Portfolio choice under dynamic investment performance criteria*

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Abstract

A new dynamic criterion of measuring performance of self-financing investment strategies is introduced. To this aim, a family of stochastic processes defined on $[0, +\infty)$ and indexed by a wealth argument is used. Optimality is associated with their martingale property along the optimal wealth trajectory. The optimal portfolios are constructed via stochastic feedback controls that are functionally related to differential constraints of fast diffusion type. A multi-asset Ito type incomplete market model is used.

1 Introduction

This paper proposes new ways of measuring the performance of investment strategies under uncertainty. Traditionally, how well the investor does is assessed through expected utility criteria, typically formulated via a deterministic, concave and increasing function of terminal wealth. A key element of this approach is the a priori choice of both the horizon and the associated risk preferences. The optimal solution (value function) has been widely studied under rather general modeling assumptions. Its fundamental properties, consequences of the dynamic programming principle, are the supermartingality for arbitrary investment strategy and martingality at an optimum. The value function, then, serves as the intermediate (indirect) utility in the relevant market environment (see, for example, [18] and [11]).

Herein, an alternative approach is proposed which offers flexibility with regards to the aforementioned a priori choices while preserving the natural optimality properties of the value function process (martingality at an optimum and supermartingality away from it). In contrast to the existing framework, the

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utility is specified for today and not for a (possibly remote) future time. The performance measurement criterion is defined in terms of a family of stochastic processes defined on $[0, \infty)$ and indexed by a wealth argument. We call it a *forward performance process*.

Several difficulties are encountered due to the fact that the associated stochastic optimization problems are posed "inversely in time" and, thus, existing techniques in portfolio choice might not be directly applicable. Herein, we develop a technique that can be used for a large class of models and produces a rich family of explicit solutions. The approach is based on the compilation of appropriately constructed differential and stochastic inputs. The differential input is determined by the investor's dynamic preference profile and satisfies a fully nonlinear differential constraint. The stochastic input consists of three processes that capture the changes in the market environment. An important ingredient of the method is the introduction of the process that rescales the differential input's time argument.

The initial utility is taken to be a concave and increasing function of wealth. The model is incomplete, non Markovian and may include many securities. The approach is general enough so that it allows for measuring investment performance with regards to a benchmark as well as when the investor might have different views about upcoming market behavior.

The risk tolerance process plays a fundamental role in the analysis. It is defined as the local risk tolerance function with its space and time arguments evaluated, respectively, at (benchmarked) wealth and the process that rescales time. The former function satisfies a fast diffusion type differential constraint while its reciprocal, the investor's local risk aversion, solves a porous medium equation.

The proposed method provides closed form solutions to the optimal allocation problems. Despite the non Markovian nature of the model, optimal allocations turn out to be stochastic feedback functionals of current wealth levels. Rescaling of time in the differential input is a key element for this local dependence. The optimal policies have a very appealing form. Specifically, they consist of two portfolios that are, respectively, proportional to (benchmarked) wealth and the (benchmarked) risk tolerance processes. The proportionality coefficients are processes depending only on the market parameters. This two fund separation result holds for arbitrary initial data and provides a rather universal, and at the same time, intuitive structure of the optimal strategies. It is worth mentioning that in traditional expected utility models the form of the optimal portfolios might not be very transparent since they are implicitly deduced through martingale representation results in the dual domain. Finally, the form of the optimal portfolios, together with the differential properties of the local risk tolerance, enable us to construct a system of stochastic differential equations that is satisfied by the optimal wealth and the associated risk tolerance process. This autonomous system also comes as a surprise given the non Markovian character of the model.

The paper is organized as follows. In section 2, we introduce the model and the notion of forward performance process. In section 3, we present a special class of forward solutions, namely those that are decreasing with time. An extended family of forward processes is presented in section 4. In section 5 we focus on the risk tolerance function and its differential properties. We conclude with section 6 in which we present and analyze the optimal investment strategies.

2 The model

The market environment consists of one riskless and k risky securities. The risky securities are stocks and their prices are modelled as Ito processes. Namely, for $i = 1, ..., k$, the price S^i of the i^{th} risky asset solves

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right) \tag{1}$$

with $S_0^i > 0$. The process $W = \left(W^1, ..., W^d\right)$ is a standard $d-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, it is assumed that the underlying filtration, $\mathcal{F}_t$, coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma\left(W_s : \ 0 \leq s \leq t\right)$.

The coefficients μ^i and σ^i, $i = 1, ..., k$, follow $\mathcal{F}_t-$adapted processes with values in $\mathcal{R}$ and $\mathcal{R}^d$, respectively. For brevity, we write $\sigma = \sigma_t$ to denote the volatility matrix, i.e., the $d \times k$ random matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. We may, then, alternatively write (1) as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right).$$

The riskless asset, the savings account, has the price process B satisfying

$$dB_t = r_t B_t dt$$

with $B_0 = 1$, and for a nonnegative, $\mathcal{F}_t-$adapted interest rate process r_t. Also, we denote by μ_t the $k \times 1$ vector with the coordinates μ_t^i and by $\mathbf{1}$ the $k-$dimensional vector with every component equal to one. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that the volatility vectors are such that

$$\mu_t - r_t \mathbf{1} \in Lin\left(\sigma_t^T\right),$$

i.e., the linear space generated by the columns of σ_t^T. This implies that

$$\sigma_t^T \left(\sigma_t^T\right)^+ \left(\mu_t - r_t \mathbf{1}\right) = \mu_t - r_t \mathbf{1}$$

and therefore the vector

$$\lambda_t = \left(\sigma_t^T\right)^+ \left(\mu_t - r_t \mathbf{1}\right) \tag{2}$$

is a solution to the equation

$$\sigma_t^T x = \mu_t - r_t \mathbf{1}.$$

The matrix $\left(\sigma_t^T\right)^+$ is the Moore-Penrose pseudo-inverse[1] of the matrix σ_t^T.

Occasionally, we will be referring to λ_t as the market price of risk. It easily follows that

$$\sigma_t \sigma_t^+ \lambda_t = \lambda_t \tag{3}$$

and hence $\lambda_t \in Lin\left(\sigma_t\right)$. We assume throughout that the process λ_t is bounded by a deterministic constant $c > 0$, i.e., for all $t \geq 0$,

$$\left|\lambda_t\right| \leq c. \tag{4}$$

Starting at $t = 0$ with an initial endowment $x \in \mathcal{R}$, the investor invests at any time $t > 0$ in the risky and riskless assets. The present value of the amounts invested are denoted by π_t^0 and π_t^i , $i = 1, ..., k$, respectively.

The present value of her investment is, then, given by $X_t^\pi = \sum_{i=0}^k \pi_t^i$. We will refer to X^π as the discounted wealth. The investment strategies will play the role of control processes and are taken to satisfy the standard assumption of being self-financing. Using (1) we, then, deduce that the discounted wealth satisfies

$$dX_t^\pi = \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot \left(\lambda_t dt + dW_t\right) = \sigma_t \pi_t \cdot \left(\lambda_t dt + dW_t\right), \tag{5}$$

where the (column) vector, $\pi_t = \left(\pi_t^i;\ i = 1, ..., k\right)$.

The set of admissible strategies, $\mathcal{A}$, consists of all self-financing $\mathcal{F}_t-$adapted processes π_t such that $E_{\mathbb{P}} \int_0^s \left|\sigma_t \pi_t\right|^2 dt < \infty$, for $s \geq 0$. The initial datum is taken to be a concave and increasing function of wealth, $u_0 : \mathcal{R} \to \mathcal{R}$ with $u_0 \in \mathcal{C}^4\left(\mathcal{R}\right)$.

Definition 1 *An $\mathcal{F}_t-$adapted process $U_t\left(x\right)$ is a forward performance if for $t \geq 0$ and $x \in \mathcal{R}$:*

i) the mapping $x \to U_t\left(x\right)$ is concave and increasing,

ii) for each self-financing strategy, π, $E_{\mathbb{P}}\left(U_t\left(X_t^\pi\right)\right)^+ < \infty$, and

$$E_{\mathbb{P}}\left(U_s\left(X_s^\pi\right)|\mathcal{F}_t\right) \leq U_t\left(X_t^\pi\right), \qquad s \geq t, \tag{6}$$

iii) there exists a self-financing strategy, π^*, for which

$$E_{\mathbb{P}}\left(U_s\left(X_s^{\pi^*}\right)|\mathcal{F}_t\right) = U_t\left(X_t^{\pi^*}\right), \qquad s \geq t \tag{7}$$

and

iv) at $t = 0$, $U_0\left(x\right) = u_0\left(x\right)$.

[1] The Moore-Penrose pseudo-inverse matrix, denoted by A^+, of a $d \times k$ matrix A is the unique $k \times d$ matrix satisfying $AA^+A = A$, $A^+AA^+ = A^+$, $\left(AA^+\right)^T = AA^+$ and $\left(A^+A\right)^T = A^+A$. This concept was developed, independently, by Moore in 1920 and Penrose in 1955 (see [26]; also [17],). One of the properties of A^+, used in (3), is that $\left(A^T\right)^+ = \left(A^+\right)^T$.

The concept of forward performance process was introduced by the authors in [21] (see, also, [22]). The model therein is incomplete binomial and the initial data is taken to be exponential. The exponential case was subsequently and extensively analyzed for the multi-asset model considered herein in [24] (the reader interested in the associated forward indifference prices may see [20], [23] and [32]). Related to our work is the recent paper [5] in which the authors consider random horizon choices, aiming at alleviating the dependence of the value function on a fixed (and deterministic) horizon. Their model is more general than ours, in terms of the assumptions on the price dynamics, but the focus in [5] is primarily on horizon effects. Horizon issues were also considered in [8] for the special case of lognormal dynamics.

It is worth observing the following differences and similarities between the forward performance process and the traditional value function. Namely, the process $U_t(x)$ is defined for all $t \geq 0$, while the value function, which we denote by $V_t(x;T)$, is defined only on $[0,T]$. In the classical set up $V_T(x;T) \in \mathcal{F}_0$, due to the deterministic choice of the terminal utility. If the terminal utility is taken to be state-dependent, $V_T(x;T) \in \mathcal{F}_T$, (see, for example, [12], [28] as well as [3], [6] and [10]), the traditional and new formulations are, essentially, identical in $[0,T]$.

We conclude by mentioning that there is a vast literature on the specification, construction and properties of the traditional value function that is based on duality methods and is applicable for very general models. However, these techniques might not be of direct use in our case. We, also, note that forward formulations of optimal control problems have been proposed and analyzed in the past. For deterministic models we refer the reader, among others, to [16], [27] and [30]. In stochastic settings, forward optimality has been studied, primarily under Markovian assumptions, in [15] via the associated martingale problems and construction of the Nisio semigroup (see, also [19]).

3 Decreasing performance processes

In this section we focus on forward performance processes for which the mapping $t \to U_t(x)$ is decreasing[2]. Their structure will subsequently help us to construct a much richer class of examples with various desirable features, among others, flexibility across units and measure changes.

Time-decreasing forward performance processes are represented via a deterministic function, say $u(x,t)$, of wealth and time with the time argument replaced by an increasing process. This process, say A, depends on the market coefficients and not on the investor's preferences. In contrast, the function u is not affected by market changes. Rather it depends on the initial datum u_0 and, for $t > 0$, satisfies a (market independent) differential constraint. In the class of concave functions we consider, this constraint yields solutions that are

[2] While preparing this manuscript, the authors became aware of a related paper in which this class of solutions is analyzed via duality methods (see [1]).

decreasing in time which, together with the fact that A is increasing, implies the time monotonicity of the forward performance processes.

Monotone forward processes were first produced by the authors in [21] (see, also, [20]) for the special choice of exponential initial data. While the model is different than the one considered herein, we present the relevant results for completeness and motivation.

Example: We trade a single stock whose levels are denoted by $S_t > 0$, $t = 0, 1, ...$ and define the variables ξ_{t+1} as $\xi_{t+1} = \frac{S_{t+1}}{S_t}$, $\xi_{t+1} = \xi^d_{t+1}$or ξ^u_{t+1} with $0 < \xi^d_{t+1} < 1 < \xi^u_{t+1}$. We also trade a riskless bond paying zero interest rate.

A non-traded risk factor might be present whose values are denoted by Y_t, $(Y_t \neq 0)$, $t = 0, 1, ...$ We then view $\{(S_t, Y_t) : t = 0, 1, ...\}$ as a two-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\mathbb{P}$ being the historical measure. The filtration $\mathcal{F}_t$ is generated by the random variables S_i and Y_i for $i = 0, 1, ..., t$.

We denote by X_t, $t = 0, 1, ...$, the investor's wealth process associated with a multi-period self-financing portfolio. We take α_t, $t = 1, ...$, to be the number of shares of the traded asset held in this portfolio over the interval $[t-1, t)$. Then, we have, for $s = t+1, t+2, ...$, the binomial analogue of (5), namely, $X_s = X_t + \sum_{i=t+1}^{s} \alpha_i (S_i - S_{i-1})$ with $X_t = x \in \mathcal{R}$. The initial datum is given by $u_0(x) = -e^{-x}$, $x \in \mathcal{R}$.

Proposition 2 *Consider, for* $i = 1, ..$, *the sets* $B_i = \{\omega : \xi_i(\omega) = \xi^u_i\}$ *and the corresponding risk neutral probabilities* $q_i = \frac{1-\xi^d_i}{\xi^u_i - \xi^d_i}$. *Let*

$$u(x,t) = -e^{-x+t} \tag{8}$$

and introduce the process

$$A_t = \sum_{i=1}^{t} h_i \tag{9}$$

with $A_0 = 0$, *where*

$$h_i = q_i \log \frac{q_i}{\mathbb{P}(B_i | \mathcal{F}_{i-1})} + (1-q_i) \log \frac{1-q_i}{1-\mathbb{P}(B_i | \mathcal{F}_{i-1})}.$$

Then

$$U_t(x) = u(x, A_t) \qquad t = 0, 1, ... \tag{10}$$

is a forward performance process.

We stress that while the form of (10) is to some extent expected due to the specific choice of initial data, rescaling the time argument is by no means routine.

Next, we use the insights gained by the binomial model and seek a representation similar to (10) for the forward solutions in the Ito-type model considered herein assuming, at the same time, more general initial data.

Proposition 3 *Let the process λ be as in (2) and define*

$$A_t = \int_0^t |\lambda_s|^2 \, ds, \qquad t \geq 0. \tag{11}$$

Let, also, $u \in \mathcal{C}^{4,1}(\mathcal{R} \times [0, +\infty))$ be a concave function of wealth satisfying $u_t u_{xx} = \frac{1}{2} u_x^2$ and $u(x,0) = u_0(x)$. Then, the time-decreasing process

$$U_t(x) = u(x, A_t) \tag{12}$$

is a forward performance.

Sketch of the proof: As mentioned earlier, time-decreasing solutions help us build a larger family of performance processes, presented in Theorem 4. Therefore, we only present the main steps of the proof, considering for simplicity the case of a single stock (see (1)) and a riskless bond paying zero interest rate. The wealth process X satisfies $dX_t = \sigma_t \pi_t (\lambda_t dt + dW_t)$ (cf. (5)) with $X_0 = x$ and π standing for a generic admissible portfolio strategy. The initial datum is a concave function $u_0 \in \mathcal{C}^4(\mathcal{R})$.

We look for a forward solution of the form $U_t(x) = u(x, A_t)$ for some concave and increasing (in the spatial argument) function $u(x,t)$, $u \in \mathcal{C}^{4,1}(\mathcal{R} \times [0, +\infty))$ with $u(x, 0) = u_0(x)$. For reasons that will be apparent in the sequel, we choose $A_t = \int_0^t \lambda_s^2 ds$. For an arbitrary control π, we, then, have

$$\begin{aligned} dU_t(X_t) &= u_x(X_t, A_t) \sigma_t \pi_t dW_t \\ &+ \left(u_t(X_t, A_t) \lambda_t^2 + u_x(X_t, A_t) \sigma_t \pi_t \lambda_t + \frac{1}{2} u_{xx}(X_t, A_t) \sigma_t^2 \pi_t^2 \right) dt \\ &= u_x(X_t, A_t) \sigma_t \pi_t dW_t \\ &+ \lambda_t^2 \left(u_t(X_t, A_t) + u_x(X_t, A_t) \alpha_t + \frac{1}{2} u_{xx}(X_t, A_t) \alpha_t^2 \right) dt \end{aligned}$$

with $\alpha = \sigma \pi \lambda^{-1}$. We readily see that, due to the concavity assumption on u, the process $U_t(X_t^\pi)$ would be a supermartingale if the above drift remains non positive. Because of its quadratic form, the appropriate drift sign is guaranteed if $u_t(x,t) u_{xx}(x,t) \geq \frac{1}{2} u_x^2(x,t)$, $(x,t) \in \mathcal{R} \times (0, +\infty)$. Let us now take that the latter inequality holds as equality and consider the control policy

$$\pi_t^* = -\sigma_t^{-1} \lambda_t \frac{u_x(X_t^*, A_t)}{u_{xx}(X_t^*, A_t)}, \tag{13}$$

with X^* being the wealth associated to π^*. We assume the appropriate regularity conditions that guarantee existence and uniqueness of the solution to the wealth equation if π^* is used. Then, the drift term vanishes yielding

$$dU_t(X_t^*) = u_x(X_t^*, A_t) \sigma_t \pi_t^* dW_t,$$

and using Definition 1 we deduce that $u(x, A_t)$ satisfies the properties (6) and (7). The fact that $u(x, A_t)$ is monotone in x follows from the related assumption on u. To establish its time monotonicity, we use the choice of A_t and that $u_t < 0$. The latter assertion follows from the differential constraint and the concavity property of u.

4 An extended class of performance processes

Extending the methodology introduced in the previous section, we construct forward performance processes for the multi-dimensional model (cf. (1)). We recall that performance is measured in terms of (discounted) wealth and that the investor's initial preference is represented by a concave and increasing function of his wealth, $u_0 : \mathcal{R} \to \mathcal{R}$ with $u_0 \in \mathcal{C}^4(\mathcal{R})$.

The construction consists of the compilation of two inputs which we call, respectively, the *differential* and the *stochastic input*. The differential one is given by a (deterministic) concave and increasing function, $u(x,t)$, of space and time. It is taken to satisfy $u(x,0) = u_0(x)$ and, for $t > 0$, a fully nonlinear differential constraint (cf. (18)). Herein, it is assumed that both u_0 and u are defined, and take values in $\mathcal{R}$. In concrete applications, the domain and range of u will depend on those of u_0 as well as the investor's feasibility (state and control) constraints. The analysis for finite domains and, more generally, for when state and control constraints are binding is substantially more difficult and will be carried out it in future work. The differential input is not affected by market changes but depends exclusively on the investor's initial preferences. In contrast, the stochastic input solely refers to the market. It consists of three $\mathcal{F}_t-$adapted process, denoted by Y, Z and A. The processes Y and Z are taken to solve

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \tag{14}$$

and

$$dZ_t = Z_t \phi_t \cdot dW_t, \tag{15}$$

with $Y_0 = Z_0 = 1$ and the coefficients δ and ϕ being $\mathcal{F}_t-$adapted and bounded (by a deterministic constant).

In the definition (20) below, Y normalizes the wealth argument while Z appears as a multiplicative factor. One might think of Y as a *benchmark* (or a numeraire) in relation to which one might wish to measure the performance of our investment strategies. The process Z, on the other hand, can be thought as a device offering flexibility to our solutions in terms of the asset returns. This might be needed if the investor has different views about the future market movements or faces trading constraints. In such cases, the returns need to be modified which essentially points to a change of measure, away from the historical one. This is naturally done through an exponential martingale. For reasons we just discussed, we will refer to Z, as the *market view process.* We assume throughout that δ_t and ϕ_t belong to $Lin(\sigma_t)$ which implies that

$$\sigma_t \sigma_t^+ \delta_t = \delta_t \quad \text{and} \quad \sigma_t \sigma_t^+ \phi_t = \phi_t. \tag{16}$$

The third component, A, of the stochastic input is a positive and non-decreasing process with zero initial value. It rescales the time argument in the differential input. As the analysis will show, the time rescaling plays a pivotal role in the construction of performance processes as well as in the characterization of the optimal allocation and wealth processes.

While the processes Y and Z are exogenously given, the process A has to be appropriately specified; see, for example, (9) and (11). In what follows we take

$$A_t = \int_0^t |\lambda_s + \phi_s - \delta_s|^2 \, ds. \tag{17}$$

Theorem 4 *Let A, Y and Z be defined as in (17), (14) and (15), with δ and ϕ satisfying (16). Let $u : \mathcal{R} \times [0, \infty) \to \mathcal{R}$ be a concave and increasing function of the spatial argument with $u \in \mathcal{C}^{4,1}(\mathcal{R} \times [0, \infty))$, and satisfying the differential constraint*

$$u_t u_{xx} = \frac{1}{2} u_x^2 \tag{18}$$

and the initial datum

$$u(x, 0) = u_0(x). \tag{19}$$

Then, the process $U_t(x)$ defined by

$$U_t(x) = u\left(\frac{x}{Y_t}, A_t\right) Z_t \tag{20}$$

is a forward performance.

Corollary 5 *If $\delta \equiv \phi \equiv 0$, $A_t = \int_0^t |\lambda_s|^2 \, ds$ and, thus, $U_t(x) = u(x, A_t)$ as in (12).*

In section 6 we explore the structure of optimal allocations. As the results therein show, it is the function r defined in (38), and not u, that emerges as the key underlying differential quantity (recall also (13)). This motivates us to question whether one should start with the differential input u and then define r, or model r directly. Results related to this direction are presented in the next section.

Computationally, if r is known then u can be constructed by successive integration, provided certain time-dependent quantities are correctly specified. But going beyond purely technical issues, it is worth noticing that (18) takes the form of a *transport* equation, namely,

$$u_t + \frac{1}{2} r(x, t) u_x = 0. \tag{21}$$

While this formulation might appear tautological, it expresses the invariance of the differential input along the characteristic curves whose slope is equal to (half of) the risk tolerance (38). To gain some intuition, consider an infinitesimal time interval $(0, \varepsilon)$. Then (21) can be approximately written as

$$u\left(x + \frac{1}{2} r_0(x) \varepsilon, \varepsilon\right) = u(x, 0) = u_0(x),$$

expressing that the points $(x, 0)$ and $\left(x + \frac{1}{2} r_0(x) \varepsilon, \varepsilon\right)$ are allocated the same (deterministic) differential performance level. In other words, in order to maintain the performance level across different times, e.g. at $t = 0$ and $t = \varepsilon$, one needs to move to higher wealth levels, namely, from x to $x+\frac{1}{2}r_0(x)\varepsilon$. One might interpret the infinitesimal amount $\frac{1}{2}r_0(x)\varepsilon$ as the compensation required by the investor in order to satisfy his impatience in the time interval $(0,\varepsilon)$ (for the notion of impatience, we refer the reader, among others to the seminal papers [7], [13] and [14]). Globally, the method of characteristics yields $u(\hat{x}(t),t)=u_0(x)$ where $\hat{x}(t)$ satisfies

$$\frac{d\hat{x}(t)}{dt}=\frac{1}{2}r(\hat{x}(t),t),$$

with $\hat{x}(0)=x$.

5 The local risk tolerance function

In this section we take a closer look at the *local risk tolerance* $r:\mathcal{R}\times[0,\infty)\to\mathcal{R}^+$, defined as

$$r(x,t)=-\frac{u_x(x,t)}{u_{xx}(x,t)}.$$

Note that it is the function r, and not u, that appears in the optimal portfolios (13) and (40). Further analysis shows that r has the following interesting, if not remarkable, differential property[3].

Proposition 6 *Let $u\in\mathcal{C}^{4,1}(\mathcal{R}\times[0,\infty))$ satisfy (18) and (19). Then, the associated local risk tolerance $r(x,t)$, defined above, satisfies*

$$r_t+\frac{1}{2}r^2r_{xx}=0 \qquad \text{and} \qquad r(x,0)=-\frac{u_0'(x)}{u_0''(x)}. \tag{22}$$

Proof. Differentiating (18) yields

$$u_{tx}=u_x-\frac{1}{2}u_x\left(\frac{u_xu_{xxx}}{u_{xx}^2}\right)$$

and, in turn,

$$u_{txx}=u_{xx}-\frac{1}{2}u_{xx}\left(\frac{u_xu_{xxx}}{u_{xx}^2}\right)-\frac{1}{2}u_x\left(\frac{u_xu_{xxx}}{u_{xx}^2}\right)_x.$$

Moreover,

$$r_x=-1+\frac{u_xu_{xxx}}{u_{xx}^2} \qquad \text{and} \qquad r_{xx}=\left(\frac{u_xu_{xxx}}{u_{xx}^2}\right)_x.$$

Consequently

$$r_t+\frac{1}{2}r^2r_{xx}=-\frac{u_{tx}}{u_{xx}}+\frac{u_xu_{txx}}{u_{xx}^2}+\frac{1}{2}\left(\frac{u_x}{u_{xx}}\right)^2\left(\frac{u_xu_{xxx}}{u_{xx}^2}\right)_x=0$$

and the statement follows. ■

[3] A similar differential formula has appeared in [2], [4] and [9].

In addition to the local risk tolerance, a quantity of interest is its reciprocal, denoted by $\gamma(x,t)$, and referred to as the *local risk aversion.* Using its definition and (22) we deduce the following.

Proposition 7 *The local risk aversion $\gamma(x,t) : \mathcal{R} \times [0,\infty) \to \mathcal{R}^+$, defined as*

$$\gamma(x,t) = \frac{1}{r(x,t)}, \tag{23}$$

satisfies

$$\gamma_t = \frac{1}{2}\left(\frac{1}{\gamma}\right)_{xx} \qquad \text{and} \qquad \gamma(x,0) = -\frac{u_0''(x)}{u_0'(x)} \ . \tag{24}$$

Readers with expertise in nonlinear partial differential equations may find the form of (22) and (24) familiar. Indeed, the constraint (22) satisfied by r is similar to a *fast diffusion equation (FDE)* while the one satisfied by γ (cf. (24)) is of *porous medium type (PME);* see, for example, [29]. We remark that, seen as partial differential equations, (22) and (24) are "ill-posed" and, thus, global solutions for arbitrary initial conditions might not exist. In addition, the exponent of the *PME* (24) is beyond the range for which global regularity has been established. Herein, however, we refrain from studying these equations since we only use (22) as a sufficient condition for our candidate solution[4]

Remark: One might wonder if local risk tolerance functions satisfying (22) actually exist. The answer is affirmative. As a matter of fact, an interesting and very rich class of such solutions is given by the two parameter family

$$r(x,t;\alpha,\beta) = \sqrt{\alpha^2 x^2 + \beta^2 e^{-\alpha^2 t}}, \qquad (x,t) \in \mathcal{R} \times [0,\infty) \tag{25}$$

with α, β positive constants. Notice that for $\alpha = 0$, $r(x,t;0,\beta) = \beta$ which yields exponential[5] differential input, $u(x,t) = -e^{-\frac{x}{\beta}+\frac{t}{2}}$ for $(x,t) \in \mathcal{R} \times [0,\infty)$.

Respectively, if $\beta = 0$ and $\alpha = 1$, $r(x,t;\alpha,0) = |x|$, implying the logarithmic input $u(x,t) = \log x - \frac{t}{2}$ for $(x,t) \in \mathcal{R}^+ \times [0,\infty)$, while when $\beta = 0$ and $\alpha \neq 1, 0$, (25) corresponds to $u(x,t) = \frac{x^\gamma}{\gamma} e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t}$ for $\gamma = \frac{\alpha-1}{\alpha}$, for $(x,t) \in \mathcal{R}^+ \times [0,\infty)$.

For the general case $\alpha, \beta > 0$, an extensive study of globally defined risk tolerance functions can be found in [31].

6 Optimal investment strategies

We focus on the structure and properties of the optimal portfolios associated with the forward performance processes U, constructed in the previous section (cf. 20). We recall that $\pi^* = \left(\pi^{1,*}, ..., \pi^{k,*}\right)$ represents the vector of the optimal discounted allocations in the k risky assets, X^* the associated optimal

[4] For preliminary results on analytic properties of the solutions to (22) for a certain class of initial data, see [25].

[5] We refer the reader to [24] for a detailed exposition and results on this case. See, also [32].

discounted wealth (cf. (5)) and $\pi^{0,*} = X^* - \mathbf{1} \cdot \pi^*$ the optimal discounted allocation in the riskless asset.

We will represent the results in the benchmarked form. For this, we introduce the *optimal benchmarked portfolio* and the associated *optimal benchmarked wealth*,

$$\tilde{\pi}_t^* \equiv \frac{1}{Y_t}\,\pi_t^* \qquad \text{and} \qquad \tilde{X}_t^* \equiv \frac{X_t^*}{Y_t}. \tag{26}$$

The dynamics of the latter is given by (37), rewritten below for convenience,

$$d\tilde{X}_t^* = \left(\sigma_t \tilde{\pi}_t^* - \tilde{X}_t^* \delta_t\right) \cdot \left(\left(\lambda_t - \delta_t\right) dt + dW_t\right). \tag{27}$$

We, also, introduce the *risk tolerance process* (at benchmarked wealth)

$$\tilde{R}_t^* = r\left(\tilde{X}_t^*, A_t\right) \tag{28}$$

with r defined in (22), and $\tilde{X}^*$ in (26) and A in (17).

The following result yields the vector of the optimal asset allocations and the associated performance process. It follows directly from the proof of Theorem 4 and, specifically, equality (40).

Theorem 8 *The optimal benchmarked portfolio process $\tilde{\pi}_t^*$, $t > 0$, is given in the feedback form*

$$\tilde{\pi}_t^* = \Pi_t^*\left(\tilde{X}_t^*\right),$$

where

$$\Pi_t^*\left(x\right) = x\sigma^+ \delta_t + r\left(x, A_t\right) \sigma_t^+ \left(\lambda_t + \phi_t - \delta_t\right), \tag{29}$$

while $\tilde{X}^*$ solves (27) with $\tilde{\pi}^*$ being used and A as in (17). Therefore,

$$\tilde{\pi}_t^* = \sigma_t^+ \delta_t \tilde{X}_t^* + \sigma_t^+ \left(\lambda_t + \phi_t - \delta_t\right) \tilde{R}_t^*, \tag{30}$$

with $\tilde{R}_t^*$ as in (28). The associated optimal performance, $U_t^* = U_t\left(\tilde{X}_t^*\right)$ satisfies

$$dU_t^* = \left(Z_t u_x\left(\tilde{X}_t^*, A_t\right) \tilde{R}_t^* \left(\lambda_t + \phi_t - \delta_t\right) + U_t^* \phi_t\right) \cdot dW_t, \tag{31}$$

with $U_0^* = u_0\left(x\right)$.

The above results yield a remarkably simple but very intuitive representation of the optimal asset allocation. We first observe that, despite the lack of Markovian assumptions, the optimal portfolios turn out to be local functionals of the benchmarked wealth. Moreover, the vector of optimal allocations can be expressed as the sum of two portfolios, given by

$$\tilde{\pi}_t^{*,X} = \sigma_t^+ \delta_t \tilde{X}_t^* \qquad \text{and} \qquad \tilde{\pi}_t^{*,R} = \sigma_t^+ \left(\lambda_t + \phi_t - \delta_t\right) \tilde{R}_t^*. \tag{32}$$

This structural decomposition of the optimal portfolios holds *independently* of the assumptions on the differential input. The first component, $\tilde{\pi}^{*,X}$, depends functionally only on the benchmarked wealth and not on the risk tolerance while the situation is reversed for the second term, $\tilde{\pi}^{*,R}$.

Further analysis yields that the processes $\left(\tilde{X}^*, \tilde{R}^*\right)$ solve an *autonomous system* of stochastic differential equations. The key ingredient for proving this result is the differential constraint (22) satisfied by the local risk tolerance.

Proposition 9 *Let r satisfy (22) and let A be as in (17). Then, the processes $\left(\tilde{X}^*, \tilde{R}^*\right)$ solve, for $t > 0$, the system*

$$d\tilde{X}_t^* = \tilde{R}_t^* \left(\lambda_t + \phi_t - \delta_t\right) \cdot \left(\left(\lambda_t - \delta_t\right) dt + dW_t\right) \tag{33}$$

and

$$d\tilde{R}_t^* = r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^*,$$

with $\tilde{X}_0^* = x$, $\tilde{R}_0^* = r_0(x)$.

Proof. Using (27) we deduce that

$$d\tilde{X}_t^* = \tilde{R}_t^* \left(\lambda_t + \phi_t - \delta_t\right) \cdot \left(\left(\lambda_t - \delta_t\right) dt + dW_t\right).$$

Moreover,

$$\begin{aligned} d\tilde{R}_t^* = dr\left(\tilde{X}_t^*, A_t\right) &= r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* \\ &+ r_t\left(\tilde{X}_t^*, A_t\right) dA_t + \frac{1}{2} r_{xx}\left(\tilde{X}_t^*, A_t\right) d\left\langle \tilde{X}^* \right\rangle_t \\ &= r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* + \left(r_t\left(\tilde{X}_t^*, A_t\right) + \frac{1}{2} r_{xx}\left(\tilde{X}_t^*, A_t\right)\left(\tilde{R}_t^*\right)^2\right) dA_t \\ &= r_x\left(\tilde{X}_t^*, A_t\right) d\tilde{X}_t^* + \left(r_t\left(\tilde{X}_t^*, A_t\right) + \frac{1}{2} r^2\left(\tilde{X}_t^*, A_t\right) r_{xx}\left(\tilde{X}_t^*, A_t\right)\right) dA_t, \end{aligned}$$

because $dA = d\left\langle \tilde{X}^* \right\rangle$. Using (22) eliminates the last term above and we conclude. ∎

We conclude this section looking at the special cases leading to $\tilde{\pi}^{*,X} = 0$ and $\tilde{\pi}^{*,R} = 0$, respectively. It turns out that they correspond to $\delta = 0$ and $\lambda + \phi - \delta = 0$.

Case 1: $\delta \equiv 0$. Then, (cf. (14)) $Y_t = Y_0 = 1$, $t \geq 0$. The optimal portfolio components are given, for $t > 0$, by

$$\pi_t^{*,X} = 0 \qquad \text{and} \qquad \pi_t^{*,R} = \sigma_t^+ \left(\lambda_t + \phi_t\right) r\left(X_t^*, A_t\right), \tag{34}$$

with $A_t = \int_0^t \left|\lambda_s + \phi_s\right|^2 ds$. For arbitrary differential input nothing else can be said about the behavior of the optimal portfolio.

The (sub)case $\lambda + \phi \equiv 0$, however, deserves special attention. Observe that $\pi_t^{*,R} = 0$, $t > 0$, under *any* form of differential input. In other words, for arbitrary preferences, it is optimal for the investor to invest *zero wealth into each risky asset.* This result comes as a surprise given the non zero returns of the risky assets. Notice that such a solution seems to capture quite accurately the strategy of a derivatives trader for whom the underlying objective is to hedge as opposed to the asset manager whose objective is to invest. Naturally, under this static strategy, the forward performance process remains unchanged. Observe that the process A satisfies $A_t = 0$, $t \geq 0$, which results in dependence of the processes $U_t(x)$ and U_t^* only on the initial data $u_0(x)$ and not on $u(x,t)$, $t > 0$.

Case 2: $\lambda + \phi - \delta \equiv 0$. In this case, the portfolio component $\tilde{\pi}^{*,R}$ vanishes and, thus, any dependence on the risk tolerance dissipates. The investor invests the amounts $\sigma_t^+ \delta_t$ of his (benchmarked) wealth to the risky assets and puts the rest in the riskless bond. In other words,

$$\tilde{\pi}_t^{*,X} = \sigma_t^+ \delta_t \tilde{X}_t^* \quad \text{and} \quad \tilde{\pi}_t^{*,R} = 0, \quad t > 0$$

and thus the investor allocates in the riskless asset the amount

$$\pi_t^{*,0} = p_t X_t^* \quad \text{with} \quad p_t = 1 - \sigma_t^+ \delta_t \cdot \mathbf{1} \tag{35}$$

and $\mathbf{1} = (1, ..., 1)$. Equation (33) yields $d\tilde{X}_t^* = 0$ and thus at the optimum, the (absolute) wealth X^* *follows* the benchmark; see (36),

$$X_t^* = xY_t \quad \text{and} \quad \tilde{R}_t^* = \tilde{R}_0^* = r_0(x). \tag{36}$$

Equality (35) shows that depending on the level of the *weight process* p, the investor allocates arbitrarily small or large proportions of his wealth in the riskless asset. There are two extreme cases. If p=0 the investor allocates zero wealth in the riskless asset. However, when p=1, the optimal allocation consists of putting all wealth in the riskless asset.

7 Appendix

Proof of Theorem 4: To prove integrability of $U_t(X_t^\pi)^+$, we first observe that the concavity of u together with (18) yields $u_t < 0$, and thus $u(x,t) \leq u(x,0)$. We also have $u(x,0)^+ \leq ax^+ + b$ for some positive constants a and b. We easily get

$$E_{\mathbb{P}}\left(U(X_t^\pi)^+\right) = E_{\mathbb{P}}\left(u\left(\frac{X_t^\pi}{Y_t}, A_t\right)^+ Z_t\right) \leq E_{\mathbb{P}}\left(\left(a\left(\frac{X_t^\pi}{Y_t}\right)^+ + b\right) Z_t\right)$$

$$= aE_{\mathbb{P}}\left((X_t^\pi)^+ \frac{Z_t}{Y_t}\right) + b \leq \left(E_{\mathbb{P}}(X_t^\pi)^2\right)^{\frac{1}{2}} \left(E_{\mathbb{P}}\left(\frac{Z_t}{Y_t}\right)^2\right)^{\frac{1}{2}} + b.$$

Moreover, for any admissible policy π

$$E_{\mathbb{P}}(X_t^\pi)^2 \leq 2E_{\mathbb{P}}\left(x^2 + \left(\int_0^t \sigma_s \pi_s \cdot (\lambda_s dt + dW_s)\right)^2\right)$$

$$\leq 2x^2 + 4E_{\mathbb{P}}\left(\int_0^t \sigma_s\pi_s \cdot \lambda_s dt\right)^2 + 4E_{\mathbb{P}}\left(\int_0^t \sigma_s\pi_s \cdot dW_s\right)^2$$

$$\leq 2x^2 + 4\left(E_{\mathbb{P}}\int_0^t |\sigma_s\pi_s|^2 ds\right)\left(E_{\mathbb{P}}\int_0^t |\lambda_s|^2 ds\right) + 4E_{\mathbb{P}}\int_0^t |\sigma_s\pi_s|^2 ds < \infty$$

and

$$E_{\mathbb{P}}\left(\frac{Z_t}{Y_t}\right)^2 < \infty,$$

where we used that the process δ and φ are bounded by a (deterministic) constant.

We continue with the derivation of the semimartingale representation for the process $U_t(X_t)$, where X_t satisfies (5) for a fixed π; to ease the notation we henceforth skip the $\pi-$superscript notation. We are going to show that $U_t(X_t)$ is a supermartingale and that there exists a π^* such that $U_t(X_t^*)$ is a local martingale, where X^* is the associated discounted wealth. To this end, applying Ito's formula, and using the regularity assumptions of u, yields

$$dU_t(X_t) = d\left(u\left(\frac{X_t}{Y_t}, A_t\right)Z_t\right)$$

$$= \left(du\left(\frac{X_t}{Y_t}, A_t\right)\right)Z_t + u\left(\frac{X_t}{Y_t}, A_t\right)dZ_t + d\left\langle u\left(\frac{X}{Y}, A\right), Z\right\rangle_t.$$

Moreover,

$$du\left(\frac{X_t}{Y_t}, A_t\right) = u_x\left(\frac{X_t}{Y_t}, A_t\right)d\left(\frac{X_t}{Y_t}\right) + u_t\left(\frac{X_t}{Y_t}, A_t\right)dA_t + \frac{1}{2}u_{xx}\left(\frac{X_t}{Y_t}, A_t\right)d\left\langle\frac{X}{Y}\right\rangle_t$$

and

$$d\left(\frac{X_t}{Y_t}\right) = \left(\frac{1}{Y_t}\sigma_t\pi_t - \frac{X_t}{Y_t}\delta_t\right)\cdot\left(\left(\lambda_t - \delta_t\right)dt + dW_t\right). \tag{37}$$

Consequently,

$$d\left\langle u\left(\frac{X}{Y}, A\right), Z\right\rangle_t = u_x\left(\frac{X_t}{Y_t}, A_t\right)d\left\langle\frac{X}{Y}, Z\right\rangle_t$$

$$= u_x\left(\frac{X_t}{Y_t}, A_t\right)\left(\frac{1}{Y_t}\sigma_t\pi_t - \frac{X_t}{Y_t}\delta_t\right)\cdot Z_t\phi_t dt,$$

$$u\left(\frac{X_t}{Y_t}, A_t\right)dZ_t = u\left(\frac{X_t}{Y_t}, A_t\right)Z_t\phi_t \cdot dW_t = U_t(X_t)\phi_t \cdot dW_t$$

and

$$\left(du\left(\frac{X_t}{Y_t}, A_t\right)\right)Z_t = u_x\left(\frac{X_t}{Y_t}, A_t\right)Z_t\left(\frac{1}{Y_t}\sigma_t\pi_t - \frac{X_t}{Y_t}\delta_t\right)\cdot\left(\left(\lambda_t - \delta_t\right)dt + dW_t\right)$$

$$+u_t\left(\frac{X_t}{Y_t}, A_t\right)Z_t\left|\lambda_t + \phi_t - \delta_t\right|^2 dt$$

$$+\frac{1}{2}u_{xx}\left(\frac{X_t}{Y_t},A_t\right)Z_t\left|\frac{1}{Y_t}\sigma_t\pi_t-\frac{X_t}{Y_t}\delta_t\right|^2dt.$$

Combining the above we deduce

$$dU_t\left(X_t\right)=\left(u_x\left(\frac{X_t}{Y_t},A_t\right)\left(\frac{Z_t}{Y_t}\sigma_t\pi_t-\frac{X_tZ_t}{Y_t}\delta_t\right)+U_t\left(X_t\right)\phi_t\right)\cdot dW_t$$

$$+u_x\left(\frac{X_t}{Y_t},A_t\right)Z_t\left(\frac{1}{Y_t}\sigma_t\pi_t-\frac{X_t}{Y_t}\delta_t\right)\cdot\left(\lambda_t+\phi_t-\delta_t\right)dt$$

$$+u_t\left(\frac{X_t}{Y_t},A_t\right)Z_t\left|\lambda_t+\phi_t-\delta_t\right|^2dt+\frac{1}{2}u_{xx}\left(\frac{X_t}{Y_t},A_t\right)Z_t\left|\frac{1}{Y_t}\sigma_t\pi_t-\frac{X_t}{Y_t}\delta_t\right|^2dt.$$

To simplify the following expressions we introduce the notation

$$r\left(x,t\right)=-\frac{u_x\left(x,t\right)}{u_{xx}\left(x,t\right)}. \tag{38}$$

Using (16) and (17) yields

$$dU_t\left(X_t\right)=\left(u_x\left(\frac{X_t}{Y_t},A_t\right)\left(\frac{Z_t}{Y_t}\sigma_t\pi_t-\frac{X_tZ_t}{Y_t}\delta_t\right)+U_t\left(X_t\right)\phi_t\right)\cdot dW_t$$

$$+\frac{1}{2}u_{xx}\left(\frac{X_t}{Y_t},A_t\right)Z_t\left|\frac{1}{Y_t}\sigma_t\pi_t-\left(\frac{X_t}{Y_t}\delta_t+r\left(\frac{X_t}{Y_t},A_t\right)\left(\lambda_t+\phi_t-\delta_t\right)\right)\right|^2dt$$

$$+\left(u_t\left(\frac{X_t}{Y_t},A_t\right)-\frac{1}{2}u_{xx}\left(\frac{X_t}{Y_t},A_t\right)r^2\left(\frac{X_t}{Y_t},A_t\right)\right)\left|\lambda_t+\phi_t-\delta_t\right|^2dt$$

$$=\left(u_x\left(\frac{X_t}{Y_t},A_t\right)\left(\frac{Z_t}{Y_t}\sigma_t\pi_t-\frac{X_tZ_t}{Y_t}\delta_t\right)+U_t\left(X_t\right)\phi_t\right)\cdot dW_t$$

$$+\frac{1}{2}u_{xx}\left(\frac{X_t}{Y_t},A_t\right)Z_t\left|\frac{1}{Y_t}\sigma_t\pi_t-\left(\frac{X_t}{Y_t}\delta_t+r\left(\frac{X_t}{Y_t},A_t\right)\left(\lambda_t+\phi_t-\delta_t\right)\right)\right|^2dt$$

$$+\left(u_t\left(\frac{X_t}{Y_t},A_t\right)-\frac{1}{2}\frac{u_x^2\left(\frac{X_t}{Y_t},A_t\right)}{u_{xx}\left(\frac{X_t}{Y_t},A_t\right)}\right)\left|\lambda_t+\phi_t-\delta_t\right|^2dt.$$

Because u satisfies (18), the last term above vanishes. Thus, for any control policy π,

$$dU_t\left(X_t^\pi\right)=\left(u_x\left(\frac{X_t^\pi}{Y_t},A_t\right)\left(\frac{Z_t}{Y_t}\sigma_t\pi_t-\frac{X_t^\pi Z_t}{Y_t}\delta_t\right)+U_t\left(X_t^\pi\right)\phi_t\right)\cdot dW_t \tag{39}$$

$$+\frac{1}{2}u_{xx}\left(\frac{X_t^\pi}{Y_t},A_t\right)Z_t\left|\frac{1}{Y_t}\sigma_t\pi_t-\left(\frac{X_t^\pi}{Y_t}\delta_t+r\left(\frac{X_t^\pi}{Y_t},A_t\right)\left(\lambda_t+\phi_t-\delta_t\right)\right)\right|^2dt$$

where X_t^π is the associated wealth process. On the other hand, u is concave and $Z_t>0$, $t\geq0$. Therefore, the process $U_t\left(X_t^\pi\right)$ is a supermartingale.

Next, we choose the control π^* such that the above drift is eliminated, i.e.

$$\frac{1}{Y_t}\sigma_t\pi_t^* = \frac{X_t^*}{Y_t}\delta_t + r\left(\frac{X_t^*}{Y_t}, A_t\right)(\lambda_t + \phi_t - \delta_t). \tag{40}$$

Observe that the process X_t^* must satisfy

$$\begin{aligned} dX_t^* &= \sigma_t\pi_t^* \cdot (\lambda_t dt + dW_t) \\ &= X_t^* dM_t + r\left(\frac{X_t^*}{Y_t}, A_t\right) Y_t dN_t, \end{aligned} \tag{41}$$

where

$$dM_t = \delta_t \cdot (\lambda_t dt + dW_t),$$

while

$$dN_t = (\lambda_t + \phi_t - \delta_t) \cdot (\lambda_t dt + dW_t).$$

Throughout we make the standard growth and continuity assumption concerning the function r to guarantee existence and uniqueness of solution to the above equation. Consequently, the optimal policy π^* is uniquely determined by (40). To prove admissibility of π^*, note that there exist positive constants C_1 and C_2 such that

$$|\sigma_t\pi_t^*|^2 \leq C_1 (X_t^*)^2 + C_2 Y_t^2,$$

for all $t \geq 0$. Consequently, for all $s > 0$

$$E_{\mathbb{P}} \int_0^s |\sigma_t\pi_t^*|^2 \, dt \leq C_1 E_{\mathbb{P}} \int_0^s (X_t^*)^2 \, dt + C_2 E_{\mathbb{P}} \int_0^s Y_t^2 dt.$$

Obviously, the second term on the right hand side above is finite. To prove that the first one is finite as well we use standard arguments applied to the equation (41). Therefore, the process $U_t(X_t^*)$ is well defined and a local martingale which concludes the proof.

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