Utility Valuation of Credit Derivatives and Application to CDOs

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Abstract

We study the impact of risk-aversion on the valuation of credit derivatives. Using the technology of utility-indifference pricing in intensity-based models of default risk, we analyze resulting yield spreads in both simple single-name credit derivatives, and complex multi-name securities, particularly CDOs. We introduce the *diversity coefficient* that characterizes the effects of defaultable investment opportunities. The impact of risk-averse valuation on CDO tranche spreads is also expressed in terms of implied correlations.

Keywords: Credit derivatives, indifference pricing.

Contents

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1 Introduction

Defaultable instruments, or credit-linked derivatives, are financial securities that pay their holders amounts that are contingent on the occurrence (or not) of a *default event* such as the bankruptcy of a firm or non-repayment of a loan. The market in credit-linked derivative products has grown astonishingly, from \$631.5 billion global volume in the first half of 2001, to above \$12 trillion through the first half of 2005[1]. The growth from mid-2004 through mid-2005 alone was 128%. They now account for approximately 10% of the total OTC derivatives market.

Despite the popularity and ever-increasing complexity of credit risk structured products, the quantitative technology for their valuation (and hedging) has lagged behind. This is largely due to the high-dimensionality of the basket derivatives, which are typically written on hundreds of underlying names; consequently, the computational efficiency of any valuation procedure severely limits model choice.

A major limitation of many approaches is the inability to capture and explain high premiums observed in credit derivatives markets for unlikely events, for example the spreads quoted for senior tranches of CDOs written on investment grade firms. The approach explored here is to explain such phenomena as a consequence of tranche holders' risk aversion, and to quantify this through the mechanism of utility-indifference valuation.

Valuation Mechanisms

In complete financial market environments, such as in the classical Black-Scholes model, the payoffs of derivative securities can be replicated by trading strategies in the underlying securities, and their prices are naturally deduced from the value of these associated portfolios. However, once non-traded risks such as unpredictable defaults are considered, perfect replication and, therefore, risk elimination breaks down, and alternative ways are needed for the quantification of risk and assignation of price.

One approach is to use market derivatives data, when available, to identify which of the many feasible arbitrage free pricing measures is consistent with market prices. In a different direction, valuation of claims involving nontradable risks can be based on optimality of decisions once this claim is incorporated in the investor's portfolio. Naturally, the risk attitude of the individual needs to be taken into account, and this is typically modeled by a concave and increasing utility function U. In a static framework, prices are determined

[1]Source: ISDA data reported at *http://www.credit-deriv.com/globalmarket.htm*

through the *certainty equivalent*, otherwise known as the principle of equivalent utility [5, 15]. The utility-based value of the claim, written on the risk Y and yielding payoff $C(Y)$, is

$$\nu(C) = U^{-1}\left(\mathbb{E}_{\mathbb{P}}\left\{U\left(C(Y)\right)\right\}\right).$$

Note that the arbitrage free price and the certainty equivalent are very different. The first is *linear* and uses the *risk neutral* measure. The certainty equivalent price is *nonlinear* and uses the *historical* assessment of risks.

Prompted by the ever-increasing number of applications (event risk sensitive claims, insurance plans, mortgages, weather derivatives, etc.), considerable effort has been invested in analyzing the utility-based valuation mechanism. Due to the prevalence of instruments dependent on non-market risks (like default), there is a great need for building new dynamic pricing rules. These rules should identify and price unhedgeable risks and, at the same time, build optimal risk monitoring policies. In this direction, a dynamic utility-based valuation theory has been developed producing so-called *indifference prices*. The mechanism is to find the price at which the writer (buyer) of the claim is indifferent in terms of maximum expected utility between holding or not holding the derivative. Specification of the indifference price requires understanding how investors act optimally with or without the derivative. These issues are naturally addressed through stochastic optimization problems of utility maximization. We refer to [22, 23] and [7] as classical references in this area. The indifference approach was initiated for European claims by Hodges and Neuberger [18] and further extended by Davis *et al.* [9].

Credit Derivatives

As well as single-name securities such as *credit default swaps* (CDSs), in which there is a relatively liquid market, basket, or multi-name products have generated considerable OTC activity. Typical of these are *collateralized debt obligations* (CDOs) whose payoffs depend on the default events of a basket portfolio of up to 300 firms over a number of years. As long as there are no defaults, investors in CDO tranches enjoy high yields, but as defaults start occurring they affect first the high-yield equity tranche, then the mezzanine tranches and perhaps on to the senior and super-senior tranches. See Davis and Lo [8] or Elizalde [13] for a concise introduction. Even more exotic contracts are CDO^2s, which depend on baskets of CDO tranches, though the market in these has thinned considerably since 2005.

The focus of modeling in the credit derivatives industry has been on *correlation* between default times. Partly this is due to the adoption of the one-factor Gaussian copula model as industry standard and the practice (up till recently) of analyzing tranche prices through implied correlation. This revealed that traded prices of senior tranches could only be realised through these models with an implausibly high correlation parameter, the so-called correlation smile. After the simultaneous downgrades of Ford and General Motors in May 2005, the standard copula model sometimes could not even fit the data.

Rather than focusing on models with "enough correlation" to reproduce market observations via standard no-arbitrage pricing, the goal of this article is to understand the effects of risk aversion on valuation of basket credit derivatives. In particular, how does risk aversion value portfolios that are sensitive to the potential default of a number of firms, and so to correlation between these events? Does the nonlinearity of the indifference pricing mechanism enhance the impact of correlation? It seems natural that some of the prices or spreads seen in credit markets are due more to "crash-o-phobia" in a relatively illiquid market, with the effect enhanced nonlinearly in baskets. When super-senior tranches offer non-trivial spreads (albeit a few basis points) for protection against the default risk of $15 - 30\%$ of investment grade US firms over the next five years, they are ascribing a seemingly large probability to "the end of the world as we know it". We seek to capture this directly as an effect of risk aversion leading to effective or perceived correlation, opposed to a mechanism of high direct correlation.

Taking the opposite angle, the method of indifference pricing should be attractive to participants in this still quite illiquid OTC market. It is a direct way for them to quantify the default risks they face in a portfolio of complex instruments, when calibration data is scarce. Unlike well-developed equity and fixed income derivatives markets where the case for traditional arbitrage-free valuation is more compelling, the potential for utility valuation to account for high-dimensionality in a way that is consistent with investors' fears of a cascade of defaults is a case for its application here.

For recent applications of indifference pricing to credit risk, see also Collin-Dufresne and Hugonnier [6], Bielecki *et al.* [3, 4], and Shouda [30].

We begin with single-name credit derivatives in Section 2. The general multi-name problem is discussed in Section 3, where the *diversity coefficient* that characterizes the effects of defaultable investment opportunities is introduced. We analyze a symmetric model applied to CDO tranche valuation in Section 4, and conclude in Section 5.

2 Indifference Valuation: Single Name

We start with single name defaultable bonds to illustrate the approach. Default occurs as in intensity-based models introduced by, among others, Artzner and Delbaen [1] , Madan and Unal [27], Lando [24] and Jarrow and Turnbull [20]. However our valuation mechanism incorporates information from the firm's stock price S. Unlike in a traditional structural approach, default occurs at a non-predictable stopping time τ with stochastic intensity process $\lambda \geq 0$, which is correlated with the firm's stock price. These are sometimes called *hybrid* models (see, for example, [26]). The process S could alternatively be taken as the price of another firm or index used to hedge the default risk. Of course the choice of the investment opportunity set affects the ensuing indifference price.

The stock price S is taken to be a geometric Brownian motion, and the intensity process is $\lambda(Y_t)$, where $\lambda(\cdot)$ is a non-negative, locally Lipschitz, smooth and bounded function, and Y is a correlated diffusion:

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t^{(1)}, \\ dY_t &= b(Y_t)\, dt + a(Y_t)\, \left(\rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\right). \end{aligned}$$

The coefficients a and b are taken to be Lipschitz functions with sublinear growth. The processes W^1 and W^2 are independent standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, I\!\!P)$, and we denote by $\mathcal{F}_t$ the augmented σ-algebra generated by $((W_u^1, W_u^2);\, 0 \leq u \leq t)$. The parameter $\rho \in (-1, 1)$ measures the instantaneous correlation between shocks to the stock price S and shocks to the intensity-driving process Y. In applications, it is natural to expect that $\lambda(\cdot)$ and ρ are specified in a way such that the intensity tends to rise when the stock price falls.

There also exists a standard exponential random variable ξ, independent of the Brownian motions. The default time τ of the firm is defined by

$$\tau = \inf\left\{ t : \int_0^t \lambda(Y_s)\, ds = \xi \right\},$$

the first time the cumulated intensity reaches the random draw ξ.

Maximal Expected Utility Problem

Let $T < \infty$ denote our finite fixed horizon, chosen later to coincide with the expiration date of the derivatives contracts of interest. The investor's control process is π_t, the dollar amount held in the stock at time t, until $\tau \wedge T$. In $t < \tau \wedge T$, his wealth process X follows

$$\begin{aligned} dX_t &= \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)\, dt \\ &= (rX_t + \pi_t(\mu - r))\, dt + \sigma \pi_t\, dW_t^{(1)}. \end{aligned}$$

The control process π is called admissible if it is $\mathcal{F}_t$-measurable and satisfies the integrability constraint $E\{\int_0^T \pi_s^2\, ds\} < \infty$. The set of admissible policies is denoted by $\mathcal{A}$.

If the default event occurs before T, the investor can no longer trade the firm's stock. He has to liquidate holdings in the stock and deposit in the bank account, so the effect is to reduce his investment opportunities. For simplicity, we assume he receives full pre-default market value on his stock holdings on liquidation, though one might extend to consider some loss, or jump downwards in the stock price at the default time. Therefore, given that $\tau < T$, for $\tau \le t \le T$, we have

$$X_t = X_\tau e^{r(t-\tau)},$$

as the bank account is the only remaining investment.

We shall work with exponential utility of discounted (to time zero) wealth. We are first interested in the optimal investment problem up to time T of the investor who does not hold any derivative security. At time zero, the maximum expected utility payoff then takes the form

$$\sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ -e^{-\gamma(e^{-rT} X_T)} \mathbf{1}_{\{\tau > T\}} + (-e^{-\gamma(e^{-r\tau} X_\tau)}) \mathbf{1}_{\{\tau \le T\}} \right\}.$$

We switch to the discounted variable $X_t \mapsto e^{-rt} X_t$ and excess growth rate $\mu \mapsto \mu - r$, and, with a slight abuse, we use the same notation.

We consider the stochastic control problem initiated at time $t \le T$, and define the default time τ_t by

$$\tau_t = \inf\left\{ s \ge t : \int_t^s \lambda(Y_u)\, du = \xi \right\},$$

the first time the cumulated intensity reaches an independent standard exponential random variable ξ.

In the absence of the defaultable claim, the investor's value function is given by

$$M(t,x,y) = \sup_{\pi\in\mathcal{A}} I\!E\left\{-e^{-\gamma X_T}\mathbf{1}_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}})\mathbf{1}_{\{\tau_t\le T\}} \mid X_t = x, Y_t = y\right\}. \tag{1}$$

Proposition 1 *The value function $M : [0,T]\times I\!R\times I\!R \to I\!R^-$ is the unique viscosity solution in the class of functions that are concave and increasing in x, and uniformly bounded in y of the HJB equation*

$$M_t + \mathcal{L}_y M + \max_\pi\left(\frac{1}{2}\sigma^2\pi^2 M_{xx} + \pi(\rho\sigma a(y)M_{xy} + \mu M_x)\right) + \lambda(y)(-e^{-\gamma x} - M) = 0, \tag{2}$$

with $M(T,x,y) = -e^{-\gamma x}$ and

$$\mathcal{L}_y = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}.$$

PROOF: The proof follows by extension of the arguments used in Theorem 4.1 of Duffie and Zariphopoulou [12] and is omitted. □

Bond Holder's Problem and Indifference Price

We now consider the same problem from the point of view of an investor who owns a (defaultable) bond of the firm, which pays \$1 on date T if the firm has survived till then. Defining $c = e^{-rT}$, we have the bond-holder's value function

$$H(t,x,y) = \sup_{\pi\in\mathcal{A}} I\!E\left\{-e^{-\gamma(X_T+c)}\mathbf{1}_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}})\mathbf{1}_{\{\tau_t\le T\}} \mid X_t = x, Y_t = y\right\}. \tag{3}$$

As in Proposition 1 for the plain investor's value function M, we have the following HJB characterization.

Proposition 2 *The value function $H : [0,T]\times I\!R\times I\!R \to I\!R^-$ is the unique viscosity solution in the class of functions that are concave and increasing in x, and uniformly bounded in y of the HJB equation*

$$H_t + \mathcal{L}_y H + \max_\pi\left(\frac{1}{2}\sigma^2\pi^2 H_{xx} + \pi(\rho\sigma a(y)H_{xy} + \mu H_x)\right) + \lambda(y)(-e^{-\gamma x} - H) = 0, \tag{4}$$

with $H(T,x,y) = -e^{-\gamma(x+c)}$.

The indifference value of the defaultable bond, from the point of view of the bond holder, is the reduction in his initial wealth level such that his maximum expected utility H is the same as the plain investor's value function M.

Definition 1 *The buyer's indifference price $p_0(T)$ (at time zero) of a defaultable bond with expiration date T is defined by*

$$M(0,x,y) = H(0,x-p_0,y). \tag{5}$$

Remark 1 *(i) As is well-known, the indifference price under exponential utility does not depend on the investor's initial wealth x, which is an attractive feature of using this utility function.*

(ii) The indifference price at times $0 < t < T$ can be defined similarly, with minor modifications to the previous calculations, in particular with quantities discounted to time t dollars.

2.1 Variational Results

In this section, we present some simple bounds for the value functions and the indifference price introduced above.

Proposition 3 *The value functions M and H satisfy, respectively*

$$\begin{aligned} -e^{-\gamma x} &\leq M(t,x,y) \leq -e^{-\gamma x-\frac{\mu^2}{2\sigma^2}(T-t)}, \qquad (6)\\ -e^{-\gamma x}+(e^{-\gamma x}-e^{-\gamma(x+c)})I\!P\{\tau_t>T \mid Y_t=y\} &\leq H(t,x,y) \leq -e^{-\gamma(x+c)-\frac{\mu^2}{2\sigma^2}(T-t)}. \qquad (7)\end{aligned}$$

Proof: We start with establishing (6). We first observe that the function $\tilde{M}(t,x,y)=-e^{-\gamma x}$ is a (viscosity) subsolution of the HJB equation (2). Moreover, $\tilde{M}(T,x,y)=M(T,x,y)$. The lower bound then follows from the comparison principle for viscosity solutions for the class of functions described in Proposition 1.

Similarly, testing the function

$$\tilde{M}(t,x,y)=-e^{-\gamma x-\frac{\mu^2}{2\sigma^2}(T-t)}$$

yields

$$\begin{aligned}&\tilde{M}_t+\mathcal{L}_y\tilde{M}+\max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2\tilde{M}_{xx}+\pi(\rho\sigma a(y)\tilde{M}_{xy}+\mu\tilde{M}_x)\right)\\ &+\lambda(y)\left(-e^{-\gamma x}+e^{-\gamma x-\frac{\mu^2}{2\sigma^2}(T-t)}\right)=\lambda(y)e^{-\gamma x}\left(e^{-\frac{\mu^2}{2\sigma^2}(T-t)}-1\right)\leq 0.\end{aligned}$$

Therefore, $\tilde{M}$ is a supersolution, with $\tilde{M}(T,x,y)=M(T,x,y)$, and the upper bound follows.

Next, we establish (7). To obtain the lower bound, we follow the sub-optimal policy of investing exclusively in the default-free bank account (that is, taking $\pi\equiv 0$). Then

$$\begin{aligned} H(t,x,y) &\geq I\!E\left\{-e^{-\gamma(x+c)}\mathbf{1}_{\{\tau_t>T\}}+(-e^{-\gamma x})\mathbf{1}_{\{\tau_t\leq T\}} \mid X_t=x, Y_t=y\right\}\\ &= -e^{-\gamma(x+c)}I\!P\{\tau_t>T\mid Y_t=y\}+(-e^{-\gamma x})I\!P\{\tau_t\leq T\mid Y_t=y\}\\ &= -e^{-\gamma x}+(e^{-\gamma x}-e^{-\gamma(x+c)})I\!P\{\tau_t>T\mid Y_t=y\},\end{aligned}$$

and the lower bound follows. The upper bound is established by testing the function

$$\tilde{H}(t,x,y)=-e^{-\gamma(x+c)-\frac{\mu^2}{2\sigma^2}(T-t)}$$

in the HJB equation (4) for H, and showing that $\tilde{H}$ is a (viscosity) supersolution. □

Remark 2 *The bounds given above reflect that, in the presence of default, the value functions are bounded between the solutions of two extreme cases. For example, the lower bounds correspond to a degenerate market (only the bank account available for trading in $[0,T]$), while the upper bounds correspond to the standard Merton case with no default risk.*

2.2 Reduction to Reaction-Diffusion Equations

The HJB equation (2) can be simplified by the familiar distortion scaling

$$M(t,x,y) = -e^{-\gamma x} u(t,y)^{1/(1-\rho^2)}, \tag{8}$$

with $u : [0,T] \times I\!R \to I\!R^+$ solving the reaction-diffusion equation

$$\begin{aligned} u_t + \widetilde{\mathcal{L}}_y u - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) u + (1-\rho^2)\lambda(y) u^{-\theta} &= 0, \\ u(T,y) &= 1, \end{aligned} \tag{9}$$

where

$$\theta = \frac{\rho^2}{1-\rho^2}, \quad \text{and} \quad \widetilde{\mathcal{L}}_y = \mathcal{L}_y - \frac{\rho\mu}{\sigma} a(y) \frac{\partial}{\partial y}.$$

Similar equations arise in other utility problems in incomplete markets, for example, in portfolio choice with recursive utility [32], valuation of mortgage-backed securities [33] and life-insurance problems [2]. One might work first with (9) and then provide the verification results for the HJB equation (2), since the solutions of (2) and (9) are related through (8). It is worth noting, however, that the reaction-diffusion equation (9) does not belong to the class of such equations with Lipschitz reaction term. Therefore, more detailed analysis is needed for establishing existence, uniqueness and regularity results. In the context of a portfolio choice problem with stochastic differential utilities, the analysis can be found in [32]. The equation at hand is slightly more complicated than the one analyzed there, in that the reaction term has the multiplicative intensity factor. Because $\lambda(\cdot)$ is taken to be bounded and Lipschitz, an adaptation of the arguments in [32] can be used to show that the reaction-diffusion problem (9) has a unique bounded and smooth solution. Furthermore, using (8) and the bounds obtained for M in Proposition 3, we have

$$e^{-(1-\rho^2)\frac{\mu^2}{2\sigma^2}(T-t)} \leq u(t,y) \leq 1.$$

For the bond holder's value function, the transformation

$$H(t,x,y) = -e^{-\gamma(x+c)} w(t,y)^{1/(1-\rho^2)}$$

reduces to

$$\begin{aligned} w_t + \widetilde{\mathcal{L}}_y w - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) w + (1-\rho^2) e^{\gamma c} \lambda(y) w^{-\theta} &= 0, \\ w(T,y) &= 1, \end{aligned} \tag{10}$$

which is a similar reaction-diffusion equation as (9). The only difference is the coefficient $e^{\gamma c} > 1$ in front of the reaction term. Existence of a unique smooth and bounded solution follows similarly.

The following lemma gives a relationship between u and w.

Lemma 1 *Let u and w be solutions of the reaction-diffusion problems (9) and (10). Then*

$$u(t,y) \leq w(t,y) \qquad \text{for } (t,y) \in [0,T] \times I\!R.$$

PROOF: We have $u(T,y) = w(T,y) = 1$. Moreover, because $e^{\gamma c} > 1$ and $\lambda > 0$,

$$(1-\rho^2)e^{\gamma c}\lambda(y)w^{-\theta} > (1-\rho^2)\lambda(y)w^{-\theta},$$

which yields

$$w_t + \widetilde{\mathcal{L}}_y w - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right) w + (1-\rho^2)\lambda(y)w^{-\theta} < 0.$$

Therefore, w is a supersolution of (9), and the result follows. □

From this, we easily obtain the following sensible bounds on the indifference value of the defaultable bond, and the yield spread.

Proposition 4 *The indifference bond price p_0 in (5) is given by*

$$p_0(T) = e^{-rT} - \frac{1}{\gamma(1-\rho^2)} \log\left(\frac{w(0,y)}{u(0,y)}\right), \tag{11}$$

and satisfies $p_0(T) \leq e^{-rT}$. The yield spread defined by

$$\mathcal{Y}_0(T) = -\frac{1}{T}\log(p_0(T)) - r$$

is non-negative for all $T > 0$.

Remark 3 *We denote the seller's indifference price by $\tilde{p}_0(T)$. In order to construct it, we replace c by $-c$ in the definition (3) of the value function H and in the ensuing transformations. If $\tilde{w}$ is the solution of*

$$\tilde{w}_t + \widetilde{\mathcal{L}}_y \tilde{w} - (1-\rho^2)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)\tilde{w} + (1-\rho^2)e^{-\gamma c}\lambda(y)\tilde{w}^{-\theta} = 0, \tag{12}$$

with $\tilde{w}(T,y) = 1$, then

$$\tilde{p}_0(T) = e^{-rT} - \frac{1}{\gamma(1-\rho^2)}\log\left(\frac{u}{\tilde{w}}\right).$$

Using comparison results, we obtain $u > \tilde{w}$ as $e^{-\gamma c} < 1$. Therefore $\tilde{p}_0(T) \leq e^{-rT}$ and the seller's yield spread is non-negative for all $T > 0$.

We note that an alterative approach to direct analysis of the primal problem is to study the dual problem of relative entropy minimization (see, for example, [10, 21]). This approach is taken in [3], and reviewed for the present class of models in [31].

2.3 Constant Intensity Case

We study explicitly the case of constant intensity, when the default time τ is independent of the level of the firm's stock price S and is simply an exponential random variable with parameter λ. This simplified structure will employed in the multi-name models that we analyze for CDO valuation in Section 3.

Proposition 5 *When λ is constant, the indifference price $p_0(T)$ (at time zero) of the defaultable bond expiring on date T is given by*

$$p_0(T) = e^{-rT} - \frac{1}{\gamma}\log\left(\frac{e^{-\alpha T} + \frac{\lambda}{\alpha}e^{\gamma c}\left(1 - e^{-\alpha T}\right)}{e^{-\alpha T} + \frac{\lambda}{\alpha}\left(1 - e^{-\alpha T}\right)}\right), \tag{13}$$

where

$$\alpha = \frac{\mu^2}{2\sigma^2} + \lambda. \tag{14}$$

Proof: We construct the explicit solutions of the HJB equations solved by the two value functions M and H. When λ is constant, the y variable disappears from the calculations, and the HJB equation (2) for the Merton value function reduces to

$$\begin{aligned} M_t - \frac{\mu^2}{2\sigma^2}\frac{M_x^2}{M_{xx}} + \lambda(-e^{-\gamma x} - M) &= 0 \\ M(T,x) &= -e^{-\gamma x}. \end{aligned}$$

Substituting $M(t,x) = -e^{-\gamma x}m(t)$, we obtain $m' - \alpha m + \lambda = 0$, with $m(T) = 1$, and α as above. The unique solution is

$$m(t) = e^{-\alpha(T-t)} + \frac{\lambda}{\alpha}\left(1 - e^{-\alpha(T-t)}\right).$$

Similarly, the defaultable bond holder's value function $H(t,x)$ satisfies the same equation as M with terminal condition $H(T,x) = -e^{-\gamma(x+c)}$. Substituting $H(t,x) = -e^{-\gamma(x+c)}h(t)$, we obtain $h' - \alpha h + \lambda e^{\gamma c} = 0$, with $h(T) = 1$. The unique solution is

$$h(t) = e^{-\alpha(T-t)} + \frac{\lambda e^{\gamma c}}{\alpha}\left(1 - e^{-\alpha(T-t)}\right).$$

Finally, the indifference price of the defaultable bond at time zero is

$$p_0(T) = e^{-rT} - \frac{1}{\gamma}\log\left(\frac{h(0)}{m(0)}\right),$$

which leads to formula (13). □

Remark 4 *The seller's indifference price is given by*

$$\tilde{p}_0(T) = e^{-rT} + \frac{1}{\gamma}\log\left(\frac{e^{-\alpha T} + \frac{\lambda}{\alpha}e^{-\gamma c}\left(1 - e^{-\alpha T}\right)}{e^{-\alpha T} + \frac{\lambda}{\alpha}\left(1 - e^{-\alpha T}\right)}\right).$$

A plot of the yield spreads $\mathcal{Y}_0(T) = -\frac{1}{T}\log(p_0(T)/e^{-rT})$ for the buyer, and similarly $\tilde{\mathcal{Y}}_0(T)$ for the seller, for various risk aversion coefficients, is shown in Figure 1. Observe that both spread curves are, in general, sloping, so the spreads are not flat even though we started with a constant intensity model. While the seller's curve is upward sloping, the buyer's may become downward sloping when the risk aversion is large enough.

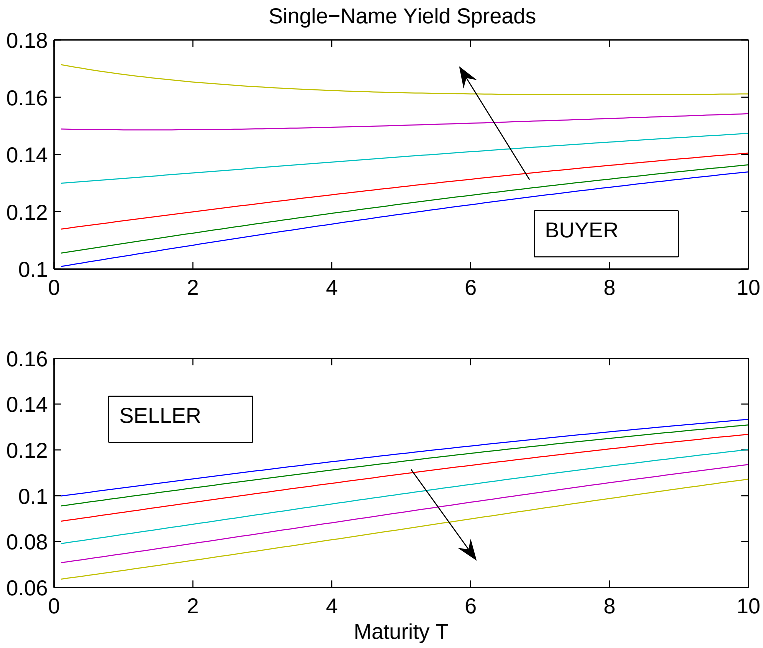

Figure 1: *Single name buyer's and seller's indifference yield spreads. The parameters are* $\lambda = 0.1$, *along with* $\mu = 0.09, r = 0.03$ *and* $\sigma = 0.15$. *The curves correspond to different risk aversion parameters* γ *and the arrows show the direction of increasing* γ *over the values* $(0.01, 0.1, 0.25, 0.5, 0.75, 1)$.

The short term limit of the yield spread is nonzero, as we would expect in the presence of non-predictable defaults. For the buyer's yield spread, we have

$$\lim_{T \downarrow 0} \mathcal{Y}_0(T) = \frac{(e^{\gamma} - 1)}{\gamma} \lambda,$$

which is larger than λ since $\gamma > 0$. This is amplified as γ becomes larger. In other words, the buyer values the claim *as though* the intensity were larger than the historically estimated value λ. The seller, on the other hand, values short-term claims as though the intensity were lower, since

$$\lim_{T \downarrow 0} \tilde{\mathcal{Y}}_0(T) = \frac{(1 - e^{-\gamma})}{\gamma} \lambda \leq \lambda.$$

The long time limit for both buyer's and seller's spread is simply α,

$$\lim_{T \to \infty} \mathcal{Y}_0(T) = \lim_{T \to \infty} \tilde{\mathcal{Y}}_0(T) = \frac{\mu^2}{2\sigma^2} + \lambda,$$

which is always larger than λ. Both long-term yield spreads converge to the intensity plus a term proportional to the square of the Sharpe ratio of the firm's stock.

3 Multi-Name Credit Derivatives

We now tackle the problem of indifference valuation of multi-name credit derivatives. The main instrument we have in mind is a *collateralized debt obligation* (CDO), for which the dimension of the underlying basket may be on the order of one hundred to three hundred names. (A simple two-name claim is analyzed in [31]). As with other approaches to these problems, particularly copula models, it becomes necessary to make huge simplifications, typically involving some sort of symmetry assumption, in order to be able to handle the high-dimensional computational challenge. We will assume throughout this section that intensities are constant (and consequently that the default times of the firms are independent), and specialize later to the completely homogeneous case in Section 4. However, the tranche spreads produced by utility valuation (Section 4.1.3) demonstrate the impact of risk-aversion even when *a priori* default time correlation is not modeled.

We suppose that there are N firms, whose stock prices processes $(S^{(i)})$ follow geometric Brownian motions:

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = (r + \mu_i)\, dt + \sigma_i\, dW_t^{(i)}, \tag{15}$$

where $(W^{(i)})$ are (in general correlated) Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the μ_i are excess growth rates and $\sigma_i > 0$ are the volatilities. We will assume constant correlations

$$\mathbb{E}\{dW_t^{(i)}\, dW_t^{(j)}\} = \rho_{ij}\, dt, \qquad i \neq j,$$

for $\rho_{ij} \in (-1, 1)$.

The ith firm has default time τ_i which is assumed exponential with parameter $\lambda_i > 0$, and the τ_i are mutually independent, and independent of the Brownian motions. Each firm's stock is available for trading by the investor until it defaults, when the holding in that stock has to be liquidated and re-invested in the remaining stocks (if any) and the bank account.

3.1 Merton Cascade Problem and Diversity Coefficient

We begin with the Merton problem without the credit derivative. The investor invests $\$\pi_t^{(i)}$ in the ith stock at times $t < \tau_i \wedge T$, so his *discounted* wealth process X evolves according to

$$dX_t = \begin{cases} \sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \mu_i\, dt + \sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \sigma_i\, dW_t^{(i)}, & t < \bar{\tau} \wedge T \\ 0 & \bar{\tau} \wedge T \leq t \leq T \end{cases}, \tag{16}$$

where $\bar{\tau} = \max_i\{\tau_i\}$.

In the case of heterogeneous dynamics and default intensities, it is necessary to keep track of which firms have defaulted and which are still alive. When there are $1 \leq n \leq N$ firms left, the index sets

$$I_n^k = \{i_1(n,k), i_2(n,k), \cdots, i_n(n,k)\}, \quad k = 1, 2, \cdots, \binom{n}{k}$$

describe all possible combinations of firms that have not yet defaulted.

When there are no firms left, we have the Merton value function

$$M^{(0)}(t, x) = -e^{-\gamma x}. \tag{17}$$

For $n \geq 1$, we define the Merton value functions for the investor starting at time $t \leq T$ with initial wealth x, when the firms in I_n^k are healthy and the others have defaulted. We have

$$M^{(I_n^k)}(t,x) = \sup_{\{\pi^{(i)} | i \in I_n^k\}} I\!\!E \left\{ -e^{-\gamma X_T} \mid X_t = x \right\}, \tag{18}$$

where it is understood that $\pi^{(i)} \equiv 0$ for $i \notin I_n^k$ in the dynamics of X.

Definition 2 *Let $\Sigma(I_n^k)$ be the $n \times n$ covariance matrix with entries*

$$(\Sigma(I_n^k))_{jm} = \sigma_{i_j} \sigma_{i_m} \rho_{i_j i_m}, \quad \textit{where} \quad i_j = i_j(n,k), i_m = i_m(n,k),$$

and let $\mu(I_n^k)$ be the $n \times 1$ vector of excess growth rates corresponding to the firms in the index set I_n^k. The diversity coefficient *D of the subset of firms I_n^k is defined by*

$$D(I_n^k) = \mu(I_n^k)^T \Sigma(I_n^k)^{-1} \mu(I_n^k), \tag{19}$$

which is a positive scalar under the assumptions on our model coefficients.

Remark 5 *Suppose an investor starts with the n stocks in I_n^k available for trading. As is well-known, under exponential utility, the optimal investment in the Merton problem, in the absence of defaults, is to hold the fixed amounts given by the vector*

$$\pi^\star = \frac{1}{\gamma} \Sigma(I_n^k)^{-1} \mu(I_n^k),$$

in each stock. Since default comes as a surprise independently of the stock price processes in the constant intensity framework of this section, the optimal strategy under default risk is also to hold these same amounts (the Merton ratios) in each stock until it defaults. The quantity $D(I_n^k) = \gamma \mu(I_n^k)^T \pi^\star$ is therefore proportional to the expected change in value of the optimal portfolio in the standard default-free Merton case. It is natural then to think of $D(I_n^k)$ as a measure of the potential return from the investment opportunity set offered by the diversity of the stocks in I_n^k. In the case of only one stock, it is the square of the Sharpe ratio of that stock.

The Merton value functions $M^{(I_n^k)}(t,x)$ solve the following system of HJB PDEs

$$\begin{aligned} M_t^{(I_n^k)} - \frac{1}{2} D(I_n^k) \frac{(M_x^{(I_n^k)})^2}{M_{xx}^{(I_n^k)}} + \sum_{j \in I_n^k} \lambda_j \left(M^{(I_n^k \setminus \{j\})} - M^{(I_n^k)} \right) &= 0, \\ M^{(I_n^k)}(T,x) &= -e^{-\gamma x}, \end{aligned} \tag{20}$$

where $M^{(I_n^k \setminus \{j\})}$ is the Merton value function when firm j has dropped out. Here the value function $M^{(I_n^k)}$ is coupled through its PDE to the n value functions corresponding to the subsets of size $n-1$ when one of the firms in I_n^k has defaulted. The initial value function for $n = 0$ is given by (17). The market information is contained in the coefficient $D(I_n^k)$.

We construct the solution of (20) in the standard way, by first making the transformation

$$M^{(I_n^k)}(t,x) = -e^{-\gamma x} v^{(I_n^k)}(t). \tag{21}$$

Substituting into (20), we require the $v^{(I_n^k)}$ to satisfy the system of ODEs

$$\begin{aligned} \frac{d}{dt}v^{(I_n^k)} - \alpha(I_n^k)v^{(I_n^k)} + \sum_{j \in I_n^k} \lambda_j v^{(I_n^k \setminus \{j\})} &= 0, \\ v^{(I_n^k)}(T) &= 1, \end{aligned} \tag{22}$$

where

$$\alpha(I_n^k) = \frac{1}{2}D(I_n^k) + \sum_{j \in I_n^k} \lambda_j, \tag{23}$$

the multi-dimensional analog of α in (14), and the $n = 0$ starting function is $v^{(0)}(t) \equiv 1$.

Proposition 6 *The system of equations (22) have unique smooth non-negative solutions on* $[0, T]$ *and so the Merton value functions defined by (18) are given by (21).*

PROOF: The ODE system for the v's is linear and its solution can be constructed recursively using the formula for the solution of (22):

$$v^{(I_n^k)}(t) = e^{-\alpha(I_n^k)(T-t)} + \sum_{j \in I_n^k} \left(\lambda_j \int_t^T e^{-\alpha(I_n^k)(s-t)} v^{(I_n^k \setminus \{j\})}(s)\, ds \right).$$

Clearly, starting with $v^{(0)}(t) \equiv 1$, the successive $v^{(I_n^k)}$ remain non-negative, smooth and bounded. It follows that (21) gives a smooth solution of (20) and can be identified with the unique viscosity solution of that system, and so with the value function of the stochastic control problem (18). $\square$

3.2 Multi-Name Claims

As an example of a basket claim, we consider a contract that pays at expiration time T an amount depending on which firms have survived, that is, it has a has a European-style payoff function $h(I_n^k)$, and does not depend on the timing of defaults. We tackle the more realistic case of CDO tranches in Section 4.

When there are no firms left, we have the claim holder's value function

$$H^{(0)}(t, x) = -e^{-\gamma x + ch(\emptyset)}, \tag{24}$$

where $\emptyset$ denotes the empty set, and $c = e^{-rT}$. For $n \geq 1$, we define the value functions for the investor starting at time $t \leq T$ with initial wealth x, when the firms in I_n^k are healthy and the others have defaulted. We have

$$H^{(I_n^k)}(t, x) = \sup_{\{\pi^{(i)} | i \in I_n^k\}} \left\{ -e^{-\gamma X_T + ch(I_n^k)} \mid X_t = x \right\}, \tag{25}$$

where it is understood that $\pi^{(i)} \equiv 0$ for $i \notin I_n^k$ in the dynamics (16) of X.

Proposition 7 *The claim holder's value functions are given by*

$$H^{(I_n^k)}(t,x) = -e^{-\gamma x} w^{(I_n^k)}(t),$$

where the $w^{(I_n^k)}$ *are the unique non-negative smooth solution of the system of linear ODEs:*

$$\begin{aligned} \frac{d}{dt} w^{(I_n^k)} - \alpha(I_n^k) w^{(I_n^k)} + \sum_{j \in I_n^k} \lambda_j w^{(I_n^k \setminus \{j\})} &= 0, \\ w^{(I_n^k)}(T) &= e^{-\gamma ch(I_n^k)}. \end{aligned} \tag{26}$$

Proof: The arguments are identical as with the Merton cascade problem, with the minor alteration in the terminal conditions. In particular, $H^{(I_n^k)}$ satisfy the same system of PDEs as (20), but with the terminal condition

$$H^{(I_n^k)}(T,x) = -e^{-\gamma x + ch(I_n^k)}.$$

□

The buyer's indifference price p_0 of the credit derivative at time zero, when all N firms are alive is defined by

$$H^{(I^{(N)})}(0, x - p_0) = H^{(I^{(N)})}(0, x),$$

and therefore

$$p_0 = \frac{1}{\gamma} \log \left(\frac{v^{(I^{(N)})}(0)}{w^{(I^{(N)})}(0)} \right).$$

4 Indifference Valuation of CDOs: Symmetric Model

We next consider the indifference valuation of CDOs. The underlying is a portfolio of N defaultable bonds with total face value Q. The holder of a CDO tranche insures losses due to defaults between certain bounds (the attachment points characterizing the tranche). For some other approaches based on arbitrage pricing using intensity-based or copula models, we refer, for instance to [8, 11, 16, 17, 19]. A multi-dimensional structural model for loss distributions is studied in [14].

We are interested in the valuation of a CDO tranche with attachment points K_L and K_U. These are percentages of the total portfolio value (notional) insured by the holder. For example $K_L = 0\%, K_U = 3\%$ corresponds to the equity tranche, and $K_L = 3\%, K_U = 7\%$ is typically the first mezzanine tranche. Recall Q denotes the total notional.

The tranche holder receives a yield $R\%$ (assumed paid continuously in our framework) on his part of the notional, which is initially $(K_U - K_L)Q$, but decreases as the losses arrive, until his tranche is blown. At each loss between the limits of his tranche's responsibility, he pays "notional minus recovery".

We want to find the yield R such that he is indifferent between holding the tranche or not.

4.1 Symmetric Model, Constant Intensities, Equal Notionals

Recall from Section 3 that the stock prices processes of the N firms are correlated geometric Brownian motions, described by (15). The ith firm has default time τ_i which is assumed exponential with parameter λ_i, and the τ_i are *independent*. All the "effective correlation" will be generated by the utility indifference valuation mechanism.

For computational tractability, we need to assume a large degree of symmetry (or exchangeability) between the firms. One way is to assume $\mu_i \equiv \mu$ and $\sigma_{ij} = \delta_{ij}\,\sigma$ and that the Brownian motions have correlation structure

$$\mathbb{E}\{dW^{(i)}dW^{(j)}\} = \rho\, dt \qquad i \neq j, \tag{27}$$

and also that $\lambda_i \equiv \lambda$. This is similar to what is assumed in the "industry standard" one-factor Gaussian copula, reviewed in Section 4.2.

We shall assume the second condition, but instead of the first, we suppose that the diversity coefficient D in Definition 2, depends only on the dimension (*i.e.* the number of stocks left).

Assumption 1 *(i) When there are $n \leq N$ stocks, labelled by the index set I_n^k, the diversity coefficient $D(I_n^k)$, defined in equation (19) is a function only of n:*

$$D(I_n^k) = D(n).$$

(ii) The intensities are identical across firms: $\lambda_i \equiv \lambda$.

(iii) The firms in the CDO have equal notionals.

As D is like the square of the (multi-dimensional) Sharpe ratio, it is natural to assume that it increases with n. The main inputs into the model, aside from the risk-aversion coefficient γ, are an "average" intensity λ and a diversity function $D(n)$.

Remark 6 *In the case of symmetric stock price dynamics described previously ($\mu_i \equiv \mu$, $\sigma_{ij} = \delta_{ij}\,\sigma$ and correlation structure (27)), we have*

$$D(n) = \frac{\mu^2 n}{\sigma^2(1 + (n-1)\rho)}, \tag{28}$$

which is illustrated in Figure 2 for various ρ. For small ρ, the diversity coefficient grows rapidly with the number of firms. As the correlation increases, $D(n)$ levels off sooner, and in the extreme $\rho = 1$, the curve is flat.

The third assumption on the CDO structure is common in standardized contracts such as the CDX and iTraxx. For example, in the CDX CDOs, each of the 125 firms is protected up to losses of \$80,000, for a total notional of $Q =$ \$10 million.

Under these conditions, we only need to keep track of the number of firms alive at each time, not which particular ones.

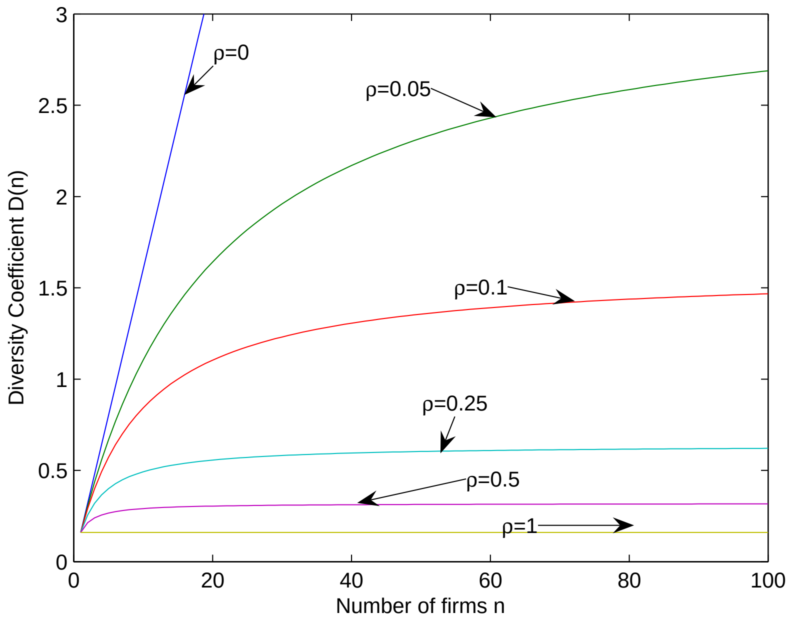

Figure 2: *Diversity coefficient in (28) for various correlations* ρ. *The other parameters are* $\mu = 0.06$ *and* $\sigma = 0.15$.

4.1.1 Merton Problem

Let $M^{(n)}(t,x)$ denote the value function when there are $n \in \{0, 1, \cdots, N\}$ firms alive. Clearly,

$$M^{(0)}(t,x) = -e^{-\gamma x}.$$

In general, for $1 \le n \le N$, $M^{(n)}(t,x)$ solves

$$\begin{aligned} M_t^{(n)} - \frac{1}{2} D(n) \frac{(M_x^{(n)})^2}{M_{xx}^{(n)}} + n\lambda \left(M^{(n-1)} - M^{(n)} \right) &= 0, \\ M^{(n)}(T,x) &= -e^{-\gamma x}, \end{aligned} \tag{29}$$

the analog of (20) under the symmetry assumptions.

In the next proposition, we construct an explicit solution for $M^{(n)}$.

Proposition 8 *The Merton value functions are given by*

$$M^{(n)}(t,x) = -e^{-\gamma x} v_n(t), \tag{30}$$

where

$$v_n(t) = c_0^{(n)} + \sum_{j=1}^{n} c_j^{(n)} e^{-\alpha_j (T-t)} \tag{31}$$

and

$$\alpha_n := \frac{1}{2} D(n) + n\lambda. \tag{32}$$

The coefficients $c_j^{(n)}$ are found from the recursion relations

$$\begin{aligned} c_0^{(n)} &= \frac{n\lambda}{\alpha_n} c_0^{(n-1)}, \qquad n = 2, \cdots, N && (33) \\ c_j^{(n)} &= \frac{n\lambda}{(\alpha_n - \alpha_j)} c_j^{(n-1)}, \qquad j = 1, \cdots, n-1 && (34) \\ c_n^{(n)} &= 1 - \sum_{j=0}^{n-1} c_j^{(n)}, \end{aligned}$$

with initial data

$$c_0^{(1)} = \frac{\lambda}{\alpha_1}.$$

Proof: Inserting the form (30) into (29) leads to the system of ODEs

$$\begin{aligned} v_n' - \alpha_n v_n + n\lambda v_{n-1} &= 0 \\ v_n(T) &= 1. \end{aligned}$$

Inserting the expression (31) leads to the recurrence relations (33). □

Remark 7 *(i) Note that the v_n are independent of the risk-aversion coefficient γ.*

(ii) The recursion relations can of course be solved explicitly, for example,

$$c_0^{(n)} = \frac{\lambda^n n!}{(\alpha_1 \alpha_2 \cdots \alpha_n)},$$

but it is computationally more stable to generate them recursively.

4.1.2 Tranche Holder's Problem

We assume a fractional recovery q, meaning each default results in a loss to the portfolio of $(1-q)(Q/N)$. The tranche holder (or protection seller) pays out this amount if the loss is within his tranche (up to the limit of the tranche). Our state variables are t, x and n, the number of firms currently healthy. It is convenient, and more standard, to define the portfolio loss when there are n firms remaining, namely

$$\ell_n = (1-q)\frac{(N-n)}{N}. \tag{35}$$

Given the lower and upper tranche attachment points K_L and K_U, we define

$$F(\ell) = (K_U - \ell)^+ - (K_L - \ell)^+. \tag{36}$$

Then $F(\ell_n) \times Q$ is the remaining notional the tranche holder insures when there are n firms still alive, and the capital on which he receives the tranche premium. The structure of the function F, being the difference of two put option payoffs, shows the tranche holder as having a "put spread" on the portfolio loss process with "strikes" K_L and K_U.

Let $H^{(n)}(t,x)$ denote the tranche holders value function when n firms are left. We assume he receives a cash flow at the rate Re^{rt} at time t on his remaining notional, where R is the tranche premium to be found. The e^{rt} factor is convenient to cancel the time variable out of the ODEs that follow.

Then, his discounted wealth process X (depending on n) follows

$$dX_t = \begin{cases} \left(\sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \mu_i + RQF(\ell_n)\right) dt + \sum_i \pi_t^{(i)} \mathbf{1}_{\{\tau_i > t\}} \sigma_i \, dW_t^{(i)}, & t < \bar{\tau} \wedge T \\ 0 & \bar{\tau} \wedge T \leq t \leq T \end{cases} . \tag{37}$$

To describe the protection payments made by the tranche holder when losses hit his tranche, we define

$$f_n = F(\ell_n) - F(\ell_{n-1}).$$

Then Qf_n is the payment made by the tranche holder if the number of firms remaining drops from n to $n-1$ (that is, the loss increases from ℓ_n to ℓ_{n-1}).

The HJB equation for $H^{(n)}(t,x)$ is

$$\begin{aligned} H_t^{(n)} - \frac{1}{2} D(n) \frac{(H_x^{(n)})^2}{H_{xx}^{(n)}} + RQF(\ell_n) H_x^{(n)} + n\lambda \left(H^{(n-1)}(t, x - Qf_n) - H^{(n)} \right) &= 0, \\ H^{(n)}(T,x) &= -e^{-\gamma x}. \end{aligned} \tag{38}$$

In the next proposition, we construct an explicit solution for $H^{(n)}$.

Proposition 9 *The tranche holder's value functions are given by*

$$H^{(n)}(t,x) = -e^{-\gamma x} w_n(t), \tag{39}$$

where

$$w_n(t) = d_0^{(n)} + \sum_{j=1}^{n} d_j^{(n)} e^{-\beta_j (T-t)} \tag{40}$$

and

$$\beta_n = \frac{1}{2} D(n) + n\lambda + \gamma RQF_n. \tag{41}$$

The coefficients $d_j^{(n)}$ are found from the recursion relations

$$\begin{aligned} d_0^{(n)} &= \frac{q_n}{\beta_n} d_0^{(n-1)} \\ d_j^{(n)} &= \frac{q_n}{(\beta_n - \beta_j)} d_j^{(n-1)}, \qquad j = 1, \cdots, n-1 \\ d_n^{(n)} &= 1 - \sum_{j=0}^{n-1} d_j^{(n)}, \end{aligned} \tag{42}$$

with initial data

$$d_0^{(1)} = \frac{q_1}{\beta_1},$$

and where $q_n = n\lambda e^{\gamma Q f_n}$.

PROOF: Inserting the form (39) into (38) leads to the system of ODEs

$$\begin{aligned} w_n' - \beta_n w_n + q_n w_{n-1} &= 0 \\ w_n(T) &= 1. \end{aligned}$$

Inserting the expression (40) leads to the recurrence relations (42). □

4.1.3 Indifference Tranche Spread

Having solved for v and w in Propositions 8 and 9, the indifference tranche spread at time zero, when all N firms in the CDO are alive, is found by solving for R such that

$$w_N(0) = v_N(0).$$

Note that while the v_n are independent of the risk-aversion coefficient γ, the w_n from the tranche holder's problem depend on γ through the combination γQ. The indifference tranche spread is a mapping

$$R = R(N; \lambda, D(\cdot); \gamma Q; K_L, K_U, T).$$

In Figure 3, we illustrate, in the case $N = 25$, the types of spreads obtained for various tranches as the risk-aversion coefficient varies. We use throughout the diversity coefficient D given in (28), and fix the attachment points as those corresponding to CDX CDOs:

Tranche	K_L	K_U
Equity	0%	3%
Mezzanine 1	3%	7%
Mezzanine 2	7%	10%
Senior	10%	15%
Super-Senior	15%	30%

Usually the premium for the equity tranche is quoted as an upfront fee that is paid in addition to 500 basis points, but for simplicity, we just treat the first tranche like the others. The same qualitative features as in Figure 3 can be seen in Figure 4 for the case of $N = 100$ firms, particularly the retarded sensitivity of the senior tranches to increases in risk-aversion.

4.2 Implied Correlations

In recent years, market tranche premia have been analyzed in terms of an implied correlation parameter ϱ, which is backed out from the one-factor Gaussian copula model. This model has come close to being regarded as an industry standard, mainly for its intuitive simplicity and computational tractability in high dimensions, although its limitations, especially since the simultaneous downgrading of Ford and General Motors in May 2005, are well-known. One might hope that implied correlation could play the role that implied volatility plays in studies of market equity option prices, but the base Gaussian copula model used for tranche pricing is far less stable in terms of its inputs than the Black-Scholes formula for option pricing. For example, the tranche premium is not guaranteed to be monotonic in the

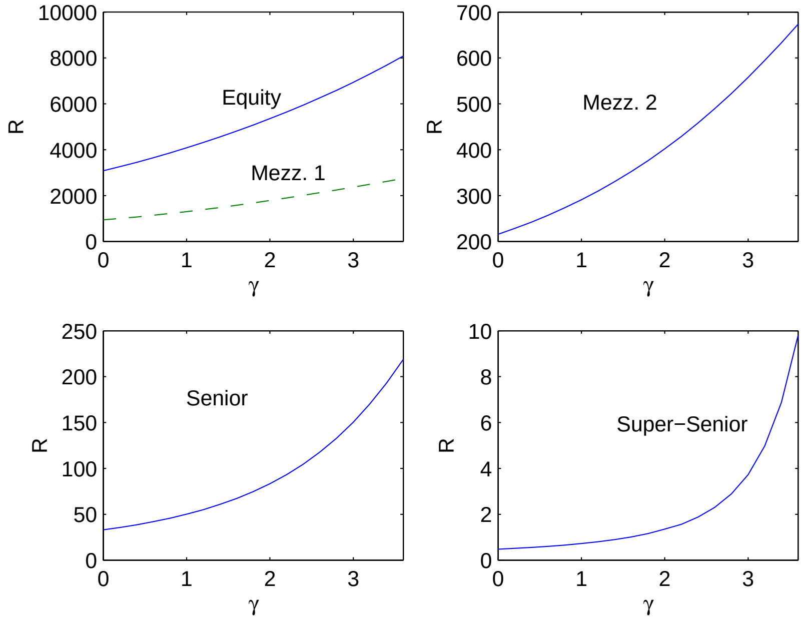

Figure 3: *CDO indifference tranche spreads (in basis points) as functions of the risk-aversion coefficient* γ*. Number of firms* $N = 25$*, and* $\lambda = 0.015$*. The diversity coefficient* D *is given by (28) with* $\mu = 0.07$*,* $\sigma = 0.15$ *and* $\rho = 0.3$*. The recovery is* $q = 40\%$*, the interest rate* $r = 3\%$ *and* $T = 5$ *years. The notional is normalized to* 1 *unit per firm, so* $Q = N$*.*

correlation parameter for some tranches, and the outputs are quite sensitive to the input intensity parameter. See, for example, [28, Chapter 3] for a survey.

Nonetheless, a partial consensus has emerged regarding an observed structural pattern, namely the implied correlation smile: the implied correlation is different for each tranche, but is often higher for the equity and senior tranches, and lower for the mezzanine tranches, resulting in a U-shaped smile curve when plotted against the attachment points. Some examples are given in [19] and [28].

The copula model is specified under a market-determined risk-neutral probability measure $I\!P^\star$, and the arbitrage-free value of any claim is given by its expected discounted cash flow under this measure. The single-name risk-neutral default probabilities $p(t)$ are assumed homogeneous and given by

$$p(t) = I\!P^\star\{\tau_i \leq t\} = 1 - e^{-\lambda^\star t},$$

where $\lambda^\star > 0$ is the risk-neutral intensity, assumed given from an underlying index, or through averaging the intensities of the CDO names, found from their individual CDS spreads.

Let L_t be the number of defaulted firms at time t. In the one-factor Gaussian copula model, the loss distribution is given by

$$I\!P^\star\{L_t = k\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \binom{N}{k} \beta(t,y)^k (1 - \beta(t,y))^{N-k} e^{-y^2/2}\, dy, \quad k = 0, 1, 2 \cdots, N$$

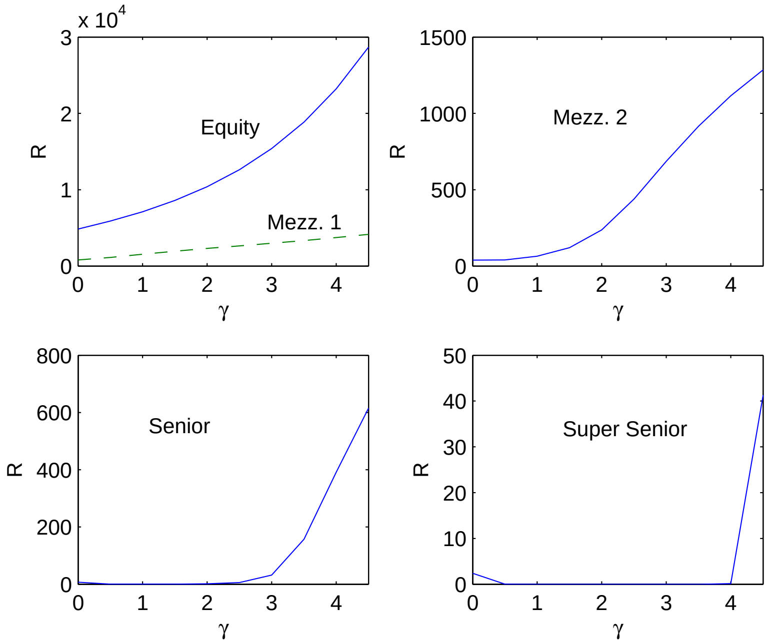

Figure 4: *CDO indifference tranche spreads (in basis points) as functions of the risk-aversion coefficient γ. Number of firms $N = 100$, all other parameters as in Figure 3.*

where

$$\beta(t, y) = \mathcal{N}\left(\frac{\mathcal{N}^{-1}(p(t)) - \sqrt{\varrho}\, y}{\sqrt{1-\varrho}}\right). \tag{43}$$

Here, $\mathcal{N}$ is the standard normal cumulative distribution function, and $\varrho \in (0,1)$ is called the correlation coefficient. See [29, Chapter 10], for example, for a derivation. We denote by $\mathcal{X}_t$ the fraction of the total portfolio value lost by time t:

$$\mathcal{X}_t = \frac{(1-q)}{N} L_t,$$

where q is the fractional recovery parameter. For our purposes, we shall use the Vasicek large portfolio approximation for $N \to \infty$, namely

$$\mathbb{P}^\star\{\mathcal{X} \leq x\} \approx \mathcal{N}\left(\frac{1}{\sqrt{\varrho}}\left[\sqrt{1-\varrho}\,\mathcal{N}^{-1}(x) - \mathcal{N}^{-1}(p(t))\right]\right),$$

which is computationally efficient and reasonably accurate for moderate values of N.

The premium payments are made at regular payment dates $t_k = k\Delta t$, with J payment dates in total. For CDX CDOs, the payments are quarterly, twenty in all over five years. The fair value of the premium leg, comprising payments to the protection seller (the tranche holder) on his remaining notional, is given by

$$V_{pr} = R\Delta t Q \sum_{k=1}^{J} e^{-rt_k} \mathbb{E}^\star\{F(\mathcal{X}_{t_k})\},$$

where the put spread payoff F was defined in (36).

We will make the standard assumption that the insurance payouts in the floating leg are also made at the same payment dates, covering all the losses since the previous payment date. In practice, they are made close to when the loss actually occurs. See, for example, [25, Chapter 8] for details. The fair value of the floating leg is given by

$$V_{fl} = Q \sum_{k=1}^{J} e^{-rt_k} \mathbb{E}^\star \left\{ \left(F(\mathcal{X}_{t_k}) - F(\mathcal{X}_{t_{k-1}}) \right) \right\},$$

and the fair tranche premium R is such that these two are equal. Hence

$$R = \frac{\sum_{k=1}^{J} e^{-rt_k} \mathbb{E}^\star \left\{ \left(F(\mathcal{X}_{t_k}) - F(\mathcal{X}_{t_{k-1}}) \right) \right\}}{\Delta t \sum_{k=1}^{J} e^{-rt_k} \mathbb{E}^\star \{ F(\mathcal{X}_{t_k}) \}}. \tag{44}$$

The implied correlation is simply the value of the the parameter ϱ in (43) that equates R is (44) to a market tranche value. As mentioned previously, such a $\varrho \in (0,1)$ may not exist, or may not be unique.

Figure 5 gives an example of how indifference valuation may lead to an implied correlation "smile" pattern. Our purpose is not to propose this mechanism for calibration of market data, but rather to demonstrate that it can produce non-trivial tranche spreads for the senior tranches starting with a simple model with independent default times. Since indifference valuation arises from comparison between expected utilities under the historical measure, and the copula model requires specification of a risk-neutral intensity $\lambda^\star$, the latter parameter has to be chosen before implied correlations can be computed. Typically, one would expect $\lambda^\star > \lambda$, reflecting a positive market price for default risk. In Figure 5, we have chosen $\lambda^\star$ arbitrarily for the illustration. Investigation of different choices, as well as sensitivity analyses with respect to other parameters are beyond the scope of the current article.

5 Conclusion

The preceding analysis demonstrates that utility valuation produces non-trivial CDO tranche spreads and implied correlations within even the simplest of intensity-based models of default. It also incorporates equity market information (growth rates, volatilities of the non-defaulted firms) as well as investor risk aversion to provide a relative value mechanism for multi-name credit derivatives.

Many issues remain for investigation: the effects of the various input parameters; the choice and estimation of different diversity coefficients that contain the impact of defaults on the diminishing investment opportunity set; the effect of time-varying or stochastic risk-aversion and stochastic (and thereby correlated) intensities; efficient computation and analysis of the fully heterogeneous case; and large portfolio ($N \to \infty$) asymptotics of the indifference values. A related problem is optimal static-dynamic *hedging* of CDO tranches, combining dynamic trading strategies in the underlying firms' stocks, and static positions in CDSs.

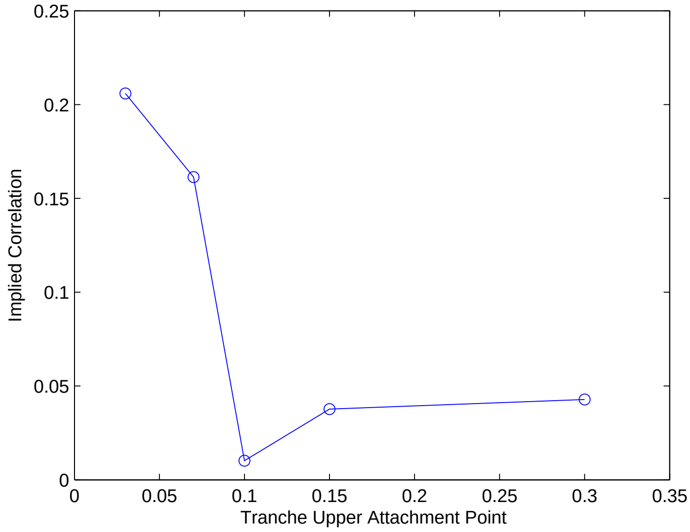

Figure 5: *Implied correlations from CDO indifference tranche spreads with* $\gamma = 1.7$. *The other parameters are as in Figure 3, and* $\lambda^{\star} = 0.0255$.

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