Optimal asset allocation in a stochastic factor model - an overview and open problems

Thaleia Zariphopoulou*

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Abstract

This paper provides an overview of the optimal investment problem in a market in which the dynamics of the risky security are affected by a correlated stochastic factor. The performance of investment strategies is measured using two criteria. The first criterion is the traditional one, formulated in terms of expected utility from terminal wealth while the second is based on the recently developed forward investment performance approach.

1 Introduction

The aim herein is to present an overview of results and open problems arising in optimal investment models in which the dynamics of the underlying stock depend on a correlated stochastic factor. Stochastic factors have been used in a number of academic papers to model the time-varying predictability of stock returns, the volatility of stocks as well as stochastic interest rates (see, for example, [1], [15], [42] and other references discussed in the next section). The performance of the investment decisions is, typically, measured via an expected utility criterion which is often formulated in a finite trading horizon.

From the technical point of view, a stochastic factor model is the simplest and most direct extension of the celebrated Merton model ([66] and [67]), in which stock dynamics are taken to be lognormal. However, as it is discussed herein, very little is known about the maximal expected utility as well as the form and properties of the optimal policies once the lognormality assumption is relaxed and correlation between the stock and the factor is introduced. This is despite the Markovian nature of the problem at hand, the advances in the theories of fully nonlinear pdes and stochastic control, and the computational

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tools that exist today. Specifically, results on the validity of the Dynamic Programming Principle, regularity of the value function, existence and verification of optimal feedback controls, representation of the value function and numerical approximations are still lacking. The only cases that have been extensively analyzed are the ones of special utilities, namely, the exponential, power and logarithmic. In these cases, convenient scaling properties reduce the associated Hamilton-Jacobi-Bellman (HJB) equation to a quasilinear one. The analysis, then, simplifies considerably both from the analytic as well as the probabilistic points of view.

The lack of rigorous results for the value function when the utility function is general limits our understanding of the optimal policies. Informally speaking, the first-order conditions in the HJB equation yield that the optimal feedback portfolio consists of two components. The first is the so-called myopic portfolio and has the same functional form as the one in the classical Merton problem. The second component, usually referred to as the excess hedging demand, is generated by the stochastic factor. Conceptually, very little is understood about this term. In addition, the sum of the two components may become zero which implies that it is optimal for a risk averse investor not to invest in a risky asset with positive risk premium. A satisfactory explanation for this counter intuitive phenomenon - related to the so-called market participation puzzle - is also lacking.

Besides these difficulties, there are other issues that limit the development of an optimal investment theory in complex market environments. One of them is the "static" choice of the utility function at the specific investment horizon. Indeed, once the utility function is chosen, no revision of risk preferences is possible at any earlier trading time. In addition, once the horizon is chosen, no investment performance criteria can be formulated for horizons longer than the initial one. These limitations have been partly addressed by allowing infinite horizon, long-term growth criteria, random horizon, recursivity and others.

Herein, we discuss a new approach that complements the existing ones. The alternative criterion has the same fundamental requirements as the classical value function process but allows for both revision of preferences and arbitrary trading horizons. It is given by a stochastic process, called the forward investment performance, defined for all times. A stochastic partial differential equation emerges which is the "forward" analogue of the HJB equation. The key new element is the performance volatility process which, in contrast to the classical formulation, is not a priori given.

The special case of zero-volatility deserves special attention as it yields useful insights for the optimal portfolios. It turns out that for this class of risk preferences, the non-myopic component always disappears, independently of the dynamics of the stochastic factor. This result might give an answer to the market participation puzzle mentioned earlier. In addition, closed form solutions can be found for the performance process as well as the associated optimal wealth and portfolio processes for general preferences and arbitrary factor dynamics.

Two classes of non-zero volatility processes and their associated optimal portfolios are, also, discussed. While from the technical point of view these cases reduce to the zero-volatility case, they provide useful results on the structure of optimal investments when the investor has alternative views for the upcoming market movements or wishes to measure performance in reference to a different numeraire/benchmark.

We finish this section mentioning that there is a very rich body of research for the analysis of the classical expected utility models based on duality techniques. This powerful approach is applicable to general market models and yields elegant results for the value function and the optimal wealth. The optimal portfolios can be then characterized via martingale representation results for the optimal wealth process (see, among others, [48], [57], [58], [80] and [81]). However, little can be said about the structure and properties of the optimal investments. Because of their volume as well as their different nature and focus, these results are not discussed herein.

The paper is organized as follows. In section 2 we present the market model. In section 3, we discuss the existing results in the classical (backward) formulation. We present some examples and state some open problems. In section 4 we present the alternative (forward) investment performance criterion and analyze, in some detail, the zero-volatility case. We also present the non-zero volatility cases, concrete examples and some open problems.

2 The model

The market consists of a risky and a riskless asset. The risky asset is a stock whose price S_t, $t \geq 0$, is modelled as a diffusion process solving

$$dS_t = \mu\left(Y_t\right) S_t dt + \sigma\left(Y_t\right) S_t dW_t^1, \tag{1}$$

with $S_0 > 0$.The stochastic factor Y_t, $t \geq 0$, satisfies

$$dY_t = b\left(Y_t\right) dt + d\left(Y_t\right)\left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right), \tag{2}$$

with $Y_0 = y$, $y \in \mathbb{R}$. The process $W_t = \left(W_t^1, W_t^2\right)$, $t \geq 0$, is a standard $2-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The underlying filtration is $\mathcal{F}_t = \sigma\left(W_s :\ 0 \leq s \leq t\right)$. It is assumed that $\rho \in (-1, 1)$.

The market coefficients $f = \mu, \sigma, b$ and d satisfy the global Lipschitz and linear growth conditions

$$\left|f\left(y\right) - f\left(\bar{y}\right)\right| \leq K\left|y - \bar{y}\right| \quad \text{ and } \quad f^2\left(y\right) \leq K\left(1 + y^2\right), \tag{3}$$

for $y, \bar{y} \in \mathbb{R}$. Moreover, it is assumed that the non degeneracy condition $\sigma\left(y\right) \geq l > 0$, $y \in \mathbb{R}$, holds.

The riskless asset, the savings account, offers constant interest rate $r > 0$.

We introduce the process

$$\lambda\left(Y_t\right) = \frac{\mu\left(Y_t\right) - r}{\sigma\left(Y_t\right)}. \tag{4}$$

We will occasionally refer to it as the market price of risk.

Starting with an initial endowment x, the investor invests at future times in the riskless and risky assets. The present value of the amounts allocated in the two accounts are denoted, respectively, by π_t^0 and π_t. The present value of her investment is, then, given by $X_t^\pi = \pi_t^0 + \pi_t$, $t > 0$. We will refer to X_t^π as the discounted wealth. Using (1) we easily deduce that it satisfies

$$dX_t^\pi = \sigma\left(Y_t\right)\pi_t\left(\lambda\left(Y_t\right)dt + dW_t^1\right). \tag{5}$$

The investment strategies will play the role of control processes and are taken to satisfy the standard assumption of being self-financing. Such a portfolio, π_t, is deemed admissible if, for $t > 0$, $\pi_t \in \mathcal{F}_t$, $E_{\mathbb{P}}\left(\int_0^t \sigma^2\left(Y_s\right)\pi_s^2 ds\right) < \infty$ and the associated discounted wealth satisfies the state constraint $X_t^\pi \in \mathbb{D}$, $t \geq 0$, for some acceptability domain $\mathbb{D} \subseteq \mathbb{R}$. We will denote the set of admissible strategies by $\mathcal{A}$.

The form of the spatial domain $\mathbb{D}$ and the consequences of this choice to the structure of the optimal portfolios are subjects of independent interest and will not be discussed herein. Frequently, portfolio constraints are also present which complicate the analysis further. For the model at hand, we will not allow for such generality as the focus is mainly on the choice and impact of risk preferences on investment decisions. To ease the notation, however, we will carry out the $\mathbb{D}-$notation and make it more specific when appropriate.

Stochastic factors have been used in portfolio choice to model asset predictability and stochastic volatility. The predictability of stock returns was first discussed in [34], [35] and [38]; see also [13], [14], [17] and [18]. More complex models were analyzed in [1] and [12]. The role of stochastic volatility in investment decisions was studied in [3], [22], [38], [39], [42], [76], [84] and others. Models that combine predictability and stochastic volatility, as the one herein, were analyzed, among others, in [51], [56], [64], [77] and [93].

In a different modeling direction, stochastic factors have been incorporated in asset allocation models with stochastic interest rates (see, for example, [15], [16], [19], [24], [25], [28], [29], [79] and [89]). From the technical point of view, the analysis is not much different as long as the model remains Markovian. However, various technically interesting questions arise (see, for example, [54], [56] and [87]).

3 The backward formulation

The traditional criterion for optimal portfolio choice has been based on maximal expected utility[1] (see, for example, [66] and [67]). The key ingredients are the choices of the trading horizon, $[0, T]$, and the investor's utility, u_T, at terminal time T. The utility function reflects the risk attitude of the investor at time

[1] See, for example, the review article [96].

T and is an increasing and concave function of his wealth[2]. It is important to observe that once these choices are made, the risk preferences cannot be revised. In addition, no investment decisions can be assessed for times beyond T.

The objective is to maximize the expected utility of terminal wealth over the set of admissible strategies. The solution, known as the value function, is defined as

$$V(x, y, t; T) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(u_T(X_T) \middle| X_t = x, Y_t = y \right), \tag{6}$$

for $(x, y, t) \in \mathbb{D} \times \mathbb{R} \times [0, T]$ and $\mathcal{A}$ being the set of admissible strategies. For conditions on the asymptotic behavior of u_T in infinite and semi-infinite domains see [80] and [81].

As solution of a stochastic optimization problem, the value function is expected to satisfy the Dynamic Programming Principle (DPP), namely,

$$V(x, y, t; T) = \sup_{\mathcal{A}} E_{\mathbb{P}} \left(V(X_s, Y_s, s; T) \middle| X_t = x, Y_t = y \right), \tag{7}$$

for $t \leq s \leq T$. This is a fundamental result in optimal control and has been proved for a wide class of optimization problems. For a detailed discussion on the validity (and strongest forms) of the DPP in problems with controlled diffusions, we refer the reader to [37] (see, also, [8], [32], [60] and [62]). Key issues are the measurability and continuity of the value function process as well as the compactness of the set of admissible controls. It is worth mentioning that a proof specific to the problem at hand has not been produced to date. Recently, a weak version of the DPP was proposed in [11] where conditions related to measurable selection and boundness of controls are relaxed.

Besides its technical challenges, the DPP exhibits two important properties of the value function process. Specifically, $V(x, Y_s, s; T)$, $s \in [t, T]$, is a supermartingale for an arbitrary investment strategy and becomes a martingale at an optimum (provided certain integrability conditions hold). One may, then, view $V(x, Y_s, s; T)$ as the intermediate (indirect) utility in the relevant market environment. It is worth noticing, however, that the notions of utility and risk aversion for times $t \in [0, T)$ are tightly connected to the investment opportunities the investor has in the specific market. Observe that the DPP yields a backward in time algorithm for the computation of the maximal utility, starting at expiration with u_T and using the martingality property to compute the solution for earlier times. For this, we refer to this formulation of the optimal portfolio choice problem as *backward.*

The Markovian assumptions on the stock price and stochastic factor dynamics allow us to study the value function via the associated HJB equation, stated in (8) below. Fundamental results in the theory of controlled diffusions yield that if the value function is smooth enough then it satisfies the HJB equation. Moreover, optimal policies may be constructed in a feedback form from the

[2] The quadratic utility represents an exception as it is not globally increasing. This utility, albeit popular for tractability reasons, yields non intuitive optimal portfolios and is not discussed herein.

first-order conditions in the HJB equation, provided that the candidate feedback process is admissible and the wealth SDE has a strong solution when the candidate control is used. The latter usually requires further regularity on the value function. In the reverse direction, a smooth solution of the HJB equation that satisfies the appropriate terminal and boundary (or growth) conditions may be identified with the value function, provided the solution is unique in the appropriate sense. These results are usually known as the "verification theorem" and we refer the reader to [37], [60] and [92] for a general exposition on the subject.

In maximal expected utility problems, it is rarely the case that the arguments in either direction of the verification theorem can be established. Indeed, it is very difficult to show a priori regularity of the value function, with the main difficulties coming from the lack of global Lipschitz regularity of the coefficients of the controlled process with respect to the controls and from the non-compactness of the set of admissible policies. It is, also, very difficult to establish existence, uniqueness and regularity of the solutions to the HJB equation. This is caused primarily by the presence of the control policy in the volatility of the controlled wealth process which makes the classical assumptions of global Lipschitz conditions of the equation with regards to the non linearities fail. Additional difficulties come from state constraints and the non-compactness of the admissible set.

To our knowledge, regularity results for the value function (6) for general utility functions have not been obtained to date except for the special cases of homothetic preferences (see, for example, [36], [56], [68], [77] and [93]). The most general result in this direction, and in a much more general market model, was recently obtained in [59] where it is shown that the value function is twice differentiable in the spatial argument but without establishing its continuity.

Because of lack of general rigorous results, we proceed with an informal discussion about the optimal feedback policies.

For the model at hand, the associated HJB equation turns out to be

$$V_t + \max_{\pi}\left(\frac{1}{2}\sigma^2(y)\,\pi^2 V_{xx} + \pi\left(\mu(y)\,V_x + \rho\sigma(y)\,d(y)\,V_{xy}\right)\right) \tag{8}$$

$$+\frac{1}{2}d^2(y)\,V_{yy} + b(y)\,V_y = 0,$$

with $V(x,y,T;T) = u_T(x)\,,\ (x,y,t) \in \mathbb{D}\times\mathbb{R}\times[0,T]\,.$

The verification results would yield that under appropriate regularity and growth conditions, the feedback policy

$$\pi_s^* = \pi^*\left(X_s^*, Y_s, s; T\right), \quad t \le s \le T,$$

with $\pi^* : \mathbb{D}\times\mathbb{R}\times[0,T]$ given by

$$\pi^*(x,y,t;T) = -\frac{\lambda(y)}{\sigma(y)}\frac{V_x(x,y,t;T)}{V_{xx}(x,y,t;T)} - \rho\frac{d(y)}{\sigma(y)}\frac{V_{xy}(x,y,t;T)}{V_{xx}(x,y,t;T)} \tag{9}$$

and X_s^*, $t \leq s \leq T$, solving

$$dX_s^* = \sigma(Y_s)\,\pi(X_s^*, Y_s, s; T)\left(\lambda(Y_s)\,ds + dW_s^1\right), \tag{10}$$

is admissible and optimal.

Some answers to the questions related to the characterization of the solutions to the HJB equation may be given if one relaxes the requirement to have classical solutions. An appropriate class of weak solutions turns out to be the so called viscosity solutions ([26], [62], [63] and [88]). The analysis and characterization of the value function in the viscosity sense has been carried out for the special cases of power and exponential utility (see, for example, [93]). However, proving that the value function is the unique viscosity solution of (8) has not been addressed.

A key property of viscosity solutions is their robustness (see [63]). If the HJB has a unique viscosity solution (in the appropriate class), robustness is used to establish convergence of numerical schemes for the value function and the optimal feedback laws. Such numerical studies have been carried out successfully for a number of applications. However, for the model at hand, no such studies are available. Numerical results using Monte Carlo techniques have been obtained in [30] for a model more general than the one herein.

Besides the technically challenging issues that problem (6) gives rise to, there is a number of very interesting questions on the economic properties of the optimal portfolios. From (9) one sees that the optimal feedback portfolio functional consists of two terms, namely,

$$\pi^{*,m}(x, y, t; T) = -\frac{\lambda(y)}{\sigma(y)}\frac{V_x(x, y, t; T)}{V_{xx}(x, y, t; T)} \tag{11}$$

and

$$\pi^{*,h}(x, y, t; T) = -\rho\frac{d(y)}{\sigma(y)}\frac{V_{xy}(x, y, t; T)}{V_{xx}(x, y, t; T)}. \tag{12}$$

The first component, $\pi^{*,m}(x, y, t; T)$, is known as the *myopic* investment strategy. It corresponds functionally to the investment policy followed by an investor in markets in which the investment opportunity set remains constant through time. The myopic portfolio is always positive for a nonzero market price of risk.

The second term, $\pi^{*,h}(x, y, t; T)$, is called the *excess hedging demand.* It represents the additional investment caused by the presence of the stochastic factor. It does not have a constant sign, for the signs of the correlation coefficient ρ and the mixed derivative V_{xy} are not definite. The excess risky demand vanishes in the uncorrelated case, $\rho = 0$, and when the volatility of the stochastic factor process is zero, $d(y) = 0$, $y \in \mathbb{R}$. In the latter case, using a simple deterministic time-rescaling argument reduces the problem to the classical Merton one. Finally, $\pi^{*,h}(x, y, t; T)$ vanishes for the case of logarithmic utility (see (13)). Despite the nomenclature "hedging demand", a rigorous study for the precise characterization and quantification of the risk that is not hedged has not been carried out. Indeed, in contrast to derivative valuation where the notion of imperfect hedge is well defined, such a notion has not been established in the area of investments (see [85] for a special case).

The total allocation in the risky asset might become zero even if the risk premium is not zero. This phenomenon, related to the so called market participation puzzle, appears at first counter intuitive, for classical economic ideas suggest that a risk averse investor should always retain nonzero holdings in an asset that offers positive risk premium. We refer the reader to, among others, [4], [20] and [43].

Important questions arise on the dependence, sensitivity and robustness of the optimal feedback portfolio in terms of the market parameters, the wealth, the level of the stochastic factor and the risk preferences. Such questions are central in financial economics and have been studied, primarily in simpler models in which intermediate consumption is also incorporated (see, among others, [2], [52], [61], [75] and [78]). For diffusion models with and without a stochastic factor qualitative results can be found in [30], [51], [64], [90] and, recently, in [9] (see, also, [65] for a general incomplete market discrete model). However, a qualitative study for general utility functions and/or arbitrary factor dynamics has not been carried out to date.

Some open problems

Problem 1: What are the weakest conditions on the market coefficients and the utility function so that the Dynamic Programming Principle holds?

Problem 2: What are the weakest conditions on the market coefficients and the utility function so that existence and uniqueness of viscosity solutions to the HJB equation hold?

Problem 3: Study the regularity of the value function and establish the associated verification theorem.

Problem 4: Develop numerical schemes for the value function and the optimal feedback policies for general utility functions.

Problem 5: Study the behavior of the optimal portfolio in terms of market inputs, the horizon length and risk preferences for general utility functions and arbitrary stochastic factor dynamics. Compute and analyze the distribution of the optimal wealth and portfolio processes as well as their moments.

3.1 The CARA, CRRA and logarithmic cases

We provide examples for the most frequently used utilities, namely, the exponential, power and logarithmic ones. They have convenient homogeneity properties which, in combination with the linearity of the wealth dynamics in the control policies, enable us to reduce the HJB equation to a quasilinear one. Under a "distortion" transformation (see, for example, [93]) the latter can be linearized and solutions in closed form can be produced using the Feynman-Kac formula. The smoothness of the value function and, in turn, the verification of the optimal feedback policies follows easily.

Multi-factor models for these preferences have been analyzed by various authors. The theory of BSDE has been successfully used to characterize and represent the solutions of the reduced HJB equation (see [33]). The regularity of its solutions has been studied using PDE arguments by [77] and [68], for power and exponential utilities, respectively. Finally, explicit solutions for a three factor model can be found in [64].

Exponential case: We have $u_T(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$ and $\gamma > 0$.

This case has been extensively studied not only in optimal investment models but, also, in indifference pricing where valuation is done primarily under exponential preferences (see [21] for a concise collection of relevant references). The value function is multiplicatively separable and given, for $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$, by

$$V(x, y, t; T) = -e^{-\gamma x} h(y, t; T)^{\delta}, \qquad \delta = \frac{1}{1-\rho^2},$$

where $h : \mathbb{R} \times [0, T] \to \mathbb{R}$ solves

$$h_t + \frac{1}{2}\sigma^2(y) h_{yy} + \left(b(y) - \rho \frac{d(y)}{\sigma(y)} \right) h_y = \frac{1}{2}\left(1-\rho^2\right) \lambda^2(y) h,$$

with $h(x, y, T; T) = 1$. The optimal feedback investment strategy is independent of the wealth level and given by

$$\pi^*(x, y, t; T) = \frac{\lambda(y)}{\sigma^2(y)} + \frac{\rho}{1-\rho^2} \frac{d(y)}{\sigma(y)} \frac{h_y(y, t; T)}{h(y, t; T)}.$$

The optimal wealth and portfolio process follow directly from (9) and (10). Namely, for $t \le s \le T$,

$$\pi_s^* = \pi^*(x, Y_s, s; T) = \frac{\lambda(Y_s)}{\sigma^2(Y_s)} + \frac{\rho}{1-\rho^2} \frac{d(Y_s)}{\sigma(Y_s)} \frac{h_y(Y_s, s; T)}{h(Y_s, s; T)}$$

and

$$X_s^* = x + \int_t^s \sigma(Y_u) \lambda(Y_u) \pi_u^* du + \int_t^s \sigma(Y_u) \pi_u^* dW_u^1.$$

A well known criticism of the exponential utility is that the optimal portfolio does not depend on the investor's wealth. While this property might be desirable in asset equilibrium pricing, it appears to be problematic and counter intuitive for investment problems. We note, however, that this property is directly related to the choice of the savings account as the numeraire. If the benchmark changes, the optimal portfolio ceases to be independent of wealth (see (57)).

The next two utilities are defined on the half-line and the stochastic optimization problem is a state-constraint one. We easily deduce from the form of the optimal portfolios that the non-negativity wealth constraint is always satisfied.

Power case: We have $u_T(x) = \frac{1}{\gamma} x^{\gamma}, \quad 0 < \gamma < 1, \gamma \neq 0$.

The value function is multiplicatively separable and given, for $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$, by

$$V(x, y, t; T) = \frac{1}{\gamma} x^\gamma f(y, t; T)^\delta, \qquad \delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma},$$

where $f : \mathbb{R} \times [0, T] \to \mathbb{R}^+$ solves the linear parabolic equation

$$f_t + \frac{1}{2} d^2(y) f_{yy} + \left(b(y) + \rho \frac{\gamma}{1-\gamma} \lambda(y) d(y) \right) f_y$$

$$+ \frac{\gamma}{2(1-\gamma)} \frac{\lambda^2(y)}{\delta} f = 0,$$

with $f(x, y, T; T) = 1$. The optimal policy feedback function is linear in wealth,

$$\pi^*(x, y, t; T) = \frac{1}{1-\gamma} \frac{\lambda(y)}{\sigma(y)} x + \frac{\rho}{(1-\gamma)+\rho^2\gamma} \frac{d(y)}{\sigma(y)} \frac{f_y(y, t; T)}{f(y, t; T)} x.$$

The optimal investment and wealth processes are, in turn, given by

$$\pi_s^* = m_s X_s^*$$

and

$$X_s^* = x \exp\left(\int_t^s \left(\sigma^2(Y_u) \lambda(Y_u) m_u - \frac{1}{2} \sigma^2(Y_u) m_u^2 \right) du + \int_t^s \sigma(Y_u) m_u dW_u^1 \right),$$

with

$$m_s = \frac{1}{1-\gamma} \frac{\lambda(Y_s)}{\sigma(Y_s)} + \frac{\rho}{(1-\gamma)+\rho^2\gamma} \frac{d(Y_s)}{\sigma(Y_s)} \frac{f_y(Y_s, s; T)}{f(Y_s, s; T)}.$$

The range of the risk aversion parameter can be relaxed to include negative values. Its choice plays important role in the boundary and asymptotic behavior of the value function as well as the long-term behavior of the optimal wealth and portfolio processes (see [51] and [64]). Verification results for weak conditions on the risk premium can be found, among others, in [55] and [56].

Logarithmic utility: We have $u_T(x) = \ln x$, $x > 0$.

The value function is additively separable, namely,

$$V(x, y, t; T) = \ln x + h(y, t; T),$$

with $h : \mathbb{R} \times [0, T] \to \mathbb{R}^+$ solving

$$h_t + \frac{1}{2} d^2(y) h_{yy} + b(y) h_y + \frac{1}{2} \lambda^2(y) h = 0$$

and $h(y, T; T) = 1$. The optimal portfolio takes the simple linear form

$$\pi^*(x, y, t; T) = \frac{\lambda(y)}{\sigma(y)} x. \tag{13}$$

In turn, the optimal investment and wealth processes are given, for $t \leq s \leq T$, by

$$\pi_s^* = \frac{\lambda\left(Y_s\right)}{\sigma\left(Y_s\right)} X_s^* \qquad \text{and} \qquad X_s^* = x \exp\left(\int_t^s \frac{1}{2}\lambda^2\left(Y_u\right) du + \int_t^s \lambda\left(Y_u\right) dW_u^1\right).$$

The logarithmic utility plays a special role in portfolio choice. Because of the additively separable form of the value function, the optimal portfolio is always myopic. It is known as the "growth optimal portfolio" and has been extensively studied in general market settings (see, for example, [6] and [50]). The associated optimal wealth is the so-called "numeraire portfolio". It has also been extensively studied, for it is the numeraire with regards to which all wealth processes are supermartingales under the historical measure (see, among others, [40] and [41]).

4 The forward formulation

As discussed in the previous section, the main feature of the expected utility approach is the a priori choice of the utility at the end of the trading horizon. Direct consequences of this choice are, from one hand, the lack of flexibility to revise the risk preferences at other times and, from the other, the inability to assess the performance of investment strategies beyond the prespecified horizon.

Addressing these limitations has been the subject of a number of studies and various approaches have been proposed. With regards to the horizon length, the most popular alternative has been the formulation of the investment problem in $[0, +\infty)$ and incorporating either intermediate consumption or optimizing the investor's long-term optimal behavior (see, among others, [47], [48] and [86]). Investment models with random horizon have also been examined ([23]). The revision of risk preferences has been partially addressed by recursive utilities (see, for example, [31], [82] and [83]).

Next, we present another alternative approach which addresses both shortcomings of the expected utility approach. The associated criterion is developed in terms of a family of stochastic processes defined on $[0, \infty)$ and indexed by the wealth argument. It will be called *forward performance process.* Its key properties are the martingality at an optimum and supermartingality away from it. These are in accordance with the analogous properties of the value function process that stem out from the Dynamic Programming Principle (cf. (7)). However, in contrast to the existing framework, the risk preferences are specified for today[3] and not for a (possibly remote) future time.

We recall that $\mathcal{F}_t$, $t \geq 0$, is the filtration generated by $W_t = \left(W_t^1, W_t^2\right)$, $t \geq 0$, and $\mathcal{A}$ the set of admissible policies. As in the previous section, we use $\mathbb{D}$ to denote the generic admissible space domain.

[3] The choice of the initial condition gives rise to interesting mathematical and modeling questions (see, for example, [73] and references therein).

Definition 1 *An $\mathcal{F}_t-$adapted process $U(x,t)$ is a forward performance if for $t \geq 0$ and $x \in \mathbb{D}$:*

i) the mapping $x \rightarrow U(x,t)$ is concave and increasing,

ii) for each portfolio process $\pi \in \mathcal{A}$, $E_{\mathbb{P}}\left(U\left(X_t^{\pi},t\right)\right)^{+} < \infty$, and

$$E_{\mathbb{P}}\left(U\left(X_s^{\pi},s\right)|\mathcal{F}_t\right) \leq U\left(X_t^{\pi},t\right), \qquad s \geq t, \tag{14}$$

iii) there exists a portfolio process $\pi^* \in \mathcal{A}$, for which

$$E_{\mathbb{P}}\left(U\left(X_s^{\pi^*},s\right)|\mathcal{F}_t\right) = U\left(X_t^{\pi^*},t\right), \qquad s \geq t, \tag{15}$$

and

iv) at $t=0$, $U(x,0) = u_0(x)$, where $u_0 : \mathbb{D} \rightarrow \mathbb{R}$ is increasing and concave.

The concept of forward performance process was introduced in [69] (see, also, [70]). The model therein is incomplete binomial and the initial data is taken to be exponential. The exponential case was subsequently and extensively analyzed in [71] and [95].

Ideas related to the forward approach can also be found in [23] where the authors consider random horizon choices, aiming at alleviating the dependence of the value function on a fixed deterministic horizon. Their model is more general in terms of the assumptions on the price dynamics but the focus in [23] is primarily on horizon effects. Horizon issues were also considered in [44] for the special case of lognormal stock dynamics.

It is worth observing the following differences and similarities between the forward performance process and the traditional value function. Namely, the process $U(x,t)$ is defined for all $t \geq 0$, while the value function $V(x,y,t;T)$, is defined only on $[0,T]$. In the classical set up discussed in the previous section, $V(x,y,T;T) \in \mathcal{F}_0$, due to the deterministic choice of the terminal utility u_T. If the terminal utility is taken to be state-dependent, $V(x,y,T;T) \in \mathcal{F}_T$, (see, for example, [49], [81] as well as [10], [27] and [46]), the traditional and new formulations are, essentially, identical in $[0,T]$.

Recently, it was shown in [74] that a sufficient condition for a process $U(x,t)$ to be a forward performance is that it satisfies a stochastic partial differential equation (see (18) below). For completeness, we state the result for a general incomplete market model with k risky stocks whose prices are modelled as Ito processes driven by a $d-$dimensional Brownian motion. We use σ_t, $t \geq 0$, to denote their $d \times k$ random volatility matrix and μ_t the $k-$dim vector with coordinates the mean rate of return of each stock. It is assumed that the volatility vectors are such that $\mu_t - r_t\mathbf{1} \in Lin\left(\sigma_t^T\right)$, where $Lin\left(\sigma_t^T\right)$ denotes the linear space generated by the columns of σ_t^T. This implies that $\sigma_t^T\left(\sigma_t^T\right)^{+}\left(\mu_t - r_t\mathbf{1}\right) = \mu_t - r_t\mathbf{1}$ and, therefore, the market price of risk vector

$$\lambda_t = \left(\sigma_t^T\right)^{+}\left(\mu_t - r_t\mathbf{1}\right) \tag{16}$$

is a solution to the equation $\sigma_t^T x = \mu_t - r_t\mathbf{1}$. The matrix $\left(\sigma_t^T\right)^{+}$ is the Moore-Penrose pseudo-inverse of the matrix σ_t^T. It easily follows that, for $t \geq 0$,

$$\sigma_t\sigma_t^{+}\lambda_t = \lambda_t. \tag{17}$$

It is assumed from now on that there exists a deterministic constant $c \geq 0$ such that, for $t \geq 0$, $\lambda(Y_t) \leq c$.

Proposition 2 *Let $U(x,t) \in \mathcal{F}_t$ be such that the mapping $x \to U(x,t)$ is increasing and concave. Let, also, $U(x,t)$ be a solution to the stochastic partial differential equation*

$$dU(x,t) = \frac{1}{2} \frac{\left| U_x(x,t)\lambda_t + \sigma_t \sigma_t^+ a_x(x,t) \right|^2}{U_{xx}(x,t)} dt + a(x,t) \cdot dW_t, \tag{18}$$

where $a(x,t) \in \mathcal{F}_t$. Then $U(x,t)$ is a forward performance process.

It might seem that all Definition 1 produces is a criterion that is dynamically consistent across time. Indeed, internal consistency is an ubiquitous requirement and needs to be ensured in any proposed criterion. It is satisfied, for example, by the traditional value function. However, the new criterion allows for much more flexibility as it is manifested by the volatility process $a(x,t)$ introduced above. Characterizing the appropriate class of admissible volatility processes is, in our view, an interesting and challenging question.

The forward performance SPDE (18) poses several challenges. It is fully nonlinear and *not* (degenerate) elliptic; the latter is a direct consequence of the "forward in time" nature of the involved stochastic optimization problem. Thus, existing results of existence, uniqueness and regularity of weak (viscosity) solutions are not directly applicable. An additional difficulty comes from the fact that the volatility coefficient may depend on the second order derivative of U. In such cases, it might not be possible to reduce the SPDE, using the method of stochastic characteristics, into a PDE with random coefficients.

For the model at hand, the coefficients appearing in (18) take the form

$$\sigma_t = (\sigma(Y_t), 0)^T \ , \ \sigma_t^+ = \left(\frac{1}{\sigma(Y_t)}, 0 \right) \ \text{ and } \ \lambda_t = \left(\frac{\mu(Y_t) - r}{\sigma(Y_t)}, 0 \right)^T.$$

We easily see that (17) is trivially satisfied.

Proposition 3 *i) Let $U(x,t) \in \mathcal{F}_t$ be such that the mapping $x \to U(x,t)$ is increasing and concave. Let, also, $U(x,t)$ be a solution to the stochastic partial differential equation*

$$dU(x,t) = \frac{1}{2} \frac{\left(\lambda(Y_t) U_x(x,t) + a_x^1(x,t) \right)^2}{U_{xx}(x,t)} dt + a^1(x,t) dW_t^1 + a^2(x,t) dW_t^2,$$

where $a(x,t) = \left(a^1(x,t), a^2(x,t)\right)^T$, with $a^i(x,t) \in \mathcal{F}_t$, $i = 1,2$. Then $U(x,t)$ is a forward performance process.

ii) Let $U(x,t)$ be a solution to the SPDE (18) such that, for each $t \geq 0$, the mapping $x \to U(x,t)$ is increasing and concave. Consider the process π_t^*, $t \geq 0$, given by

$$\pi_t^* = -\frac{\lambda(Y_t) U_x(X_t^*, t) + a_x^1(X_t^*, t)}{\sigma(Y_t) U_{xx}(X_t^*, t)} \tag{19}$$

where X_t^*, $t \geq 0$, *solves*

$$dX_t^* = \sigma\left(Y_t\right)\pi_t^*\left(\lambda\left(Y_t\right)dt + dW_t^1\right) \tag{20}$$

with $X_0^* = x$. *If* $\pi_t^* \in \mathcal{A}$ *and (20) has a strong solution, then* π_t^* *and* X_t^* *are optimal.*

Remark: The same stochastic partial differential equation emerges in the classical formulation of the optimal portfolio choice problem. Indeed, assuming for the moment that the appropriate regularity assumptions hold, expanding the process $V\left(x, Y_t, t; T\right)$ (cf. (2) and (6)), yields,

$$dV\left(x, Y_t, t\right) = \left(V_t\left(x, Y_t, t\right) + \frac{1}{2}d^2\left(Y_t\right)V_{yy}\left(x, Y_t, t\right) + b\left(Y_t\right)V_y\left(x, Y_t, t\right)\right)dt$$

$$+\rho d\left(Y_t\right)V_y\left(x, Y_t, t\right)dW_t^1 + \sqrt{1-\rho^2}d\left(Y_t\right)V_y\left(x, Y_t, t\right)dW_t^2.$$

Using that $V\left(x, y, t; T\right)$ solves the HJB equation and rearranging terms, we deduce that

$$dV\left(x, Y_t, t\right) = \frac{1}{2}\frac{\left(\lambda\left(Y_t\right)V_x\left(x, Y_t, t\right) + \rho d\left(Y_t\right)V_{xy}\left(x, Y_t, t\right)\right)^2}{V_{xx}\left(x, t\right)}dt$$

$$+\rho d\left(Y_t\right)V_y\left(x, Y_t, t\right)dW_t^1 + \sqrt{1-\rho^2}d\left(Y_t\right)V_y\left(x, Y_t, t\right)dW_t^2.$$

The above SPDE corresponds to the volatility choice, for $0 \leq t < T$,

$$a^1\left(x, t\right) = \rho d\left(Y_t\right)V_y\left(x, Y_t, t\right) \quad \text{and} \quad a^2\left(x, t\right) = \sqrt{1-\rho^2}d\left(Y_t\right)V_y\left(x, Y_t, t\right).$$

Notice that in the backward optimal investment model, there is *no* freedom in choosing the volatility coefficients, for they are uniquely obtained from the Ito decomposition of the value function process.

4.1 The zero volatility case

An important class of forward performance processes are the ones that are *decreasing* in time. They yield an intuitively rich family of performance criteria which compile in a transparent way the dynamic risk profile of the investor and the information coming from the evolution of the investment opportunity set. This section is dedicated to the representation of these processes and the construction of the associated optimal wealth and portfolios. These issues have been extensively studied in [72] and [73], and we refer the reader therein for the proofs of the results that follow.

The *local risk tolerance* function $r\left(x, t\right), t \geq 0$, defined below, plays a crucial role in the representation of the optimal investment and wealth processes. It represents the dynamic counterpart of the static risk tolerance function, $r_T\left(x\right) = -\frac{u_T'(x)}{u_T''(x)}$. Observe that similarly to its static analogue, it is chosen exogenously to the market. However, now it is time-dependent and solves the autonomous fast diffusion equation (25)[4]. The reciprocal of the risk tolerance, the *local risk aversion,* $\gamma = r^{-1}$ solves the porous medium equation (26). We recall that $u_0(x)$ is the initial condition of the forward performance process. It is assumed that $u_0 \in \mathcal{C}^4(\mathbb{D})$.

Theorem 4 *Let λ be as in (4) and define the time-rescaling process*

$$A_t = \int_0^t \lambda(Y_s)^2 \, ds, \quad t \geq 0. \tag{21}$$

Let, also, $u \in \mathcal{C}^{4,1}(\mathbb{D} \times (0, +\infty))$ be a concave and increasing in the spatial argument function satisfying

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}, \tag{22}$$

and $u(x, 0) = u_0(x)$. Then, the time-decreasing process

$$U_t(x) = u(x, A_t) \tag{23}$$

is a forward performance.

Proposition 5 *Let the local risk tolerance function $r : \mathbb{D} \times [0, +\infty) \to \mathbb{R}_0^+$ be defined by*

$$r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)}, \tag{24}$$

with u solving (22). Then, r satisfies

$$r_t + \frac{1}{2} r^2 r_{xx} = 0, \tag{25}$$

with $r(x, 0) = -\frac{u_0'(x)}{u_0''(x)}$. Its reciprocal, $\gamma = r^{-1}$, solves

$$\gamma_t + \frac{1}{2} \left(\frac{1}{\gamma} \right)_{xx} = 0, \tag{26}$$

with $\gamma(x, 0) = -\frac{u_0''(x)}{u_0'(x)}$.

An analytically explicit construction of the function u was recently developed in [73]. A strictly increasing space-time harmonic function, $h : \mathbb{R} \times [0, +\infty) \to \mathbb{D}$, solving the backward heat equation

$$h_t + \frac{1}{2} h_{xx} = 0, \tag{27}$$

plays a key role. This function is always globally defined but its range varies as *Range* $(h) = \mathbb{D}$, with $\mathbb{D}$ being the domain of u.

[4] See [7] and [45] for a similar equation arising in the traditional Merton problem.

It was shown in [73] that there is a one-to-one correspondence between strictly increasing solutions of (27) and strictly increasing and concave solutions of (22) (see, Propositions 9, 13 and 14 therein). Pivotal role in the analysis is played by a positive Borel measure, ν, through which the function h is represented in an integral form (see (32) and (38) below). This representation stems from classical results of Widder for the solutions of the (backward) heat equation (see [91]). We note that in the applications at hand, these results are not directly applicable, for the range of h is not always constrained to the positive semi-axis. Indeed, we will see that h is used to represent the optimal wealth (cf. (43)), which, in unconstrained problems, may take arbitrary values.

The results that follow correspond to the infinite domain case, $\mathbb{D}=\mathbb{R}$. To ease the presentation we introduce the following sets,

$$\mathcal{B}^{+}(\mathbb{R})=\left\{\nu\in\mathcal{B}(\mathbb{R}):\ \forall B\in\mathcal{B},\ \nu(B)\geq 0\quad\text{and}\quad\int_{\mathbb{R}}e^{yx}\nu(dy)<\infty,\ x\in\mathbb{R}\right\},\tag{28}$$

$$\mathcal{B}_{0}^{+}(\mathbb{R})=\left\{\nu\in\mathcal{B}^{+}(\mathbb{R})\ \text{ and }\nu(\{0\})=0\right\},\tag{29}$$

$$\mathcal{B}_{+}^{+}(\mathbb{R})=\left\{\nu\in\mathcal{B}_{0}^{+}(\mathbb{R}):\nu((-\infty,0))=0\ \right\}\tag{30}$$

and

$$\mathcal{B}_{-}^{+}(\mathbb{R})=\left\{\nu\in\mathcal{B}_{0}^{+}(\mathbb{R}):\ \nu((0,+\infty))=0\ \right\}.\tag{31}$$

We start with representation results for strictly increasing solutions of (27) with unbounded range.

Proposition 6 *i) Let $\nu\in\mathcal{B}^{+}(\mathbb{R})$ and $C\in\mathbb{R}$. Then, the function h defined, for $(x,t)\in\mathbb{R}\times[0,+\infty)$, by*

$$h(x,t)=\int_{\mathbb{R}}\frac{e^{yx-\frac{1}{2}y^{2}t}-1}{y}\nu(dy)+C,\tag{32}$$

is a strictly increasing solution to (27).

Moreover, if $\nu(\{0\})>0$, or $\nu\in\mathcal{B}_{+}^{+}(\mathbb{R})$ and $\int_{0^+}^{+\infty}\frac{\nu(dy)}{y}=+\infty$, or $\nu\in\mathcal{B}_{-}^{+}(\mathbb{R})$ and $\int_{-\infty}^{0^-}\frac{\nu(dy)}{y}=-\infty$, then Range$(h)=(-\infty,+\infty)$, for $t\geq 0$. On the other hand, if $\nu\in\mathcal{B}_{+}^{+}(\mathbb{R})$ with $\int_{0^+}^{+\infty}\frac{\nu(dy)}{y}<+\infty$ (resp. $\nu\in\mathcal{B}_{-}^{+}(\mathbb{R})$ with $\int_{-\infty}^{0^-}\frac{\nu(dy)}{y}>-\infty$), then Range$(h)=\left(C-\int_{0^+}^{+\infty}\frac{\nu(dy)}{y},+\infty\right)$ (resp. Range$(h)=\left(-\infty,C-\int_{-\infty}^{0^-}\frac{\nu(dy)}{y}\right)$), for $t\geq 0$.

ii) Conversely, let $h:\mathbb{R}\times[0,+\infty)\to\mathbb{R}$ be a strictly increasing solution to (27). Then, there exists $\nu\in\mathcal{B}^{+}(\mathbb{R})$ such that h is given by (32).

Moreover, if Range$(h)=(-\infty,+\infty)$, $t\geq 0$, then it must be either that $\nu(\{0\})>0$, or $\nu\in\mathcal{B}_{+}^{+}(\mathbb{R})$ and $\int_{0^+}^{+\infty}\frac{\nu(dy)}{y}=+\infty$, or $\nu\in\mathcal{B}_{-}^{+}(\mathbb{R})$ and $\int_{-\infty}^{0^-}\frac{\nu(dy)}{y}=$

$-\infty$. *On the other hand, if* $Range\,(h) = (x_0, +\infty)$ *(resp.* $Range\,(h) = (-\infty, x_0)$*),* $t \geq 0$ *and* $x_0 \in \mathbb{R}$*, then it must be that* $\nu \in \mathcal{B}_+^+(\mathbb{R})$ *with* $\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} < +\infty$ *(resp.* $\nu \in \mathcal{B}_-^+(\mathbb{R})$ *with* $\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} > -\infty$*).*

The next proposition yields the one-to-one correspondence between the solutions h and u. Without loss of generality, we will normalize the values

$$h\,(0,0) = 0, \tag{33}$$

choosing $C = 0$, and[5]

$$u\,(0,0) = 0 \quad \text{and} \quad u_x\,(0,0) = 1. \tag{34}$$

Proposition 7 *i) Let* $\nu \in \mathcal{B}^+(\mathbb{R})$ *and* $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *be as in (32) with the measure* ν *being used. Assume that* h *is of full range, for each* $t \geq 0$*, and let* $h^{(-1)} : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *be its spatial inverse. Then, the function* u *defined for* $(x,t) \in \mathbb{R} \times [0,+\infty)$ *and given by*

$$u\,(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}\,(x,s)\,,s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz, \tag{35}$$

is an increasing and strictly concave solution of (22) satisfying (34).

Moreover, for $t \geq 0$*, the Inada conditions,*

$$\lim_{x \to -\infty} u_x\,(x,t) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} u_x\,(x,t) = 0, \tag{36}$$

are satisfied.

ii) Conversely, let u *be an increasing and strictly concave function satisfying, for* $(x,t) \in \mathbb{R} \times [0,+\infty)$*, (22) and (34), and the Inada conditions (36), for* $t \geq 0$*. Then, there exists* $\nu \in \mathcal{B}^+(\mathbb{R})$*, such that* u *admits representation (35) with* h *given by (32), for* $(x,t) \in \mathbb{R} \times [0,+\infty)$*. Moreover,* h *is of full range, for each* $t \geq 0$*, and satisfies (33).*

The cases of semi-finite domain, $\mathbb{D} = \mathbb{R}^+$, $\mathbb{R}_0^+, \mathbb{R}^-$ and $\mathbb{R}_0^-$ deserve special attention as they are used in the popular choices of power and logarithmic risk preferences. In these cases, the support of the measure is constrained to the half-line. The representation results above need to be modified for semi-infinite domains. Various cases emerge, depending on certain characteristics of the measure ν which affect the boundary behavior of the solution u. The arguments are both computationally cumbersome and long. For completeness we state one of these cases and we refer the reader to [73] for the others. To this end, we assume that

$$\nu \in \mathcal{B}_+^+(\mathbb{R}) \qquad \text{and} \qquad \int_{0^+}^{+\infty} \frac{\nu\,(dy)}{y} < +\infty, \tag{37}$$

[5] The first equality is imposed in an ad hoc way. The second one, however, is in accordance with (33). For details see the proof of Proposition 9 in [73].

with $\mathcal{B}_+^+(\mathbb{R})$ given in (30). Choosing for convenience $C = \int_{0^+}^{+\infty} \frac{1}{y}\nu(dy)$ in (32) yields[6] the solution to (27)

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2 t}}{y}\nu(dy), \tag{38}$$

with *Range* $(h) = (0, +\infty)$.

Proposition 8 *i) Let ν satisfy (37) and, in addition, $\nu((0,1]) = 0$. Let, also, $h : \mathbb{R} \times [0, +\infty) \to (0, +\infty)$ be as in (38) and $h^{(-1)} : (0, +\infty) \times [0, +\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined, for $(x,t) \in (0, +\infty) \times [0, +\infty)$, by*

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz, \tag{39}$$

is an increasing and strictly concave solution of (22) with

$$\lim_{x\to 0} u(x,t) = 0, \quad \text{for} \ \ t \geq 0. \tag{40}$$

Moreover, for $t \geq 0$, the Inada conditions

$$\lim_{x\to 0} u_x(x,t) = +\infty \quad \text{and} \quad \lim_{x\to +\infty} u_x(x,t) = 0 \tag{41}$$

are satisfied.

ii) Conversely, let u, defined for $(x,t) \in (0,+\infty) \times [0,+\infty)$, be an increasing and strictly concave function satisfying (22), (40) and the Inada conditions (41). Then, there exists $\nu \in \mathcal{B}^+(\mathbb{R})$ satisfying (37) and $\nu((0,1]) = 0$, such that u admits representation (39) with h given by (38), for $(x,t) \in \mathbb{R} \times [0, +\infty)$.

Note that the above results yield implicit representation constraints for the initial condition u_0. For example, from (35) we must have $u_0'(x) = e^{-h^{(-1)}(x,0)}$, $x \in \mathbb{R}$, with the integrand $e^{-h^{(-1)}(x,0)}$ specified from $h(x,0) = \int_{\mathbb{R}} \frac{e^{yx}-1}{y}\nu(dy)$. This, in turn, yields that the inverse of u_0' must be represented as

$$(u_0')^{(-1)}(x) = \int_{\mathbb{R}} \frac{e^{-y\ln x} - 1}{y}\nu(dy), \quad x > 0.$$

Characterizing the set of admissible initial data and provide an intuitively meaningful interpretation is, in our view, an interesting question.

We continue with the construction of the optimal wealth and portfolio processes for the class of time decreasing performance processes. As the theorem below shows, the optimal processes can be calculated in closed form.

[6] One may alternatively represent h as $h(x,t) = \int_0^{+\infty} e^{yx-\frac{1}{2}y^2 t}\mu(dy)$ with $\mu(dy) = \frac{\nu(dy)}{y}$. Note that $\mu \in \mathcal{B}^+(\mathbb{R})$. Such a representation was used in [5].

Theorem 9 *i) Let h be a strictly increasing solution to (27), for $(x,t) \in \mathbb{R}\times [0,+\infty)$, and assume that the associated measure ν satisfies, for $t>0$,*

$$\int_{\mathbb{R}} e^{yx+\frac{1}{2}y^2 t}\nu\left(dy\right) < \infty. \tag{42}$$

Let also A_t be as in (21) and M_t, $t \geq 0$, given by

$$M_t = \int_0^t \lambda\left(Y_s\right) dW_s^1.$$

Define the processes X_t^* and π_t^* by

$$X_t^* = h\left(h^{(-1)}\left(x,0\right) + A_t + M_t, A_t\right) \tag{43}$$

and

$$\pi_t^* = \frac{\lambda\left(Y_t\right)}{\sigma\left(Y_t\right)} h_x\left(h^{(-1)}\left(x,0\right) + A_t + M_t, A_t\right), \tag{44}$$

$t \geq 0$, $x \in \mathbb{R}$ with h as above and $h^{(-1)}$ standing for its spatial inverse. Then, the portfolio π_t^* is admissible and generates X_t^*, i.e.,

$$X_t^* = x + \int_0^t \sigma\left(Y_s\right)\pi_s^*\left(\lambda\left(Y_s\right)ds + dW_s^1\right). \tag{45}$$

ii) Let u be the associated with h increasing and strictly concave solution to (22). Then, the process $u\left(X_t^*, A_t\right)$, $t \geq 0$, satisfies

$$du\left(X_t^*, A_t\right) = u_x\left(X_t^*, A_t\right)\sigma(Y_t)\pi_t^* dW_t^1, \tag{46}$$

with X_t^* and π_t^* as in (43) and (44). Therefore, the processes X_t^* and π_t^* are optimal.

The optimal portfolio π_t^* may be also represented in terms of the risk tolerance process, R_t^*, defined as

$$R_t^* = r\left(X_t^*, A_t\right), \tag{47}$$

with X_t^* solving (45) and r as in (24). Indeed, one can show that the local risk tolerance function satisfies, for $(x,t) \in \mathbb{D}\times[0,+\infty)$,

$$r\left(x,t\right) = h_x\left(h^{(-1)}\left(x,t\right), t\right). \tag{48}$$

Therefore, (44) yields

$$\pi_t^* = \frac{\lambda\left(Y_t\right)}{\sigma\left(Y_t\right)} R_t^*. \tag{49}$$

One then sees that under the investment performance criterion (23), the investor will *always* follow a myopic strategy. The excess hedging demand component disappears as long as the volatility performance process remains zero.

4.2 The CARA, CRRA and generalized CRRA cases

Case 1: Let $\nu = \delta_0$, where δ_0 is a Dirac measure at 0. Then, from (32) we obtain $h(x,t) = x$ and, thus, (35) yields $u(x,t) = 1 - e^{-x+\frac{t}{2}}$. The optimal performance process is

$$U(x,t) = 1 - e^{-x+\frac{A_t}{2}}.$$

Formulae (45) and (44) yield, respectively,

$$X_t^* = x + A_t + M_t \quad \text{and} \quad \pi_t^* = \frac{\lambda(Y_t)}{\sigma(Y_t)}.$$

This class of forward performance processes is analyzed in detail in [71] (see, also, [95]).

Case 2: Let $\nu = \delta_\gamma$, $\gamma > 1$. Then (38) yields $h(x,t) = \frac{1}{\gamma} e^{\gamma x - \frac{1}{2}\gamma^2 t}$. Since $\nu((0,1]) = 0$, u is given by (39) and, therefore, $u(x,t) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}t}$. The forward performance process is

$$U_t(x) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma-1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}A_t}, \; t \geq 0.$$

The optimal wealth and portfolio processes are given, respectively, by

$$X_t^* = x \exp\left(\gamma\left(1 - \frac{\gamma}{2}\right) A_t + \gamma M_t\right) \quad \text{and} \quad \pi_t^* = \gamma \frac{\lambda(Y_t)}{\sigma(Y_t)} X_t^*.$$

For the cases $\nu = \delta_\gamma$ with $\gamma = 1$, $\gamma \in (0,1)$ and $\gamma = -\frac{1}{2k+1}$, $k > 0$, see [94].

Case 3: Let $\nu = \frac{b}{2}(\delta_a + \delta_{-a})$, $a, b > 0$, and $\delta_{\pm a}$ are Dirac measures at $\pm a$, $a \neq 1$. We, then, have $h(x,t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax)$ and, from (35),

$$u(x,t)$$

$$= \frac{\sqrt[a]{\alpha}}{\alpha^2 - 1} e^{\frac{1-a}{2}t} \frac{b^2 e^{-\alpha t} + a(1+\alpha)\left(\alpha x^2 + x\sqrt{\alpha^2 x^2 + b^2 e^{-\alpha^2 t}}\right)}{\left(\alpha x + \sqrt{\alpha^2 x^2 + b^2 e^{-\alpha^2 t}}\right)^{1+\frac{1}{\alpha}}} - \frac{\sqrt[a]{\alpha}}{\alpha^2 - 1} b^{1-\frac{1}{a}}.$$

Equalities (43) and (44) yield the optimal wealth and portfolio processes

$$X_t^* = \frac{b}{a} e^{-\frac{1}{2}a^2 A_t} \sinh\left(a\left(h^{(-1)}(x,0)\right) + A_t + M_t\right)$$

and

$$\pi_t^* = b \frac{\lambda(Y_t)}{\sigma(Y_t)} e^{-\frac{1}{2}a^2 A_t} \cosh\left(a\left(h^{(-1)}(x,0)\right) + A_t + M_t\right).$$

The case $a = 1$ deserves special attention as it corresponds to the generalized logarithmic case (see [94] for details).

4.3 Two special cases of volatilities

We focus on the case that the volatility coefficient a is a local affine function of U and xU_x. These examples can be reduced to the zero-volatility case but in markets with modified risk premia.

4.3.1 The "market-view" case: $\left(\alpha^1(x,t), a^2(x,t)\right) = U(x,t)\left(\varphi_t^1, \varphi_t^2\right)$, $\varphi_t^1, \varphi_t^2 \in \mathcal{F}_t$

We assume that the processes φ_t^1, φ_t^2 are bounded by a (deterministic) constant.

The forward performance SPDE, (18), becomes

$$dU(x,t) = \frac{1}{2}\left(\lambda(Y_t) + \varphi_t^1\right)^2 \frac{\left(U_x(x,t)\right)^2}{U_{xx}(x,t)} dt + U(x,t)\left(\varphi_t^1 dW_t^1 + \varphi_t^2 dW_t^2\right). \quad (50)$$

We introduce the process

$$U(x,t) = u\left(x, A_t^{\varphi}\right) M_t, \quad (51)$$

with u as in (22), the process A_t^{φ}, $t \geq 0$, defined as

$$A_t^{\varphi} = \int_0^t \left(\lambda(Y_s) + \varphi_s^1\right)^2 ds \quad (52)$$

and the exponential martingale M_t, $t \geq 0$, solving

$$dM_t = M_t\left(\varphi_t^1 dW_t^1 + \varphi_t^2 dW_t^2\right) \quad \text{with} \quad M_0 = 1,$$

One may interpret M_t as a device that offers the flexibility to modify our views on asset returns, changing the original market risk premium, $\lambda(Y_t)$, to $\lambda_t^M = \lambda(Y_t) + \varphi_t^1$.

The optimal allocation vector, $\pi_t^*, t > 0$, has the same functional form as (9) but for a different time-rescaling process, namely,

$$\pi_t^* = -\frac{\lambda_t^M}{\sigma(Y_t)} \frac{u_x\left(X_t^*, A_t^{\varphi}\right)}{u_{xx}\left(X_t^*, A_t^{\varphi}\right)} = \frac{\lambda_t^M}{\sigma(Y_t)} r\left(X_t^*, A_t^{\varphi}\right),$$

with A^{φ} as in (52) and r as in (24). The optimal wealth process solves

$$dX_t^* = r\left(X_t^*, A_t\right)\lambda_t^M\left(\lambda(Y_t)dt + dW_t^1\right).$$

It is worth noticing that if we choose $\varphi_t^1 = -\lambda(Y_t)$, $t \geq 0$, solutions become static, independently of the choice of the second volatility component. Indeed, the time-rescaling process vanishes, $A_t^{\varphi} = 0$, $t > 0$. In turn, the forward performance process becomes constant, $U(x,t) = u_0(x)$, $t > 0$ and the optimal investment and wealth processes degenerate,

$$\pi_t^* = 0 \quad \text{and} \quad X_t^* = x, \quad t \geq 0.$$

An optimal policy is to allocate zero wealth in the risky asset.

4.3.2 The "benchmark" case: $\left(\alpha^1(x,t), a^2(x,t)\right) = (-\delta_t x U_x(x,t), 0)$, $\delta_t \in \mathcal{F}_t$

It is assumed that δ_t, $t \geq 0$, is bounded by a deterministic constant. The forward performance SPDE, (18), becomes

$$dU(x,t) = \frac{1}{2}\frac{\left(U_x(x,t)\left(\lambda(Y_t) - \delta_t\right) - xU_{xx}\delta_t\right)^2}{U_{xx}(x,t)}dt - xU_x\delta_t dW_t^1. \tag{53}$$

Let A_t^δ, $t \geq 0$, be

$$A_t^\delta = \int_0^t \left(\lambda(Y_s) - \delta_s\right)^2 ds, \tag{54}$$

and consider the process N_t, $t \geq 0$, solving

$$dN_t = N_t\delta_t\left(\lambda(Y_t)\,dt + dW_t^1\right) \qquad \text{with} \quad N_0 = 1. \tag{55}$$

One can then show that the process

$$U(x,t) = u\left(\frac{x}{N_t}, A_t^\delta\right), \tag{56}$$

with u as in (22), is a forward performance.

One may interpret the auxiliary process N_t, $t \geq 0$, as a benchmark with respect to which the performance of investment policies is measured. It is, then, natural to look at the benchmarked optimal portfolio and wealth processes, $\tilde{\pi}_t^*$ and $\tilde{X}_t^*$, $t \geq 0$, defined, respectively, as

$$\tilde{\pi}_t^* = \frac{\pi_t^*}{N_t} \quad \text{and} \quad \tilde{X}_t^* = \frac{X_t^*}{N_t}.$$

Using (19) and (56) we obtain, setting $\lambda_t^N = \lambda(Y_t) - \delta_t$,

$$\tilde{\pi}_t^* = \frac{\delta_t}{\sigma(Y_t)}\tilde{X}_t^* - \frac{\lambda_t^N}{\sigma(Y_t)}\frac{u_x\left(\tilde{X}_t^*, A_t^\delta\right)}{u_{xx}\left(\tilde{X}_t^*, A_t^\delta\right)} \tag{57}$$

$$= \frac{\delta_t}{\sigma(Y_t)}\tilde{X}_t^* + \frac{\lambda_t^N}{\sigma(Y_t)}r\left(\tilde{X}_t^*, A_t^\delta\right),$$

with A_t^δ, $t \geq 0$ as in (54), r as in (24) and $\tilde{X}_t^*$ solving

$$d\tilde{X}_t^* = \tilde{R}_t^*\lambda_t^N\left(\lambda(Y_t)dt + dW_t^1\right).$$

The optimal portfolio process is represented as the sum of two funds, say $\tilde{\pi}^{*,X}$ and $\tilde{\pi}_t^{*,R}$, defined as

$$\tilde{\pi}^{*,X} = \frac{\delta_t}{\sigma(Y_t)}\tilde{X}_t^* \qquad \text{and} \quad \tilde{\pi}_t^{*,R} = \frac{\lambda_t^N}{\sigma(Y_t)}r\left(\tilde{X}_t^*, A_t^\delta\right).$$

The first component is independent of the risk preferences, depends linearly on wealth and vanishes if $\delta_t = 0$. The situation is reversed for the other component in that it depends only on the investor's risk preferences and vanishes when $\delta_t = \lambda(Y_t)$. The latter condition corresponds to the case when the stock becomes the benchmark itself. Note that, even for exponential preferences, the optimal portfolio may depend on the wealth if performance is measured in terms of a benchmark different than the savings account.

Some open problems

Problem 1: Characterize the class of volatility processes for which the SPDE (18) has a solution which satisfies the requirements of a forward performance process.

Problem 2: Prove a verification theorem for the forward stochastic optimization problem (18).

Problem 3: Characterize the family of initial risk preferences $u_0(x)$ for which a forward performance process exists.

Problem 4: Infer the investor's initial risk preferences from his desirable investment targets.

Problem 5: Study the invariance and consistency of the forward performance process and the associated optimal portfolios in terms of different numeraires and benchmarks.

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