

Stochastic partial differential equations and portfolio choice*

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Abstract

We introduce a stochastic partial differential equation which describes the evolution of the investment performance process in portfolio choice models. The equation is derived for two formulations of the investment problem, namely, the traditional one (based on maximal expected utility of terminal wealth) and the recently developed forward formulation. The novel element in the forward case is the volatility process which is up to the investor to choose. We provide various examples for both cases and discuss the differences and similarities between the different forms of the equation as well as the associated solutions and optimal processes.

1 Introduction

We introduce a stochastic partial differential equation (SPDE) arising in optimal portfolio selection problems which describes the evolution of the value function process. The SPDE is expected to hold under mild conditions on the asset price dynamics and in general incomplete markets.

The aim herein is not to study questions on the existence, uniqueness and regularity of the solution of the investment performance SPDE. These questions are very challenging due to the possible degeneracy and full nonlinearity of the equation as well as other difficulties stemming from the market incompleteness.

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They require extensive study and effort and are being currently investigated by the authors and others. We stress that similar questions have not yet been established even for the simplest possible extension of the classical Merton model in incomplete markets, namely, a single factor model with Markovian dynamics (see [45]).

Abstracting from technical considerations, we provide various representative examples with explicit solutions to the SPDE for two different formulations of the optimal investment problem. The first problem is the classical one in which one maximizes the expected utility of terminal wealth. This problem has been extensively analyzed either in its primal formulation via the associated Hamilton-Jacobi-Bellman (HJB) equation in Markovian models or in its dual formulation. However, once one departs from the complete market setting very little, if anything, can be said about the properties of the maximal expected utility, especially with regards to its regularity, the optimal policies and related verification results.

The investment performance SPDE offers an alternative way to examine the evolution of this process beyond the class of Markovian models. One might think of it as the non-Markovian analogue of the HJB equation. Besides providing information for the maximal expected investment performance, it also provides the optimal investment strategy in a generalized stochastic feedback form. Analyzing the SPDE could perhaps lead to a better understanding of the nature and properties of the value function as well as the optimal wealth and optimal investment processes.

The second problem for which we provide the associated SPDE arises in an alternative approach for portfolio choice that is based on the so-called forward investment performance criterion. In this approach, developed by the authors during the last years, the investor does not choose her risk preferences at a single point in time but has the flexibility to revise them dynamically. Recall that in the classical problem once the trading horizon is chosen the investor not only can he not revise his preferences but he cannot extend his utility beyond the initially chosen horizon either. For the new problem, the SPDE plays a very important role, for it exposes in a very transparent way how this flexibility is being modeled. Indeed, this is done via the volatility component of the forward performance process. This input is up to the investor to choose. It represents his uncertainty about the upcoming changes - from one trading period to the next - of the shape of his current risk preferences.

As expected, the SPDEs in the traditional and the forward formulations have similar structure (see (12) and (28)). However, there are fundamental differences. In the first case, a terminal condition is imposed (see (13)) which is in most cases deterministic. In other words, the solution is progressively measurable with regards to the market filtration in $[0, T)$ but degenerates to a deterministic function at the end of the horizon. In contrast, in the second formulation an initial condition is imposed and the solution does not degenerate at any future time. Moreover, as we will discuss later on, in the classical utility problem the investor's volatility is uniquely determined while in the forward case it is not.

A common characteristic of the traditional and forward SPDEs is the form of the drift. Their drifts are uniquely determined once the volatility and the market inputs are specified. One could say that there is similarity between the investment performance SPDEs and the ones appearing in term structure models.

The paper is organized as follows. In section 2 we describe the investment model. In section 3 we recall the classical maximal expected utility problem and derive the associated SPDE. In section 4, we provide examples from models with Markovian dynamics driven by stochastic factors. We also examine the power and logarithmic cases. In section 5, we recall the forward portfolio choice problem and, in analogy to the classical case, we derive the associated SPDE. We finish the section by discussing the connection between the forward investment problem and the traditional expected utility maximization one. In section 6 we provide several examples. We start with the zero volatility case. We then examine two families of non-zero volatility models. The first family incorporates non-zero performance volatilities that model different market views and investment in terms of a benchmark (or different numeraire choice) while the second family corresponds to the forward analogue of the stochastic factor Markovian model.

2 The investment model

The market environment consists of one riskless and k risky securities. The risky securities are stocks and their prices are modelled as Ito processes. Namely, for $i = 1, \dots, k$, the price S_t^i , $t \geq 0$, of the i^{th} risky asset satisfies

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^d \sigma_t^{ji} dW_t^j \right), \quad (1)$$

with $S_0^i > 0$, for $i = 1, \dots, k$. The process $W_t = (W_t^1, \dots, W_t^d)$, $t \geq 0$, is a standard d -dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, it is assumed that the underlying filtration, \mathcal{F}_t , coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$.

The coefficients μ_t^i and $\sigma_t^i = (\sigma_t^{1i}, \dots, \sigma_t^{di})$, $i = 1, \dots, k$, $t \geq 0$, are \mathcal{F}_t -progressively measurable processes with values in \mathbb{R} and \mathbb{R}^d , respectively. For brevity, we use σ_t to denote the volatility matrix, i.e. the $d \times k$ random matrix (σ_t^{ji}) , whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. Alternatively, we write (1) as

$$dS_t^i = S_t^i (\mu_t^i dt + \sigma_t^i \cdot dW_t).$$

The riskless asset, the savings account, has the price process B_t , $t \geq 0$, satisfying

$$dB_t = r_t B_t dt \quad (2)$$

with $B_0 = 1$, and for a nonnegative, \mathcal{F}_t -progressively measurable interest rate process r_t . Also, we denote by μ_t the $k \times 1$ vector with the coordinates μ_t^i and by $\mathbf{1}$ the k -dimensional vector with every component equal to one. The market coefficients, μ_t, σ_t and r_t , are taken to be bounded (by a deterministic constant).

We assume that the volatility vectors are such that

$$\mu_t - r_t \mathbf{1} \in \text{Lin}(\sigma_t^T),$$

where $\text{Lin}(\sigma_t^T)$ denotes the linear space generated by the columns of σ_t^T . This implies that $\sigma_t^T (\sigma_t^T)^+ (\mu_t - r_t \mathbf{1}) = \mu_t - r_t \mathbf{1}$ and, therefore, the vector

$$\lambda_t = (\sigma_t^T)^+ (\mu_t - r_t \mathbf{1}) \quad (3)$$

is a solution to the equation $\sigma_t^T x = \mu_t - r_t \mathbf{1}$. The matrix $(\sigma_t^T)^+$ is the Moore-Penrose pseudo-inverse of the matrix σ_t^T . It easily follows that

$$\sigma_t \sigma_t^+ \lambda_t = \lambda_t \quad (4)$$

and, hence, $\lambda_t \in \text{Lin}(\sigma_t)$. We assume throughout that the process λ_t is bounded by a deterministic constant $c > 0$, i.e., for all $t \geq 0$, $|\lambda_t| \leq c$.

Starting at $t = 0$ with an initial endowment $x \in \mathbb{R}^+$, the investor invests at any time $t > 0$ in the risky and riskless assets. The present value of the amounts invested are denoted, respectively, by π_t^0 and π_t^i , $i = 1, \dots, k$.

The present value of her aggregate investment is, then, given by $X_t^\pi = \sum_{i=0}^k \pi_t^i$, $t > 0$. We will refer to X^π as the discounted wealth generated by the (discounted) strategy $(\pi_t^0, \pi_t^1, \dots, \pi_t^k)$. The investment strategies will play the role of control processes. Their admissibility set is defined as

$$\mathcal{A} = \{ \pi : \pi_t \text{ is self-financing and } \mathcal{F}_t \text{ - progressively measurable} \quad (5)$$

$$\text{with } \mathbb{E} \left(\int_0^t |\sigma_s \pi_s|^2 ds < \infty \right) \text{ and } X_t^\pi \geq 0, t \geq 0 \}.$$

Using (1) and (2) we deduce that the discounted wealth satisfies, for $t > 0$,

$$X_t^\pi = x + \sum_{i=1}^k \int_0^t \pi_s^i (\mu_s^i - r_s) ds + \sum_{i=1}^k \int_0^t \pi_s^i \sigma_s^i \cdot dW_s.$$

Writing the above in vector notation and using (3) and (4) yields

$$dX_t^\pi = \pi_t \cdot (\mu_t - r_t \mathbf{1}) dt + \sigma_t \pi_t \cdot dW_t = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t), \quad (6)$$

where the (column) vector, $\pi_t = (\pi_t^i; i = 1, \dots, k)$.

3 The backward formulation of the portfolio choice problem and the associated SPDE

The traditional criterion for optimal portfolio choice has been based on maximal expected utility (see the seminal paper [32]). The key ingredients are the choices of the trading horizon $[0, T]$ and the investor's utility, $u_T : \mathbb{R}^+ \rightarrow \mathbb{R}$ at terminal time T . The utility function reflects the risk attitude of the investor at time T and is an increasing and concave function of his wealth.

The objective is to maximize the expected utility of terminal wealth over admissible strategies. We will denote the set of such strategies by \mathcal{A}_T , a straightforward restriction of \mathcal{A} on $[0, T]$. The maximal expected utility is defined as

$$V(x, t; T) = \sup_{\mathcal{A}_T} E_{\mathbb{P}}(u_T(X_T^\pi) | \mathcal{F}_t, X_t = x), \quad (7)$$

for $(x, t) \in \mathbb{R}^+ \times [0, T]$. The function u_T satisfies the standard Inada condition (see, for example, [27] and [28]). We introduce the T -notation throughout this section to highlight the dependence of all quantities on the investment horizon at which the investor's risk preferences are chosen.

As solution of a stochastic optimization problem, $V(x, t; T)$ is expected to satisfy the Dynamic Programming Principle (DPP), namely,

$$V(x, t; T) = \sup_{\mathcal{A}_T} E_{\mathbb{P}}(V(X_s^\pi, s; T) | \mathcal{F}_t, X_t = x), \quad (8)$$

for $t \leq s \leq T$. This is a fundamental result in optimal control and has been proved for a wide class of optimization problems. For a detailed discussion on the validity (and strongest forms) of the DPP in problems with controlled diffusions, we refer the reader to [15] and [43] (see, also, [7], [11] and [30]). Key issues are the measurability and continuity of the value function process as well as the compactness of the set of admissible controls. It is worth mentioning that a proof specific to the problem at hand has not been produced to date¹.

Besides its technical challenges, the DPP exhibits two important properties of the solution. Specifically, $V(X_t^\pi, t; T)$, is a supermartingale for an arbitrary investment strategy and becomes a martingale at an optimum (provided certain integrability conditions hold). Observe also that the DPP yields a backward in time algorithm for the computation of the maximal utility, starting at expiration with u_T and using the martingality property to compute the solution for earlier times. For this, we refer to this formulation of the optimal portfolio choice problem as *backward*.

Regularity results for the process $V(x, t; T)$ have not been produced to date except for special cases. To the best of our knowledge, the most general result for arbitrary utilities can be found in [29].

We continue with the derivation of the SPDE for the value function process. For the moment, the discussion is informal, for general regularity results are

¹Recently, a weak version of the DPP was proposed in [6] where conditions related to measurable selection and boundness of controls are relaxed.

lacking. To this end, let us assume that $V(x, t; T)$ admits the Itô decomposition

$$dV(x, t; T) = b(x, t; T) dt + a(x, t; T) \cdot dW_t \quad (9)$$

for some coefficients $b(x, t; T)$ and $a(x, t; T)$ which are \mathcal{F}_t -progressively measurable processes.

Let us also assume that the mapping $x \rightarrow V(x, t; T)$ is strictly concave and increasing and that $V(x, t; T)$ is smooth enough so that the Ito-Ventzell formula can be applied to $V(X_t^\pi, t; T)$ for any strategy $\pi \in \mathcal{A}_T$. We then obtain

$$\begin{aligned} dV(X_t^\pi, t; T) &= b(X_t^\pi, t; T) dt + a(X_t^\pi, t; T) \cdot dW_t \\ &+ V_x(X_t^\pi, t; T) dX_t^\pi + \frac{1}{2} V_{xx}(X_t^\pi, t; T) d\langle X^\pi \rangle_t + a_x(X_t^\pi, t; T) \cdot d\langle W, X^\pi \rangle_t \\ &= (b(X_t^\pi, t; T) + \sigma_t \pi_t \cdot (V_x(X_t^\pi, t; T) \lambda_t + a_x(X_t^\pi, t; T))) dt \\ &+ \frac{1}{2} V_{xx}(X_t^\pi, t; T) |\sigma_t \pi_t|^2 dt + (a(X_t^\pi, t; T) + V_x(X_t^\pi, t; T) \sigma_t \pi_t) \cdot dW_t \\ &= (b(X_t^\pi, t; T) + \sigma_t \pi_t \cdot \sigma_t \sigma_t^+ (V_x(X_t^\pi, t; T) \lambda_t + a_x(X_t^\pi, t; T)) \\ &+ \frac{1}{2} V_{xx}(X_t^\pi, t; T) |\sigma_t \pi_t|^2) dt + (a(X_t^\pi, t; T) + V_x(X_t^\pi, t; T) \sigma_t \pi_t) \cdot dW_t. \end{aligned}$$

From the DPP we know that the process $V(X_t^\pi, t; T)$ is a supermartingale for arbitrary admissible policies and becomes a martingale at an optimum.

Let us now choose as control policy the process

$$\pi_t^* = \pi_t^*(X_t^*, t; T)$$

where the feedback process in the right hand side (denoted by a slight abuse of notation by $\pi_t^*(x, t; T)$) is given by

$$\pi_t^*(x, t; T) = -\sigma_t^+ \frac{V_x(x, t; T) \lambda_t + \sigma_t \sigma_t^+ a_x(x, t; T)}{V_{xx}(x, t; T)}, \quad (10)$$

where X_t^* is the wealth process generated by (6) with π_t^* being used. It is assumed that $\pi_t^* \in \mathcal{A}_T$. It is easy to check that π_t^* is the point at which the quadratic expression appearing in the drift above achieves its maximum and, moreover, that the maximum value at this point is given by

$$\begin{aligned} &\frac{1}{2} V_{xx}(X_t^*, t; T) |\sigma_t \pi_t^*|^2 + \sigma_t \pi_t^* \cdot \sigma_t \sigma_t^+ (V_x(X_t^*, t; T) \lambda_t + a_x(X_t^*, t; T)) \\ &= -\frac{1}{2} \frac{|V_x(X_t^*, t; T) \lambda_t + \sigma_t \sigma_t^+ a_x(X_t^*, t; T)|^2}{V_{xx}(X_t^*, t; T)}. \end{aligned}$$

We, then, deduce that the drift coefficient $b(x, t; T)$ must satisfy

$$b(x, t; T) = \frac{1}{2} \frac{|V_x(x, t; T) \lambda_t + \sigma_t \sigma_t^+ a_x(x, t; T)|^2}{V_{xx}(x, t; T)}. \quad (11)$$

Combining the above leads to the SPDE

$$dV(x, t; T) = \frac{1}{2} \frac{|V_x(x, t; T) \lambda_t + \sigma_t \sigma_t^+ a_x(x, t; T)|^2}{V_{xx}(x, t; T)} dt + a(x, t; T) \cdot dW_t \quad (12)$$

with

$$V(x, T; T) = u_T(x). \quad (13)$$

To the best of our knowledge, the above SPDE has not been derived to date. For deterministic terminal utilities, the volatility $a(x, t; T)$ is present because of the stochasticity of the investment opportunity set, as the examples in the next section show.

The optimal feedback portfolio process π_t^* consists of two terms, namely,

$$\pi_t^{*,m} = -\frac{V_x(x, t; T)}{V_{xx}(x, t; T)} \sigma_t^+ \lambda_t \quad \text{and} \quad \pi_t^{*,h} = -\sigma_t^+ \frac{a_x(x, t; T)}{V_{xx}(x, t; T)}.$$

The first component, $\pi_t^{*,m}$, known as the *myopic* investment strategy, resembles the investment policy followed by an investor in markets in which the investment opportunity set remains constant through time. The second term, $\pi_t^{*,h}$, is called the *excess hedging demand* and represents the additional (positive or negative) investment generated by the volatility $a(x, t; T)$ of the performance process V .

4 Examples: Markovian stochastic factor models

Stochastic factors have been used in portfolio choice to model asset predictability and stochastic volatility. The predictability of stock returns was first discussed in [13], [14] and [16] (see also [1], [8], [42] and others). The role of stochastic volatility in investment decisions was first studied in [3], [16], [17], [20] (see also [9], [24], [31] and others). There is a vast literature on such models and we refer the reader to the review paper [45] for detailed bibliography, exposition of existing results and open problems.

4.1 Single stochastic factor models

There is only one risky asset whose price S_t , $t \geq 0$, is modelled as a diffusion process solving

$$dS_t = \mu(Y_t) S_t dt + \sigma(Y_t) S_t dW_t^1, \quad (14)$$

with $S_0 > 0$. The stochastic factor Y_t , $t \geq 0$, satisfies

$$dY_t = b(Y_t) dt + d(Y_t) \left(\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2 \right), \quad (15)$$

with $Y_0 = y$, $y \in \mathbb{R}$. It is assumed that $\rho \in (-1, 1)$.

The market coefficients $f = \mu, \sigma, b$ and d satisfy the standard global Lipschitz and linear growth conditions $|f(y) - f(\bar{y})| \leq K|y - \bar{y}|$ and $f^2(y) \leq$

$K(1+y^2)$, for $y, \bar{y} \in \mathbb{R}$. Moreover, it is assumed that the non-degeneracy condition $\sigma(y) \geq l > 0$, $y \in \mathbb{R}$, holds.

It is also assumed that the riskless asset (cf. (2)) offers constant interest rate $r > 0$.

The coefficients appearing in (12) take the form

$$\sigma_t = (\sigma(Y_t), 0)^T, \quad \sigma_t^+ = \left(\frac{1}{\sigma(Y_t)}, 0 \right) \quad \text{and} \quad \lambda_t = \left(\frac{\mu(Y_t) - r}{\sigma(Y_t)}, 0 \right)^T. \quad (16)$$

We easily see that condition (4) is trivially satisfied.

The value function is defined as

$$v(x, y, t; T) = \sup_{\mathcal{A}_T} E_{\mathbb{P}}(u_T(X_T) | X_t = x, Y_t = y), \quad (17)$$

for $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$ and \mathcal{A}_T being the set of admissible strategies.

Regularity results for the value function (17) for general utility functions have not been obtained to date except for the special cases of homothetic preferences (see, for example, [26], [33], [40] and [44]). We, thus, proceed with an informal discussion for the associated HJB equation, the form of the process V and the optimal policies. To this end, the HJB turns out to be

$$v_t + \max_{\pi} \left(\frac{1}{2} \sigma^2(y) \pi^2 v_{xx} + \pi (\mu(y) v_x + \rho \sigma(y) d(y) v_{xy}) \right) + \frac{1}{2} d^2(y) v_{yy} + b(y) v_y = 0, \quad (18)$$

with $v(x, y, T; T) = u_T(x)$, $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$.

Its solution yields the process V , namely, for $0 \leq t \leq T$,

$$V(x, t; T) = v(x, Y_t, t; T). \quad (19)$$

We next show that $V(x, t; T)$ above solves the SPDE (12) with volatility vector

$$a(x, t; T) = (a_1(x, t; T), a_2(x, t; T))$$

with

$$a_1(x, t; T) = \rho d(Y_t) v_y \quad \text{and} \quad a_2(x, t; T) = \sqrt{1 - \rho^2} d(Y_t) v_y, \quad (20)$$

where the arguments of v_y have been suppressed for convenience.

Indeed, using (19), (16), (20) and Itô's formula yields

$$\begin{aligned} & dV(x, t; T) \\ &= \left(v_t(x, Y_t, t; T) + \frac{1}{2} d(Y_t)^2 v_{yy}(x, Y_t, t; T) + b(Y_t) v_y(x, Y_t, t; T) \right) dt \\ & \quad + \left(\rho d(Y_t) v_y(x, Y_t, t; T), \sqrt{1 - \rho^2} d(Y_t) v_y(x, Y_t, t; T) \right) \cdot dW_t \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \max_{\pi} \left(\frac{1}{2} \sigma^2 (Y_t) \pi^2 v_{xx} (x, Y_t, t; T) \right. \\
&\quad \left. + \pi (\mu (Y_t) v_x (x, Y_t, t; T) + \rho \sigma (Y_t) d (Y_t) v_{xy} (x, Y_t, t; T)) \right) dt \\
&\quad + \left(\rho d (Y_t) v_y (x, Y_t, t; T), \sqrt{1 - \rho^2} d (Y_t) v_y (x, Y_t, t; T) \right) \cdot dW_t \\
&= \frac{1}{2} \frac{(\mu (Y_t) v_x (x, Y_t, t; T) + \rho \sigma (Y_t) d (Y_t) v_{xy} (x, Y_t, t; T))^2}{v_{xx} (x, Y_t, t; T)} dt \\
&\quad + \left(\rho d (Y_t) v_y (x, Y_t, t; T), \sqrt{1 - \rho^2} d (Y_t) v_y (x, Y_t, t; T) \right) \cdot dW_t \\
&= \frac{1}{2} \frac{|V_x (x, t; T) \lambda_t + \sigma_t \sigma_t^+ a_x (x, t)|^2}{V_{xx} (x, t; T)} dt + a (x, t; T) \cdot dW_t.
\end{aligned}$$

The optimal feedback portfolio process for $t \leq s \leq T$ is obtained from (10). We easily deduce that it coincides with the feedback policy (evaluated at Y_s) derived from the first order conditions in the HJB equation. Indeed, for $t \leq s \leq T$,

$$\begin{aligned}
\pi_s^* (x, s; T) &= -\sigma_s^+ \frac{V_x (x, s; T) \lambda_s + \sigma_s \sigma_s^+ a_x (x, s; T)}{V_{xx} (x, s; T)} \\
&= -\frac{\lambda (Y_s) v_x (x, Y_s, s; T)}{\sigma (Y_s) v_{xx} (x, Y_s, s; T)} - \rho \frac{d (Y_s) v_{xy} (x, Y_s, s; T)}{\sigma (Y_s) v_{xx} (x, Y_s, s; T)} \\
&= \pi^* (x, Y_s, s; T)
\end{aligned} \tag{21}$$

where $\pi^* : \mathbb{R}^+ \times \mathbb{R} \times [0, T]$ is given by

$$\pi^* (x, y, s; T) = \arg \max_{\pi} \left(\frac{1}{2} \sigma^2 (y) \pi^2 v_{xx} + \pi (\mu (y) v_x + \rho \sigma (y) d (y) v_{xy}) \right).$$

Next, we present two cases for which the above results are rigorous. Specifically, we provide examples for the most frequently used utilities, the power and the logarithmic ones. These utilities have convenient homogeneity properties which, in combination with the linearity of the wealth dynamics in the control policies, enable us to reduce the HJB equation to a quasilinear one. Under a "distortion" transformation (see, for example, [44]) the latter can be linearized and solutions in closed form can be produced using the Feynman-Kac formula. The smoothness of the value function and, in turn, the verification of the optimal feedback policies follows (see, among others, [24], [25], [26], [31] and [44]).

4.1.1 The CRRA case: $u_T(x) = \frac{1}{\gamma}x^\gamma$, $0 < \gamma < 1$, $\gamma \neq 0$.

The value function v (cf. (17)) is multiplicatively separable and given, for $(x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times [0, T]$, by

$$v(x, y, t; T) = \frac{1}{\gamma}x^\gamma f(y, t; T)^\delta, \quad \delta = \frac{1 - \gamma}{1 - \gamma + \rho^2\gamma}, \quad (22)$$

where $f : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^+$ solves the linear parabolic equation

$$\begin{aligned} f_t + \frac{1}{2}d^2(y) f_{yy} + \left(b(y) + \rho \frac{\gamma}{1 - \gamma} \lambda(y) d(y) \right) f_y \\ + \frac{\gamma}{2(1 - \gamma)} \frac{\lambda^2(y)}{\delta} f = 0, \end{aligned} \quad (23)$$

with $f(x, y, T; T) = 1$. The value function process $V(x, t; T)$ is given by

$$V(x, t; T) = \frac{1}{\gamma}x^\gamma f(Y_t, t; T)^\delta,$$

with f solving (23) and δ as in (22). Direct calculations show that it satisfies the SPDE (12) with volatility components (cf. (20))

$$\begin{cases} a_1(x, t; T) = \rho \frac{\delta}{\gamma} x^\gamma d(Y_t) f_y(Y_t, t; T) f(Y_t, t; T)^{\delta-1} \\ a_2(x, t; T) = \sqrt{1 - \rho^2} \frac{\delta}{\gamma} x^\gamma d(Y_t) f_y(Y_t, t; T) f(Y_t, t; T)^{\delta-1}. \end{cases} \quad (24)$$

The optimal feedback portfolio process is given for $t \leq s \leq T$ by (10), namely,

$$\begin{aligned} \pi_s^*(x, s; T) &= -\sigma_s^+ \frac{V_x(x, s; T) \lambda_s + \sigma_s \sigma_s^+ a_x(x, s; T)}{V_{xx}(x, s; T)} \\ &= \left(\frac{\lambda(Y_s)}{\sigma(Y_s)(1 - \gamma)} + \rho \frac{d(Y_s)}{\sigma(Y_s)(1 - \gamma + \rho^2\gamma)} \frac{f_y(Y_s, s; T)}{f(Y_s, s; T)} \right) x, \end{aligned}$$

which is in accordance with (21) and (22).

4.1.2 The logarithmic case: $u_T(x) = \ln x$, $x > 0$.

The value function is additively separable, namely,

$$v(x, y, t; T) = \ln x + h(y, t; T),$$

with $h : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^+$ solving

$$h_t + \frac{1}{2}d^2(y) h_{yy} + b(y) h_y + \frac{1}{2}\lambda^2(y) h = 0 \quad (25)$$

and $h(y, T; T) = 1$. The value function process $V(x, t; T)$ is, in turn, given by

$$V(x, t; T) = \ln x + h(Y_t, t; T).$$

We easily deduce that it satisfies the SPDE (12) with volatility vector

$$a_1(x, t; T) = \rho h_y(Y_t, t; T) \quad \text{and} \quad a_2(x, t; T) = \sqrt{1 - \rho^2} h_y(Y_t, t; T).$$

Observe that because the volatility process does not depend on wealth, the excess risky demand is zero, as (10) indicates. Indeed, the optimal portfolio process is always myopic, given, for $t \leq s \leq T$, by $\pi_s^* = \frac{\lambda(Y_s)}{\sigma(Y_s)} X_s^*$ with

$$X_s^* = x \exp \left(\int_t^s \frac{1}{2} \lambda^2(Y_u) du + \int_t^s \lambda(Y_u) dW_u^1 \right).$$

The logarithmic utility plays a special role in portfolio choice. The optimal portfolio is known as the "growth optimal portfolio" and has been extensively studied in general market settings (see, for example, [4] and [23]). The associated optimal wealth is the so-called "numeraire portfolio". It has also been extensively studied, for it is the numeraire with regards to which all wealth processes are supermartingales under the historical measure (see, among others, [18] and [19]).

4.2 Multi-stochastic factor models

Multi-stochastic factor models for homothetic preferences have been analyzed by various authors. The theory of BSDE has been successfully used to characterize and represent the solutions of the reduced HJB equation (see [12]). The regularity of its solutions has been studied using PDE arguments in [33] and [40], for power and exponential utilities, respectively. Finally, explicit solutions for a three factor model can be found in [31]. Working as in the previous examples one obtains that the value function process satisfies the SPDE (12) and that the optimal policies can be expressed in the stochastic feedback form (10). These calculations are routine but tedious and are omitted for the sake of the presentation.

5 The forward formulation of the portfolio choice problem and the associated SPDE

In the classical expected utility models of terminal wealth, discussed in the previous section, one chooses the investment horizon $[0, T]$ and then assigns the utility function $u_T(x)$ at the end of it (cf. (7)). Once these choices are made, the investor's risk preferences cannot be revised. In addition, no investment decisions can be assessed for trading times beyond T .

Recently, the authors proposed an alternative approach to optimal portfolio choice which is based on the so-called forward investment performance criterion (see, for example, [38]). In this approach the investor does not choose her risk preferences at a single point in time but has the flexibility to revise them dynamically for all trading times. Investment strategies are chosen from the

set \mathcal{A} defined in (5). A strategy is deemed optimal if it generates a wealth process whose average performance is maintained over time. In other words, the average performance of this strategy at any future date, conditional on today's information, preserves the performance of this strategy up until today. Any strategy that fails to maintain the average performance over time is, then, sub-optimal.

Next, we recall the definition of the forward investment performance. This criterion was first introduced in [34] (see also [35]) in the context of an incomplete binomial model and subsequently studied in [36], [37] and [38]. A rich class of such processes which are monotone in time was recently completely analyzed in [39].

We note that the definition below is slightly different than the original one in that the initial condition is not explicitly included. As the example in section 6.1 shows, not all strictly increasing and concave solutions can serve as initial conditions, even for the special class of time monotone investment performance processes (see (40) in Remark 2). Characterizing the set of appropriate initial conditions is a challenging question and is currently being investigated by the authors. Another difference is that, herein, we only allow for policies that keep the wealth nonnegative. Going beyond such strategies raises very interesting questions on possible arbitrage opportunities which are left for future study.

Definition 1 *An \mathcal{F}_t -progressively measurable process $U(x, t)$ is a forward performance if for $t \geq 0$ and $x \in \mathbb{R}^+$:*

- i) the mapping $x \rightarrow U(x, t)$ is strictly concave and increasing,*
- ii) for each $\pi \in \mathcal{A}$, $E(U(X_t^\pi, t))^+ < \infty$, and*

$$E(U_s(X_s^\pi) | \mathcal{F}_t) \leq U_t(X_t^\pi), \quad s \geq t, \quad (26)$$

- iii) there exists $\pi^* \in \mathcal{A}$, for which*

$$E(U(X_s^{\pi^*}, s) | \mathcal{F}_t) = U_t(X_t^{\pi^*}, t), \quad s \geq t. \quad (27)$$

It might seem that all this definition produces is a criterion that is dynamically consistent across time. Indeed, internal consistency is an ubiquitous requirement and needs to be ensured in any proposed criterion. It is satisfied, for example, by the traditional value function process. However, the new criterion allows for much more flexibility as it is manifested by the volatility process $a(x, t)$ introduced below. The volatility process is the *novel* element in the new approach of optimal portfolio choice.

We continue with the derivation of the SPDE associated with the forward investment performance process. As in section 3 we proceed with informal arguments and present rigorous results in the upcoming examples. To this end, we consider a process, say $U(x, t)$, that is \mathcal{F}_t -progressively measurable and satisfies condition (i) of the above definition. We also assume that the mapping

$x \rightarrow U(x, t)$ is smooth enough so that the Ito-Ventzell formula can be applied to $U(X_t^\pi, t)$, for any strategy $\pi \in \mathcal{A}$ and that $E(U(X_t^\pi, t))^+ < +\infty$, $t \geq 0$.

Let us now assume that $U(x, t)$ satisfies the SPDE

$$dU(x, t) = \frac{1}{2} \frac{|U_x(x, t) \lambda_t + \sigma_t \sigma_t^+ a_x(x, t)|^2}{U_{xx}(x, t)} dt + a(x, t) \cdot dW_t, \quad (28)$$

where the volatility $a(x, t)$ is an \mathcal{F}_t -progressively measurable, d -dimensional and continuously differentiable in the spatial argument process.

We first show that under appropriate integrability conditions $U(X_t^\pi, t)$ is a supermartingale for every admissible portfolio strategy. Indeed, denote the above drift coefficient by

$$b(x, t) = \frac{1}{2} \frac{|U_x(x, t) \lambda_t + \sigma_t \sigma_t^+ a_x(x, t)|^2}{U_{xx}(x, t)}$$

and rewrite (28) as

$$dU(x, t) = b(x, t) dt + a(x, t) \cdot dW_t.$$

Consider the wealth process X^π (cf. (6)) generated using an admissible strategy π . Applying the Ito-Ventzell formula to $U(X_t^\pi, t)$ yields

$$\begin{aligned} dU(X_t^\pi, t) &= b(X_t^\pi, t) dt + a(X_t^\pi, t) \cdot dW_t \\ &+ U_x(X_t^\pi, t) dX_t^\pi + \frac{1}{2} U_{xx}(X_t^\pi, t) d\langle X^\pi \rangle_t + a_x(X_t^\pi, t) \cdot d\langle W, X^\pi \rangle_t \\ &= \left(b(X_t^\pi, t) + \sigma_t \pi_t \cdot (U_x(X_t^\pi, t) \lambda_t + a_x(X_t^\pi, t)) + \frac{1}{2} U_{xx}(X_t^\pi, t) |\sigma_t \pi_t|^2 \right) dt \\ &\quad + (a(X_t^\pi, t) + U_x(X_t^\pi, t) \sigma_t \pi_t) \cdot dW_t \\ &= \left(b(X_t^\pi, t) + \sigma_t \pi_t \cdot \sigma_t \sigma_t^+ (U_x(X_t^\pi, t) \lambda_t + a_x(X_t^\pi, t)) + \frac{1}{2} U_{xx}(X_t^\pi, t) |\sigma_t \pi_t|^2 \right) dt \\ &\quad + (a(X_t^\pi, t) + U_x(X_t^\pi, t) \sigma_t \pi_t) \cdot dW_t \\ &= \frac{1}{2} U_{xx}(X_t^\pi, t) \left| \sigma_t \pi_t + \sigma_t \sigma_t^+ \frac{U_x(X_t^\pi, t) \lambda_t + a_x(X_t^\pi, t)}{U_{xx}(X_t^\pi, t)} \right|^2 dt \\ &\quad + (a(X_t^\pi, t) + U_x(X_t^\pi, t) \sigma_t \pi_t) \cdot dW_t \end{aligned} \quad (29)$$

and we conclude using the concavity assumption on $U(x, t)$.

We next assume that the stochastic differential equation

$$dX_t^* = - \frac{U_x(X_t^*, t) \lambda_t + \sigma_t \sigma_t^+ a_x(X_t^*, t)}{U_{xx}(X_t^*, t)} \cdot (\lambda_t dt + dW_t) \quad (30)$$

has a nonnegative solution X_t^* , $t \geq 0$, with $X_0 = x$, $x \in \mathbb{R}^+$ and that the strategy π_t^* , $t \geq 0$, defined by

$$\pi_t^* = -\sigma_t^+ \frac{U_x(X_t^*, t) \lambda_t + a_x(X_t^*, t)}{U_{xx}(X_t^*, t)} \quad (31)$$

is admissible. Notice that X_t^* corresponds to the wealth generated by this investment strategy.

From (29) we then see that $U(X_t^*, t)$ is a martingale (under appropriate integrability conditions).

Using the supermartingality property of $U(X_t^\pi, t)$ and the martingality property of $U(X_t^*, t)$ we easily deduce that if $U(x, t)$ solves (28) then it is a forward investment performance process. Moreover, the processes X_t^* and π_t^* , given in (30) and (31), are optimal.

The analysis of the forward performance SPDE (28) is a formidable task. The reasons are threefold. Firstly, it is degenerate and fully nonlinear. Moreover, it is formulated forward in time, which might lead to "ill-posed" behavior. Secondly, one needs to specify the appropriate class of admissible volatility processes, namely, volatility inputs that yield solutions that satisfy the requirements of Definition 1. This question is challenging both from the modelling as well as the technical points of view. Thirdly, as it was mentioned earlier, one also needs to specify the appropriate class of initial conditions.

Addressing these issues is an ongoing research project of the authors and not the scope of this paper. Herein, we only construct explicit solutions for different choices of the volatility process $a(x, t)$. These choices provide a rich class of forward performance processes which exhibit several interesting modelling features.

The initial condition represents the investor's current performance criterion. The volatility process $a(x, t)$ represents the uncertainty about the future shape of this criterion. From the modelling perspective, one can see an analogy to term structure, where the initial condition is the current forward curve as traded in the market, while the volatility captures the way the curve moves from one day to the next. However the analogy stops here. One has to develop different methods to recover the initial condition and to specify the volatility $a(x, t)$ for the investment problem governed by (28).

We stress the fundamental difference between the volatility processes of the investment performance criteria in the backward and the forward formulation. In the backward case, the volatility is uniquely determined through the backward construction of the maximal expected utility. The investor does not have the flexibility to choose this process (see for instance example 4.1.1 and the volatility components (24)). In contrast, in the forward case, the volatility process is chosen by the investor; as examples in section 6.2 show.

5.1 Stochastic optimization and forward investment performance process

The intuition behind Definition 1 comes from the analogous martingale and supermartingale properties that the traditional maximal expected utility has, as seen from (8).

However, there are two important observations to make. Firstly, the analogous equivalence between stochastic optimization and the martingality and supermartingality of the solution in the forward formulation of the problem has

not yet been established. Specifically, one could define the forward performance process via the (forward) stochastic optimization problem

$$U(x, t) = \sup_{\mathcal{A}} E(U(X_s^\pi, s) | \mathcal{F}_t, X_t^\pi = x),$$

for all $0 \leq t \leq s$ and with the appropriate initial condition. Characterizing its solutions poses a number of challenging questions, some of them being currently investigated by the authors². From a different perspective, one could seek an *axiomatic* construction of a forward performance process. Results in this direction, as well as on the dual formulation of the problem, can be found in [46] for the exponential case.

The second observation is on the relation between the forward performance process and the classical maximal expected utility. One would expect that in a finite trading horizon, say $[0, T]$, the following holds. Define as utility at terminal time the random variable $u_T(x) \in \mathcal{F}_T$ given by

$$u_T(x) = U(x, T),$$

with $U(x, T)$ being the value of the forward performance process at T . Solve, for $0 \leq t \leq T$, the stochastic optimization problem of state-dependent utility

$$V(x, t; T) = \sup_{\mathcal{A}_T} E(u_T(X_T) | \mathcal{F}_t, X_t = x) \quad (32)$$

(see, among others, [10], [21], [22] and [41]). Then, for $0 \leq t \leq T$, the classical value function process and the forward investment performance process coincide,

$$U(x, t) = V(x, t; T).$$

6 Examples

In this section we provide examples of forward investment performance processes which satisfy the SPDE (28). We first look at the case of zero volatility. We then examine two families with non-zero volatility. The examples in the first family (examples 6.2.1, 6.2.2 and 6.2.3) build on the zero volatility case. The second family yields the forward analogue of the stochastic factor model presented for the backward case in example 4.1.

We note that the fact that a process solves the SPDE (28) does not automatically guarantee that it satisfies Definition 1, for there are additional conditions to be verified. One can show that the solutions presented in examples 6.2.1, 6.2.2 and 6.2.3 indeed satisfy these conditions; we refer the reader to [39] for all technical details.

²While preparing this revised version, the authors came across the revised version of [2] where similar questions are studied for the nonnegative wealth case.

6.1 The case of zero volatility: $a(x, t) \equiv 0$

When $a(x, t) \equiv 0$ the SPDE (28) reduces to

$$dU(x, t) = \frac{1}{2} |\lambda_t|^2 \frac{U_x(x, t)^2}{U_{xx}(x, t)} dt.$$

In [38] it was shown that its solution is given by the time-monotone process

$$U(x, t) = u(x, A_t), \quad (33)$$

where $u : \mathbb{R}^+ \times [0, +\infty) \rightarrow \mathbb{R}$ is increasing and strictly concave in x , and satisfies the fully nonlinear equation

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}. \quad (34)$$

The process A_t is given in (39) below. In a more recent paper, see [39], it was shown that there is a one-to-one correspondence between increasing and strictly concave solution to (34) and strictly increasing positive solutions $h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}^+$ to the heat equation

$$h_t + \frac{1}{2} h_{xx} = 0. \quad (35)$$

Specifically, we have the representation

$$h(x, t) = \int_{0^+}^{+\infty} \frac{e^{yx - \frac{1}{2}y^2t}}{y} \nu(dy) \quad (36)$$

of strictly increasing solutions of (35) and

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x, s) + \frac{s}{2}} h_x \left(h^{(-1)}(x, s), s \right) ds + \int_0^x e^{-h^{(-1)}(z, 0)} dz \quad (37)$$

for strictly concave and increasing solutions of (34). The measure ν appearing above is a positive Borel measure that satisfies appropriate integrability conditions. We refer the reader to [39] for a detailed study of this measure and the interplay between its support and various properties of the functions h and u .

The optimal wealth and portfolio processes are given in closed form, namely,

$$X_t^* = h \left(h^{(-1)}(x, 0) + A_t + M_t, A_t \right) \quad \text{and} \quad \pi_t^* = h_x \left(h^{(-1)}(X_t^*, A_t), A_t \right) \sigma_t^+ \lambda_t, \quad (38)$$

where the market input processes A_t and M_t , $t \geq 0$, are defined as

$$A_t = \int_0^t |\lambda_s|^2 ds \quad \text{and} \quad M_t = \int_0^t \lambda_s \cdot dW_s. \quad (39)$$

Remark 2 *Formulae (37) and (36) indicate that not all concave and increasing functions can serve as initial conditions. Indeed, from (33), (37), (36) and (39), we see that the initial condition $U(x, 0)$ must be represented as*

$$U(x, 0) = \int_0^x e^{-h^{(-1)}(z, 0)} dz \quad \text{with} \quad h(x, t) = \int_{0^+}^{+\infty} \frac{e^{yx}}{y} \nu(dy). \quad (40)$$

Remark 3 One could choose the volatility to be $a(x, t) \equiv k_t$, with the process k_t being \mathcal{F}_t -progressively measurable and independent of x , U and its derivatives). This case essentially reduces to the one with zero volatility. Indeed, we easily conclude that the process

$$U(x, t) = u(x, A_t) + \int_0^t k_s \cdot dW_s, \quad (41)$$

with u and A_t , $t \geq 0$, as in (34) and (39) solves (28). The optimal investment and wealth processes remain the same as in (38) above.

6.1.1 Single stochastic factor models

This is the same model as in section 4.1. Using (33) we see that the forward performance process is given by

$$U(x, t) = u\left(x, \int_0^t \lambda^2(Y_s) ds\right) \quad (42)$$

with u solving (34). Using (38) and (37) one deduces that the optimal portfolio process is given by

$$\pi_t^* = -\frac{u_x(X_t^*, A_t)}{u_{xx}(X_t^*, A_t)} \sigma_t^+ \lambda_t. \quad (43)$$

The calculations are tedious and can be found in section 4 of [39].

Notice that the optimal portfolio (43) is purely myopic even though the investment opportunity set is stochastic. This is because the investor's performance process was chosen to be zero. This is in contrast with the one in (21). We also observe that the investment performance process U in (42) is of bounded variation while the one in (19) is not.

6.2 Cases of non-zero volatility

We start with three cases of non-zero volatility. Two auxiliary process are involved, φ_t and δ_t , $t \geq 0$. They are both independent of wealth and are taken to be \mathcal{F}_t -progressively measurable and bounded by a deterministic constant. In addition, it is assumed that δ_t satisfies, similarly to the process λ_t , the condition $\sigma_t \sigma_t^+ \delta_t = \delta_t$ (cf. (4)) and, thus, $\delta_t \in \text{Lin}(\sigma_t)$. Because the third case is the combination of the first two, we provide the complete calculations therein.

We conclude with a fourth case which is the forward analogue of example 4.1.

6.2.1 The market view case: $a(x, t) = U(x, t) \varphi_t$

The forward performance SPDE (28) simplifies to

$$dU(x, t) = \frac{1}{2} \frac{|U_x(x, t) (\lambda_t + \sigma_t \sigma_t^+ \varphi_t)|^2}{U_{xx}(x, t)} dt + U(x, t) \varphi_t \cdot dW_t.$$

It turns out (see example 6.2.3 for $\delta_t \equiv 0, t \geq 0$) that the process

$$U(x, t) = u(x, A_t) Z_t,$$

with u satisfying (34), $Z_t, t \geq 0$, solving

$$dZ_t = Z_t \varphi_t \cdot dW_t \text{ with } Z_0 = 1 \quad (44)$$

and $A_t = \int_0^t |\lambda_s + \sigma_s \sigma_s^+ \varphi_s|^2 ds$ satisfies (28).

One may interpret Z_t as a device that offers the flexibility to modify our views on asset returns. For this reason, we call this case the "market-view" case.

Using (31) we obtain that the optimal allocation vector, $\pi_t^*, t \geq 0$, has the same functional form as (43) but for a different time-rescaling process, namely,

$$\pi_t^* = -\frac{u_x(X_t^*, A_t)}{u_{xx}(X_t^*, A_t)} \sigma_t^+ (\lambda_t + \varphi_t),$$

with $A_t, t \geq 0$, as above. It is worth noticing that if the process φ_t is chosen to satisfy $\varphi_t = -\lambda_t$, solutions become static. Specifically, the time-rescaling process vanishes, $A_t = 0$, and, in turn, the forward performance process becomes constant across times. The optimal investment and wealth processes degenerate $\pi_t^* = 0$ and $X_t^* = x, t \geq 0$. In other words, even for non-zero λ_t , the optimal policy is to allocate *zero* wealth in every risky asset and at all times.

6.2.2 The benchmark case: $a(x, t) = x U_x(x, t) \delta_t$

The forward performance SPDE (28) simplifies to

$$dU(x, t) = \frac{1}{2} \frac{|U_x(x, t)(\lambda_t - \delta_t) - x U_{xx}(x, t) \delta_t|^2}{U_{xx}} dt - x U_x(x, t) \delta_t \cdot dW_t.$$

One can verify (see example 6.2.3 for $\varphi_t \equiv 0, t \geq 0$) that the process

$$U(x, t) = u\left(\frac{x}{Y_t}, A_t\right),$$

with $Y_t, t \geq 0$, solving

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \quad \text{with } Y_0 = 1 \quad (45)$$

and $A_t = \int_0^t |\lambda_s - \delta_s|^2 ds$ satisfies (28).

One may interpret the auxiliary process Y_t as a benchmark (or numeraire) with respect to which the performance of investment policies is measured.

Next, we define the benchmarked optimal portfolio and wealth processes,

$$\tilde{\pi}_t^* = \frac{\pi_t^*}{Y_t} \quad \text{and} \quad \tilde{X}_t^* = \frac{X_t^*}{Y_t}. \quad (46)$$

Then, $\tilde{\pi}_t^*$ is given by (cf. (49) for $\varphi_t \equiv 0$),

$$\tilde{\pi}_t^* = \tilde{X}_t^* \sigma_t^+ \delta_t - \frac{u_x(\tilde{X}_t^*, A_t)}{u_{xx}(\tilde{X}_t^*, A_t)} \sigma_t^+ (\lambda_t - \delta_t),$$

with A_t as above with \tilde{X}_t^* solving

$$d\tilde{X}_t^* = -\frac{u_x(\tilde{X}_t^*, A_t)}{u_{xx}(\tilde{X}_t^*, A_t)} (\lambda_t - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t).$$

6.2.3 The combined market view/benchmark case: $a(x, t) = -xU_x(x, t)\delta_t + U(x, t)\varphi_t$

The forward performance SPDE (28) becomes

$$\begin{aligned} dU(x, t) &= \frac{1}{2} \frac{|U_x(x, t)(\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t) - xU_{xx}(x, t)\delta_t|^2}{U_{xx}} dt \\ &\quad + (-xU_x(x, t)\delta_t + U(x, t)\varphi_t) \cdot dW_t. \end{aligned} \quad (47)$$

We introduce the time-rescaling process $A_t = \int_0^t |\lambda_s + \sigma_s \sigma_s^+ \varphi_s - \delta_s|^2 ds$.

We define the process

$$U(x, t) = u\left(\frac{x}{Y_t}, A_t\right) Z_t \quad (48)$$

with u solving (34) and Y_t and Z_t as in (45) and (44). We claim that it solves (47). Indeed, expanding yields

$$dU(x, t) = \left(du\left(\frac{x}{Y_t}, A_t\right)\right) Z_t + u\left(\frac{x}{Y_t}, A_t\right) dZ_t + d\left\langle u\left(\frac{x}{Y_t}, A_t\right), Z_t \right\rangle.$$

Moreover,

$$du\left(\frac{x}{Y_t}, A_t\right) = u_x\left(\frac{x}{Y_t}, A_t\right) d\left(\frac{x}{Y_t}\right) + u_t\left(\frac{x}{Y_t}, A_t\right) dA_t + \frac{1}{2} u_{xx}\left(\frac{x}{Y_t}, A_t\right) d\left\langle \frac{x}{Y_t} \right\rangle_t,$$

and

$$d\left(\frac{x}{Y_t}\right) = -\frac{x}{Y_t} \delta_t \cdot ((\lambda_t - \delta_t) dt + dW_t).$$

Consequently,

$$\begin{aligned} d\left\langle u\left(\frac{x}{Y_t}, A_t\right), Z_t \right\rangle_t &= -\frac{x}{Y_t} u_x\left(\frac{x}{Y_t}, A_t\right) Z_t \delta_t \cdot \varphi_t dt, \\ u\left(\frac{x}{Y_t}, A_t\right) dZ_t &= u\left(\frac{x}{Y_t}, A_t\right) Z_t \varphi_t \cdot dW_t = U(x, t) \varphi_t \cdot dW_t \end{aligned}$$

and

$$\begin{aligned} \left(du \left(\frac{x}{Y_t}, A_t \right) \right) Z_t &= -\frac{x}{Y_t} u_x \left(\frac{x}{Y_t}, A_t \right) Z_t \delta_t \cdot ((\lambda_t - \delta_t) dt + dW_t) \\ + u_t \left(\frac{x}{Y_t}, A_t \right) Z_t &|\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t|^2 dt + \frac{1}{2} u_{xx} \left(\frac{x}{Y_t}, A_t \right) Z_t \left| -\frac{x}{Y_t} \delta_t \right|^2 dt. \end{aligned}$$

We then deduce that

$$\begin{aligned} dU(x, t) &= -\frac{x}{Y_t} u_x \left(\frac{x}{Y_t}, A_t \right) Z_t \delta_t \cdot (\lambda_t + \varphi_t - \delta_t) dt \\ + u_t \left(\frac{x}{Y_t}, A_t \right) Z_t &|\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t|^2 dt + \frac{1}{2} u_{xx} \left(\frac{x}{Y_t}, A_t \right) Z_t \left| -\frac{x}{Y_t} \delta_t \right|^2 dt \\ &+ \left(-\frac{x}{Y_t} u_x \left(\frac{x}{Y_t}, A_t \right) Z_t \delta_t + U(x, t) \varphi_t \right) \cdot dW_t \\ &= -x U_x(x, t) \delta_t \cdot (\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t) dt \\ &+ \frac{1}{2} \frac{U_x^2(x, t)}{U_{xx}(x, t)} |\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t|^2 dt + \frac{1}{2} x^2 U_{xx}(x, t) |\delta_t|^2 dt \\ &\quad (-x U_x(x, t) \delta_t + U(x, t) \varphi_t) \cdot dW_t \\ &= \frac{1}{2} \frac{|U_x(x, t) (\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t) - x U_{xx}(x, t) \delta_t|^2}{U_{xx}(x, t)} dt \\ &\quad + (-x U_x(x, t) \delta_t + U(x, t) \varphi_t) \cdot dW_t, \end{aligned}$$

where we used (34). The optimal benchmarked policies $\tilde{\pi}_t^*$ and \tilde{X}_t^* are defined as in (46). Using (21) and (48) we deduce, after some routine but tedious calculations, that

$$\tilde{\pi}_t^* = \tilde{X}_t^* \sigma_t^+ \delta_t - \frac{u_x(\tilde{X}_t^*, A_t)}{u_{xx}(\tilde{X}_t^*, A_t)} \sigma_t^+ (\lambda_t + \varphi_t - \delta_t) \quad (49)$$

where \tilde{X}_t^* solves

$$d\tilde{X}_t^* = -\frac{u_x(\tilde{X}_t^*, A_t)}{u_{xx}(\tilde{X}_t^*, A_t)} (\lambda_t + \sigma_t \sigma_t^+ \varphi_t - \delta_t) \cdot ((\lambda_t - \delta_t) dt + dW_t). \quad (50)$$

6.2.4 Single stochastic factor models

This is the forward analogue of example 4.1 Consider a single stock and a stochastic factor satisfying (14) and (15). Let $w : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ satisfying the (forward) HJB equation

$$w_t + \max_{\pi} \left(\frac{1}{2} \sigma^2(y) \pi^2 w_{xx} + \pi (\mu(y) w_x + \rho \sigma(y) d(y) w_{xy}) \right) \quad (51)$$

$$+ \frac{1}{2} d^2(y) w_{yy} + b(y) w_y = 0,$$

for an appropriate initial condition $w(y, 0)$. Then, for $t \geq 0$, the process

$$U(x, t) = w(x, Y_t, t) \quad (52)$$

satisfies the SPDE (28) with volatility vector $a(x, t) = (a_1(x, t), a_2(x, t))$ with

$$a_1(x, t) = \rho d(Y_t) w_y(y, t) \quad \text{and} \quad a_2(x, t) = \sqrt{1 - \rho^2} d(Y_t) w_y(y, t).$$

Notice the main differences between forward investment process (52) and (48). Firstly, they are constructed by deterministic functions, w and u , that solve different pdes. Secondly, the investment performance process in (52) does not involve time-rescaling while the one in (48) does.

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