Derivatives pricing, investment management and the term structure of exponential utilities: The case of binomial model

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Abstract

The main aim of this paper is to systematically analyze the relationships between the optimality of investment decisions and derivatives pricing in a context of an incomplete model. The concept of utility of wealth is used to define the notion of optimality. The focus is on the important links with the classical arbitrage free theory of complete models. To avoid technicalities and at the same time explain the analogies and the differences due to the presence of incompleteness we only consider the exponential utility. Motivated by the same reasons of clarity and simplicity, we introduce incompleteness be means of a simple binomial model with one riskless and two risky assets, of which only one is traded.

1 Introduction

Derivatives pricing and investment management seem to have little in common. Even at the organizational level they belong to two quite separate parts of financial markets. The so-called *sell side,* represented mainly by the investment banks, among other things offers derivatives products to their customers. Some of them are wealth managers, belonging to the so-called *buy side* of financial markets.

So far the only universally accepted method of derivative pricing is based upon the idea of risk replication. Models have been developed which allow for perfect replication of option payoffs via implementation of a replicating and self financing strategy. We call them complete. The option price is calculated as the cost of this replication. Adjustments to the price are later made to cover for risks due to the unrealistic representation of reality.

More accurate description of the market is given by the so-called incomplete models in which not all risk in a derivative product can be eliminated by dynamic hedging. However, this potential model advantage is hampered by another difficulty. Namely, the concept of price for a derivative contract is not uniquely defined. Many approaches have been proposed and extensively studied, however, up until now no clear consensus has emerged.

On the other side of the spectrum of financial markets there are fund managers. They have developed their own methods for development and implementation of their investment decisions. They may often use derivative products to improve their performance, however, their focus is on investment strategy with view to optimize returns rather than on risk replication. Therefore it should not come as a surprise that the models they use are very different from the models used in derivatives pricing.

The main aim of this paper is to work towards convergence of the methodologies used in these apparently quite distant areas. The idea is to associate the concept of price for a derivative contract with a rather natural to a fund manager constraint, that is, maximization of utility of wealth. We choose to work with exponential utility and a very simple model structure in order to eliminate all technical difficulties and to concentrate exclusively on the most important links between the two areas.

The classical one-period binomial model and its multi-period generalization the Cox-Ross-Rubinstein model are the simplest examples of complete arbitrage free pricing models. Many textbooks and university courses on mathematical finance use these models to present the fundamental ideas without getting into difficulties of the general martingale theory necessary for the more realistic asset price dynamics. Throughout this paper we adopt the same method of presentation. Our incomplete models are very simple and hence mathematics behind them mostly trivial. The emphasis is on explaining the fundamental ideas and compare them with the classical theory.

2 One-period binomial model

In this section we introduce a simple one-period binomial model with one riskless and two risky assets, of which only one is traded. By construction, the model is incomplete and our aim is to develop a coherent methodology for investment management and derive from it a pricing methodology for derivative contracts. Our investment management methodology is based on maximization of utility of wealth. There are number of constraints we want to impose on our investment decision process and derivatives pricing method. In particular, we want our investment decisions not to depend on units in which wealth is expressed. This is mainly because we also need to make sure that our pricing method is consistent with the absence of arbitrage and is also numeraire independent. Obviously, we want our pricing concept to have a clear intuitive meaning, so effort is made to interpret the results and draw analogies with the classical arbitrage free theory of complete models whenever possible.

2.1 Indifference price representation

Consider a single period model in a market environment with one riskless and two risky assets. Only one of the risky assets is traded. For simplicity, assume zero interest rate. The current values of the traded and non-traded risky assets are denoted, respectively, by S_0 and Y_0. At the end of the period T, the value of the traded asset is S_T with $S_T = S_0\xi$, $\xi = \xi^d, \xi^u$ and $0 < \xi^d < 1 < \xi^u$. Similarly, the value of the non-traded asset Y_T satisfies $Y_T = Y_0\eta$, $\eta = \eta^d, \eta^u$,with $\eta^d < \eta^u, (Y_0, Y_T \neq 0)$.

We introduce randomness into our single period model by means of the probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$, where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\mathbb{P}$ is a probability measure on the $\sigma-$algebra $\mathcal{F}_T = 2^\Omega$ of all subsets of Ω.For each $i = 1, .., 4$ we assume that $p_i = \mathbb{P}\{\omega_i\} > 0$ and we model the upwards and the downwards movement of the two risky assets S_T and Y_T by setting their values as follows

$$
\begin{array}{ll}
S_T(\omega_1) = S_0\xi^u, \quad Y_T(\omega_1) = Y_0\eta^u & S_T(\omega_3) = S_0\xi^d, \quad Y_T(\omega_3) = Y_0\eta^u \\
S_T(\omega_2) = S_0\xi^u, \quad Y_T(\omega_2) = Y_0\eta^d & S_T(\omega_4) = S_0\xi^d, \quad Y_T(\omega_4) = Y_0\eta^d.
\end{array}
$$

The measure $\mathbb{P}$ represents the so-called historical measure.

Observe that the σ-algebra $\mathcal{F}_T$ coincides with the σ-algebra $\mathcal{F}_T^{(S,Y)}$ generated by the random variables S_T and Y_T. In what follows we will also need the σ-algebra $\mathcal{F}_T^S$ generated exclusively by the random variable S_T.

Consider a portfolio consisting of α shares of stock and the amount β invested in the riskless asset. Its current value $X_0 = x$ is equal to $\beta + \alpha S_0 = x$. Its wealth X_T, at the end of the period $[0, T]$, is given by

$$X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0). \tag{1}$$

Now introduce a claim, settling at time T and yielding payoff C_T. In pricing of C_T, we need to specify our risk preferences. We choose to work with exponential utility of the form

$$U(x) = -e^{-\gamma x}, \ x \in \mathbb{R} \text{ and } \gamma > 0. \tag{2}$$

Optimality of investments, which will ultimately yield the indifference price of C_T, is examined via the value function

$$V^{C_T}(x) = \sup_\alpha E_\mathbb{P}\left(-e^{-\gamma(X_T - C_T)}\right) = e^{-\gamma x} \sup_\alpha E_\mathbb{P}\left(-e^{-\gamma\alpha(S_T - S_0) + \gamma C_T}\right). \tag{3}$$

Below, we recall the definition of indifference prices.

Definition 1 *The indifference price of the claim $C_T = c(S_T, Y_T)$ is defined as the amount $\nu(C_T)$ for which the two value functions V^{C_T} and V^0, defined in (3) and corresponding, respectively, to the claims C_T and 0 coincide. Namely, $\nu(C_T)$ is the amount which satisfies*

$$V^0(x) = V^{C_T}(x + \nu(C_T)), \tag{4}$$

for all initial wealth levels x.

Looking at the classical *arbitrage free* pricing theory, we recall that derivative valuation has two fundamental components which do not depend on specific model assumptions. Namely, the price is obtained as a *linear* functional of the (discounted) payoff representable via the (unique) *risk neutral* equivalent martingale measure.

Our goal is to understand how these two components, namely, the linear valuation operator and the risk neutral pricing measure change when markets become incomplete. In the context of pricing by *indifference*, we will look for a *valuation functional* and a naturally related with it *pricing measure* under which the price is given as

$$\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T). \tag{5}$$

Before we determine the fundamental features that $\mathcal{E}$ and $\mathbb{Q}$ should have, let us look at some representative cases.

Examples: i) First, we consider a claim of the form $C_T = c\,(S_T)$. Intuitively, the indifference price should coincide with the arbitrage free price, for there is no risk that cannot be hedged. Indeed, one can construct a nested complete one-period binomial model and show that

$$\nu(c(S_T)) = E_{\mathbb{Q}^*}(c(S_T)), \tag{6}$$

with $\mathbb{Q}^*$ being the relevant risk neutral measure. The indifference price mechanism reduces to the arbitrage free one and any effect on preferences dissipates.

ii) Next, we look at a claim of the form $C_T = c\,(Y_T)$ and assume for simplicity that the random variables S_T and Y_T are independent under the measure $\mathbb{P}$. In this case, intuitively, the presence of the traded asset should *not* affect the price. Indeed, working directly with the value function (3) and the definition (4) it is straightforward to deduce that

$$\nu(c(Y_T)) = \frac{1}{\gamma}\log E_{\mathbb{P}}(e^{\gamma c(Y_T)}). \tag{7}$$

The indifference price coincides with the classical actuarial valuation principle, the so-called *certainty equivalent* value which is nonlinear in the payoff and the involved measure is the historical one.

iii) Finally, we examine a claim of the form $C_T = c_1\,(S_T) + c_2\,(Y_T)$. One could be, wrongly, tempted to price C_T by first pricing $c_1\,(S_T)$ by arbitrage, next pricing $c_2\,(Y_T)$ by certainty equivalent, and adding the results. Intuitively, this should work when S_T and Y_T are independent. However, this cannot possibly work under strong dependence between the two variables, for example, when Y_T is a function of S_T. In general,

$$\nu(c_1(S_T) + c_2(Y_T)) \neq E_{\mathbb{Q}^*}(c_1(S_T)) + \frac{1}{\gamma}\log E_{\mathbb{P}}(e^{\gamma c_2(Y_T)}).$$

The above illustrative examples indicate certain fundamental characteristics the $\mathcal{E}$ and $\mathbb{Q}$ should have. First of all, we observe that a *nonlinear* valuation functional must be sought. Clearly, any effort to represent indifference prices as expected payoffs under an appropriately chosen measure should be abandoned. Indeed, *no linear pricing mechanism can be compatible with the concept of indifference based valuation as defined in (4).* Note that this fundamental observation comes in contradistinction to the central direction of existing approaches in incomplete models that yield prices as expected payoffs under an optimally chosen measure.

We also see that risk preferences may affect the valuation device given their inherent role in price specification. However, intuitively speaking, we would prefer to isolate the pricing measure independently on the risk preferences. Finally, the pricing measure and the valuation device should ideally be the same for all claims to be priced.

The next Proposition yields the indifference price in the desired form (5).

Proposition 2 *Let $\mathbb{Q}$ be a measure under which the traded asset is a martingale and, at the same time, the conditional distribution of the nontraded asset, given the traded one, is preserved with respect to the historical measure $\mathbb{P}$, i.e.*

$$\mathbb{Q}(Y_T\,|S_T\,) = \mathbb{P}(Y_T\,|S_T\,). \tag{8}$$

Let $C_T = c(S_T, Y_T)$ be the claim to be priced under exponential preferences with risk aversion coefficient γ. Then, the indifference price of C_T is given by

$$\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\,|S_T\,\right)\right). \tag{9}$$

Proof. We prove the above result by constructing the indifference price via its definition (4). We start with the specification of the value functions V^0 and V^{C_T}. We represent the payoff C_T as a random variable defined on Ω with values $C_T\,(\omega_i) = c_i \in \mathbb{R}$, for $i = 1, .., 4$. Elementary arguments lead to

$$V^{C_T}\,(x) = e^{-\gamma x}\sup_{\alpha}\Big(-e^{-\gamma\alpha S_0(\xi^u-1)}\left(e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2\right)$$
$$-e^{-\gamma\alpha S_0\left(\xi^d-1\right)}\left(e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4\right)\Big).$$

Maximizing over α leads to the optimal number of shares $\alpha^{C_T,*}$, given by

$$\alpha^{C_T,*} = \frac{1}{\gamma S_0\left(\xi^u-\xi^d\right)}\log\frac{\left(\xi^u-1\right)\left(p_1+p_2\right)}{\left(1-\xi^d\right)\left(p_3+p_4\right)} \tag{10}$$
$$+\frac{1}{\gamma S_0\left(\xi^u-\xi^d\right)}\log\frac{\left(e^{\gamma c_1}p_1+e^{\gamma c_2}p_2\right)\left(p_3+p_4\right)}{\left(e^{\gamma c_3}p_3+e^{\gamma c_4}p_4\right)\left(p_1+p_2\right)}.$$

Further straightforward, albeit tedious, calculations yield

$$V^{C_T}\,(x) = -e^{-\gamma x}\frac{1}{q^q\left(1-q\right)^{1-q}}\left(e^{\gamma c_1}p_1+e^{\gamma c_2}p_2\right)^q\left(e^{\gamma c_3}p_3+e^{\gamma c_4}p_4\right)^{1-q}, \tag{11}$$

where

$$q=\frac{1-\xi^{d}}{\xi^{u}-\xi^{d}}. \tag{12}$$

For $C_T=0$, the value function takes the form

$$V^{0}\left(x\right)=-e^{-\gamma x}\left(\frac{p_1+p_2}{q}\right)^{q}\left(\frac{p_3+p_4}{1-q}\right)^{1-q}. \tag{13}$$

From the definition of the indifference price (4) and the representations (11), (13) of the relevant value functions, it follows that

$$\nu(C_T)=q\frac{1}{\gamma}\log\frac{e^{\gamma c_1}p_1+e^{\gamma c_2}p_2}{p_1+p_2}+(1-q)\,\frac{1}{\gamma}\log\frac{e^{\gamma c_3}p_3+e^{\gamma c_4}p_4}{p_3+p_4}. \tag{14}$$

Next, we show that the above price admits the probabilistic representation (9). We first consider the terms involving the historical probabilities in (14) and we note that they can be actually written in terms of the *conditional historical* expectations, namely,

$$\frac{e^{\gamma c_1}p_1+e^{\gamma c_2}p_2}{p_1+p_2}=E_{\mathbb{P}}\left(e^{\gamma C_T}\left|A\right.\right)$$

and

$$\frac{e^{\gamma c_3}p_3+e^{\gamma c_4}p_4}{p_3+p_4}=E_{\mathbb{P}}\left(e^{\gamma C_T}\left|A^c\right.\right),$$

where $A=\{\omega_1,\omega_2\}=\{\omega:S_T(\omega)=S_0\xi^u\}$. It is important to observe that conditioning is taken with respect to the *terminal* values of the traded asset.

We continue with the specification of the pricing measure defined in (12). For this, we denote (with a slight abuse of notation) by q_1,q_2,q_3,q_4 the elementary probabilities of the sought measure $\mathbb{Q}$. Straightforward calculations yield that

$$q_1+q_2=q \tag{15}$$

with q as in (12). To compute q_1, we look at the conditional historical probability of $\{Y_T=Y_0\eta^u\}$, given $\{S_T=S_0\xi^u\}$, and we impose (8), yielding

$$\frac{p_1}{p_1+p_2}=\frac{q_1}{q}.$$

The probabilities q_2,q_3 and q_4, computed in a similar manner, are written below in a concise form

$$q_i=q\frac{p_i}{p_1+p_2},\ \ i=1,2\ \ \text{and}\ \ q_i=(1-q)\frac{p_i}{p_3+p_4}\ ,\ \ i=3,4. \tag{16}$$

It easily follows that the nonlinear terms in (14) can be compiled as

$$\frac{1}{\gamma}\left(\log\frac{e^{\gamma c_1}p_1+e^{\gamma c_2}p_2}{p_1+p_2}\right)I_A+\frac{1}{\gamma}\left(\log\frac{e^{\gamma c_3}p_3+e^{\gamma c_4}p_4}{p_3+p_4}\right)I_{A^c}$$

$$= \frac{1}{\gamma} E_{\mathbb{P}} \left(e^{\gamma C_T} \left| A \right. \right) I_A + \frac{1}{\gamma} E_{\mathbb{P}} \left(e^{\gamma C_T} \left| A^c \right. \right) I_{A^c} = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \left| S_T \right. \right).$$

Therefore, taking the expectation with respect to $\mathbb{Q}$ yields

$$E_{\mathbb{Q}} \left(\frac{1}{\gamma} \log E_{\mathbb{Q}} (e^{\gamma C_T} \left| S_T \right.) \right)$$

$$= E_{\mathbb{Q}} \left(\frac{1}{\gamma} \left(\log \frac{e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2}{p_1 + p_2} \right) I_A + \frac{1}{\gamma} \left(\log \frac{e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4}{p_3 + p_4} \right) I_{A^c} \right)$$

$$= q \frac{1}{\gamma} \log \frac{e^{\gamma c_1} p_1 + e^{\gamma c_2} p_2}{p_1 + p_2} + (1-q) \frac{1}{\gamma} \log \frac{e^{\gamma c_3} p_3 + e^{\gamma c_4} p_4}{p_3 + p_4} = \nu(C_T),$$

where we used (14) to conclude. ■

We next discuss the key ingredients and highlight the intuitively natural features of the probabilistic pricing formula (9).

Interpretation of the Indifference Price: Valuation is done via a *two-step nonlinear* procedure and under a single pricing measure.

i) *Valuation procedure*: In the first step, risk preferences are injected into the valuation process. The original derivative payoff is being distorted to the *preference adjusted* payoff

$$\tilde{C}_T = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \left| S_T \right. \right).$$

This new payoff has *actuarial* type characteristics and reflects the weight that risk aversion carries in the utility based methodology. However, certainty equivalent is not applied in a naive way. Indeed, we do not consider any classical actuarial type functional,

$$\tilde{C}_T \neq \frac{1}{\gamma} \log E_{\mathbb{P}} \left(e^{\gamma C_T} \right) \text{ and } \tilde{C}_T \neq \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \right).$$

Rather, the indifference price mechanism creates a *conditional certainty equivalent* of the payoff, under the pricing measure $\mathbb{Q}$.

In the second step, the pricing procedure picks up arbitrage free pricing characteristics. It prices the preference adjusted payoff $\tilde{C}_T$, dependent only on S_T, through an arbitrage free method. The same pricing measure is being used in both steps.

The price is then given by

$$\nu(C_T) = \mathcal{E}_{\mathbb{Q}}(C_T) = E_{\mathbb{Q}}(\tilde{C}_T).$$

It is important to observe that the two steps are *not* interchangeable and of entirely different nature. The first step prices in a nonlinear way as opposed to the second step that uses linear, arbitrage free, valuation principles. In a sense, this is entirely justifiable: *the unhedgeable risks are identified, isolated and priced in the first step and, thus, the remaining risks become hedgeable. One should then use a nonlinear valuation device for the unhedgeable risks and, linear pricing for the hedgeable ones.* A natural consequence of this is that risk preferences enter exclusively in the conditional certainty equivalent term, the only term related to unhedgeable risks. Both steps are generic and valid for any payoff.

ii) *Pricing measure*: One pricing measure is used throughout. Its essential role is not to alter the conditional distribution of risks, given the ones we can trade, from their respective historical values.

Naturally, there is no dependence on the payoff. The most interesting part however is its independence on risk preferences. This universality is expected and quite pleasing. It follows from the way we identified the pricing measure, via (8), a selection criterion that is obviously independent of any risk attitude. Finally, the distorted payoff $\tilde{C}_T$ can be computed under both the historical and the pricing measure, indeed, we have

$$\tilde{C}_T = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \left| S_T \right. \right) = \frac{1}{\gamma} \log E_{\mathbb{P}} \left(e^{\gamma C_T} \left| S_T \right. \right).$$

The remainder of this section is dedicated to a comparison of our representation of the indifference prices and of the associated value functions with the well known representations obtained by Rouge and El Karoui (2000) and Delbaen et al.(2002) (see also Kabanov and Stricker (2003)). The technical arguments are not difficult and therefore the discussion is provided in a casual fashion. The conclusions are presented in Proposition 4.

In the aforementioned works, it has been established that the indifference price solves a stochastic optimization problem. The objective therein is to maximize, over all martingale measures, the expected payoff of the claim, reduced by an entropic penalty term (see (19) below). This representation is a direct result of the choice of exponential preferences and of the duality approach used on the primary expected utility problem.

A martingale measure that naturally arises in this analysis is the so-called *minimal relative entropy measure,* denoted herein by $\tilde{\mathbb{Q}}$. It is defined as the minimizer of the relative entropy, namely,

$$H\left(\tilde{\mathbb{Q}} \left| \mathbb{P} \right. \right) = \min_{Q \in \mathcal{Q}_e} H\left(Q \left| \mathbb{P} \right. \right) \tag{17}$$

where

$$H\left(Q \left| \mathbb{P} \right. \right) = E_{\mathbb{P}} \left(\frac{dQ}{d\mathbb{P}} \ln \frac{dQ}{d\mathbb{P}} \right). \tag{18}$$

For an extensive study of this measure, we refer to Frittelli (2000a, 2000b), Rheinlander () etc.

Under general model assumptions, the following result was established by Rouge and El Karoui (2000) and by Delbaen et al. (2002).

Proposition 3 *The indifference price $\nu(C_T)$ is given by*

$$\nu\left(C_T\right) = \sup_{Q \in \mathcal{Q}_e} \left(E_Q\left(C_T\right) - \frac{1}{\gamma}\left(H\left(Q|\, \mathbb{P}\right) - H\left(\tilde{\mathbb{Q}}\,|\mathbb{P}\right)\right)\right). \tag{19}$$

where $\mathcal{Q}_e$ is the set of martingale measures equivalent to $\mathbb{P}$.

The above formula has several attractive features. It is valid for general models and arbitrary payoffs. The entropic penalty directly quantifies the effects of incompleteness on the prices. The formula also exposes the limiting behavior of the price as the investor becomes risk neutral, namely, as γ converges to zero. Finally, it highlights in an intuitively pleasing way the monotonicity of the price in terms of risk preferences and its convergence to the arbitrage-free price as the market becomes complete.

This representation has, however, some shortcomings. It provides the price via a new optimization problem, a fact that does allow for a universal analogue to its arbitrage free counterpart. It also yields a pricing measure that has the undesirable feature of depending on the specific payoff. Finally, the price formula (19) considerably obstructs the analysis and study of certain important aspects of indifference valuation, as for example, its numeraire independence and its consistency with the evolution of risk preferences across time.

We start our comparative analysis by exploring the relation between the pricing measure $\mathbb{Q}$, used in (9), and of the minimal entropy measure $\tilde{\mathbb{Q}}$, appearing in (19). We can readily see that the two measures *coincide.*

In fact, consider the relative entropy (18) and look at its minimizers. For the simple model at hand, if $\hat{Q}$ is an arbitrary martingale measure defined by the elementary probabilities $\hat{q}_i$, $i = 1, ..., 4$, then

$$H\left(\hat{Q}\,|\mathbb{P}\right) = \sum_{i=1}^{4} \hat{q}_i \log \frac{\hat{q}_i}{p_i}.$$

Simple calculations yield that the manimizing elementary probabilities, say $\tilde{q}_i$ $i = 1, ..., 4$ are given by

$$\tilde{q}_i = q\frac{p_i}{p_1 + p_2}, \;\; i = 1, 2 \;\; \text{and} \;\; \tilde{q}_i = (1 - q)\frac{p_i}{p_3 + p_4}, \;\; i = 3, 4$$

and, thus, they are equal to the q_i, $i = 1, ..., 4$ of $\mathbb{Q}$ (see (16)).

Therefore,

$$H\left(\mathbb{Q}\,|\mathbb{P}\right) = H\left(\tilde{\mathbb{Q}}\,|\mathbb{P}\right) = \sum_{i=1}^{4} q_i \log \frac{q_i}{p_i}. \tag{20}$$

Next, we recall that the pricing measure $\mathbb{Q}$ was shown to satisfy the intuitively pleasing property $\mathbb{Q}\left(Y_T\,|S_T\right) = \mathbb{P}\left(Y_T\,|S_T\right)$. The measure $\tilde{\mathbb{Q}}$, therefore, is the unique martingale measure under which the conditional distribution of unhedgeable risks, given the hedgeable ones, remains the same as the one taken under the historical measure. The minimal entropy measure is not a mere technical element arising in the dual analysis; rather, it is a natural choice for a pricing measure as the one that allocates the same conditional weights to unhedgeable risks as the historical measure $\mathbb{P}$.

We also observe that the minimal relative entropy measure is *not* a maximizer in the pricing formula (19). Indeed, if this were the case, the indifference price would have been the expected value of the payoff under $\tilde{\mathbb{Q}}$, an obviously incorrect conclusion. This can happen only if the market is complete in which case, the minimal relative entropy measure coincides with the unique risk neutral one and the indifference price reduces to the arbitrage free price.

We can use the above observations to deduce alternative formulae for the involved value functions (3). These representations, first produced by Delbaen et al. (2002), are interesting on their own right. As we will see in subsequent sections, they offer valuable insights for specification of the dynamic risk preferences and are instrumental in the construction of indifference prices in more complex model environments.

To this end, we first observe that

$$-\log\left(\frac{p_1+p_2}{q}\right)^q\left(\frac{p_3+p_4}{1-q}\right)^{1-q}=\sum_{i=1}^{4}q_i\log\frac{q_i}{p_i},$$

which, in view of (20), implies that the left hand side represents the minimal relative entropy. Combining this with (13) yields the representation (21). This structural result is intuitively pleasing. It reflects how risk preferences are dynamically adjusted via the optimal investments. In fact, the value function V^0 is directly obtained from the terminal utility U by a mere translation of the wealth argument. In a sense, the entropy $H(\mathbb{Q}\,|\mathbb{P})$ represents the wealth value adjustment due to the magnitude of the investment opportunities or, using the continuous time models language, the size of the corresponding Sharpe ratio. Naturally, it is not related to any contingent payoff.

A similar representation can be derived for the value function V^{C_T}, see (22) below. It follows directly from the definitions of the indifference price and the value functions (see, respectively, (3) and (4)). Formula (22) shows that V^{C_T} can be obtained from the terminal utility through two wealth adjustments, one that is related to the indifference price and the other, already appearing in the absence of the claim, reflecting the magnitude of investment opportunities.

We summarize the above results below.

Proposition 4 *Let $\nu(C_T)$ be the indifference price of the claim, $\mathbb{Q}$ the pricing measure introduced in (8) and $H\left(\mathbb{Q}\,|\mathbb{P}\right)$ its associated entropy.*

i) The minimal relative entropy measure $\tilde{\mathbb{Q}}$ satisfies

$$\mathbb{Q}\equiv\tilde{\mathbb{Q}}$$

and therefore,

$$\tilde{\mathbb{Q}}\left(Y_T\left|S_T\right.\right)=\mathbb{P}\left(Y_T\left|S_T\right.\right).$$

ii) The value functions V^0 and V^{C_T} are represented, respectively, by

$$V^0(x)=-e^{-\gamma x-H(\mathbb{Q}|\mathbb{P})}=U\left(x+\frac{1}{\gamma}H\left(\mathbb{Q}\left|\mathbb{P}\right.\right)\right) \tag{21}$$

and

$$V^{C_T}(x)=-e^{-\gamma x-H(\mathbb{Q}|\mathbb{P})+\gamma\nu(C_T)}=U\left(x+\frac{1}{\gamma}H\left(\mathbb{Q}\left|\mathbb{P}\right.\right)-\nu(C_T)\right) \tag{22}$$

with U as in (2).

iii) The indifference price satisfies

$$\nu\left(C_T\right)=\sup_{Q\in\mathcal{Q}_e}\left(E_Q\left(C_T\right)-\frac{1}{\gamma}\left(H\left(Q\right|\mathbb{P}\right)-H\left(\mathbb{Q}\left|\mathbb{P}\right)\right)\right)=\mathcal{E}_{\mathbb{Q}}\left(C_T\right),$$

where the nonlinear pricing functional $\mathcal{E}_{\mathbb{Q}}$ is given by

$$\mathcal{E}_{\mathbb{Q}}\left(C_T\right)=E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)\right).$$

2.2 Properties of the indifference prices

The previous analysis produced the *nonlinear* price representation

$$\nu\left(C_T\right)=\mathcal{E}_{\mathbb{Q}}\left(C_T\right)=E_{\mathbb{Q}}\left(\tilde{C}_T\right),$$

where the preference adjusted payoff $\tilde{C}_T$ is the *conditional certainty equivalent*

$$\tilde{C}_T=\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\left|S_T\right.\right) \tag{23}$$

and the pricing measure $\mathbb{Q}$ is the *minimal relative entropy* measure (see Proposition 1). This pricing formula yields a direct constitutive analogue to the linear pricing rule of the complete models.

Our next task is to explore the structural properties of the indifference prices, their behavior with respect to various inputs as well as their differences and similarities to the arbitrage free prices.

Throughout we occasionally adopt the notation $\nu(C_T;\gamma)$. This is done for clarity and it is omitted whenever there is no ambiguity.

i) Behavior with respect to risk aversion coefficient.

While risk preferences are not affecting the arbitrage free prices due to perfect risk replication, they represent an indispensable element of indifference prices. Indeed, the indifference price risk aversion coefficient γ appears in the construction of $\tilde{C}_T$. It is through this conditional preference adjusted payoff that the indifference valuation mechanism extracts and valuates the underlying unhedgeable risks.

Proposition 5 *The function*

$$\gamma \to \nu\left(C_T;\gamma\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T}\left|S_T\right.\right)\right)$$

from $\mathbb{R}_+$ into $\mathbb{R}$ is increasing and continuous. Moreover, if for all claims C_T we have

$$\nu\left(C_T;\gamma\right) = \nu\left(C_T;1\right) \tag{24}$$

then $\gamma = 1$.

Proof. Continuity follows directly from the formula and the properties of conditional expectation. To establish monotonicity, let us assume that $0 < \gamma_1 < \gamma_2$. Holder's inequality then yields

$$E_{\mathbb{Q}}\left(e^{\gamma_1 C_T}\left|S_T\right.\right) \leq \left(E_{\mathbb{Q}}\left(e^{\gamma_2 C_T}\left|S_T\right.\right)\right)^{\gamma_1/\gamma_2}$$

and, in turn,

$$\frac{1}{\gamma_1}\log E_{\mathbb{Q}}\left(e^{\gamma_1 C_T}\left|S_T\right.\right) \leq \frac{1}{\gamma_2}\log E_{\mathbb{Q}}\left(e^{\gamma_2 C_T}\left|S_T\right.\right).$$

Taking expectation, with respect to the pricing measure $\mathbb{Q}$, we deduce the first statement. Now consider the claims of the form $C_T\left(\omega_1\right) = c_1, C_T\left(\omega_i\right) = 0, i = 2,3,4$ and note that (24) leads to

$$\frac{1}{\gamma}\log\frac{e^{\gamma c_1}p_1 + p_2}{p_1+p_2} = \log\frac{e^{c_1}p_1+p_2}{p_1+p_2}$$

for all c_1. To prove the second statement it is enough to differentiate both sides with respect to c_1 and rearrange terms. ■

Proposition 6 *The following limiting relations hold*

$$\lim_{\gamma\to 0^+}\nu\left(C_T;\gamma\right) = E_{\mathbb{Q}}\left(C_T\right), \tag{25}$$

$$\lim_{\gamma\to\infty}\nu\left(C_T;\gamma\right) = E_{\mathbb{Q}}\left\|C_T\right\|_{L^{\infty}_{\mathbb{Q}\{\cdot|S_T\}}}, \tag{26}$$

Proof. To show (25), we pass to the limit as $\gamma \to 0$ in the price formula (cf.(14))

$$\nu(C_T;\gamma) = q\frac{1}{\gamma}\log\left(\frac{e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2}{p_1+p_2}\right) + (1-q)\,\frac{1}{\gamma}\log\left(\frac{e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4}{p_3+p_4}\right) \tag{27}$$

to obtain

$$\lim_{\gamma\to 0}\nu(C_T;\gamma) = q\left(\frac{p_1c_1}{p_1+p_2}+\frac{p_2c_2}{p_1+p_2}\right)+(1-q)\left(\frac{p_3c_3}{p_3+p_4}+\frac{p_4c_4}{p_3+p_4}\right).$$

On the other hand, by the properties of the pricing measure, we have

$$q\frac{p_i}{p_1+p_2}=q_i,\ \ i=1,2\ \text{ and } (1-q)\frac{p_i}{p_3+p_4}=q_i,\ \ i=3,4$$

and, in turn,

$$\lim_{\gamma\to 0}\nu(C_T;\gamma)=\sum_{i=1}^{4}q_ic_i.$$

To establish (26), we pass to the limit as $\gamma\to\infty$ in (27). We readily get that

$$\lim_{\gamma\to\infty}\nu(C_T)=q\max(c_1,c_2)+(1-q)\max(c_3,c_4)$$

and the statement follows. ■

Proposition 7 *The indifference price satisfies*

$$\lim_{\gamma\to 0}\frac{\partial\nu\left(C_T;\gamma\right)}{\partial\gamma}=\frac{1}{2}E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(C_T\left|S_T\right.\right)\right), \tag{28}$$

and thus,

$$\nu\left(C_T;\gamma\right)=E_{\mathbb{Q}}\left(C_T\right)+\frac{1}{2}\gamma E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(C_T\left|S_T\right.\right)\right)+o\left(\gamma\right). \tag{29}$$

Proof. We only show (28), since (29) is an easy consequence. We first differentiate $\nu\left(C_T;\gamma\right)$ with respect to γ, obtaining

$$\frac{\partial\nu\left(C_T;\gamma\right)}{\partial\gamma}=E_{\mathbb{Q}}\left(-\frac{1}{\gamma^2}\log E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)+\frac{1}{\gamma}\frac{E_{\mathbb{Q}}(C_Te^{\gamma C_T}\left|S_T\right.)}{E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)}\right)$$

$$=\frac{1}{\gamma}\left(E_{\mathbb{Q}}\left(\frac{E_{\mathbb{Q}}(C_Te^{\gamma C_T}\left|S_T\right.)}{E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)}\right)-\nu(C_T;\gamma)\right).$$

Therefore,

$$\lim_{\gamma\to 0}\frac{\partial\nu\left(C_T;\gamma\right)}{\partial\gamma}=\lim_{\gamma\to 0}\frac{1}{\gamma}\left(E_{\mathbb{Q}}\left(\frac{E_{\mathbb{Q}}(C_Te^{\gamma C_T}\left|S_T\right.)}{E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)}\right)-\nu(C_T;\gamma)\right)$$

$$=\lim_{\gamma\to 0}E_{\mathbb{Q}}\left(\frac{E_{\mathbb{Q}}(C_T^2e^{\gamma C_T}\left|S_T\right.)E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)-\left(E_{\mathbb{Q}}(C_Te^{\gamma C_T}\left|S_T\right.)\right)^2}{\left(E_{\mathbb{Q}}(e^{\gamma C_T}\left|S_T\right.)\right)^2}\right)$$

$$-\lim_{\gamma\to 0}\frac{\partial\nu\left(C_T;\gamma\right)}{\partial\gamma}.$$

Therefore

$$\lim_{\gamma\to 0}\frac{\partial\nu\left(C_T;\gamma\right)}{\partial\gamma}=\frac{1}{2}E_{\mathbb{Q}}\left(E_{\mathbb{Q}}(C_T^2\left|S_T\right.)-\left(E_{\mathbb{Q}}(C_T\left|S_T\right.)\right)^2\right).$$

■

Proposition 8 *The indifference price is consistent with the no arbitrage principle, namely, for all $\gamma > 0$,*

$$\inf_{Q\in\mathcal{Q}_e} E_Q\left(C_T\right) \leq \nu(C_T) \leq \sup_{Q\in\mathcal{Q}_e} E_Q\left(C_T\right), \tag{30}$$

where $\mathcal{Q}_e$ is the set of martingale measures that are equivalent to $\mathbb{P}$.

Proof. We assume, without loss of generality that $c_1 < c_2$ and that $c_3 < c_4$. The monotonicity of the price with respect to risk aversion implies

$$\lim_{\gamma\to 0}\nu(C_T;\gamma) \leq \nu(C_T;\gamma) \leq \lim_{\gamma\to\infty}\nu(C_T;\gamma)$$

and, in turn, that

$$E_{\mathbb{Q}}\left(C_T\right) \leq \nu(C_T;\gamma) \leq E_{\mathbb{Q}}\left\|C_T\right\|_{L^\infty_{\mathbb{Q}\{\cdot|S_T\}}}.$$

Taking the infimum over all martingale measures, yields

$$\inf_{Q\in\mathcal{Q}} E_Q(C_T) \leq E_{\mathbb{Q}}\left(C_T\right) \leq \nu(C_T;\gamma)$$

and the left hand side of (30) follows.

We next observe that

$$E_{\mathbb{Q}}\left\|C_T\right\|_{L^\infty_{\mathbb{Q}\{\cdot|S_T\}}} = E_{\bar{Q}}\left(C_T\right)$$

where the martingale measure $\bar{Q}$ has elementary probabilities

$$\bar{Q}\left\{\omega_1\right\} = 0,\ \bar{Q}\left\{\omega_2\right\} = q,\ \bar{Q}\left\{\omega_3\right\} = 0,\ \bar{Q}\left\{\omega_4\right\} = 1-q.$$

Observing that

$$\nu(C_T;\gamma) \leq E_{\bar{Q}}\left(C_T\right) \leq \sup_{Q\in\mathcal{Q}} E_Q(C_T)$$

we conclude. ■

ii) *Behavior with respect to payoffs*

We first explore the monotonicity, convexity and scaling behavior of the indifference prices. We note that in the next two Propositions, all inequalities among payoffs and their prices hold both under the historical and the pricing measures $\mathbb{P}$ and $\mathbb{Q}$. Since these two measures are equivalent, we skip any measure-specific notation for the ease of the presentation.

Proposition 9 *The indifference price is a nondecreasing and convex function of the claim's payoff, namely,*

$$\text{if } C_T^1 \leq C_T^2 \text{ then } \nu\left(C_T^1\right) \leq \nu\left(C_T^2\right) \tag{31}$$

and

$$\text{for } \alpha\in(0,1),\quad \nu\left(\alpha C_T^1 + (1-\alpha)C_T^2\right) \leq \alpha\nu\left(C_T^1\right) + (1-\alpha)\nu\left(C_T^2\right). \tag{32}$$

Proof. Inequality (31) follows directly from the formula (9). To establish (32), we apply Holder's inequality to obtain

$$E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\left(\alpha C_T^1+(1-\alpha)C_T^2\right)}\left|S_T\right.\right)\right)$$

$$\leq E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log\left(\left(E_{\mathbb{Q}}\left(e^{\gamma C_T^1}\left|S_T\right.\right)\right)^{\alpha}\left(E_{\mathbb{Q}}\left(e^{\gamma C_T^2}\left|S_T\right.\right)\right)^{1-\alpha}\right)\right)$$

$$=\alpha E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T^1}\left|S_T\right.\right)\right)+(1-\alpha)\,E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C_T^2}\left|S_T\right.\right)\right)$$

and the result follows. ■

Proposition 10 *The indifference price satisfies*

$$\nu\left(\alpha C_T\right)\leq\alpha\nu\left(C_T\right)\quad\textit{for }\alpha\in(0,1)\tag{33}$$

and

$$\nu\left(\alpha C_T\right)\geq\alpha\nu\left(C_T\right)\quad\textit{for }\alpha\geq 1.\tag{34}$$

Proof. To show (33) we observe

$$\nu\left(\alpha C_T\right)=E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\alpha C_T}\left|S_T\right.\right)\right)$$

$$=\alpha E_{\mathbb{Q}}\left(\frac{1}{\bar{\gamma}}\log E_{\mathbb{Q}}\left(e^{\bar{\gamma}C_T}\left|S_T\right.\right)\right)$$

where

$$\bar{\gamma}=\alpha\gamma<\gamma.$$

Using the monotonicity of the price with respect to risk aversion, we conclude. Inequality (34) follows by the same argument. ■

The following result highlights an important property of the indifference price operator. We see that any hedgeable risk is automatically scaled out from the nonlinear part of the pricing rule, and it is priced directly by arbitrage. Hedgeable risks do not differ from their conditional certainty equivalent payoffs. In this sense, we say that the pricing operator has the property of *translation invariance with respect to hedgeable risks.*

Note that this property is stronger, and more intuitive, than requiring mere invariance with respect to constant risks.

Proposition 11 *The indifference pricing operator is translation invariant with respect to hedgeable risks, namely, if* $C_T=C_T^1+C_T^2$, *with* $C_T^1=C^1(S_T)$ *and* $C_T^2=C^2(S_T,Y_T)$, *then*

$$\nu(C_T)=\mathcal{E}_{\mathbb{Q}}(C^1(S_T)+C^2(S_T,Y_T))\tag{35}$$

$$=E_{\mathbb{Q}}\left(C^1\left(S_T\right)\right)+\nu(C^2\left(S_T,Y_T\right)).$$

Proof. The price formula (9) together with the measurability properties of $C^1(S_T)$ yields

$$\nu(C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\left(C^1(S_T+C^2(S_T,Y_T)\right)}|S_T\right)\right)$$

$$= E_{\mathbb{Q}}\left(C^1(S_T)\right) + E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C^2(S_T,Y_T)}|S_T\right)\right)$$

$$= E_{\mathbb{Q}}\left(C_T^1\right) + \nu(C_T^2).$$

■

The above property yields the following conclusions for two extreme cases.

Special cases: i) Let $C_T = C^1(S_T) + C^2(S_T, Y_T)$ with Y_T depending functionally on S_T. The payoff C_T^2 is then $\mathcal{F}_T^S-$measurable and, therefore,

$$C^2(S_T, Y_T) = \tilde{C}^2(S_T, Y_T).$$

Combining the above with (35) implies

$$\nu(C_T) = E_{\mathbb{Q}}\left(C^1(S_T)\right) + \nu(C^2(S_T, Y_T))$$

$$= E_{\mathbb{Q}}\left(C^1(S_T)\right) + E_{\mathbb{Q}}(C^2(S_T, Y_T)).$$

The indifference price simplifies to the Black and Scholes one and the minimal relative entropy measure coincides with the unique risk neutral measure.

ii) Let $C_T = C^1(S_T) + C^2(Y_T)$ with Y_T and S_T independent under $\mathbb{P}$. Then,

$$\tilde{C}^2(Y_T) = \frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma C^2(Y_T)}|S_T\right) = \frac{1}{\gamma}\log E_{\mathbb{P}}\left(e^{\gamma C^2(Y_T)}\right).$$

The indifference price of C_T consists of the arbitrage free price of the first claim plus the traditional actuarial certainty equivalent price of the second,

$$\nu(C_T) = E_{\mathbb{Q}}(C^1(S_T)) + \frac{1}{\gamma}\log E_{\mathbb{P}}\left(e^{\gamma C^2(Y_T)}\right).$$

We finish this section presenting the link between the indifference pricing functional ν and the so called convex measures of risk (cf. Follmer and Schied 2002).

Definition 12 $\rho : \mathcal{F}_T \to \mathbb{R}$ *is called a convex measure of risk if it satisfies the following conditions for all* $C_T^1, C_T^2 \in \mathcal{F}_T$.

- *Convexity:* $\rho\left(\alpha C_T^1 + (1-\alpha)C_T^2\right) \leq \alpha\rho\left(C_T^1\right) + (1-\alpha)\rho\left(C_T^2\right),\ 0 \leq \alpha \leq 1.$
- *Monotonicity: If* $C_T^1 \leq C_T^2$, *then* $\rho\left(C_T^1\right) \geq \rho\left(C_T^2\right)$.
- *Translation invariance: If* $m \in \mathbb{R}$, *then* $\rho\left(C_T^1 + m\right) = \rho\left(C_T^1\right) - m.$

For any $C_T \in \mathcal{F}_T$ define

$$\rho(C_T) = \nu(-C_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{-\gamma C_T}\,|S_T\right)\right). \tag{36}$$

Proposition 13 *The mapping ρ given in (36) defines a convex measure of risk.*

Proof. All conditions follow trivially from the properties of the indifference price discussed earlier. ■

Note that the number $\nu(C_T)$ represents the *indifference value* of the payoff C_T, while the number $\rho(C_T) = \nu(-C_T)$ is usually interpreted as a *capital requirement* imposed by a supervising body for accepting the position C_T. It is interesting to observe that the concept of indifference value, deduced from the desire to behave optimally as an investor, is in the above sense consistent with a method that may be used to determine the capital amount, for a position to be acceptable to a supervising body.

2.3 Risk monitoring strategies

We now turn our attention to the important issue of managing risk generated by the derivative contract.

In complete markets, the payoff is reproduced by the associated replicating portfolio. Any risk associated with the claim is eliminated and the relevant portfolio is naturally characterized as the hedging one.

For the model at hand, any $\mathcal{F}_T^S$*-measurable* claim C_T is replicable and the familiar representation formula

$$C_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0) \tag{37}$$

holds, with

$$\nu(C_T) = E_{\mathbb{Q}}(C_T) \quad \text{and} \quad \frac{\partial \nu(C_T)}{\partial S_0} = \frac{\partial E_{\mathbb{Q}}(C_T)}{\partial S_0}. \tag{38}$$

The indifference price coincides with the arbitrage free price and its spatial derivative yields the so-called delta.

When the market is incomplete however, perfect replication is not viable and a payoff representation similar to the above cannot be obtained. However, one may still seek a constitutive analogue to (37).

We recall that the indifference price was produced via comparison of the optimal investment decisions with and without a claim. We should therefore base our study on the analysis on the relevant optimal portfolios, and the relation between the optimal wealth levels they generate and the indifference price. We start with an auxiliary structural result for the optimal policies of the underlying maximal expected utility problems (3).

Proposition 14 *Let $\nu(C_T)$ be the indifference price of the claim C_T. The optimal number of shares $\alpha^{C_T,*}$ in the optimal investment problem (3) is given by*

$$\alpha^{C_T,*} = \alpha^{0,*} + \frac{\partial \nu(C_T)}{\partial S_0}, \tag{39}$$

where

$$\alpha^{0,*} = -\frac{1}{\gamma}\frac{\partial H(\mathbb{Q}\,|\mathbb{P})}{\partial S_0} \tag{40}$$

represents the number of shares held optimally in the absence of the claim. Both optimal controls $\alpha^{C_T,*}$ and $a^{0,*}$ are wealth independent.

Proof. We first recall that $\alpha^{C_T,*}$ was provided in (10), rewritten below for convenience,

$$\alpha^{C_T,*} = \frac{1}{\gamma S_0\left(\xi^u - \xi^d\right)} \log \frac{(\xi^u - 1)(p_1 + p_2)}{\left(1 - \xi^d\right)(p_3 + p_4)}$$

$$+ \frac{1}{\gamma S_0\left(\xi^u - \xi^d\right)} \log \frac{(e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2)(p_3 + p_4)}{(e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4)(p_1 + p_2)}.$$

When the claim is not taken into account, one can easily deduce, by setting $c_i = 0$, $i = 1,..,4$ above, that the corresponding optimal policy $\alpha^{0,*}$ equals

$$\alpha^{0,*} = \frac{1}{\gamma S_0\left(\xi^u - \xi^d\right)} \log \frac{(\xi^u - 1)(p_1 + p_2)}{\left(1 - \xi^d\right)(p_3 + p_4)} \tag{41}$$

$$= \frac{1}{\gamma\left(S^u - S^d\right)} \log \frac{(1-q)(p_1 + p_2)}{q(p_3 + p_4)}.$$

Using the fact that

$$\frac{\partial q}{\partial S_0} = \frac{\partial}{\partial S_0}\frac{S_0 - S^d}{S^u - S^d} = \frac{1}{S^u - S^d}$$

and differentiating the entropy formula (20) gives

$$\frac{\partial H(\mathbb{Q}\,|\mathbb{P})}{\partial S_0} = -\log \frac{(1-q)(p_1 + p_2)}{q(p_3 + p_4)}\frac{\partial q}{\partial S_0}.$$

Differentiating in turn the price formula (9) gives

$$\frac{\partial \nu(C_T)}{\partial S_0} = \frac{1}{\gamma S_0\left(\xi^u - \xi^d\right)} \log \frac{(e^{\gamma c_1}p_1 + e^{\gamma c_2}p_2)(p_3 + p_4)}{(e^{\gamma c_3}p_3 + e^{\gamma c_4}p_4)(p_1 + p_2)}$$

which, combined with the expressions for $\alpha^{C_T,*}$ and $\alpha^{0,*}$ yields (39). ■

Next we consider the *optimal wealth* variables $X^{C_T,*}$ and $X^{0,*}$ representing, respectively, the agent's optimal wealth with and without the claim. In the first case, the agent starts with initial wealth $x + \nu(C_T)$ and buys $\alpha^{C_T,*}$ shares of stock. If the claim is not taken into account, the investor starts with x and follows the strategy $\alpha^{0,*}$. In other words,

$$X_T^{C_T,*} = x + \nu(C_T) + \alpha^{C_T,*}(S_T - S_0) \tag{42}$$

and

$$X_T^{0,*} = x + \alpha^{0,*}(S_T - S_0). \tag{43}$$

We now introduce two important quantities that will help us produce a meaningful decomposition of the claim's payoff. These are the *residual optimal wealth* and the *residual risk,* denoted respectively by L and R.

Definition 15 *The residual optimal wealth is defined as*

$$L_t = X_t^{C_T,*} - X_t^{0,*} \quad \text{for} \;\; t = 0, T. \tag{44}$$

In a complete model environment, the residual optimal wealth coincides with the value of the perfectly replicating portfolio. It is therefore a martingale under the unique risk neutral measure, and it generates the claim's payoff at expiration.

When the market is incomplete, however, several interesting observations can be made. The residual terminal optimal wealth L_T reproduces the claim only partially. In addition, it is an $\mathcal{F}_T^S-$measurable variable and retains its martingale property under all martingale measures. Its most important property, however, is that it coincides with the conditional certainty equivalent. This fact will play an instrumental role in two directions, namely, in the identification of the replicable part of the claim and in the specification of the risk monitoring policy.

Proposition 16 *i) The residual optimal wealth process satisfies:*

$$L_0 = \nu(C_T) \tag{45}$$

and

$$L_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}\left(S_T - S_0\right). \tag{46}$$

ii) Morever, L_T coincides with the conditional certainty equivalent,

$$L_T = \tilde{C}_T. \tag{47}$$

iii) Finally, the process L_t is a martingale under all equivalent martingale measures,

$$E_Q(L_T) = L_0 = \nu(C_T) \quad \text{for } Q \in \mathcal{Q}_e. \tag{48}$$

Proof. Part (i) follows easily from the definition (44), the optimal wealth representations (42) and (43) and the relation (39) between the optimal policies.

To show (47), we first recall that

$$\nu(C_T) = E_{\mathbb{Q}}(\tilde{C}_T), \tag{49}$$

which, in view of (46) yields,

$$L_T = E_{\mathbb{Q}}(\tilde{C}_T) + \frac{\partial E_{\mathbb{Q}}(\tilde{C}_T)}{\partial S_0}\left(S_T - S_0\right).$$

The claim $\tilde{C}_T$ however is $\mathcal{F}_T^S-$measurable and, thus, replicable. Its arbitrage free decomposition is

$$\tilde{C}_T = E_{\mathbb{Q}}(\tilde{C}_T) + \frac{\partial E_{\mathbb{Q}}(\tilde{C}_T)}{\partial S_0}(S_T - S_0)$$

and the identity (47) follows.

The martingale property (iii) is an easy consequence of (46). ■

Definition 17 *The residual risk R_t is defined as the difference between the payoff of the claim and the residual optimal wealth, namely,*

$$R_t = C_t - L_t \quad for \quad t = 0, T. \tag{50}$$

If perfect replication is viable, the residual risk is zero throughout and its notion degenerates. In general, it represents the component of the claim that is not replicable, given that risks that can be hedged have been already *extracted* optimally according to our utility criteria. As such, it should not generate any additional conditional certainty equivalent part nor it should, in consequence, acquire any additional indifference value.

Proposition 18 *The residual risk has the following properties:*

i) It satisfies

$$R_t = 0 \tag{51}$$

and

$$R_T = C_T - \tilde{C}_T. \tag{52}$$

ii) Its conditional certainty equivalent is zero,

$$\tilde{R}_T = 0. \tag{53}$$

iii) Its indifference price is zero,

$$\nu(R_T) = 0. \tag{54}$$

iv) It is a supermartingale under the pricing measure $\mathbb{Q}$,

$$E_{\mathbb{Q}}(R_T) \leq R_t = 0. \tag{55}$$

v) Its expected, under the historical measure $\mathbb{P}$, certainty equivalent is zero,

$$\frac{1}{\gamma}\log E_{\mathbb{P}}(e^{\gamma R_T}) = 0. \tag{56}$$

Proof. Part (i) follows directly from the definition of the residual risk and the properties of L_t, $t = 0, T$.

To show (ii), we apply directly the definition of the conditional certainty equivalent. This, together with the measurability of $\tilde{C}_T$, yields

$$\tilde{R}_T = \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma \left(C_T - \tilde{C}_T \right)} \left| S_T \right. \right)$$

$$= \frac{1}{\gamma} \log E_{\mathbb{Q}} \left(e^{\gamma C_T} \left| S_T \right. \right) - \tilde{C}_T$$

$$= \tilde{C}_T - \tilde{C}_T = 0.$$

Parts (iii) and (iv) are immediate consequences of (49), (52) and (53).

To establish (56), we recall that

$$\tilde{R}_T = \frac{1}{\gamma} \log E_{\mathbb{Q}}(e^{\gamma R_T} \left| S_T \right.) = \frac{1}{\gamma} \log E_{\mathbb{P}}(e^{\gamma R_T} \left| S_T \right.)$$

where we used (8). Using (53) and taking the expectation under $\mathbb{P}$ yields the result. ■

Being a supermartingale, the residual risk can be decomposed according to the Doob decomposition. The related components can be easily retrieved and are presented below.

Proposition 19 *The supermartingale* $R_t \quad t = 0, T$ *admits the decomposition*

$$R_t = R_t^m + R_t^d$$

where

$$R_0^m = 0 \quad \textit{and} \quad R_T^m = R_T - E_{\mathbb{Q}}(R_T), \tag{57}$$

and

$$R_0^d = 0 \quad \textit{and} \quad R_T^d = E_{\mathbb{Q}}(R_T). \tag{58}$$

The component R_t^m *is an* $\mathcal{F}_T^{(S,Y)}$*-martingale under* $\mathbb{Q}$ *while* R_t^d *is decreasing and adapted to the trivial filtration* $\mathcal{F}_0^{(S,Y)}$*.*

We are now ready to provide the *payoff decomposition* result. This result is central in the study of risks associated with the indifference valuation method since it provides in a direct manner the *constitutive analogue* of the arbitrage free payoff decomposition (37).

Theorem 20 *Let* $\tilde{C}_T$ *and* R_T *be, respectively, the conditional certainty equivalent and the residual risk associated with the claim* C_T*. Let also* R_t^m *and* R_t^d *be the Doob decomposition components (57) and (58).*

Define the process $M_t^{\tilde{C}}$ *, for* $t = 0, T,$ *by*

$$M_0^{\tilde{C}} = \nu\left(C_T\right) \quad and \quad M_T^{\tilde{C}} = \nu\left(C_T\right) + \frac{\partial \nu\left(C_T\right)}{\partial S_0}(S_T - S_0). \tag{59}$$

i) The claim C_T *admits the unique, under* $\mathbb{Q}$*, payoff decomposition*

$$\begin{aligned} C_T &= \tilde{C}_T + R_T \\ &= \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0) + R_T \\ &= M_T^{\tilde{C}} + R_T^m + R_T^d. \end{aligned}$$

ii) The indifference price process ν_t*, for* $t = 0, T$*, defined by*

$$\nu_0 = \nu(C_T) \quad and \quad \nu_T = C_T$$

is an $\mathcal{F}_T^{(S,Y)}-$ *supermartingale under* $\mathbb{Q}$*. It admits the unique decomposition*

$$\nu_t = M_t + R_t^d$$

where

$$M_t = M_t^{\tilde{C}} + R_t^m.$$

The components M_t *and* R_t^d *represent, respectively, the associated martingale and the non-increasing parts of the price process* ν_t*.*

From the application point of view, one may think of R_T and its moments as natural variables for the *quantification of errors* associated with the risk monitoring policy. As the Proposition below shows, the expected error obtains a rather intuitive form. It is proportional to the risk aversion and to the expected conditional variance of the nontradable risks. Naturally, both the expectation and the conditional variance need to be considered under the pricing measure $\mathbb{Q}$.

Proposition 21 *The expected residual risk satisfies*

$$E_{\mathbb{Q}}(R_T) = -\frac{1}{2}\gamma E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(C_T \left| S_T\right.\right)\right) + o\left(\gamma\right)$$

and

$$E_{\mathbb{Q}}(R_T) = -\frac{1}{2}\gamma E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(R_T \left| S_T\right.\right)\right) + o\left(\gamma\right).$$

Proof. The proof follows from (52), yielding

$$E_{\mathbb{Q}}(R_T) = E_{\mathbb{Q}}(C_T) - E_{\mathbb{Q}}(\tilde{C}_T),$$

and the approximation formula (29). The second equality is obvious. ■

2.4 Conditional indifference prices

From the previous analysis, we can deduce that the indifference price is not a linear function of the claim's payoff, namely, for $\alpha \neq 0, 1$,

$$\nu(\alpha C_T) \neq \alpha \nu(C_T). \tag{60}$$

Indeed, as it was established in Proposition 10, if $\alpha > 1$ (resp. $\alpha < 1$), the indifference price is a superhomogeneous (resp. subhomogeneous) function of C_T.

Following simple arguments, we easily conclude that if two payoffs, say C_T^1 and C_T^2, are considered, the indifference price functional is *nonadditive*, namely,

$$\nu(C_T^1 + C_T^2) \neq \nu(C_T^1) + \nu(C_T^2). \tag{61}$$

Extending these arguments to the case of multiple payoffs, we obtain that for, say, N payoffs

$$\nu(\sum_{i=1}^{N} C_T^i) \neq \sum_{i=1}^{N} \nu(C_T^i). \tag{62}$$

The nonlinear behavior of indifference prices is a direct consequence of the nonlinear character of the indifference valuation mechanism. Naturally, this nonlinear characteristic is inherited to the associated risk monitoring strategies. The nonadditivity property is perhaps the one that most differentiates the indifference prices and the relevant risk monitoring strategies from their complete market counterparts.

This might then look as a serious deficiency of the indifference valuation approach both for the theoretical as well as the practical point of view. However, it should be noted that the aggregate valuation of the above claims was considered as if the individual risks were priced in isolation. In practice, risks and projects need to be valuated and hedged *relative to already* undertaken risks. In complete markets, perfect risk elimination makes this relative risk positioning redundant. But, when risks cannot be eliminated one should develop a methodology that would quantify and price the incoming incremental risks, while taking into account the existing unhedgeble risk exposure.

These considerations lead us to the *conditional indifference* valuation concept.

Definition 22 *Let $C^1{}_T = C^1(S_T, Y_T)$ and $C^2{}_T = C^2(S_T, Y_T)$ be two claims that have indifference prices $\nu(C_T^1)$ and $\nu(C_T^2)$. Let V^{C^1}, V^{C^2} and $V^{C^1+C^2}$ be the value functions (22) corresponding to claims C_T^1, C_T^2 and $C_T^1 + C_T^2$.*

The conditional indifference prices $\nu\left(C_T^2/C_T^1\right)$ and $\nu\left(C_T^1/C_T^2\right)$ are defined, respectively, as the amounts satisfying,

$$V^{C_1}(x) = V^{C_1+C_2}(x + \nu\left(C_T^2/C_T^1\right)) \tag{63}$$

and

$$V^{C_2}(x) = V^{C_1+C_2}(x + \nu\left(C_T^1/C_T^2\right)) \tag{64}$$

for all wealth levels.

As the following result yields, when a new claim is being priced *relatively* to an already incorporated risk exposure, the associated indifference prices become *linear.*

Proposition 23 *Assume that the claims* ${C^1}_T = C^1(S_T, Y_T)$ *and* ${C^2}_T = C^2(S_T, Y_T)$ *have indifference prices* $\nu(C_T^1)$ *and* $\nu(C_T^2)$ *and conditional indifference prices* $\nu\left(C_T^1/C_T^2\right)$ *and* $\nu\left(C_T^2/C_T^1\right)$.

Then, the indifference price of the claim with payoff $C_T = C_T^1 + C_T^2$ *satisfies*

$$\nu\left(C_T\right) = \nu\left(C_T^1\right) + \nu\left(C_T^2/C_T^1\right) \tag{65}$$

and

$$\nu\left(C_T\right) = \nu\left(C_T^2\right) + \nu\left(C_T^1/C_T^2\right).$$

Proof. We only show the first statement since the second follows by analogous arguments. For this, we recall the representation formula (22) which yields, respectively,

$$V^{C^1}(x) = -e^{-\gamma x - H(\mathbb{Q}|\mathbb{P}) + \gamma\nu\left(C_T^1\right)}$$

and

$$V^{C^1+C^2}(x) = -e^{-\gamma x - H(\mathbb{Q}|\mathbb{P}) + \gamma\nu\left(C_T^1 + C_T^2\right)}.$$

Moreover, the same formula together with the definition of the conditional indifference price $\nu\left(C_T^2/C_T^1\right)$ implies that

$$V^{C^1}(x) = -e^{-\gamma x - H(\mathbb{Q}|\mathbb{P}) + \gamma\nu\left(C_T^1\right)}$$

$$= -e^{-\gamma\left(x + \nu\left(C_T^2/C_T^1\right)\right) - H(\mathbb{Q}|\mathbb{P}) + \gamma\nu\left(C_T^1 + C_T^2\right)} = V^{C^1+C^2}(x + \nu\left(C_T^2/C_T^1\right))$$

for all wealth levels. Equating the exponents yields (65). ■

Corollary 24 *The indifference prices* $\nu\left(C_T^1\right)$ *and* $\nu\left(C_T^2\right)$, *and their conditional counterparts* $\nu\left(C_T^1/C_T^2\right)$ *and* $\nu\left(C_T^1/C_T^2\right)$, *satisfy*

$$\nu\left(C_T^1\right) - \nu\left(C_T^2\right) = \nu\left(C_T^1/C_T^2\right) - \nu\left(C_T^2/C_T^1\right).$$

Corollary 25 *The indifference price of the claim* $C_T = C_T^1 + C_T^2$ *is given by*

$$\nu\left(C_T^1 + C_T^2\right) = \frac{1}{2}\left(\nu\left(C_T^1\right) + \nu\left(C_T^2\right)\right) + \frac{1}{2}\left(\nu\left(C_T^1/C_T^2\right) + \nu\left(C_T^2/C_T^1\right)\right).$$

Moreover,

$$\nu\left(C_T^1 + C_T^2\right) - \left(\nu\left(C_T^1\right) + \nu\left(C_T^2\right)\right)$$

$$= \frac{1}{2}\left(\nu\left(C_T^1/C_T^2\right) + \nu\left(C_T^2/C_T^1\right) - \nu\left(C_T^1\right) - \nu\left(C_T^2\right)\right).$$

The latter formula yields the *error* emerging from the *nonadditive* character of the indifference price. This error may vanish in certain cases, as the examples below demonstrate. These examples were discussed in detail in Section 2.2 in the context of the translation invariance property of indifference prices. They refer to the special cases of a complete and a fully incomplete market setting.

Special cases: i) Let $C_T = C_T^1 + C_T^2$ with $C_T^1 = C^1(S_T, Y_T)$ and $C_T^2 = C^2(S_T)$. The translation invariance property (35) implies that the price is additive. In fact,

$$\nu(C_T) = \nu(C_T^1) + \nu(C_T^2)$$

with $\nu(C_T^2) = E_{\mathbb{Q}} C^2(S_T)$. Proposition 36 then yields that

$$\nu\left(C_T^2 / C_T^1\right) = \nu\left(C_T^2\right).$$

If additionally, Y_T depends functionally on S_T, then we easily deduce that

$$\nu(C_T) = E_{\mathbb{Q}}\left(C_T^1\right) + E_{\mathbb{Q}}(C_T^2)$$

and, in turn, that

$$\nu(C_T^1/C_T^2) = \nu(C_T^1) \quad \text{and} \quad \nu(C_T^2/C_T^1) = \nu(C_T^2).$$

ii) Let $C_T = C^1(S_T) + C^2(Y_T)$ with Y_T and S_T be independent under $\mathbb{P}$. Then, it was shown that the price behaves additively, namely,

$$\nu(C_T) = \nu(C_T^1) + \nu(C_T^2)$$

with

$$\nu(C_T^1) = E_{\mathbb{Q}}(C_T^1) \quad \text{and} \quad \nu(C_T^2) = \frac{1}{\gamma} \log E_{\mathbb{P}}\left(e^{\gamma C^2(Y_T)}\right).$$

Proposition 36 then implies

$$\nu(C_T^1/C_T^2) = \nu(C_T^1) \quad \text{and} \quad \nu(C_T^2/C_T^1) = \nu(C_T^2).$$

The above examples demonstrate that the conditional indifference prices reduce to the unconditional ones if the relevant risks are either fully replicable or independent from the traded ones.

2.5 Wealths, preferences and numeraires

The results of the previous sections were derived under the assumptions of zero interest rates and constant risk aversion. In this case, the wealths at the beginning and the end of a time period are expressed in a comparable unit (spot or forward) and the possible dependence of the optimization problem on the unit choice is not apparent. Below we analyze this question by looking first at the relationship between the spot and forward units. Then we consider state dependent risk aversion in order to cover other cases on numeraires. In particular, we consider the stock itself as a numeraire and show that the indifference prices can be made numeraire independent and consistent with the static no arbitrage constraint if the appropriate dependence across units is build into the preference structure.

i) Indifference prices in spot and forward units

Consider the one period model, introduced in Section 2.1, of a market with a riskless bond and two risky assets, of which only one is traded. The dynamics of the risky assets remain unchanged but we now allow for nonzero riskless rate. The price of the riskless asset, therefore, satisfies, $B_0 = 1$ and $B_T = 1 + r$ with $r > 0$.

Because of the nonzero riskless rate, the price formula (9) cannot be directly applied. In order to produce meaningful prices, one needs to be consistent with the units in which the quantities that are used in price specification are expressed. For the case at hand, we will consider the valuation problem in *spot* and in *forward* units and will force the price to become independent of the unit choice.

We start with the formulation of the *spot indifference* price problem.

Consider a portfolio consisting of α shares of stock and the amount β invested in the riskless asset. Its current value is given by $\beta+\alpha S_0 = x$, where x represents the agent's initial wealth, $X_0 = x$. Expressed in *spot* units, that is discounted to time 0, its terminal wealth X_T^s satisfies

$$X_T^s = x + \alpha \left(\frac{S_T}{1+r} - S_0 \right). \tag{66}$$

The investor's utility is taken to be exponential with constant absolute risk aversion coefficient γ^s. It is important to note that for the utility to be well defined this coefficient needs to be expressed in *spot* units. Optimality of investments will be carried out through the relevant *spot value function* given by

$$V^{s,C_T}(x) = \sup_{\alpha} E_{\mathbb{P}} \left(-e^{-\gamma^s \left(X_T^s - \frac{C_T}{1+r} \right)} \right). \tag{67}$$

Note that the option payoff C_T is also discounted from time T to time 0. The following definition is a natural extension of Definition 1.

Definition 26 *The spot indifference price of the claim C_T is defined as the amount $\nu^s(C_T)$ for which the two spot value functions V^{s,C_T} and $V^{s,0}$, defined in (67) and corresponding to claims C_T and* 0, *coincide. Namely, it is the amount $\nu^s(C_T)$ satisfying*

$$V^{s,0}(x) = V^{s,C_T}(x + \nu^s(C_T)) \tag{68}$$

for any initial wealth x.

Proposition 27 *Let* $\mathbb{Q}^s$ *be the measure such that*

$$E_{\mathbb{Q}^s} \frac{S_T}{1+r} = S_0,$$

and

$$\mathbb{Q}^s \left(Y_T \left| S_T \right. \right) = \mathbb{P} \left(Y_T \left| S_T \right. \right). \tag{69}$$

Moreover, let $C_T = c(S_T, Y_T)$ *be the claim to be priced, in spot units, with spot risk aversion coefficient* γ^s*. Then, the spot indifference price is given by*

$$\nu^s \left(C_T \right) = \mathcal{E}_{\mathbb{Q}^s} \left(\frac{C_T}{1+r} \right) = E_{\mathbb{Q}^s} \left(\frac{1}{\gamma^s} \log E_{\mathbb{Q}^s} \left(e^{\gamma^s \frac{C_T}{1+r}} \left| S_T \right. \right) \right). \tag{70}$$

Proof. Working along similar arguments to the ones used in the proof of Proposition 1, we first establish that the spot value functions, V^{s,C_T} and $V^{s,0}$, are given by

$$V^{s,0} \left(x \right) = -e^{-\gamma x} \frac{1}{\left(q^s \right)^{q^s} \left(1 - q^s \right)^{1-q^s}} \left(p_1 + p_2 \right)^{q^s} \left(p_3 + p_4 \right)^{1-q^s}$$

and

$$V^{s,C_T} \left(x \right) = -e^{-\gamma x} \frac{1}{\left(q^s \right)^{q^s} \left(1 - q^s \right)^{1-q^s}} \left(e^{\gamma^s \frac{c_1}{1+r}} p_1 + e^{\gamma^s \frac{c_2}{1+r}} p_2 \right)^{q^s}$$
$$\times \left(e^{\gamma^s \frac{c_3}{1+r}} p_3 + e^{\gamma^s \frac{c_4}{1+r}} p_4 \right)^{1-q^s},$$

where

$$q^s = \frac{(1+r) - \xi^d}{\xi^u - \xi^d}. \tag{71}$$

Applying the definition of the spot indifference price (68), yields

$$\nu^s (C_T) = q^s (\frac{1}{\gamma^s} \log \frac{e^{\gamma^s \frac{c_1}{1+r}} p_1 + e^{\gamma^s \frac{c_2}{1+r}} p_2}{p_1 + p_2}) \tag{72}$$
$$+ (1 - q^s) (\frac{1}{\gamma^s} \log \frac{e^{\gamma^s \frac{c_3}{1+r}} p_3 + e^{\gamma^s \frac{c_4}{1+r}} p_4}{p_3 + p_4}).$$

Straightforward calculations yield that the spot pricing measure $\mathbb{Q}^s$ has elementary probabilities, denoted by q_i^s, $i = 1, .., 4$,

$$q_i^s = q^s \frac{p_1}{p_1 + p_2}, \ i = 1, 2 \quad \text{and} \quad q_i^s = (1 - q^s) \frac{p_3}{p_3 + p_4}, \ i = 3, 4. \tag{73}$$

We next introduce the *spot conditional certainty equivalent*

$$\tilde{C}_T^s = \frac{1}{\gamma^s} \log E_{\mathbb{Q}^s} \left(e^{\gamma^s \frac{C_T}{1+r}} \left| S_T \right. \right).$$

Equation (72) then yields

$$\nu^s(C_T) = E_{\mathbb{Q}^s}(\tilde{C}_T^s)$$

and (70) follows. ■

We next analyze the indifference valuation of C_T assuming that all relevant prices and value functions are expressed in *forward* units. To this end, we consider the *forward terminal wealth*

$$X_T^f = X_T^s(1+r) = x(1+r) + \alpha(S_T - S_0(1+r))$$
$$= f + \alpha(F_T - F_0), \tag{74}$$

where $f = x(1+r)$ is the forward value of the current wealth and $F_0 = S_0(1+r)$, $F_T = S_T$ is the forward price process. Implicitly, we assume existence of the forward market for the risky traded asset S, and hence of the quoted prices F_0 and F_T, for it can be replicated by trading in the spot market. The corresponding *forward value function* is

$$V^{f,C_T}(f) = \sup_\alpha E_{\mathbb{P}}\left(-e^{-\gamma^f\left(X_T^f - C_T\right)}\right). \tag{75}$$

The risk aversion coefficient γ^f is naturally expressed in *forward* units.

Definition 28 *The forward indifference price of the claim C_T is defined as the amount $\nu^f(C_T)$, expressed in the forward units, for which the two forward value functions V^{f,C_T} and $V^{f,0}$, defined in (75) and corresponding to claims C_T and* 0, *coincide. Namely, it is the amount $\nu^f(C_T)$ satisfying*

$$V^{f,0}(f) = V^{f,C_T}(f + \nu^f(C_T)) \tag{76}$$

for any initial wealth f.

Proposition 29 *Let $\mathbb{Q}^f$ be a measure under which*

$$E_{\mathbb{Q}^f} F_T = F_0$$

and

$$\mathbb{Q}^f(Y_T | F_T) = \mathbb{P}(Y_T | F_T). \tag{77}$$

Then

$$\mathbb{Q}^f = \mathbb{Q}^s. \tag{78}$$

Let $C_T = c(S_T, Y_T)$ be the claim to be priced under exponential preferences with forward risk aversion coefficient γ^f. Then, the forward indifference price of C_T is given by

$$\nu^f(C_T) = \mathcal{E}_{\mathbb{Q}^f}(C_T) = E_{\mathbb{Q}^f}\left(\frac{1}{\gamma^f}\log E_{\mathbb{Q}^f}\left(e^{\gamma^f C_T} | F_T\right)\right). \tag{79}$$

Proof. Given the deterministic interest rate assumption, the fact that the measures $\mathbb{Q}^f$ and $\mathbb{Q}^s$coincide is obvious. We next observe that the forward value function V^{f,C_T} can be written as

$$V^{f,C_T}(f) = \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma^f\left(X_T^f - C_T\right)}\right)$$

$$= \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma^f\left(x(1+r)+\alpha(S_T - S_0(1+r)) - C_T\right)}\right)$$

$$= \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\tilde{\gamma}\left(x+\alpha\left(\frac{S_T}{1+r} - S_0\right) - \frac{C_T}{1+r}\right)}\right)$$

with $\tilde{\gamma} = \gamma^f(1+r)$. Therefore, V^{f,C_T}, and in turn V^{0,C_T}, can be directly retrieved from their spot counterparts. The rest of the proof then follows easily and it is therefore omitted. ■

For the rest of the analysis, we denote by $\mathbb{Q}$ the common forward and spot pricing measure.

We are now ready to investigate when the spot and forward indifference prices are consistent with the static no arbitrage condition and *independent* of the units (spot or forward) chosen in the optimization problem. The result below gives the necessary and sufficient conditions on the spot and forward risk aversion coefficients.

Proposition 30 *The spot and the forward indifference prices are consistent with the static no arbitrage condition, that is,*

$$\nu^f(C_T) = (1+r)\,\nu^s(C_T) \tag{80}$$

if and only if the spot and forward risk aversion coefficients satisfy

$$\gamma^f = \frac{\gamma^s}{1+r}. \tag{81}$$

Proof. We first show that if (81) holds then (80) follows. Recalling (70) and (79), we deduce that, if (81) holds, then $\nu^f(C_T)$ can be written as

$$\nu^f(C_T) = (1+r)\,E_{\mathbb{Q}}\left(\frac{1}{\gamma^s}\log E_{\mathbb{Q}}\left(e^{\gamma^s\frac{C_T}{1+r}}\,|S_T\right)\right)$$

and one direction of the statement follows. We remind the reader that $\mathbb{Q} = \mathbb{Q}^s = \mathbb{Q}^f$.

We next show that for (80) to hold for all C_T we must have (81). Indeed, if the consistency relationship (80) holds, then, for all claims C_T,

$$\frac{1}{1+r}E_{\mathbb{Q}}\left(\frac{1}{\gamma^f}\log E_{\mathbb{Q}}\left(e^{\gamma^f C_T}\,|S_T\right)\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma^s}\log E_{\mathbb{Q}}\left(e^{\gamma^s\frac{C_T}{1+r}}\,|S_T\right)\right)$$

and the statement follows from (24). ■

We continue with a representation result for the spot and forward value functions. We recall that the spot and forward pricing measures reduce to the same measure $\mathbb{Q}$ which, therefore, has elementary probabilities $q_i, \quad i = 1,..,4$ given in (73). Working along similar arguments to the ones used in Section 2.1 we can easily establish that, among all equivalent martingale measures, $\mathbb{Q}$ has the minimal, relative to the historical measure $\mathbb{P}$, entropy. The minimal relative entropy is given by

$$H\left(\mathbb{Q}\,|\mathbb{P}\right) = \sum_{i=1,..,4} q_i \log\left(\frac{p_i}{q_i}\right).$$

Proposition 31 *The value functions V^{s,C_T}, V^{f,C_T} are given by*

$$V^{s,C_T}(x) = -e^{-\gamma^s(x-\nu^s(C_T))-H(\mathbb{Q}|\mathbb{P})} = U^s\left(x - \nu^s(C_T) + \frac{1}{\gamma^s}H\left(\mathbb{Q}\,|\mathbb{P}\right)\right),$$

$$V^{f,C_T}(f) = -e^{-\gamma^f\left(f-\nu^f(C_T)\right)-H(\mathbb{Q}|\mathbb{P})} = U^f\left(f - \nu^f(C_T) + \frac{1}{\gamma^f}H\left(\mathbb{Q}\,|\mathbb{P}\right)\right),$$

where $U^s(x) = -e^{-\gamma^s x}$ and $U^f(f) = -e^{-\gamma^f f}$ represent the spot and forward utility functions.

Remark 32 *Recall that the argument x in $U^s(x)$ and in $V^{s,C_T}(x)$ is expressed in the spot units, while the same argument f in $U^f(f)$ and $V^{f,C_T}(f)$ is expressed in the forward units. Therefore, the utility and the value functions represent the same utility and value, independently on the units in which optimization problem is solved, if and only if (81) holds. More generally, the indifference based valuation as well as the associated optimal investment problems can be formulated and solved in a numeraire independent fashion provided the appropriate relations are build into the preference structure. In fact, these problems can be analysed without making any reference to a unit by optimizing over unitless quantities like $\gamma^s X_T^s$ or $\gamma^f X_T^f$.*

ii) *Indifference prices and state dependent preferences*

Before we proceed with the specification of the price and the conditions for numeraire independence, we extend our previous setup to the case when the risk aversion coefficient is *random*. Specifically, we assume that it is a function of the states of the traded asset. We may then conveniently represent the risk aversion at time T as the $\mathcal{F}_T^S$-measurable random variable $\gamma_T = \gamma\left(S_T\right)$ taking the values $\gamma^u = \gamma\left(S_0\xi^u\right)$ and $\gamma^d = \gamma\left(S_0\xi^d\right)$ when the events $\{\omega : S_T\left(\omega\right) = S_0\xi^u\} = \left\{\omega^1, \omega^2\right\}$ and $\left\{\omega : S_T\left(\omega\right) = S_0\xi^d\right\} = \left\{\omega^3, \omega^4\right\}$ occur. Generally, the risk aversion γ_T is expressed in the unit which is the *reciprocal* of the wealth X_T unit.

Alternatively, one may think of the *risk tolerance* γ_T^{-1} which is obviously expressed in the units of wealth at time T. Here we assume the same model as in the previous section and choose to work with the spot units, so $X_T = X_T^s$ as in (66).

We have mentioned already that representations of the indifference prices under minimal model assumptions have been derived via the duality arguments. These results can be extended even to the cases when the risk aversion coefficient is random. The related pricing formulae, however, take a form that reveals little insights about the numeraire and units effects. For this, we seek an alternative price representation, provided below, which to the best of our knowledge is new.

We also adopt the notation $\nu\left(C_T;\gamma_T\right), V^{C_T}\left(x;\gamma_T\right)$ and $V^0\left(x;\gamma_T\right)$, for the price and the relevant value functions, so that the random nature of risk preferences is conveniently highlighted.

Proposition 33 *Assume the risk aversion γ_T coefficient of the form $\gamma_T = \gamma\left(S_T\right)$. Let $\mathbb{Q}$ be the measure defined in (8) and let $C_T = c(Y_T, S_T)$ be a claim to be priced under exponential utility with risk aversion coefficient γ_T. Then, the indifference price of C_T is given by*

$$\nu\left(C_T;\gamma_T\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma_T}\log E_{\mathbb{Q}}\left(e^{\gamma_T\frac{C_T}{1+r}}|S_T\right)\right).$$

Proof. In order to construct the indifference price, we need to compute the value functions $V^{C_T}\left(x;\gamma_T\right)$ and $V^0\left(x;\gamma_T\right)$. We recall that

$$V^{C_T}\left(x;\gamma_T\right) = \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\gamma_T\left(x+\alpha\left(\frac{S_T}{1+r}-S_0\right)-\frac{C_T}{1+r}\right)}\right) \tag{82}$$

and introduce notation

$$\delta^u = \gamma^u\left(\frac{\xi^u}{1+r}-1\right) \quad \text{and} \quad \delta^d = \gamma^d\left(1-\frac{\xi^d}{1+r}\right). \tag{83}$$

Further calculations yield

$$V^{C_T}\left(x;\gamma_T\right) = \sup_{\alpha}\phi\left(\alpha\right)$$

where

$$\begin{aligned}\phi\left(\alpha\right) = & -e^{-\alpha S_0\delta^u}\left(e^{-\gamma^u\left(x-\frac{c_1}{1+r}\right)}p_1 + e^{-\gamma^u\left(x-\frac{c_2}{1+r}\right)}p_2\right)\\ & -e^{-\alpha S_0\delta^d}\left(e^{-\gamma^d\left(x-\frac{c_3}{1+r}\right)}p_3 + e^{-\gamma^d\left(x-\frac{c_4}{1+r}\right)}p_4\right)\end{aligned}$$

with δ^u and δ^d given in (83).

Differentiating with respect to α yields that the maximum occurs at

$$\alpha^* = \frac{1}{S_0\left(\delta^u + \delta^d\right)} \log \frac{\delta^u e^{-\gamma^u x}\left(e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2\right)}{\delta^d e^{-\gamma^d x}\left(e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4\right)}. \tag{84}$$

Calculating the terms $-e^{-\alpha^* S_0 \delta^u}$ and $-e^{-\alpha^* S_0 \delta^d}$ gives

$$e^{-\alpha^* S_0 \delta^u} = \left(\frac{\delta^u}{\delta^d}\right)^{-\frac{\delta^u}{\delta^u+\delta^d}} e^{-\frac{\delta^u\left(\gamma^u-\gamma^d\right)}{\delta^u+\delta^d}x} \left(\frac{e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2}{e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4}\right)^{-\frac{\delta^u}{\delta^u+\delta^d}}$$

and

$$e^{-\alpha^* S_0 \delta^d} = \left(\frac{\delta^u}{\delta^d}\right)^{\frac{\delta^d}{\delta^u+\delta^d}} e^{-\frac{\delta^d\left(\gamma^u-\gamma^d\right)}{\delta^u+\delta^d}x} \left(\frac{e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2}{e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4}\right)^{\frac{\delta^d}{\delta^u+\delta^d}}.$$

It then follows, after tedious but routine calculations that

$$\begin{aligned} V^{C_T}\left(x;\gamma_T\right) &= \phi\left(\alpha^*\right) \\ &= -\left(\left(\frac{\delta^u}{\delta^d}\right)^{-\frac{\delta^u}{\delta^u+\delta^d}} + \left(\frac{\delta^u}{\delta^d}\right)^{\frac{\delta^d}{\delta^u+\delta^d}}\right) e^{-\frac{\delta^u\gamma^d+\delta^d\gamma^u}{\delta^u+\delta^d}x} \\ &\times \left(e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2\right)^{\frac{\delta^d}{\delta^u+\delta^d}} \left(e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4\right)^{\frac{\delta^u}{\delta^u+\delta^d}}. \end{aligned} \tag{85}$$

Substituting $C_T = 0$ in turns implies

$$\begin{aligned} V^0\left(x;\gamma_T\right) = -&\left(\left(\frac{\delta^u}{\delta^d}\right)^{-\frac{\delta^u}{\delta^u+\delta^d}} + \left(\frac{\delta^u}{\delta^d}\right)^{\frac{\delta^d}{\delta^u+\delta^d}}\right) e^{-\frac{\delta^u\gamma^d+\delta^d\gamma^u}{\delta^u+\delta^d}x} \\ &\times \left(p_1+p_2\right)^{\frac{\delta^d}{\delta^u+\delta^d}} \left(p_3+p_4\right)^{\frac{\delta^u}{\delta^u+\delta^d}}. \end{aligned} \tag{86}$$

Using (85), (86) and the definition (4) of the indifference price we get

$$\begin{aligned} \nu(C_T;\gamma_T) &= \left(\frac{\delta^u\gamma^d+\delta^d\gamma^u}{\delta^u+\delta^d}\right)^{-1} \\ \times &\left(\frac{\delta^d}{\delta^u+\delta^d}\log\frac{e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2}{p_1+p_2} + \frac{\delta^u}{\delta^u+\delta^d}\log\frac{e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4}{p_3+p_4}\right). \end{aligned} \tag{87}$$

We next observe that

$$\frac{\gamma^u\delta^d}{\delta^u\gamma^d+\delta^d\gamma^u} = \frac{1+r-\xi^d}{\xi^u-\xi^d} = \frac{S_0\left(1+r\right)-S^d}{S^u-S^d} = q \tag{88}$$

and

$$\frac{\gamma^d \delta^u}{\delta^u \gamma^d + \delta^d \gamma^u} = \frac{\xi^u - 1 - r}{\xi^u - \xi^d} = \frac{S^u - S_0 (1 + r)}{S^u - S^d} = 1 - q,$$

where $S^u = S_0 \xi^u$ and $S^d = S_0 \xi^d$. The above equalities combined with (87) then yield

$$\nu (C_T; \gamma_T) = q \frac{1}{\gamma^u} \log \frac{e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2}{p_1 + p_2} + (1 - q) \frac{1}{\gamma^d} \log \frac{e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4}{p_3 + p_4}. \tag{89}$$

Following the arguments developed in the proof of Proposition 1 we see that

$$\frac{e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2}{p_1 + p_2} = E_{\mathbb{Q}} \left(e^{\gamma_T \frac{C_T}{1+r}} | S_T = S^u \right) \tag{90}$$

and

$$\frac{e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4}{p_3 + p_4} = E_{\mathbb{Q}} \left(e^{\gamma_T \frac{C_T}{1+r}} | S_T = S^d \right) \tag{91}$$

with $\mathbb{Q}$ as above. Finally, using the equalities (90) and (91), and the expression in (89) we easily conclude. ■

We continue with a representation result for the value functions. Successively, we will interpret the terms appearing in the formulae (85) and (86). The term in the exponent simplifies to

$$\frac{\delta^u \gamma^d + \delta^d \gamma^u}{\delta^u + \delta^d} = \left(\frac{\delta^u + \delta^d}{\delta^u \gamma^d + \delta^d \gamma^u} \right)^{-1}$$

$$= \left(\frac{1}{\gamma^d} (1 - q) + \frac{1}{\gamma^u} q \right)^{-1} = \left(E_{\mathbb{Q}} \gamma_T^{-1} \right)^{-1}.$$

The terms involving the payoff can be written as

$$\left(\frac{e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2}{p_1 + p_2} \right)^{\frac{\delta^d}{\delta^u + \delta^d}} \left(\frac{e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4}{p_3 + p_4} \right)^{\frac{\delta^u}{\delta^u + \delta^d}}$$

$$= e^{\left(E_{\mathbb{Q}} \gamma_T^{-1} \right)^{-1} \nu (C_T; \gamma_T)}.$$

Finally the term $V^0 (0; \gamma_T)$ is related with the relative entropy. Indeed, observe that

$$\frac{\delta^u}{\delta^d} = \frac{\gamma^u (1 - q)}{\gamma^d q}, \quad \frac{\delta^u}{\delta^u + \delta^d} = \frac{\left(\gamma^d \right)^{-1} (1 - q)}{E_{\mathbb{Q}} \gamma_T^{-1}}, \quad \frac{\delta^d}{\delta^u + \delta^d} = \frac{\left(\gamma^u \right)^{-1} q}{E_{\mathbb{Q}} \gamma_T^{-1}}$$

and, hence, after tedious calculations, we get

$$V^0 (0; \gamma_T) = - \left(\frac{p_1 + p_2}{q^*} \right)^{q^*} \left(\frac{p_3 + p_4}{1 - q^*} \right)^{1 - q^*} = -e^{-H(\mathbb{Q}^* | \mathbb{P})}. \tag{92}$$

Herein the measure $\mathbb{Q}^*$ is defined by

$$q_i^* = q^* \frac{p_i}{p_1 + p_2}, \quad i = 1,2 \text{ and } q_i^* = (1-q^*)\frac{p_i}{p_3+p_4}, \quad i = 3,4$$

and

$$q^* = \frac{(\gamma^u)^{-1} q}{E_{\mathbb{Q}} \gamma_T^{-1}} \tag{93}$$

with q given in (88). Note that $\mathbb{Q}^*$ satisfies

$$E_{\mathbb{Q}^*} \gamma_T \left(S_T - S_0 \left(1+r\right)\right) = 0 \tag{94}$$

and gives the same conditional distribution of Y_T given S_T as the measure $\mathbb{P}$. Therefore, $\mathbb{Q}^*$ is the measure which has minimal relative to $\mathbb{P}$ entropy and satisfies (94). Alternatively one can define $\mathbb{Q}^*$ by its Radon-Nikodym density with respect to the minimal relative entropy martingele measure $\mathbb{Q}$. Namely

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(\omega_i) = \frac{q^*}{q} = \frac{(\gamma^u)^{-1}}{E_{\mathbb{Q}}\gamma_T^{-1}}, \quad i = 1,2$$

and

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(\omega_i) = \frac{1-q^*}{1-q} = \frac{\left(\gamma^d\right)^{-1}}{E_{\mathbb{Q}}\gamma_T^{-1}}, \quad i = 3,4.$$

Proposition 34 *Assume the risk aversion coefficient γ_T of the form $\gamma_T = \gamma(S_T)$. Let $\mathbb{Q}$ be the measure defined in (8) and let $C_T = c(Y_T, S_T)$ be a claim to be priced under exponential utility with risk aversion coefficient γ_T. Then, the value function $V^{C_T}(x;\gamma_T)$ defined in (82) admits the following representation*

$$V^{C_T}(x;\gamma_T) = -\exp\left(-\frac{x - \nu(C_T;\gamma_T)}{E_{\mathbb{Q}}\gamma_T^{-1}} - H\left(\mathbb{Q}^* | \mathbb{P}\right)\right) \tag{95}$$

where

$$\nu(C_T;\gamma_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma_T}\log E_{\mathbb{Q}}\left(e^{\gamma_T \frac{C_T}{1+r}} | S_T\right)\right) \tag{96}$$

and

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}}(\omega) = \frac{\gamma_T^{-1}(\omega)}{E_{\mathbb{Q}}\gamma_T^{-1}} \tag{97}$$

We finish this subsection with the analysis of the optimal policy (84). We begin by decomposing α^* into the following three components

$$\alpha^* = \alpha^{0,*} + \alpha^{1,*} + \alpha^{2,*},$$

where

$$\alpha^{0,*} = \frac{1}{S_0\left(\delta^u + \delta^d\right)} \log \frac{\delta^u (p_1+p_2)}{\delta^d (p_3+p_4)},$$

$$\alpha^{1,*} = \frac{1}{S_0\left(\delta^u + \delta^d\right)} \log \frac{e^{-\gamma^u x}}{e^{-\gamma^d x}},$$

and

$$\alpha^{2,*} = \frac{1}{S_0\left(\delta^u + \delta^d\right)} \log \frac{\left(e^{\gamma^u \frac{c_1}{1+r}} p_1 + e^{\gamma^u \frac{c_2}{1+r}} p_2\right)(p_3 + p_4)}{\left(e^{\gamma^d \frac{c_3}{1+r}} p_3 + e^{\gamma^d \frac{c_4}{1+r}} p_4\right)(p_1 + p_2)}.$$

We will represent the various quantities in terms of the measures $\mathbb{Q}$ and $\mathbb{Q}^*$. Recall that

$$q = \frac{S_0(1+r) - S^d}{S^u - S^d}, \quad \frac{\partial q}{\partial S_0} = \frac{1+r}{S^u - S^d},$$

and

$$\frac{\delta^d}{\delta^u + \delta^d} = \frac{(\gamma^u)^{-1} q}{E_{\mathbb{Q}} \gamma_T^{-1}} = q^*, \quad \frac{\delta^u}{\delta^u + \delta^d} = \frac{\left(\gamma^d\right)^{-1}(1-q)}{E_{\mathbb{Q}} \gamma_T^{-1}} = 1 - q^*.$$

It follows trivially that

$$\log \frac{\delta^u (p_1 + p_2)}{\delta^d (p_3 + p_4)} = \log \frac{(1-q^*)(p_1 + p_2)}{q^*(p_3 + p_4)}.$$

Moreover, the entropy $H(\mathbb{Q}^* | \mathbb{P})$ is given by

$$H(\mathbb{Q}^* | \mathbb{P}) = q^* \log \frac{q^*}{p_1 + p_2} + (1-q^*) \log \frac{1-q^*}{p_3 + p_4},$$

and hence

$$\frac{\partial H(\mathbb{Q}^* | \mathbb{P})}{\partial S_0} = -\left(\frac{\partial q^*}{\partial S_0}\right) \log \frac{(1-q^*)(p_1 + p_2)}{q^*(p_3 + p_4)}.$$

The sensitivity of q^* to S_0 can be easily calculated. Indeed, we get

$$\frac{\partial q^*}{\partial S_0} = \frac{\partial q}{\partial S_0} \frac{(\gamma^u)^{-1}\left(\gamma^d\right)^{-1}}{\left(E_{\mathbb{Q}} \gamma_T^{-1}\right)^2}.$$

Also, because the coefficient $S_0\left(\delta^u + \delta^d\right)$ can be written as

$$\frac{1}{S_0\left(\delta^u + \delta^d\right)} = \frac{\partial q}{\partial S_0} \frac{(\gamma^u)^{-1}\left(\gamma^d\right)^{-1}}{E_{\mathbb{Q}} \gamma_T^{-1}},$$

we finally get

$$\frac{1}{S_0\left(\delta^u + \delta^d\right)} = \frac{\partial q^*}{\partial S_0} E_{\mathbb{Q}} \gamma_T^{-1}.$$

The above formulae imply that the first term in the decomposition of the optimal policy α^* can be written as

$$\alpha^{0,*} = -\frac{\partial H\left(\mathbb{Q}^*\left|\mathbb{P}\right.\right)}{\partial S_0} E_{\mathbb{Q}}\gamma_T^{-1}.$$

Moving to the second term we notice that

$$\alpha^{*,1} = -\frac{1}{S_0\left(\delta^u + \delta^d\right)}\left(\gamma^u - \gamma^d\right)x$$

and because

$$\frac{\partial \log E_{\mathbb{Q}}\gamma_T^{-1}}{\partial S_0} = \frac{\partial \log E_{\mathbb{Q}}\gamma_T^{-1}}{\partial q}\frac{\partial q}{\partial S_0}$$

$$= \frac{1}{E_{\mathbb{Q}}\gamma_T^{-1}}\left(\left(\gamma^u\right)^{-1} - \left(\gamma^d\right)^{-1}\right)\frac{\partial q}{\partial S_0} = -\frac{1}{S_0\left(\delta^u + \delta^d\right)}\left(\gamma^u - \gamma^d\right)$$

we easilly obtain that

$$\alpha^{*,1} = \frac{\partial \log E_{\mathbb{Q}}\gamma_T^{-1}}{\partial S_0}x.$$

Obviously, the last term has to do with the indifference price sensitivity. Indeed, observe that

$$\alpha^{*,2} = E_{\mathbb{Q}}\gamma_T^{-1}\frac{\partial q^*}{\partial S_0}\frac{\partial}{\partial q^*}E_{\mathbb{Q}^*}\left(\log E_{\mathbb{Q}^*}\left(e^{\gamma_T\frac{C_T}{1+r}}\left|S_T\right.\right)\right)$$

$$= E_{\mathbb{Q}}\gamma_T^{-1}\frac{\partial}{\partial S_0}E_{\mathbb{Q}^*}\left(\log E_{\mathbb{Q}^*}\left(e^{\gamma_T\frac{C_T}{1+r}}\left|S_T\right.\right)\right)$$

$$= E_{\mathbb{Q}}\gamma_T^{-1}\frac{\partial}{\partial S_0}\frac{\nu\left(C_T;\gamma_T\right)}{E_{\mathbb{Q}}\gamma_T^{-1}}.$$

Below we summarize the results about the policy.

Proposition 35 *The optimal policy admitts the following decomposition*

$$\alpha^* = \alpha^{0,*} + \alpha^{1,*} + \alpha^{2,*}, \tag{98}$$

where

$$\alpha^{0,*} = -\frac{\partial H\left(\mathbb{Q}^*\left|\mathbb{P}\right.\right)}{\partial S_0}E_{\mathbb{Q}}\gamma_T^{-1}, \quad \alpha^{1,*} = \frac{\partial \log E_{\mathbb{Q}}\gamma_T^{-1}}{\partial S_0}x$$

and

$$\alpha^{2,*} = E_{\mathbb{Q}}\gamma_T^{-1}\frac{\partial}{\partial S_0}\frac{\nu\left(C_T;\gamma_T\right)}{E_{\mathbb{Q}}\gamma_T^{-1}}.$$

The above propositions demonstrate the effects of the random nature of risk aversion on the form of the value function, the indifference price and the optimal policy. The appearance of the expected value of the risk tolerance, the reciprocal of risk aversion, seems to indicate that this quantity rather then risk aversion is more natural from the structural and interpretation points of view. Moreover, two interesting new features appear. First of all, the optimal policy is no longer independent of the initial wealth. However, the hedging demand $\alpha^{2,*}$ due to the presence of a derivative contract remains independent on the initial wealth. So does the shares amount $\alpha^{0,*}$, which is aiming to benefit from the opportunities created by the differences in the probabilities allocated to the outcomes by the historical measure $\mathbb{P}$ and the minimal relative entropy martingale measure $\mathbb{Q}^*$. The number of shares $\alpha^{1,*}$ depends linearly on the initial wealth x with the slope $\frac{\partial \log E_{\mathbb{Q}} \gamma_T^{-1}}{\partial S_0}$ representing the relative sensitivity of the current risk tolerance to the changes in the stock price. The second interesting and new feature is the sensitivity $\frac{\partial}{\partial S_0} \frac{\nu(C_T;\gamma_T)}{E_{\mathbb{Q}} \gamma_T^{-1}}$ of the option price expressed in a unitless fashion, i.e., relatively to the current risk tolerance.

iii) *Indifference prices and general numeraires*

Recall that the wealth X_T at time T, given in (66), is expressed in the spot units. Observe that if the stock price is taken as the numeraire, the wealth will be expressed in *the number of shares of stock.* In particular, the terminal wealth X_T^S is given by

$$X_T^S = \frac{X_T}{S_T} = \frac{x}{S_T} + \alpha \left(\frac{1}{1+r} - \frac{S_0}{S_T} \right)$$

and the current wealth, equal to the number of shares at time 0, is

$$X_0^S = \frac{x}{S_0} = x^S.$$

Note that X_T is discounted to time 0, and hence X_T^S is the time 0 equivalent of the number of shares held in the portfolio at time T. The related *value function* is given by

$$V^{S,C_T}\left(x^S\right) = \sup_{\alpha} E_{\mathbb{P}} \left(-e^{-\gamma^S(S_T)\left(X_T^S - \frac{C_T}{S_T(1+r)}\right)} \right) \tag{99}$$

where $\gamma^S(S_T)$ represents the risk aversion associated with this unit. Obviously, the argument x^S refers to the number of shares.

Definition 36 *The indifference price of the claim C_T is defined as the number of shares $\nu^S(C_T)$ for which the two value functions V^{S,C_T} and $V^{S,0}$, defined in (99) and corresponding to claims C_T and 0, coincide. Namely, it is the number $\nu^s(C_T)$ satisfying*

$$V^{S,0}(x^S) = V^{S,C_T}(x^S + \nu^S(C_T)) \tag{100}$$

for any initial number of shares x^S.

Proposition 37 *Let $\mathbb{Q}^S$ be a measure under which the discounted, by the traded asset, riskless bond, B_t/S_t , $t = 0, T$ is a martingale and, at the same time, the conditional distribution of the nontraded asset, given the traded one, is preserved with respect to the historical measure $\mathbb{P}$, i.e.*

$$\mathbb{Q}^S(Y_T \,|S_T\,) = \mathbb{P}(Y_T \,|S_T\,). \tag{101}$$

Let $C_T = c(S_T, Y_T)$ be the claim to be priced under exponential preferences with state dependent risk aversion coefficient $\gamma^S\left(S_T\right)$. Then, the indifference price of C_T, quoted in the number of shares of stock, is

$$\nu^S\left(C_T\right) = E_{\mathbb{Q}^S}\left(\frac{1}{\gamma^S\left(S_T\right)}\log E_{\mathbb{Q}^S}\left(e^{\gamma^S\left(S_T\right)\frac{C_T}{S_T(1+r)}}\,|S_T\right)\right). \tag{102}$$

Proof. We start with the specification of the measure $\mathbb{Q}^S$. We recall that, given the choice of numeraire, the martingale in consideration is $B_t^S = \frac{B_t}{S_t}$, where B_t and S_t denote respectively the original bond and stock process. We denote by $q_i^S,\ i = 1, .., 4$ the elementary probabilities of $\mathbb{Q}^S$. Simple calculations yield that

$$q_i^S = q^S\frac{p_i}{p_1+p_2},\quad i = 1, 2 \text{ and } q_i^S = (1-q^S)\frac{p_i}{p_3+p_4},\quad i = 3, 4,$$

where

$$q^S = \left(\frac{1}{\xi^d} - \frac{1}{1+r}\right)\frac{\xi^u\xi^d}{\xi^u - \xi^d}.$$

Alternatively the measure $\mathbb{Q}^S$ can be defined by its Radon-Nikodym density with respect to the measure $\mathbb{Q}$, namely

$$\frac{d\mathbb{Q}^S}{d\mathbb{Q}} = \frac{S_T}{(1+r)\,S_0}.$$

Indeed, we have

$$E_{\mathbb{Q}^S}\frac{1+r}{S_T} = E_{\mathbb{Q}}\frac{S_T}{(1+r)\,S_0}\frac{1+r}{S_T} = \frac{1}{S_0}.$$

We next observe that the value function V^{S,C_T} (cf. (99)) can be written as

$$V^{S,C_T}\left(x^S\right) = \sup_{\alpha} E_{\mathbb{P}}\left(-e^{-\lambda_T\left(x+\alpha\left(\frac{S_T}{1+r}-S_0\right)-\frac{C_T}{1+r}\right)}\right),$$

where

$$\lambda_T = \frac{\gamma^S\left(S_T\right)}{S_T}. \tag{103}$$

Therefore, by (95), we get

$$V^{S,C_T}\left(x^S\right) = V^{C_T}\left(x;\lambda_T\right) = -\exp\left(-\frac{x-\nu\left(C_T;\lambda_T\right)}{E_{\mathbb{Q}}\lambda_T^{-1}} - H\left(\widetilde{\mathbb{Q}}\,|\mathbb{P}\right)\right)$$

where

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{\lambda_T^{-1}}{E_{\mathbb{Q}}\lambda_T^{-1}}.$$

The argument x in $V^{C_T}(x;\lambda_T)$ as well as the price

$$\nu(C_T;\lambda_T) = E_{\mathbb{Q}}\left(\frac{1}{\lambda_T}\log E_{\mathbb{Q}}\left(e^{\lambda_T \frac{C_T}{1+r}}|S_T\right)\right)$$

are expressed in the spot units and, hence, for all claims C_T we have

$$\nu(C_T;\gamma_T) = \nu(C_T;\lambda_T).$$

Considering as in Proposition 5 claims of the form $C_T^1(\omega_1) = c_1, C_T^1(\omega_i) = 0, i = 2,3,4$ and $C_T^2(\omega_3) = c_3, C_T^2(\omega_i) = 0, i = 1,2,4$ we get that $\gamma_T = \lambda_T$. Consequently, the risk aversion $\gamma^S(S_T)$ associated with the numeraire S must satisfy

$$\gamma^S(S_T) = \gamma_T S_T.$$

Moreover, because for any payoff say G dependent only on S_T

$$E_{\mathbb{Q}}\frac{G(S_T)}{1+r} = S_0 E_{\mathbb{Q}^S}\frac{G(S_T)}{S_T},$$

we also get that

$$\nu(C_T;\gamma_T) = E_{\mathbb{Q}}\left(\frac{1}{\gamma_T}\log E_{\mathbb{Q}}\left(e^{\gamma_T \frac{C_T}{1+r}}|S_T\right)\right)$$

$$= (1+r)S_0 E_{\mathbb{Q}^S}\left(\frac{1}{\gamma_T S_T}\log E_{\mathbb{Q}}\left(e^{\gamma_T \frac{C_T}{1+r}}|S_T\right)\right)$$

$$= (1+r)S_0 E_{\mathbb{Q}^S}\left(\frac{1}{\gamma^S(S_T)}\log E_{\mathbb{Q}^S}\left(e^{\gamma^S(S_T)\frac{C_T}{S_T(1+r)}}|S_T\right)\right).$$

The statement then follows because the quantity ($cf.$95)

$$\frac{\nu(C_T;\gamma_T)}{(1+r)S_0}.$$

is the indifference price quoted in the equivalent number of shares. ■

2.6 Functions of value and utility

As we have already indicated in the previous section, the arguments of functions of value and utility can be arbitrary. Therefore, in order to refer to the same value and utility, independently on the arguments, and also in order to eliminate static arbitrage opportunities from our model, one has to make sure that the risk aversion multiplied by wealth represents the same quantity independently on the wealth units. For this reason, in what follows, we fix the benchmark risk aversion parameter $\gamma_T = \gamma(S_T)$, as representing our aversion to risk associated with wealth expressed in the spot units, i.e., discounted to the current time. Risk aversion parameters associated with other units of wealth will be set to reflect the same value and utility. Also, in view of the representations (95) and (98), we introduce the risk tolerance parameter

$$\tau_T = \gamma_T^{-1} = \gamma^{-1}(S_T).$$

Consequently, the utility and value functions in the single-period Merton problem can be written as

$$U(x) = -e^{-\frac{x}{\tau_T}}$$

and

$$V^0\left(x;\tau_T^{-1}\right) = -e^{-\frac{x}{E_{\mathbb{Q}}\tau_T} - H(\mathbb{Q}^*|\mathbb{P})}, \tag{104}$$

where

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \frac{\tau_T}{E_{\mathbb{Q}}\tau_T},$$

and $\mathbb{Q}$ is a martingale measure with the minimal relatively to $\mathbb{P}$ entropy.

Note that the value of zero wealth, as measured by the value function at the beginning of a time period, is

$$V^0\left(0;\tau_T^{-1}\right) = -e^{-H(\mathbb{Q}^*|\mathbb{P})}.$$

Hence it depends on the entropy term $H(\mathbb{Q}^*|\mathbb{P})$ and thus on the model. On the other hand, the utility of zero wealth at the end of a time period, as measured by the utility function equals -1 and hence does not depend on the model. An investor may prefer to associate zero rather than -1 utility value with zero wealth. This requirement is easily met by adding 1 to $U(x)$. We choose not to do it because this does not change anything in our analysis and lengthens many expressions. An investor may also want to associate -1 (or 0 by analogy with the utility) value with zero wealth making it independent on the model. This can be easily achieved by normalizing the utility function (104) accordingly. Indeed, let

$$V_1(x) = -e^{-\frac{x}{E_{\mathbb{Q}}\tau_T} + H(\mathbb{Q}^*|\mathbb{P})} \tag{105}$$

represent the utility of wealth x at time T. Obviously, the associated value function $V_0(x)$ satisfies

$$V_0(x) = e^{H(\mathbb{Q}^*|\mathbb{P})}V^0\left(x;\tau_T^{-1}\right) = -e^{-\frac{x}{\tau_T}}. \tag{106}$$

Observe that in the multi-period case, studied in the following sections, one needs to reconcile single-period and multi-period concepts of the value and utility functions. Indeed, when dealing concurrently with investment problems over multiple investments horizons, one needs to identify the utility function at a given horizon with the value function obtained by solving the optimal investment problem over the next time period. Together, the functions of utility and value lead to the natural concept of a dynamic utility or of a *term strucure of* *utilies.* Such a utility is a function of wealth and of the investment horizon. In a single-period case, the dynamic utility for the beginning and the end of period, as given by the functions $V_0(x)$ and $V_1(x)$, respectively, is normalized at the beginning of the period. By analogy with the classical arbitrage free terminology, we call it the *spot utility.* On the other hand, the dynamic utility, given by

$$U_0(x) = -e^{-\frac{x}{E_{\mathbb{Q}}\tau_T} + H(\mathbb{Q}^*|\mathbb{P})}, \quad U_1(x) = -e^{-\frac{x}{\tau_T}}, \tag{107}$$

is normalized at the end of the time period. Again, by analogy with the classical terminology, we call it the *forward utility.*