Indifference valuation in incomplete binomial models*

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Abstract

The indifference valuation problem in incomplete binomial models is analyzed. The model is general in that the stochastic factor which generates the marlet incompleteness may affect the transition propabilities and/or the values of the traded asset as well as the claim's payoff. Two pricing algorithms are constructed which use, respectively, the minimal martingale and minimal entropy measures. The interplay among the different kinds of market incompleteness, the pricong measures and the price functionals is studied in detail. The dependence of the prices in the choice of the trading horizon is, also, discussed. Finally, the family of "almost complete" models is studied. It is shown that the two measures and the price functionals coincide, and that the effects of the horizon choice dissipate.

1 Introduction

This paper is a contribution to indifference valuation in incomplete binomial models under exponential preferences. Market incompleteness stems from the presence of a stochastic factor which may affect the transition probabilities of the traded asset or/and its values. It may also affect the payoff of the claim in consideration. The model is, thus, more general than the binomial models considered so far in exponential indifference valuation (see, among others, [1], [8] and [15]).

The aim is to construct valuation algorithms and provide a detailed study of their properties and structure. We construct two such algorithms. They are both iterative and resemble the ones introduced in [15] and [8]. However, the existing pricing schemes are applicable only when the stochastic factor affects

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exclusively the claim's payoff. When the factor affects the dynamics and values of the traded asset, the situation is much more complex as internal market incompleteness emerges which, thus, needs to be priced together with the one coming from the claim's payoff. The algorithms herein exhibit how the pricing of both kinds of incompleteness is carried out and the interplay among the incompleteness, the pricing measures and the pricing functionals.

In both algorithms, the indifference price is calculated via iterative pricing schemes, applied backwards in time, starting at the claim's maturity. The schemes have local and dynamic properties. Dynamically, the pricing functionals are similar in that, at each time interval, the price is computed via the single-step pricing operators, applied to the end of the period payoff. The latter turns out to be the indifference price at the next time step, yielding prices consistent across times.

Locally, valuation is executed in two steps, in analogy to the single-period counterpart (26). In the first sub-step, the end of the period payoff is altered via a non-linear functional and the conditioning on the information generated by an appropriately chosen filtration. The new intermediate payoff is in turn, priced by expectation. There are, however, important differences both between the pricing measures and the form of the non-linear price functionals. The first algorithm uses the minimal martingale measure. This measure has the intuitively pleasing property of preserving the conditional distribution of the stochastic factor, given the stock price, in terms of its historical counterpart. However, the form of the associated pricing functional has no apparent natural form. The situation is reversed the second algorithm which uses the minimal entropy measure. We show that the density of this measure has no intuitively pleasing structure in contrast to the relevant functional which does.

The forms of the (non-linear) price functionals motivate us to investigate two important questions. Firstly, we study whether these functionals provide a natural extension to the classical static certainty equivalent pricing rule. We show that both price functionals fail to provide such connection. Secondly, we study how the indifference prices are affected by the choice of the trading horizon, the point at which the underlying exponential utility is pre-specified. We show that prices are different for different horizon choices and provide this difference in closed form.

Finally, we investigate how the above results simplify when the model simplifies to the one that has been studied so far, i.e. when the stochastic factor affects solely the claim's payoff. We call such a model *reduced.* We show that, as expected, there is a unique pricing measure as the nested model is now complete. We also show that the price functionals become identical. A direct and important consequence of these simplifications is that the indifference prices become independent on the choice of the trading horizon (normalization point).

The paper is organized as follows. In section 2, we introduce the incomplete (non-reduced) model and provide auxiliary results on the two pricing measures and the exponential value function process. In section 3, we construct the two pricing algorithms and discuss their properties. We also investigate the connection of the price functionals with the static certainty equivalent. In section 4 we investigate the dependence of the indifference prices on the normalization point and, also, provide numerical results. We conclude with section 5 in which we analyze the almost complete (reduced) binomial models.

2 The model and preliminary results

In a trading horizon, $[0,T]$, two securities are available for trading, a riskless bond and a risky stock. The time T is arbitrary but fixed. The bond offers zero interest rate. The values of the stock, denoted by S_t, $t=0,1,...,T$, satisfy $S_t>0$ and are given by

$$\xi_{t+1}=\frac{S_{t+1}}{S_t},\quad \xi_{t+1}=\xi_{t+1}^d,\ \xi_{t+1}^u \quad \text{with} \quad 0<\xi_{t+1}^d<1<\xi_{t+1}^u. \tag{1}$$

Incompleteness is generated by a non-traded factor, denoted by Y_t, $t=0,1,...,T$, whose levels satisfy $Y_t\neq 0$ and are given by

$$\eta_{t+1}=\frac{Y_{t+1}}{Y_t},\ \eta_{t+1}=\eta_{t+1}^d,\ \eta_{t+1}^u \quad \text{with} \ \ 0<\eta_{t+1}^d<\eta_{t+1}^u. \tag{2}$$

We, then, view $\{(S_t,Y_t):t=0,1,...\}$ as a two-dimensional stochastic process defined on the probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$. The filtration $\mathcal{F}_t$ is generated by the random variables S_i and Y_i, or, equivalently, by ξ_i and η_i, for $i=0,1,...,t$. We, also, consider the filtration $\mathcal{F}_t^S$ generated only by S_i, for $i=0,1,...,t$. The real (historical) probability measure on Ω and $\mathcal{F}$ is denoted by $\mathbb{P}$.

We assume that the values ξ_{t+1}^d, ξ_{t+1}^u of the $\mathcal{F}_{t+1}-$measurable random variable ξ_{t+1} satisfy

$$\xi_{t+1}^d\in\mathcal{F}_t \qquad \text{and} \qquad \xi_{t+1}^u\in\mathcal{F}_t\ . \tag{3}$$

An investor starts at $t=0,1,...,T$ with initial endowment $X_t=x\in\mathbb{R}$ and trades between the stock and the bond, following self-financing strategies. The number of shares held in his portfolio over the time period $[i-1,i)$, $i=t+1,t+2,...,T$, is denoted by α_i. It is throughout assumed that $\alpha_i\in\mathcal{F}_{i-1}$. The individual's aggregate wealth is, then, given, by

$$X_s=x+\sum_{i=t+1}^{s}\alpha_i\triangle S_i, \tag{4}$$

where $\triangle S_i=S_i-S_{i-1}$ and $s=t+1,...,T$.

The performance of the implemented investment strategies is measured via an expected exponential utility criterion applied to the terminal wealth that these portfolios generate. The maximal expected utility (value function) is, then, given by the solution of the stochastic optimization problem

$$V_t(x)=\sup_{\alpha_{t+1},...,\alpha_T}E_{\mathbb{P}}\left(-e^{-\gamma X_T}\,|\mathcal{F}_t\right), \tag{5}$$

$t=0,1,...,T$ with $\gamma>0$ and X_T as in (4), $X_t=x$. This process has been extensively analyzed for general market settings (see, for example, [2], [5] and [13]).

The goal is to carry out a detailed study of the indifference prices, under the dynamic preference criterion (5). We stress that the binomial model we consider is quite more general than all such models that have been, so far, analyzed in the context of indifference valuation[1]; see, among others, [3], [8], [16] and [15]. Indeed, in these works, the nested model is complete, with the non-traded factor affecting the claim's payoff but not the transition probability or the values of the traded asset. In such "almost complete" models, considerable simplifications take place due to the lack of internal market incompleteness. We revisit these cases in Section 5.

In the extended framework herein, additional pricing features emerge. The analysis and interpretation of these new elements will employ the *minimal martingale* and the *minimal entropy* measures (see, for example, [4] and [5], respectively). For the reader's convenience, we present some auxiliary results on the densities of these measures (see Propositions 3 and 4) and a parity relation (see Proposition 6). These results were derived for the specific binomial model at hand in the companion paper [17], where we refer the reader for their proofs.

To this end, we let $\mathcal{Q}_T$ be the set of martingale measures restricted on $\mathcal{F}_T$. With a slight abuse of notation, we denote by $\mathbb{Q}$ its generic element.

We consider, for $t = 0, 1, ..., T$, the quantities

$$H_T^{mm}\left(\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = E_{\mathbb{P}}\left(-\ln\frac{\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)}{\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right)$$

and

$$H_T^{me}\left(\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = E_Q\left(\ln\frac{\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)}{\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)}\,|\mathcal{F}_t\right),$$

where $\mathbb{Q} \in \mathcal{Q}_T$ and $\mathbb{Q}\left(\cdot\,|\mathcal{F}_t\right)$ and $\mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)$ denote the restrictions of $\mathbb{Q}$ and $\mathbb{P}$ on $\mathcal{F}_t$.

The minimal martingale measure, $\mathbb{Q}^{mm}$, and the minimal entropy measure, $\mathbb{Q}^{me}$, are defined as the minimizers of H_T^{mm} and H_T^{me}, respectively, i.e.,

$$H_T^{mm}\left(\mathbb{Q}^{mm}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = \min_{Q\in\mathcal{Q}_T} H_T^{mm}\left(Q\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right)$$

and

$$H_T^{me}\left(\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right) = \min_{Q\in\mathcal{Q}_T} H_T^{me}\left(Q\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right).$$

Most of the analysis below will involve the latter entropy. To simplify the presentation we will be using the condensed notation

$$\mathcal{H}_{t,T}^{me} = H_T^{me}\left(\mathbb{Q}^{me}\left(\cdot\,|\mathcal{F}_t\right) \mid \mathbb{P}\left(\cdot\,|\mathcal{F}_t\right)\right). \tag{6}$$

We will be referring to the process $\mathcal{H}_{t,T}^{me}$ as the *minimal aggregate entropy*.

[1] While preparing the final version of this manuscript, the recent paper [7] was brought to the attention of the authors. Therein, the utility is of power type and the model more general than the one considered herein.

We introduce, for $t = 0, 1, ..., T$, the sets

$$A_t = \{\omega : \xi_t(\omega) = \xi_t^u\} \quad \text{and} \quad B_t = \{\omega : \eta_t(\omega) = \eta_t^u\}. \tag{7}$$

Note that for all $\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}_T$,

$$\mathbb{Q}(A_t | \mathcal{F}_{t-1}) = \mathbb{Q}'(A_t | \mathcal{F}_{t-1}). \tag{8}$$

Definition 1 *Let ξ_t, $t = 0, 1, .., T$, be as in (1) and consider the risk neutral probabilities*

$$q_t = \frac{1 - \xi_t^d}{\xi_t^u - \xi_t^d}.$$

The local entropy process h_t, $t = 1, ..., T$, is defined by

$$h_t = q_t \ln \frac{q_t}{\mathbb{P}(A_t | \mathcal{F}_{t-1})} + (1 - q_t) \ln \frac{1 - q_t}{1 - \mathbb{P}(A_t | \mathcal{F}_{t-1})}, \tag{9}$$

where A_t as in (7), $\mathbb{P}$ is the historical probability measure and $\mathcal{F}_t$ is the filtration generated by the random variables S_i and Y_i, for $i = 0, 1, ..., t$.

Lemma 2 *The local entropy process h_t is $\mathcal{F}_t$-predictable, i.e., for $t = 1, ..., T$, $h_t \in \mathcal{F}_{t-1}$. Moreover, for all $\mathbb{Q} \in \mathcal{Q}_T$,*

$$h_t = \mathbb{Q}(A_t | \mathcal{F}_{t-1}) \ln \frac{\mathbb{Q}(A_t | \mathcal{F}_{t-1})}{\mathbb{P}(A_t | \mathcal{F}_{t-1})} + (1 - \mathbb{Q}(A_t | \mathcal{F}_{t-1})) \ln \frac{1 - \mathbb{Q}(A_t | \mathcal{F}_{t-1})}{1 - \mathbb{P}(A_t | \mathcal{F}_{t-1})}.$$

The result below highlights an important property of the minimal martingale measure. Specifically, under this measure, the conditional distribution of the non-traded factor, given the stock price, is preserved in relation to its historical counterpart.

Proposition 3 *The minimal martingale measure $\mathbb{Q}^{mm}$ has, for $t = 1, ..., T$, the property*

$$\mathbb{Q}^{mm}\left(Y_t | \mathcal{F}_{t-1} \vee \mathcal{F}_t^S\right) = \mathbb{P}\left(Y_t | \mathcal{F}_{t-1} \vee \mathcal{F}_t^S\right), \tag{10}$$

or, equivalently,

$$\frac{\mathbb{Q}^{mm}(A_t B_t | \mathcal{F}_{t-1})}{\mathbb{P}(A_t B_t | \mathcal{F}_{t-1})} = \frac{\mathbb{Q}^{mm}(A_t B_t^c | \mathcal{F}_{t-1})}{\mathbb{P}(A_t B_t^c | \mathcal{F}_{t-1})} = \frac{\mathbb{Q}^{mm}(A_t | \mathcal{F}_{t-1})}{\mathbb{P}(A_t | \mathcal{F}_{t-1})}$$

and

$$\frac{\mathbb{Q}^{mm}(A_t^c B_t | \mathcal{F}_{t-1})}{\mathbb{P}(A_t^c B_t | \mathcal{F}_{t-1})} = \frac{\mathbb{Q}^{mm}(A_t^c B_t^c | \mathcal{F}_{t-1})}{\mathbb{P}(A_t^c B_t^c | \mathcal{F}_{t-1})} = \frac{\mathbb{Q}^{mm}(A_t^c | \mathcal{F}_{t-1})}{\mathbb{P}(A_t^c | \mathcal{F}_{t-1})},$$

with the sets A_t and B_t given in (7).

Next, we present an analogous explicit representation of the minimal entropy measure. The construction – which is to the best of our knowledge new – is based on an iterative procedure which yields the conditional distribution $\mathbb{Q}^{me}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)$ in terms of its historical analogue $\mathbb{P}\left(Y_t|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)$ and the (conditional) on $\mathcal{F}_{t-1}$ values of the minimal aggregate entropy $\mathcal{H}_{t,T}^{me}$. The latter term is constructed through an independent iterative procedure which involves the minimal martingale measure, obtained already in (10). To ease the presentation, the construction of $\mathcal{H}_{t,T}^{me}$ is given, separately, in Proposition 5.

To this end, we introduce the auxiliary quantities

$$\mathcal{J}_{\mathbb{Q}}^{(s,s+1)}(Z)=E_{\mathbb{Q}}\left(\ln E_{\mathbb{Q}}\ \left(e^Z\left|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right.\right)|\mathcal{F}_s\right)\tag{11}$$

and

$$\mathcal{J}_{\mathbb{Q}}^{(s,t)}(Z)=\mathcal{J}_{\mathbb{Q}}^{(s,s+1)}\left(\mathcal{J}_{\mathbb{Q}}^{(s+1,s+2)}\left(...\mathcal{J}_{\mathbb{Q}}^{(t-1,t)}(Z)\right)\right),\tag{12}$$

for $s=0,1,...,T-1$, $t=s+1,...,T$, Z a random variable in $(\Omega,\mathcal{F},\mathbb{P})$ and $\mathbb{Q}\in\mathcal{Q}_T$. We recall that $\mathcal{F}_s$ and $\mathcal{F}_s^S$ are the filtrations generated, respectively, by (S_i,Y_i) and S_i for $i=1,...,s$.

Proposition 4 *The minimal entropy measure, $\mathbb{Q}^{me}$, satisfies, for $t=1,...,T$,*

$$\begin{aligned}\frac{\mathbb{Q}^{me}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}&=\frac{\mathbb{Q}^{me}\left(A_t|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,uu}}}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,uu}}+\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,ud}}},\\ \frac{\mathbb{Q}^{me}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}&=\frac{\mathbb{Q}^{me}\left(A_t|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,ud}}}{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,uu}}+\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,ud}}},\\ \frac{\mathbb{Q}^{me}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)}&=\frac{\mathbb{Q}^{me}\left(A_t^c|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,du}}}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,du}}+\mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,dd}}}\end{aligned}\tag{13}$$

and

$$\frac{\mathbb{Q}^{me}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)}=\frac{\mathbb{Q}^{me}\left(A_t^c|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,dd}}}{\mathbb{P}\left(A_t^cB_t|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,du}}+\mathbb{P}\left(A_t^cB_t^c|\,\mathcal{F}_{t-1}\right)e^{-\mathcal{H}_{t,T}^{me,dd}}}$$

where A_t,B_t are as in (7) and $\mathcal{H}_{t,T}^{me,uu},\mathcal{H}_{t,T}^{me,ud},\mathcal{H}_{t,T}^{me,du},\mathcal{H}_{t,T}^{me,dd}$ are the values of the $\mathcal{F}_t-$measurable random variable $\mathcal{H}_{t,T}^{me}$, (cf. (6)), conditional on $\mathcal{F}_{t-1}$.

The above equalities can be expressed slightly differently in order to provide the direct analogues of (10). For example, (13) yields

$$\frac{\mathbb{Q}_T^{me}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}_T^{me}\left(A_t|\,\mathcal{F}_{t-1}\right)}=\frac{\dfrac{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}_{t,T}^{me,uu}}}{\dfrac{\mathbb{P}\left(A_tB_t|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}_{t,T}^{me,uu}}+\dfrac{\mathbb{P}\left(A_tB_t^c|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}e^{-\mathcal{H}_{t,T}^{me,ud}}}.$$

Notice that if $\mathcal{H}_{t,T}^{me,uu}=\mathcal{H}_{t,T}^{me,ud}$ equality (13) reduces to (10). This observation will play a key role in the analysis of the reduced binomial model.

Proposition 5 *Let $\mathbb{Q}^{mm}$ be the minimal martingale measure and h_t, $t = 0, 1, .., T$, be as in (9). Let, also, $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}, \mathcal{J}^{(t,T)}_{\mathbb{Q}^{mm}}$ be as in (11) and (12) for $\mathbb{Q} = \mathbb{Q}^{mm}$. The minimal aggregate entropy $\mathcal{H}^{me}_{t,T}$ is given by the iterative scheme*

$$\mathcal{H}^{me}_{T,T} = 0 \qquad \text{and} \qquad \mathcal{H}^{me}_{T-1,T} = h_T$$

and

$$\mathcal{H}^{me}_{t,T} = h_{t+1} - \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}} \left(-\mathcal{H}^{me}_{t+1,T} \right), \quad t = 0, 1, ..., T-2. \tag{14}$$

Moreover,

$$\mathcal{H}^{me}_{t,T} = -\mathcal{J}^{(t,T)}_{\mathbb{Q}^{mm}} \left(-\sum_{i=t+1}^{T} h_i \right). \tag{15}$$

Next, we present a parity result between the nonlinear functionals $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}$ and $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}}$ which will, in turn, play a key role in the upcoming construction of the pricing algorithms[2].

Proposition 6 *Let $\mathcal{H}^{me}_{t,T}$ be the aggregate entropy and $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}}, \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}}$ as in (11) for $\mathbb{Q} = \mathbb{Q}^{me}, \mathbb{Q}^{mm}$ and $t = 0, 1, .., T-1$. Then, for Z in $(\Omega, \mathcal{F}, \mathbb{P})$,*

$$\mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}} (Z) = \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}} \left(Z - \mathcal{H}^{me}_{t+1,T} \right) - \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{mm}} \left(-\mathcal{H}^{me}_{t+1,T} \right). \tag{16}$$

Proposition 7 *The minimal aggregate entropy $\mathcal{H}^{me}_{t,T}$ is given by the iterative scheme*

$$\mathcal{H}^{me}_{T,T} = 0 \qquad \text{and} \qquad \mathcal{H}^{me}_{T-1,T} = h_T,$$

and

$$\mathcal{H}^{me}_{t,T} = h_{t+1} + \mathcal{J}^{(t,t+1)}_{\mathbb{Q}^{me}} \left(\mathcal{H}^{me}_{t+1,T} \right), \quad t = 0, 1, ..., T-2, \tag{17}$$

with h_t, $t = 0, 1, ..., T$, defined in (9) and $\mathcal{J}^{(t,t+1)}_{\mathbb{Q}}$ as in (11) for $\mathbb{Q} = \mathbb{Q}^{me}$. Moreover,

$$\mathcal{H}^{me}_{t,T} = \mathcal{J}^{(t,T)}_{\mathbb{Q}^{me}} \left(\sum_{i=t+1}^{T} h_i \right). \tag{18}$$

Corollary 8 *The minimal martingale and the minimal entropy measures satisfy*

$$\mathbb{Q}^{mm} \left(Y_T | \, \mathcal{F}_{T-1} \vee \mathcal{F}^S_T \right) = \mathbb{Q}^{me} \left(Y_T | \, \mathcal{F}_{T-1} \vee \mathcal{F}^S_T \right) = \mathbb{P} \left(Y_T | \, \mathcal{F}_{T-1} \vee \mathcal{F}^S_T \right). \tag{19}$$

It is worth commenting on some distinct features of the minimal martingale and the minimal entropy measures. Firstly, we note that the density of the former (see (10)) has the intuitively pleasing property of preserving the conditional distribution of the non-traded factor, given the stock price, in terms of its historical counterpart. In essence, this property states that the unhedgeable

[2]For similar results in a diffusion model with stochastic volatility see [18].

risks, given the hedgeable ones, are viewed in the same manner under $\mathbb{P}$ and $\mathbb{Q}^{mm}$.

The minimal entropy measure, however, albeit its predominant role in exponential utility maximization, appears to be lacking an intuitively pleasing structure, as (13) shows. Secondly, we observe its dependence on the horizon choice, T, as reflected by the $T-$dependent values $\mathcal{H}_{t,T}^{me}$ in (13). In Section 4, we will see how the indifference prices inherit, in turn, this dependence. Note, however, that the minimal martingale measure does not depend on the specific horizon as (10) shows.

We finish with representation results for the value function process $V_t(x)$, defined in (5). The first formula is well known (see, for example, [2] and [13]) while formulae (21) and (22) are, to the best of our knowledge, new and follow from (15) and (18).

Proposition 9 *The value function satisfies, for $x \in \mathbb{R}$ and $t = 0, 1, ..., T$,*

$$V_t(x) = -e^{-\gamma x - \mathcal{H}_{t,T}^{me}} \tag{20}$$

$$= -\exp\left(-\gamma x - \mathcal{J}_{\mathbb{Q}^{me}}^{(t,T)}\left(\sum_{i=t+1}^{T} h_i\right)\right) \tag{21}$$

$$= -\exp\left(-\gamma x + \mathcal{J}_{\mathbb{Q}^{mm}}^{(t,T)}\left(-\sum_{i=t+1}^{T} h_i\right)\right), \tag{22}$$

with h_t, $t = 0, 1, ..., T$, given in (9) and $\mathcal{J}_{\mathbb{Q}}^{(t,T)}$ as in (12), for $\mathbb{Q} = \mathbb{Q}^{mm}, \mathbb{Q}^{me}$.

3 Indifference valuation algorithms

In this section, we review the notion of indifference price and provide two iterative algorithms for its construction. The claim to be priced is written, at time t_0, on both the traded stock and the non-traded factor. For simplicity, we assume that $t_0 = 0$. The claim matures at $t = 1, ..., T$, yielding payoff C_t, represented as an $\mathcal{F}_t$-measurable random variable. We are interested in computing its indifference price in reference to the exponential criterion (5). For the moment, we price a single claim and present the results on the multi-claim case afterwards. For convenience, we eliminate the "exponential" terminology. We recall the familiar definition of indifference price (see, for example, [2] and [13]).

Definition 10 *Consider a claim, written at time $t_0 = 0$ and yielding at t payoff $C_t \in \mathcal{F}_t$, $t = 0, 1, ..., T$. Let $V_t(x)$ be the value function process (5). The claim's indifference price is defined as the amount $\nu_s(C_t)$, $s = 0, 1, ..., t$, for which*

$$V_s(x - \nu_s(C_t)) = \sup_{\alpha_{s+1},...,\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t - C_t\right) | \mathcal{F}_s\right), \tag{23}$$

for all initial wealth levels $X_s = x \in \mathbb{R}$.

We remark that the alignment of the expiry of the claim with the time at which the value function process is calculated in the right hand side of the pricing condition (23) is chosen for mere convenience. Indeed, the above definition can be directly extended to times beyond the claim's maturity in that (23) can be replaced by

$$V_s\left(X_s-\nu_s\left(C_t\right)\right)=\sup_{\alpha_{s+1},\ldots,\alpha_{t'}} E_{\mathbb{P}}\left(V_{t'}\left(X_{t'}-C_t\right)\right|\mathcal{F}_s), \tag{24}$$

for $t'=t+1,...,T-1,T$. This follows easily from (23), the dynamic programming principle and the fact that $C_t\in\mathcal{F}_{t'}$. Observe, however, that this cannot be done for times t' exceeding T.

Next, we review the price representation obtained for the single-period case in [8] (see, also, [9]). Therein, the claim's indifference price is represented as a non-linear expectation of its payoff, providing the incomplete market analogue of the linear arbitrage-free pricing rule. We refer the reader to these papers for a detailed discussion on the nature and properties of the pricing formula. For indifference prices in single-period models for utilities different than the exponential, see [3].

Proposition 11 *(Single-period model) Let $\mathbb{Q}$ be the martingale measure under which the conditional distribution of the non-traded factor, given the traded asset, is preserved with respect to the historical measure $\mathbb{P}$, i.e.,*

$$\mathbb{Q}(Y_T\,|S_T\,)=\mathbb{P}(Y_T\,|S_T\,). \tag{25}$$

Let $C_T=C(S_T,Y_T)$ be the claim to be priced under exponential preferences with risk aversion coefficient γ. Then, its indifference price, $\nu_0\left(C_T\ \right)$, is given by

$$\nu_0(C_T)=\mathcal{E}_{\mathbb{Q}}(C_T)=E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}}\left(e^{\gamma C_T}\,|S_T\,\right)\right). \tag{26}$$

As the above result shows, the underlying indifference pricing blocks are the non-linear expectation $\mathcal{E}_{\mathbb{Q}}(\cdot)$ and the pricing measure $\mathbb{Q}$. For the multi-period case, we need to build their appropriate multi-period analogues. We stress that due to the inherent nonlinearities of the problem, together with the fact that the model at hand is non-reduced (i.e., the nested model is not complete), it is not at all clear how these analogues should be constructed. Notice, for example, that property (25) is satisfied by both the minimal martingale and minimal entropy measures, $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$, but only at expiration (see (19)). For times before $T-1$, the two measures differ and property (25) is held by $\mathbb{Q}^{mm}$, and not $\mathbb{Q}^{me}$, which is the natural martingale measure in exponential utility maximization. This important difference motivates us to look for algorithmic price representations under each of these two measures.

Definition 12 *Let $T>0$ and Z be a random variable in $(\Omega,\mathcal{F},\,\mathbb{P})$. For $s=0,1,...,T-1$, $t=s+1,...,T$ and $\mathbb{Q}\in\mathcal{Q}_T$, define the single- and multi-step price functionals*

$$\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}\left(Z\right)=\frac{1}{\gamma}E_{\mathbb{Q}}\left(\ln E_{\mathbb{Q}}\left(e^{\gamma Z}\left|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)\right|\mathcal{F}_s\right) \tag{27}$$

and

$$\mathcal{E}_{\mathbb{Q}}^{(s,t)}(Z)=\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}\left(...\mathcal{E}_{\mathbb{Q}}^{(t-1,t)}(Z)\right). \tag{28}$$

We caution the reader that, for $t>s+1$,

$$\mathcal{E}_{\mathbb{Q}}^{(s,t)}(Z)\neq\frac{1}{\gamma}E_{\mathbb{Q}}\left(\ln E_{\mathbb{Q}}\left(e^{\gamma Z}\left|\mathcal{F}_s\vee\mathcal{F}_t^S\right.\right)|\mathcal{F}_s\right).$$

Definition 13 *Let Z be a random variable in $(\Omega,\mathcal{F},\mathbb{P})$. For $s=0,1,...,T-1$ and $t=s+1,...,T$, define the nonlinear single- and multi-step functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{P}_{\mathbb{Q}^{mm}}^{(t,s)}$ by*

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) \tag{29}$$

and

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}(Z)=\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(...\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-1,t)}(Z)\right), \tag{30}$$

with $\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}$ given in (27) with $\mathbb{Q}=\mathbb{Q}^{mm}$.

The following lemma provides the explicit form of the multi-step functional $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}$.

Lemma 14 *Let Z be a random variable in $(\Omega,\mathcal{F},\mathbb{P})$. Then, for $s<t-1$,*

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}(Z)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}\sum_{i=s+2}^{t}h_i\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}\sum_{i=s+2}^{t}h_i\right). \tag{31}$$

Proof We establish (31) only for $s=t-2$ since the rest of the proof follows along similar arguments. We need to show that

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-2,t)}(Z)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right). \tag{32}$$

Using (29) and (30), we write

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-2,t)}(Z)=\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{P}_{\mathbb{Q}^{mm}}^{(t-1,t)}(Z)\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right) \tag{33}$$

$$-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right).$$

On the other hand, (14) yields

$$-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T} = -\frac{1}{\gamma}h_t + \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)$$

and the second term in (33) becomes

$$\begin{aligned}
-\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right) &= -\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}h_t + \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right) \\
&= -\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}h_t - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right),
\end{aligned}$$

where we used the measurability properties of h_t. Similarly, the first term in (33) becomes

$$\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(Z - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right) - \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right) - \frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right)$$

$$\begin{aligned}
&= \mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(Z - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right. \\
&\left. -\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right) - \frac{1}{\gamma}h_t + \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right) \\
&= \mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(Z - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right) - \frac{1}{\gamma}h_t\right) \\
&= \mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(Z - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T} - \frac{1}{\gamma}h_t\right)\right).
\end{aligned}$$

Combining the above, (32) follows. ■

We are now ready to provide the pricing algorithms for the indifference price. The first algorithm uses the minimal martingale measure and the pricing functionals $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ and $\mathcal{P}^{(s,t)}_{\mathbb{Q}^{mm}}$ while the second one uses the minimal entropy measure and the pricing functionals $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}$ and $\mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}$. To ease the presentation, we first state the main theorems and, then, provide their proofs and discussion.

Theorem 15 *Consider a claim written at $t_0 = 0$ and expiring at t yielding payoff $C_t \in \mathcal{F}_t$. For $t = 1, ..., T$ and $s = 0, 1, ..., t-1$, the following statements are true:*

i) The indifference price $\nu_s(C_t)$, defined in (23), is given by the algorithm

$$\nu_t(C_t) = C_t, \tag{34}$$

$$\nu_s(C_t) = \mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(\nu_{s+1}(C_t)\right), \tag{35}$$

where $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ is the single-step pricing functional defined in (29).

ii) The indifference price $\nu_s(C_t) \in \mathcal{F}_s$ is given by

$$\nu_s(C_t) = \mathcal{P}^{(s,t)}_{\mathbb{Q}^{mm}}(C_t) \tag{36}$$

$$= \mathcal{E}^{(s,t)}_{\mathbb{Q}^{mm}}\left(C_t - \frac{1}{\gamma}\mathcal{H}^{me}_{t,T} - \frac{1}{\gamma}\sum_{i=s+2}^{t} h_i\right) - \mathcal{E}^{(s,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T} - \frac{1}{\gamma}\sum_{i=s+2}^{t} h_i\right), \tag{37}$$

with the multi-step price functionals $\mathcal{P}^{(s,t)}_{\mathbb{Q}^{mm}}$ and $\mathcal{E}^{(s,t)}_{\mathbb{Q}^{mm}}$ defined, respectively, in (30) and (28) for $\mathbb{Q} = \mathbb{Q}^{mm}$.

iii) The pricing algorithm is consistent across time in that, for $0 \leq s \leq s' \leq t$, the semigroup property

$$\nu_s(C_t) = \mathcal{P}^{(s,s')}_{\mathbb{Q}^{mm}}(\mathcal{P}^{(s',t)}_{\mathbb{Q}^{mm}}(C_t)) = \mathcal{P}^{(s,s')}_{\mathbb{Q}^{mm}}(\nu_{s'}(C_t)) = \nu_s(\mathcal{P}^{(s',t)}_{\mathbb{Q}^{mm}}(C_t)) \tag{38}$$

holds.

Theorem 16 *Consider a claim written at $t_0 = 0$ and expiring at t yielding payoff $C_t \in \mathcal{F}_t$. For $t = 1, ..., T$ and $s = 0, 1, ..., t-1$, the following statements are true:*

i) The indifference price $\nu_s(C_t)$, defined in (23), is given by the algorithm

$$\nu_t(C_t) = C_t,$$

$$\nu_s(C_t) = \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}(\nu_{s+1}(C_t)), \tag{39}$$

where $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}$ is the single-step price functional defined in (27) for $\mathbb{Q} = \mathbb{Q}^{me}$.

ii) The indifference price process is given by

$$\nu_s(C_t) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}(C_t), \tag{40}$$

with the multi-step price functional $\mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}$ defined in (28) for $\mathbb{Q} = \mathbb{Q}^{me}$.

iii) The pricing algorithm is consistent across time in that, for $0 \leq s \leq s' \leq t$, the semigroup property

$$\nu_s(C_t) = \mathcal{E}^{(s,s')}_{\mathbb{Q}^{me}}(\mathcal{E}^{(s',t)}_{\mathbb{Q}^{me}}(C_t)) = \mathcal{E}^{(s,s')}_{\mathbb{Q}^{me}}(\nu_{s'}(C_t)) = \nu_s(\mathcal{E}^{(s',t)}_{\mathbb{Q}^{me}}(C_t)) \tag{41}$$

holds.

Before we provide the proof for Theorem 15 we state the following lemma.

Lemma 17 *Let $s = 0, 1, ..., T-1$, $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ be defined in (27) for $\mathbb{Q} = \mathbb{Q}^{mm}$ and Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. Then,*

$$\sup_{\alpha_{s+1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{s+1}-Z)}|\mathcal{F}_s\right) = -e^{-\gamma\left(X_s - \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}(Z)\right) - h_{s+1}}, \tag{42}$$

with h_s as in (9).

Proof With A_{s+1} as in (7) we have

$$\sup_{\alpha_{s+1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{s+1}-Z)}|\mathcal{F}_s\right)$$

$$= -e^{-\gamma X_s}\Big(\mathbb{P}(A_{s+1}|\mathcal{F}_s)e^{-\gamma\alpha_{s+1}S_s(\xi^u_{s+1}-1)}E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s\vee A_{s+1}\right)$$

$$+(1-\mathbb{P}(A_{s+1}|\mathcal{F}_s))e^{-\gamma\alpha_{s+1}S_s(\xi^d_{s+1}-1)}E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s\vee A^c_{s+1}\right)\Big).$$

Differentiating with respect to α_{s+1} yields that the optimum occurs at

$$\alpha_{s+1} = \frac{1}{\gamma S_s\left(\xi^u_{s+1}-\xi^d_{s+1}\right)}\ln\left(\frac{E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s\vee A_{s+1}\right)\mathbb{P}(A_{s+1}|\mathcal{F}_s)\left(\xi^u_{s+1}-1\right)}{E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s\vee A^c_{s+1}\right)(1-\mathbb{P}(A_{s+1}|\mathcal{F}_s))\left(1-\xi^d_{s+1}\right)}\right).$$

Using the form of the density of the minimal martingale measure (see (10)) we obtain

$$\sup_{\alpha_{s+1}} E_{\mathbb{P}}\left(-e^{-\gamma(X_{s+1}-Z)}|\mathcal{F}_s\right)$$

$$= -\exp\big(-\gamma X_s + \mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)\ln E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s\vee A_{s+1}\right)$$

$$+\,(1-\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s))\ln E_{\mathbb{P}}\left(e^{\gamma Z}|\mathcal{F}_s\vee A^c_{s+1}\right)\big)\times$$

$$\times\left(\frac{\mathbb{P}(A_{s+1}|\mathcal{F}_s)}{\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}\right)^{\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}\left(\frac{1-\mathbb{P}(A_{s+1}|\mathcal{F}_s)}{1-\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}\right)^{1-\mathbb{Q}^{mm}(A_{s+1}|\mathcal{F}_s)}.$$

Using once again the form of the density of the minimal martingale measure (10) and the definition of $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ (cf. (27)), (42) follows. ■

We are now ready to prove Theorem 15.

Proof i) Equality (34) is immediate. We prove (35) for $s=t-1$. From (20) we have

$$\sup_{\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t-C_t\right)|\,\mathcal{F}_{t-1}\right)$$

$$= \sup_{\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma\left(X_t-\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right)}\Big|\,\mathcal{F}_{t-1}\right).$$

Using Lemma 17 for $s=t-1$ and $Z=C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}$, we get

$$\sup_{\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t-C_t\right)|\,\mathcal{F}_{t-1}\right) = -e^{-\gamma\left(X_{t-1}-\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right)-h_t}.$$

Combining the above with (23) and formula (20) for V_{t-1}, we deduce

$$\nu_{t-1}\left(C_t\right) = \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)+\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}-\frac{1}{\gamma}h_t$$

$$= \mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)-\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right), \qquad (43)$$

where we used (14) for $\mathcal{H}^{me}_{t-1,T}$.

For $s=t-2$, we have

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t-C_t\right)\middle|\,\mathcal{F}_{t-2}\right)$$

$$=\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})+\alpha_t(S_t-S_{t-1})-\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right)}\middle|\,\mathcal{F}_{t-2}\right)$$

$$=\sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2}))}\right.$$

$$\left.\times\sup_{\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_t(S_t-S_{t-1})-\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right)}\middle|\,\mathcal{F}_{t-1}\right)\middle|\,\mathcal{F}_{t-2}\right).$$

Using Lemma 17 for $s=t-1$ and $Z=C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}$, and (14) and (43) we deduce

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t-C_t\right)\middle|\,\mathcal{F}_{t-2}\right)$$

$$=\sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})-\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(C_t-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)\right)-h_t}\middle|\,\mathcal{F}_{t-2}\right)$$

$$=\sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})-\left(\nu_{t-1}(C_t)+\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t,T}\right)-\frac{1}{\gamma}h_t\right)\right)}\middle|\,\mathcal{F}_{t-2}\right)$$

$$=\sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma\left(X_{t-2}+\alpha_{t-1}(S_{t-1}-S_{t-2})-\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right)\right)}\middle|\,\mathcal{F}_{t-2}\right).$$

Using Lemma 17 once again, this time for $s=t-2$ and $Z=\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}$, we obtain

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(V_t\left(X_t-C_t\right)\middle|\,\mathcal{F}_{t-2}\right)$$

$$=-e^{-\gamma\left(X_{t-2}-\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right)\right)-h_{t-1}}. \tag{44}$$

On the other hand, (20) yields,

$$V_{t-2}\left(X_{t-2}-\nu_{t-2}(C_t)\right)=-e^{-\gamma(X_{t-2}-\nu_{t-2}(C_t))-\mathcal{H}^{me}_{t-2,T}}.$$

Comparing the above to (44), using the definition of the indifference price (23) and formula (14), we deduce

$$\nu_{t-2}(C_t)=\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right)-\frac{1}{\gamma}h_{t-1}+\frac{1}{\gamma}\mathcal{H}^{me}_{t-2,T}$$

$$=\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right)-\mathcal{E}^{(t-2,t-1)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{t-1,T}\right), \tag{45}$$

and we conclude. For $s=0,...,t-3$, (35) follows along similar arguments.

ii) In view of property (31), assertions (36) and (37) are equivalent. We only show (37). For $s=t-1$, (37) follows trivially. To show (37) for $s=t-2$, we work as follows. We first observe that (14) together with the measurability properties of the local entropy process h_t yield

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}h_t\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right).$$

On the other hand, using (35) for $\nu_{t-1}(C_t)$ and (14), we deduce

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\nu_{t-1}(C_t)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}\mathcal{H}_{t-1,T}^{me}\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)\right.$$

$$\left.+\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}h_t\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}\right)-\frac{1}{\gamma}h_t\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-1,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right)\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right).$$

Combining the above with (45) yields

$$\nu_{t-2}(C_t)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(C_t-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(t-2,t)}\left(-\frac{1}{\gamma}\mathcal{H}_{t,T}^{me}-\frac{1}{\gamma}h_t\right),$$

and we deduce (37). For $s=0,...,t-3$, we work similarly.

The semigroup property (38) follows easily. ■

We continue with the proof of Theorem 16.

Proof We only need to establish that

$$\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)\right)$$

$$=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)-\frac{\mathcal{H}_{s+1,T}^{me}}{\gamma}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{\mathcal{H}_{s+1,T}^{me}}{\gamma}\right),$$

since all assertions of the theorem would follow by straightforward arguments.

To this end, let $Z=\gamma\nu_{s+1}\left(C_t\right)-\mathcal{H}_{s+1,T}^{me}$. Then (16) yields

$$\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)-\mathcal{H}_{s+1,T}^{me}\right)=\mathcal{J}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)\right)+\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\mathcal{H}_{s+1,T}^{me}\right)$$

and, in turn,

$$\frac{1}{\gamma}\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)-\mathcal{H}_{s+1,T}^{me}\right)$$

$$=\frac{1}{\gamma}\mathcal{J}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\gamma\nu_{s+1}\left(C_t\right)\right)+\frac{1}{\gamma}\mathcal{J}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\mathcal{H}_{s+1,T}^{me}\right).$$

We easily conclude. ■

Discussion on the pricing algorithms: The indifference price is calculated via the iterative pricing schemes (35) and (39), applied backwards in time, starting at the claim's maturity. The schemes have local and dynamic properties.

Dynamically, the pricing functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,t)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,t)}$ are similar. Specifically, at each time interval, say $(s,s+1)$, the price $\nu_s(C_t)$ is computed via the single-step pricing operators, $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$, applied to the end of the period payoff. The latter turns out to be the indifference price, $\nu_{s+1}(C_t)$, yielding prices consistent across time.

Locally, however, the pricing roles of $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ are very different both in structure and the associated measures. We start with the latter price functional since it has the simpler of the two forms. Valuation is executed in two steps, in analogy to the single-period counterpart (26). In the first sub-step, the end of the period payoff, $\nu_{s+1}(C_t)$, is altered via a non-linear functional and the conditioning on the information generated by $\mathcal{F}_s\vee\mathcal{F}_{s+1}^S$. The new payoff,

$$\tilde{\nu}_{s+1}(C_t)=\frac{1}{\gamma}\ln E_{\mathbb{Q}^{me}}\left(e^{\gamma\nu_{s+1}(C_t)}\left|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right.\right) \tag{46}$$

emerges which is, in turn, priced by expectation. The indifference price is, then, given by

$$\nu_s(C_t)=E_{\mathbb{Q}^{me}}(\tilde{\nu}_{s+1}(C_t)\left|\mathcal{F}_s\right.). \tag{47}$$

While structure-wise the price functional $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ has a simple and intuitive form, the employed measure $\mathbb{Q}^{me}$ does *not*, as can be seen from (13).

The situation is reversed in the first algorithm. Specifically, the pricing functional $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ has no transparent form while the used measure, $\mathbb{Q}^{mm}$, has the intuitively pleasing property (10). Indeed, $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ incorporates the minimal aggregate entropy $\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}$ in a "palindromic" manner. Namely, at each time step, the end of the period payoff $\nu_{s+1}\left(C_t\right)$ is reduced by $\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}$ and priced, yielding the indifference price

$$Z_{s+1}^1=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(\nu_{s+1}\left(C_t\right)-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right).$$

In turn, the payoff

$$Z_{s+1}^2=-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)$$

is added. Both quantities Z^1_{s+1} and Z^2_{s+1} are calculated via the two-step procedure similar to the one described in (46) and (47). Notice that due to the non-linear character of the indifference price, the entropic liability $-\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T}$ could not be factored out. That is,

$$\nu_s(C_t) = \mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(\nu_{s+1}\left(C_t\right)\right)$$

$$= \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(\nu_{s+1}\left(C_t\right) - \frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T}\right) - \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(-\frac{1}{\gamma}\mathcal{H}^{me}_{s+1,T}\right)$$

$$\neq \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(\nu_{s+1}\left(C_t\right)\right).$$

The following results follow easily from the above pricing algorithms.

Corollary 18 *Let the payoff C_t be of the form*

$$C_t = Y_t + Z_t$$

with $Y_t \in \mathcal{F}_t$ and Z_t being such that there exist $Z_s \in \mathcal{F}^S_s$ and $\alpha_i \in \mathcal{F}^S_{i-1}$, $i = s+1, ..., t$, satisfying $Z_t = Z_s + \Sigma^t_{i=s+1}\alpha_i \Delta S_i$, a.e. Then,

$$\nu_s\left(C_t\right) = \nu_s\left(Y_t + Z_t\right) = \nu_s\left(Y_t\right) + Z_s$$

$$= \mathcal{P}^{(s,t)}_{\mathbb{Q}^{mm}}\left(Y_t\right) + E_{\mathbb{Q}^{mm}}\left(\left. Z_t \right| \mathcal{F}^S_s\right) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}}\left(Y_t\right) + E_{\mathbb{Q}^{me}}\left(\left. Z_t \right| \mathcal{F}^S_s\right).$$

Next, we provide the pricing algorithms for the multi-claim case. For convenience, we assume that in the interval $[0, n+1]$ with $n+1 \leq T$, we price a collection of $n+2$ claims, $C_0, C_1, ..., C_j, ...C_{n+1}$, with each generic claim maturing at time j, $j = 0, 1, ..., n+1$ and yielding payoff $C_j \in \mathcal{F}_j$. Using repeatedly Corollary 19 we obtain the following.

Theorem 19 *Consider a collection of $n+2$ claims, written at $t_0 = 0$, yielding payoffs $C_j \in \mathcal{F}_j$, with $j = 0, 1, ..., n+1$. The following statements hold:*

i) The indifference price $\nu_s(\Sigma^{n+1}_{j=s}C_j)$, is given, for $s = 0, 1, ..., n+1$, by the iterative algorithm

$$\nu_{n+1}\left(C_{n+1}\right) = C_{n+1},$$

$$\nu_s(C_s + \Sigma^{n+1}_{j=s+1}C_j) = C_s + \mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}(C_{s+1} + \nu_{s+1}(\Sigma^{n+1}_{j=s+2}C_j))$$

$$= C_s + \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}(C_{s+1} + \nu_{s+1}(\Sigma^{n+1}_{j=s+2}C_j)),$$

with $\mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}$ and $\mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}$ as in (29) and (27).

ii) The indifference price process $\nu_s(C_s + \Sigma^{n+1}_{j=s+1}C_j) \in \mathcal{F}_s$ and satisfies, for $s = 0, 1, ..., n+1$,

$$\nu_s(C_s + \Sigma^{n+1}_{j=s+1}C_j)$$

$$= C_s + \mathcal{P}^{(s,s+1)}_{\mathbb{Q}^{mm}}\left(C_{s+1} + \mathcal{P}^{(s+1,s+2)}_{\mathbb{Q}^{mm}}\left(C_{s+2} + ...\mathcal{P}^{(n-1,n)}_{\mathbb{Q}^{mm}}\left(C_n + \mathcal{P}^{(n,n+1)}_{\mathbb{Q}^{mm}}\left(C_{n+1}\right)\right)\right)\right)$$

$$= C_s + \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^{me}}\left(C_{s+1} + \mathcal{E}^{(s+1,s+2)}_{\mathbb{Q}^{me}}\left(C_{s+2} + ...\mathcal{E}^{(n-1,n)}_{\mathbb{Q}^{me}}\left(C_n + \mathcal{E}^{(n,n+1)}_{\mathbb{Q}^{me}}\left(C_{n+1}\right)\right)\right)\right).$$

3.1 Conditional certainty equivalent and the indifference price

The form of the auxiliary single-step payoff $\tilde{\nu}_{s+1}(C_t)$, $s = 0, 1, ..., t$, introduced in (46), motivates us to ask whether there is a natural connection between it and the static certainty equivalent pricing rule. The latter is given, for a random variable Z, by

$$\mathcal{C}(Z) = -u^{-1} E_{\mathbb{P}}(u(-Z)), \tag{48}$$

with u being an increasing and concave utility function. We explore this question next.

We first introduce the auxiliary process $V_s^{-1}(x)$, $s = 0, 1, ..., T$, denoting the spatial inverse of the value function (5), and given by

$$V_s^{-1}(x) = -\frac{\ln(-x)}{\gamma} - \frac{\mathcal{H}_{s,T}^{me}}{\gamma}, \qquad x \in \mathbb{R}^-. \tag{49}$$

In the binomial model at hand, a natural extension of (48) would, then, be the *conditional certainty equivalent,* introduced below.

Definition 20 *Let Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ and $V_s(x)$ and $V_s^{-1}(x)$, $s = 0, 1, ..., T$, be, respectively, the value function process and its inverse (cf. (5) and (49)). For $\mathbb{Q} \in \mathcal{Q}_T$, define the conditional certainty equivalent $\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z)$ by*

$$\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z) = -V_{s+1}^{-1}\left(E_{\mathbb{Q}}\left(V_{s+1}(-Z) \middle| \mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right)\right). \tag{50}$$

The following Lemma follows from direct arguments.

Lemma 21 *Let the conditional certainty equivalent be defined in (48), and $\mathbb{Q}^{mm}$ and $\mathbb{Q}^{me}$ be the minimal martingale and minimal entropy measures. Then,*

$$\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(0) \neq 0 \qquad \textit{and} \qquad \mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z) \neq \mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z),$$

and

$$\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(Z + \frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) \neq \mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z) + \mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)$$

with $Z \in (\Omega, \mathcal{F}, \mathbb{P})$.

We note that there are particular cases when the above become equalities. These cases are discussed in Proposition 28 and Theorem 29.

In the next proposition we represent the single-step indifference price functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$using the above conditional certainty equivalents.

Proposition 22 *Let Z be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}$ be as in (50) for $Q = \mathbb{Q}^{mm}, \mathbb{Q}^{me}$ and $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ be as in (29) and (27). Then, for $s = 0, 1, ..., T$,*

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z) = E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)\middle|\, \mathcal{F}_s\right) - E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(0)\middle|\, \mathcal{F}_s\right). \tag{51}$$

Similarly,

$$\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z) = E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(Z + \frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\, \mathcal{F}_s\right) \tag{52}$$
$$-E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\, \mathcal{F}_s\right).$$

Proof To prove (51), we use Definition 20 to obtain

$$E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)\middle|\, \mathcal{F}_s\right) = E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^{mm}}\left(e^{\gamma Z - \mathcal{H}_{s+1,T}^{me}}\middle|\, \mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right)\middle|\, \mathcal{F}_s\right)$$
$$+E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\, \mathcal{F}_s\right) \tag{53}$$
$$= \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z - \frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) + E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\, \mathcal{F}_s\right).$$

For $Z = 0$, we have

$$E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(0)\middle|\, \mathcal{F}_s\right) = \mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\tfrac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) + E_{\mathbb{Q}^{mm}}\left(\tfrac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\, \mathcal{F}_s\right). \tag{54}$$

Subtracting (54) from (53) and using (29) yields (51).

To prove (52), we work similarly. To this end, we have

$$E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(Z + \frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\, \mathcal{F}_s\right)$$
$$= E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^{mm}}\left(e^{\gamma\left(Z + \frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right) - \mathcal{H}_{s+1,T}^{me}}\middle|\, \mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right)\middle|\, \mathcal{F}_s\right)$$
$$+E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\middle|\, \mathcal{F}_s\right)$$
$$= E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^{mm}}\left(e^{\gamma Z}\middle|\, \mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right)\middle|\, \mathcal{F}_s\right) + E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me} \mid \mathcal{F}_s\right)$$
$$= \mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z) + E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me} \mid \mathcal{F}_s\right),$$

and for $Z = 0$,

$$E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)\middle|\, \mathcal{F}_s\right) = E_{\mathbb{Q}^{me}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me} \mid \mathcal{F}_s\right).$$

Subtracting the above we conclude. ■

As the analysis above shows, there is *no* direct connection between the pricing functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}$ and the conditional certainty equivalents $\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ and $\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}$ in that

$$\nu_s(C_t) \neq E_{\mathbb{Q}^{mm}}\left(\mathcal{C}_{\mathbb{Q}^{mm}}^{(s,s+1)}(\nu_{s+1}(C_t)) \middle| \mathcal{F}_s \right) \tag{55}$$

and

$$\nu_s(C_t) \neq E_{\mathbb{Q}^{me}}\left(\mathcal{C}_{\mathbb{Q}^{me}}^{(s,s+1)}(\nu_{s+1}(C_t)) \middle| \mathcal{F}_s \right). \tag{56}$$

This is a direct consequence of the form of the dynamic risk preference process $V_s(x)$ (see (22) and (21)), as well as the measurability properties of the minimal aggregate entropy $\mathcal{H}_{s,T}^{me}$ (see, for example, (15) or (18)).

We stress, however, that such a connection is present in two cases. Specifically, it holds when the binomial model is of reduced form. This case is analyzed in detail in Section 5. It is, also, present in an alternative kind of indifference prices built in reference to a new risk preference framework in which the value function process, $V_s(x)$, is replaced by its "forward" analogue (we refer the reader to [11] for further details).

4 Risk preference normalization points and the related indifference prices

So far, we have derived indifference prices associated with an exponential utility function set at time T. An important implicit assumption in the entire construction is that the claims we consider mature *before* this exogenously chosen horizon. We will refer to the instant T as the *risk preference normalization point.*

Two questions then arise. Firstly, how the indifference prices depend on the choice of the risk preference normalization point and, secondly, can this dependence be relaxed. Herein, we only address the first question and refer the reader to [11] for the second one.

In order to emphasize the dependence on the horizon choice, we introduce the notation $V_{t,T}(x)$ and $\nu_s(C_t;T)$ for the value function and the indifference price, respectively. We will be occasionally using the terminology of the indifference price normalized at T.

Theorem 23 *Let $\hat{T}$ and T be two normalization points with $\hat{T} > T$, and let $\mathcal{H}_{s,T}^{me}$ and $\mathcal{H}_{s,\hat{T}}^{me}$ be the associated minimal aggregate entropy processes. Consider a claim written at $t_0 = 0$ and maturing at $t = 0, 1, ..., T$, yielding payoff $C_t \in \mathcal{F}_t$.*

Let $\nu_s\left(C_t;\hat{T}\right)$ and $\nu_s(C_t;T)$ be the indifference prices normalized at $\hat{T}$ and T, respectively. Then, for $0 \leq s \leq t \leq T$,

$$\nu_s\left(C_t;\hat{T}\right) = \nu_s(C_t - Z_t;T) + Z_s \tag{57}$$

where, for $u = s, ..., t$*,*

$$Z_u = \frac{1}{\gamma}\left(\mathcal{H}^{me}_{u,\hat{T}} - \mathcal{H}^{me}_{u,T}\right). \tag{58}$$

Proof Consider the normalization point $\hat{T}$. Then, (20) yields

$$\sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,\hat{T}}\left(X_t - C_t\right)\Big|\,\mathcal{F}_s\right) =$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - C_t\right) - \mathcal{H}^{me}_{t,\hat{T}}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - C_t\right) - \left(\mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T}\right) - \mathcal{H}^{me}_{t,T}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - \left(C_t - \frac{1}{\gamma}\left(\mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T}\right)\right)\right) - \mathcal{H}^{me}_{t,T}\right)\Big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(-\exp\left(-\gamma\left(X_t - \left(C_t - Z_t\right)\right) - \mathcal{H}^{me}_{t,T}\right)\big|\,\mathcal{F}_s\right)$$

$$= \sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,T}\left(X_t - \left(C_t - Z_t\right)\right)\big|\,\mathcal{F}_s\right),$$

where we used (58). From (23), we have

$$\sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,\hat{T}}\left(X_t - C_t\right)\Big|\,\mathcal{F}_s\right) = -\exp\left(-\gamma\left(x - \nu_s\left(C_t; \hat{T}\right)\right) - \mathcal{H}^{me}_{s,\hat{T}}\right)$$

and, similarly,

$$\sup_{\alpha_{s+1},\ldots,\alpha_t} E_{\mathbb{P}}\left(V_{t,T}\left(X_t - \left(C_t - Z_t\right)\right)\big|\,\mathcal{F}_s\right) = -\exp\left(-\gamma\left(x - \nu_s\left(C_t - Z_t; T\right)\right) - \mathcal{H}^{me}_{s,T}\right).$$

Combining the above we easily conclude. ■

Corollary 24 *Consider the claim* $Z_t \in \mathcal{F}_t$ *written at* $t_0 = 0$ *and yielding at* t *payoff*

$$Z_t = \frac{1}{\gamma}\left(\mathcal{H}^{me}_{t,\hat{T}} - \mathcal{H}^{me}_{t,T}\right),$$

for $\hat{T} > T$*. Then, for* $0 \leq t \leq T$*, and* $s \leq t$

$$Z_s = \nu_s\left(-Z_t; T\right) = -\mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}_T}\left(-Z_t\right). \tag{59}$$

On the other hand,

$$Z_s = \nu_s\left(Z_t; \hat{T}\right) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^{me}_{\hat{T}}}\left(Z_t\right). \tag{60}$$

4.1 Numerical results

We study numerically the dependence of the indifference price on the risk preference normalization point as well as on the risk aversion. We consider a (non-reduced) incomplete model in which the stochastic factor affects both the claim's payoff and the transition probabilities of the stock price process.

Specifically, we assume that the values ξ_t^u, ξ_t^d, η_t^u and η_t^d, $t = 0, 1, ..., T$ (see (1) and (2)) are given by

$$\xi_t^u = 1 + \mu dt + \sigma\sqrt{dt} \quad \text{and} \quad \xi_t^d = 1 + \mu dt - \sigma\sqrt{dt},$$

and

$$\eta_t^u = 1 + bdt + a\sqrt{dt} \quad \text{and} \quad \eta_t^u = 1 + bdt + a\sqrt{dt},$$

with the constants σ, μ, a and b satisfying $-\sigma < \mu\sqrt{dt} < \sigma$ and $-a < b\sqrt{dt} < a$. The time increment dt is given by $dt = \frac{1}{N}T$ where T and N represent, respectively, the backward normalization point and the number of periods in $[0, T]$. For $t = 0, 1, ..., T$, we choose

$$\mathbb{P}(Y_t = Y_t^u | \mathcal{F}_{t-1}) = 0.5,$$

$$\mathbb{P}(S_t = S_t^u | \mathcal{F}_{t-1}) = \begin{cases} 0.75, & Y_{t-1} \geq Y_0, \\ 0.5, & \text{otherwise} \end{cases}$$

and $Cor\,(\Delta S_t, \Delta Y_t | \mathcal{F}_{t-1}) = 0.5$. We consider a call option written on the stochastic factor. The model parameters are chosen as $\sigma = 0.2$, $a = 0.5$, $b = \mu = 0$ and $S_0 = Y_0 = K = 10$.

Figures 1 and 2 show, respectively, the dependence of the option's price on the risk preference normalization time, T, and the risk aversion coefficient, γ.

In Figure 1, γ is fixed at 0.2. The number of time steps, N, varies from 60 to 155 in 5 unit increments, and T varies from 0.083 to 0.215. The contract's expiration is fixed at 0.083 years. In Figure 2, $N = 115$, T=0.4792, the contracts expiration is set at 0.25 years and γ varies from 0.001 to 0.901, with 0.045 increments.

Finally, Figure 3 incorporates changes in both T and γ. Therein, N and T are varying as in Figure 1.

As discussed earlier, the indifference price changes with the risk preference normalization point. For the chosen example, the price decreases as the normalization point moves further away from the contract's expiration. This dependence dissipates considerably when the normalization point is set at more than twice the contract's expiration time. It is worth noticing that Figure 1 suggests that the price has a finite limit as $T \to \infty$. Two interesting questions – that would require an additional theoretical investigation – arise, namely, whether the limit coincides with a price obtained from any known pricing methodology and whether such a limit exists for other, more general, contingent claims.

Figure 2 shows the dependence of the indifference price on the risk aversion for a fixed backward normalization point T. The latter is taken to be different that the claim's expiration. One easily sees the well known result that the price is monotone with respect to γ.

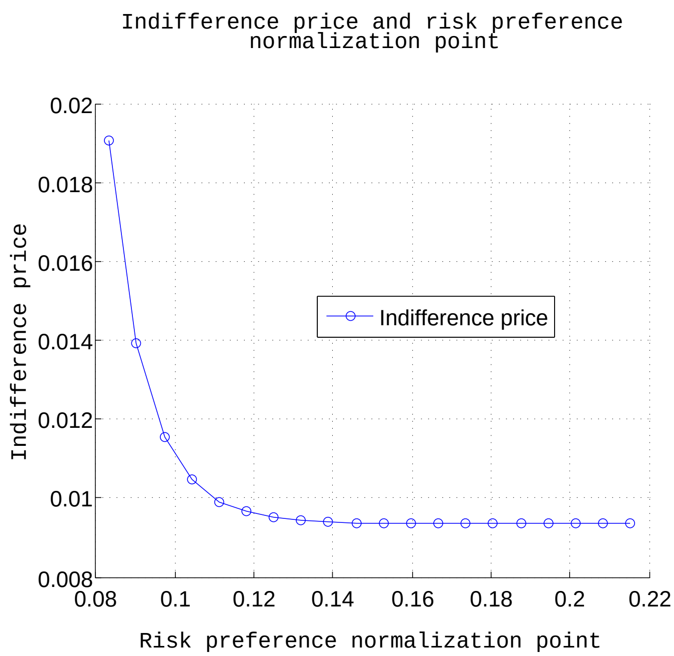

Figure 1: Dependence of the indifference price on the risk preference normalization point

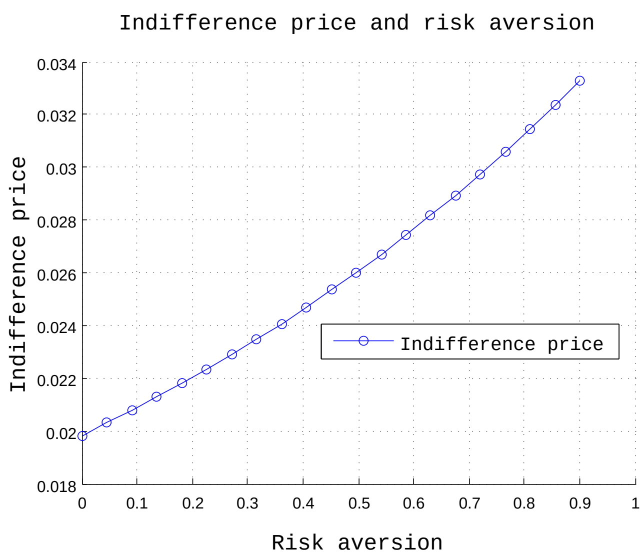

Figure 2: Dependence of the indifference price on the risk aversion

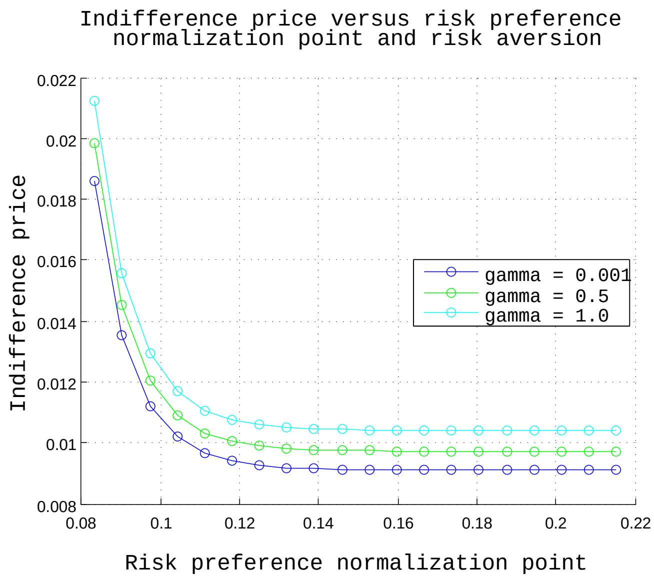

Figure 3: Dependence of the indifference price on the risk preference normalization point and the risk aversion

Figure 3 displays results for various levels of risk aversion, namely, when $\gamma = 0.001$, 0.5, and 1.0. The graph highlights the interplay between the risk preference normalization point and the risk aversion. As it is known, when the risk aversion approaches zero, the indifference price becomes linear. Based on the latter observation, one may wrongly expect that the dependence on T vanishes for small values of γ. This is not, however, what the graph shows. For example, when $\gamma = 0.001$, significant dependence on T is still present on the price. In our opinion, this dependence is attributed to the fact that while the pricing functional (27) is independent of the horizon choice, the associated pricing measure, the minimal entropy one, is not as (13) shows. As discussed earlier, this dependence is reversed if one uses the pricing algorithm in Theorem 15, in that now the pricing functional (29) depends on the normalization point while the pricing measure, the minimal martingale one, is not as (10) shows. In both cases, the corresponding dependences prevail even if the risk aversion coefficient becomes very small.

5 Reduced incomplete binomial models

We focus on an important special case of the incomplete binomial model introduced in Section 2. Specifically, we assume that neither the values nor the transition probabilities of the stock price process S_t are affected by the non-traded factor process Y_t, i.e. for $t = 0, 1, ..., T-1$,

$$\xi_{t+1}^u \in \mathcal{F}_t^S \quad \text{and} \quad \xi_{t+1}^d \in \mathcal{F}_t^S, \tag{61}$$

and

$$\mathbb{P}\left(\xi_{t+1} = \xi_{t+1}^u \left| \mathcal{F}_t \right.\right) = \mathbb{P}\left(\xi_{t+1} = \xi_{t+1}^u \left| \mathcal{F}_t^S \right.\right). \tag{62}$$

We will call such an incomplete binomial model *reduced.* Notice that under (61) and (62) the nested model becomes complete and market incompleteness is generated only through the presence of the non-traded risk factor in the claim's payoff. To our knowledge, this is the only case analyzed so far in exponential indifference pricing in binomial models (see, among others, [1], [8], [16] and [15])).

As it is expected, the minimal martingale and minimal entropy measures must coincide since there is now a unique (nested) martingale measure. We denote this measure by $\mathbb{Q}\left(\cdot\right| \mathcal{F}_t)$, $t = 0, 1, ..., T$. The interesting fact is that the minimal aggregate entropy looses its non-linear character and reduces to a mere conditional expectation of the aggregate local entropy.

Lemma 25 *Under assumptions (61) and (62), the local entropy process is $\mathcal{F}_t^S$-predictable, i.e., $h_t \in \mathcal{F}_{t-1}^S$, $t = 1, ..., T$.*

Proposition 26 *In the reduced binomial model, the minimal martingale and minimal entropy measures coincide, i.e. for* $t = 0, 1, ..., T,$

$$\mathbb{Q}\left(\cdot \middle| \mathcal{F}_t\right) = \mathbb{Q}^{mm}\left(\cdot \middle| \mathcal{F}_t\right) = \mathbb{Q}^{me}\left(\cdot \middle| \mathcal{F}_t\right). \tag{63}$$

Moreover, the minimal aggregate entropy $\mathcal{H}_{t,T}^{me}$ *becomes*

$$\mathcal{H}_{t,T}^{me} = E_{\mathbb{Q}}\left(\sum_{i=t+1}^{T} h_i \middle| \mathcal{F}_t^S\right), \tag{64}$$

with h_i*,* $i = t+1, ..., T$*, as in (9).*

The proof of the above results can be found in [17]. We only comment that the key step is to combine the reduced measurability of the local entropy process with Proposition 5 to establish that the conditional entropic terms appearing in (13) satisfy, for $t = 0, 1, ..., T,$

$$\mathcal{H}_{t,T}^{me,uu} = \mathcal{H}_{t,T}^{me,ud} \quad \text{and} \quad \mathcal{H}_{t,T}^{me,du} = \mathcal{H}_{t,T}^{me,dd}.$$

Combining (64) with Proposition 9, we deduce the following result.

Proposition 27 *Under assumptions (61) and (62), the value function process* $V_t(x)$ *is* $\mathcal{F}_t^S-$*adapted and given by*

$$V_t(x) = -\exp\left(-\gamma x - E_{\mathbb{Q}}\left(\sum_{i=t+1}^{T} h_i \middle| \mathcal{F}_t^S\right)\right),$$

with $\mathbb{Q}$ *as in (63) and* h *as in (9).*

The next result shows that in the reduced binomial model, the pricing functionals $\mathcal{P}_{\mathbb{Q}}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}$ coincide. Moreover, they are equal to the conditional certainty equivalent $\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}$.

Proposition 28 *Let* $\mathbb{Q}$ *be as in (63) and* Z *be a random variable in* $(\Omega, \mathcal{F}, \mathbb{P})$*. For* $s = 0, 1, ..., T-1$*, the following statements are true.*

i) The single-step pricing functionals $\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}$ *and* $\mathcal{E}_{\mathbb{Q}^{me}}^{(s,s+1)}(Z)$ *(cf. (29) and (27)) coincide*

$$\mathcal{P}_{\mathbb{Q}}^{(s,s+1)}(Z) = \mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z).$$

ii) Moreover, the conditional certainty equivalence defined in (50) satisfies

$$E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z) \middle| \mathcal{F}_s\right) = \mathcal{P}_{\mathbb{Q}}^{(s,s+1)}(Z) = \mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z).$$

Proof i) From (29) we have

$$\mathcal{P}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)-\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right).$$

Property (64) implies $\mathcal{H}_{s+1,T}^{me}\in\mathcal{F}_{s+1}^{S}$, for $s=0,1,...,T-1$, and, thus,

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(Z-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)=\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}(Z)-E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right).$$

Similarly,

$$\mathcal{E}_{\mathbb{Q}^{mm}}^{(s,s+1)}\left(-\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\right)=-E_{\mathbb{Q}^{mm}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right).$$

Combining the above with (63) we easily conclude.

ii) We only show that

$$E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z)\Big|\,\mathcal{F}_s\right)=\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z),$$

since the rest of the statements follow easily. To this end, using (63), and (20) and (49) in (20), we deduce

$$E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(Z)\Big|\,\mathcal{F}_s\right)=E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}}\left(e^{\gamma Z-\mathcal{H}_{s+1,T}^{me}}\Big|\,\mathcal{F}_s\vee\mathcal{F}_{s+1}^{S}\right)\Big|\,\mathcal{F}_s\right)$$

$$+E_{\mathbb{Q}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right).$$

Using that in the reduced model $\mathcal{H}_{s+1,T}^{me}\in\mathcal{F}_{s+1}^{S}$, we obtain

$$E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}}\left(e^{\gamma Z-\mathcal{H}_{s+1,T}^{me}}\Big|\,\mathcal{F}_s\vee\mathcal{F}_{s+1}^{S}\right)\Big|\,\mathcal{F}_s\right)$$

$$=\mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(Z)-E_{\mathbb{Q}}\left(\frac{1}{\gamma}\mathcal{H}_{s+1,T}^{me}\mid\mathcal{F}_s\right),$$

and the assertion follows. ■

We are now ready to state the main theorems of this section. The first theorem, a direct consequence of the above result, states that in the reduced model the two pricing algorithms (presented on Theorems 15 and 16) coincide. It also states that the single-step indifference price functional yields a natural stochastic extension of the classical certainty equivalent rule.

The second theorem shows that in the reduced model the indifference price is *not* affected by the risk preference normalization point. The intuition behind this property is the following. In the general model, there are two sources of market incompleteness, one coming from the payoff and the other from the model itself. The latter affects the form of the value function which is also affected by the choice of the normalization point. Once the internal incompleteness is removed, the measurability of key quantities (e.g. the minimal aggregate entropy) reduces and scaling simplifications take place.

Theorem 29 *In the reduced binomial model, the indifference price $\nu_s(C_t)$ satisfies, for $s = 0, 1, ..., t$,*

$$\nu_t(C_t) = C_t,$$

$$\nu_s(C_t) = \mathcal{P}_{\mathbb{Q}}^{(s,s+1)}(\nu_{s+1}(C_t)) = \mathcal{E}_{\mathbb{Q}}^{(s,s+1)}(\nu_{s+1}(C_t))$$

$$= E_{\mathbb{Q}}\left(\mathcal{C}_{\mathbb{Q}}^{(s,s+1)}(\nu_{s+1}(C_t))\Big| \mathcal{F}_s\right).$$

Theorem 30 *In the reduced binomial model, the indifference prices are invariant with respect to the choice of the normalization point. Specifically, consider a claim written at $t_0 = 0$ and maturing at t yielding payoff $C_t \in \mathcal{F}_t$. Let T, $\hat{T}$ be two normalization points with $\hat{T} > T$ and $\nu_s(C_t; T)$, $\nu_s\left(C_t; \hat{T}\right)$, $0 \leq s \leq t \leq T$, be the associated indifference prices. Then,*

$$\nu_s(C_t; T) = \nu_s\left(C_t; \hat{T}\right). \tag{65}$$

Proof Using (64) we have $\mathcal{H}_{s,T'}^{me} \in \mathcal{F}_s^S$, for $T' = T$, $\hat{T}$ and $s \leq t \leq T$. Therefore, the claim $Z_t = \mathcal{H}_{t,\hat{T}}^{me} - \mathcal{H}_{t,T}^{me} \in \mathcal{F}_t^S$ and, in turn, (59) implies, for $s = 0, 1, ..., t$, $Z_s = E_{\mathbb{Q}}(Z_t | \mathcal{F}_s)$. The parity equality (57), then, yields

$$\nu_{t-1}\left(C_t; \hat{T}\right) = \nu_{t-1}(C_t - Z_t; T) + Z_{t-1}$$

$$= \nu_{t-1}(C_t; T) - E_{\mathbb{Q}}(Z_t | \mathcal{F}_{t-1}) + Z_{t-1} = \nu_{t-1}(C_t; T),$$

with $\mathbb{Q}$ as in (63). Similarly, for $s = t - 2$, we deduce, using (57),

$$\nu_{t-2}\left(C_t; \hat{T}\right) = \nu_{t-2}\left(\nu_{t-1}(C_t; T); \hat{T}\right)$$

$$= \nu_{t-2}(\nu_{t-1}(C_t; T) - Z_{t-1}; T) + Z_{t-2}$$

$$= \nu_{t-2}(\nu_{t-1}(C_t; T); T) - E_{\mathbb{Q}}(Z_{t-1} | \mathcal{F}_{t-2}) + Z_{t-2}$$

$$= \nu_{t-2}(\nu_{t-1}(C_t; T); T) = \nu_{t-2}(C_t; T).$$

Proceeding iteratively and using similar to the above arguments, we obtain (65) for $s = 0, 1, ..., t - 2$. ■

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