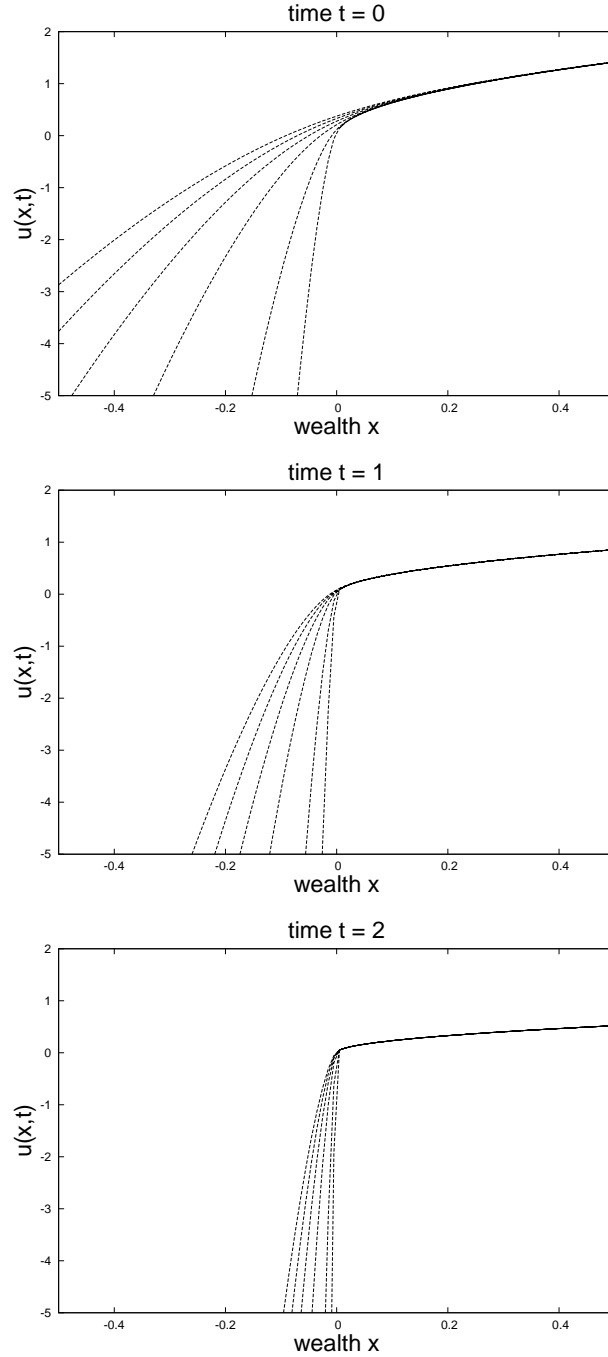
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Figure 4: **Convergence to the power case.** We choose $\alpha = 4$. For times $t = 0, 1, 2$, the three panels demonstrate the convergence, as $\beta \to 0$, of the differential input $u(x, t; \alpha, \beta)$, given in (3.12), for $M = 2^{\frac{1}{\sqrt{\alpha}}}$ and $N = 0$. The curve of solid line corresponds to the power differential input $u^p(x, t) = \lim_{\beta \to 0} u(x, t; \alpha, \beta) = \frac{1}{\gamma} x^\gamma e^{-\frac{1}{2} \frac{\gamma}{1-\gamma} t}$. The curves of dotted lines correspond to $u(x, t; \alpha, \beta)$ for $\beta = 1 \times 10^{-1}$, 6×10^{-2}, 3×10^{-2}, 1×10^{-2}, 1×10^{-3} and 1×10^{-4}, respectively.
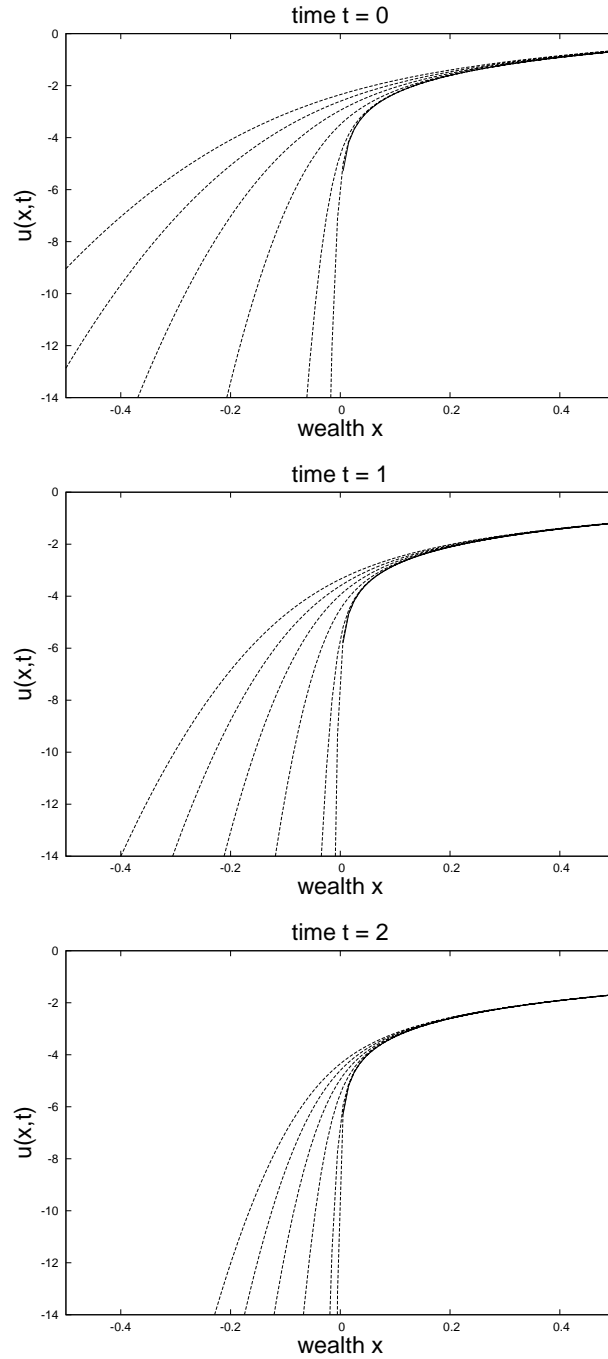
Figure 5: **Convergence to the logarithmic case.** For times $t = 0, 1, 2$, the three panels demonstrate the convergence, as $\beta \to 0$, of the differential input $u(x, t; \alpha, \beta)$, given in (3.13), for $M = 2$ and $N = -\left(\frac{1}{2} + \log 2\right)$. The curve of solid line corresponds to the logarithmic differential input $u^l(x, t) = \lim_{\beta \to 0} u(x, t; 1, \beta) = \log(x) - \frac{1}{2}t$. The curves of dotted lines correspond to $u(x, t; \alpha, \beta)$ for $\beta = 1 \times 10^{-1}$, 6×10^{-2}, 3×10^{-2}, 1×10^{-2}, 1×10^{-3} and 1×10^{-4}, respectively.