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Forward indifference valuation of American options

Tim Leung[a]*, Ronnie Sircar[b1] and Thaleia Zariphopoulou[c,d,e2]

[a]*IEOR Department, Columbia University, New York, NY 10027, USA;* [b]*ORFE Department, Princeton University, E-Quad, Princeton, NJ 08544, USA;* [c]*The Oxford-Man Institute of Quantitative Finance and Mathematical Institute, University of Oxford, Oxford, UK;* [d]*Department of Mathematics, The University of Texas at Austin, Austin, TX 78712, USA;* [e]*Department of Information, Risk and Operations Management, The University of Texas at Austin, Austin, TX 78712, USA*

We analyse the valuation of American options under the forward performance criterion introduced by Musiela and Zariphopoulou [Quant. Finance 9 (2008), pp. 161–170]. In this framework, the performance criterion evolves forward in time without reference to a specific future time horizon, and may depend on the stochastic market conditions. We examine two applications: the valuation of American options with stochastic volatility and the modelling of early exercises of American-style employee stock options. We work with the assumption that forward indifference prices have sufficient regularity to be solutions of variational inequalities, and provide a comparative analysis between the classical and forward indifference valuation approaches. In the case of exponential forward performance, we derive a duality formula for the forward indifference price. Furthermore, we study the marginal forward performance price, which is related to the classical marginal utility price introduced by Davis (*Mathematics of Derivatives Securities*, Cambridge University Press, 1997, pp. 227–254). We prove that, under arbitrary time-monotone forward performance criteria, the marginal forward indifference price of any claim is always independent of the investor's wealth and is represented as the expected discounted pay-off under the minimal martingale measure.

Keywords: forward performance; marginal forward performance price; indifference pricing; American options; employee stock options; stochastic control with optimal stopping

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1. Introduction

Utility maximization theory has been central to quantifying rational investment decisions and risk-averse valuations of assets at least since the work of von Neumann and Morgenstern in the 1940s. In the Merton problem of continuous-time portfolio optimization [28], utility is defined at some fixed time horizon in the future when investment decisions are assessed in terms of the expected utility of terminal wealth. For portfolios involving derivatives and associated utility indifference pricing problems, derivative pay-offs or random endowments may be realized at random times, which requires the specification of utility at other times, not just at a single terminal time.

This consideration is particularly important for investment and valuation problems involving defaultable securities or American options.

One way to address this issue is to consider the definition of utility at the time of a random cash flow as analogous to specifying what the investor does with the endowment thereafter. Any answer to the latter question necessarily involves details of the market in which he might invest, and utilities and markets are inextricably linked. Some examples of works based on this idea include [24,32] for utility indifference pricing of American options, and [19,26] for defaultable securities. This approach allows for comparing utilities of wealth at different times. However, as is common in classical utility indifference pricing, the investor's risk preferences at intermediate times and the optimal investment decisions still directly depend on an *a priori* chosen investment horizon.

This issue of horizon dependence has been addressed by one of the authors and Musiela through the construction of the forward performance criterion (see e.g. [29]). In this approach, the investor's utility is specified at an initial time, and his risk preferences at subsequent times evolve forward without reference to any specific ultimate time horizon. This results in a stochastic utility process, called the *forward performance process*, whose evolution depends on the random market conditions. In a related study [18], Henderson and Hobson analysed the optimal timing of asset sale based on the so-called horizon-unbiased utility functions which have no preferred horizon for the associated dynamic portfolio optimization problem. Hence, these approaches necessarily connect risk preferences with market models. The risk profile of a given investor is no longer considered separately from his investment opportunities and the market. This is entirely natural: the current economic crisis has clearly shown increased risk aversion in investors as the market has fallen.

In this paper, we develop an indifference valuation methodology based on the forward performance criterion. Specifically, we study the valuation of a long position in an American option in an incomplete diffusion market model. Our main objective is to analyse the optimal trading and exercise strategies that maximize the option holder's forward performance coming from both the dynamic portfolio and the option pay-off upon exercise. In Section 2, we formulate the combined stochastic control and optimal stopping problem faced by the option holder. Then, we define the holder's *forward indifference price* for the American option by comparing the optimal expected forward performance with and without the derivative (see Definition 2). The analysis of the indifference price will yield a number of useful mathematical characterizations and financial interpretations for optimal trading and exercise strategies.

In Section 3, we discuss the exponential forward indifference valuation of an American option in a stochastic volatility model. Using the analytical properties of the exponential forward performance, we show that the forward indifference price is wealth independent. By applying a transformation to the associated Hamilton–Jacobi–Bellman (HJB) variational inequality, we state the variational inequality that the forward indifference price, if it has sufficient regularity, satisfies. Due to the nonlinearity of these variational inequalities, the questions of existence, uniqueness, smoothness are open challenging issues, which we do not address herein. In the case with exponential forward performance, we derive a duality formula for the forward indifference price. This is useful for the comparative analysis between the forward and classical exponential indifference prices. For instance, we show that the forward indifference price representation involves a relative entropy minimization (up to a stopping time) with respect to the minimal martingale measure (MMM), as opposed to the minimal entropy martingale measure (MEMM) that arises in the classical exponential utility indifference price (see, among others, [8,35] for

European claims and [25] for American claims). We also present this contrasting difference in the asymptotic results of indifference prices.

Another application studied in this paper is the modelling of early exercises of employee stock options (ESOs), which are American-style call options written on the firm's stock granted to the employee as a form of compensation. In Section 4, we assume a forward performance criterion for the employee and investigate the impact of various factors, such as wealth and risk tolerance, on the employee's exercise timing. In particular, we find that the employee tends to exercise the ESO earlier when his wealth approaches zero.

Lastly, in Section 5, we introduce an alternative valuation mechanism for American options based on the *marginal* forward performance. In the classical utility framework, as introduced by Davis [7], the marginal utility price represents the per-unit price that a risk-averse investor is willing to pay for an infinitesimal position in a contingent claim. In general, the marginal utility price is closely linked to both the investor's utility function and the market set-up, and it only becomes wealth independent under very special circumstances (see [23] for details). We adapt the classical definition to our forward performance framework and give a definition of the *marginal forward indifference price*. We show that, in contrast to the classical marginal utility price, the marginal forward indifference price under time-monotone criteria turns out to be independent of both the holder's wealth and the forward performance criterion, and is equivalent to pricing linearly under the *MMM*. Section 6 concludes the paper and discusses extensions for future research.

2. Forward investment performance measurement and indifference valuation

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a filtration $(\mathcal{F}_t)_{t \geq 0}$ that satisfies the usual conditions of right continuity and completeness. In addition, all stochastic processes considered in this paper are continuous-path processes. The financial market consists of two liquidly traded assets, namely, a riskless money market account and a stock. The money market account has the price process B that satisfies

$$dB_t = r_t B_t \, dt \tag{1}$$

with $B_0 = 1$, where $(r_t)_{t \geq 0}$ is a non-negative $\mathcal{F}_t$-adapted interest rate process. We shall use B as the numeraire throughout.

The discounted stock price S is modelled as a continuous Itô process satisfying

$$dS_t = S_t \sigma_t (\lambda_t \, dt + dW_t) \tag{2}$$

with $S_0 > 0$, where $(W_t)_{t \geq 0}$ is an $\mathcal{F}_t$-adapted standard Brownian motion. The Sharpe ratio $(\lambda_t)_{t \geq 0}$ is a bounded $\mathcal{F}_t$-adapted process, and the volatility coefficient $(\sigma_t)_{t \geq 0}$ is strictly positive bounded $\mathcal{F}_t$-adapted process. Moreover, we assume that a strong solution exists for the stochastic differential equation (SDE) (2).

Starting with initial endowment $x \in \mathbb{R}$, the investor dynamically rebalances his portfolio allocations between the stock and the money market account. Under the self-financing trading condition, the discounted wealth satisfies

$$dX_t^\pi = \pi_t \sigma_t (\lambda_t \, dt + dW_t), \tag{3}$$

where $(\pi_t)_{t \geq 0}$ represents the discounted cash amount invested in stock. The set of admissible strategies $\mathcal{Z}$ consists of all self-financing $\mathcal{F}_t$-adapted processes $(\pi_t)_{t \geq 0}$ such that $\mathbb{E}\{\int_0^s \sigma_t^2 \pi_t^2 \, dt\} < \infty$ for each $s > 0$. For $0 \leq t \leq s$, we denote by $\mathcal{Z}_{t,s}$ the set of admissible strategies over the period $[t, s]$.

In the standard Merton portfolio optimization problem, risk preferences are modelled by a deterministic utility function $\hat{U}(\cdot)$ defined at some fixed terminal time T. Starting with $\mathcal{F}_t$-measurable wealth X_t at time $t \leq T$, the Merton value process is given by

$$M_t(X_t) = \underset{\pi \in \mathcal{Z}_{t,T}}{\text{ess sup}}\, \mathbb{E}\{\hat{U}(X_T^{\pi})|\mathcal{F}_t\}. \tag{4}$$

When the dynamic programming principle holds, the Merton problem can be written as

$$M_t(X_t) = \underset{\pi \in \mathcal{Z}_{t,s}}{\text{ess sup}}\, \mathbb{E}\{M_s(X_s^{\pi})|\mathcal{F}_t\}, \quad 0 \leq t \leq s \leq T. \tag{5}$$

Some well-known examples when (5) holds include (i) markets with Markovian dynamics where the optimal portfolio allocation can be found by solving a HJB equation; (ii) when the utility is of exponential type, in which case (5) holds under quite general semimartingale models (see [25,27]) and (iii) when the expected utility is replaced by a dynamic time-consistent concave utility functional, defined, for instance, from a backward stochastic differential equation in Itô markets (see [6,22]). The dynamic programming principle (5) is taken as the defining characteristic of the forward performance criterion.

In the forward performance framework, the investor's utility function $u_0(x)$ is defined at the initial time 0, and his performance criterion evolves forward in time. We adapt the definition of the forward performance process given by Musiela and Zariphopoulou [29]:

DEFINITION 1. An $\mathcal{F}_t$-adapted process $(U_t(x))_{t \geq 0}$ is a forward performance process if:

1. it satisfies the initial datum $U_0(x) = u_0(x)$, $x \in \mathbb{R}$, where $u_0 : \mathbb{R} \mapsto \mathbb{R}$ is an increasing and strictly concave function of x;
2. for each $t \geq 0$, the mapping $x \mapsto U_t(x)$ is increasing and strictly concave in $x \in \mathbb{R}$ and
3. for $0 \leq t \leq s < \infty$, we have

$$U_t(X_t) = \underset{\pi \in \mathcal{Z}_{t,s}}{\text{ess sup}}\, \mathbb{E}\{U_s(X_s^{\pi})|\mathcal{F}_t\} \tag{6}$$

for any $\mathcal{F}_t$-measurable initial wealth X_t.

In related studies, condition 3 is also referred to as the horizon-unbiased condition in [18] and the self-generating condition in [38].

As with the classical utility maximization problem, the existence and characterization of the optimal strategy in (6) are challenging questions and depend on the market structure and utility function used. Related research for forward performance processes includes [9,31,38] (for exponential preferences). In this paper, however, our analysis will focus on a class of *explicit* forward performance processes (see Theorem 3), whose optimal strategies have been completely characterized in the recent papers [4] and [30]. Our objective is to apply forward performance to the indifference pricing of American options and investigate some properties of the forward indifference prices.

2.1. *Forward indifference price*

We introduce the forward indifference valuation from the perspective of the holder of an American option. The option pay-off is modelled by an $\mathcal{F}_t$-adapted bounded process denoted by $(g_t)_{0 \leq t \leq T}$, with a finite expiration date T. The collection of admissible exercise

times is the set of stopping times with respect to $\mathcal{F}_{0,T} = (\mathcal{F}_t)_{0 \leq t \leq T}$ that take values in $[0,T]$. For $0 \leq t \leq s \leq T$, we denote by $\mathcal{T}_{t,s}$ the set of stopping times taking values in $[t,s]$.

The option holder chooses his dynamic trading strategy π and exercise time τ, in order to maximize his expected forward performance from both investing in the market and receiving the option pay-off. This leads to a combined stochastic control and optimal stopping problem. Specifically, we define

$$V_t(X_t) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\{U_\tau(X_\tau^\pi + g_\tau)|\mathcal{F}_t\}, \quad t \in [0,T], \tag{7}$$

which is the holder's value process based on a forward performance starting at time t with wealth X_t.

In the classical case with a terminal utility function $\hat{U}$, the holder's optimal investment problem is to solve

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\{M_\tau(X_\tau^\pi + g_\tau)|\mathcal{F}_t\},$$

where M is the solution to the Merton problem defined in (4). In this formulation, M plays the role of intermediate utility at stopping times $\tau \leq T$ and, therefore, specifies that option proceeds received at any exercise time τ are reinvested following the Merton optimal strategy up till time T. By contrast, the forward performance process U specifies utilities at all times, without reference to any specific horizon.

The holder's forward indifference price p_t for the American option g is defined as the discounted cash amount such that the option holder is indifferent between two positions: optimal investment with an American option position, and optimal investment without the American option but instead with extra initial wealth p_t.

DEFINITION 2. The holder's forward indifference price process $(p_t)_{0 \leq t \leq T}$ for the American option is defined by the equation

$$V_t(X_t) = U_t(X_t + p_t), \quad t \in [0,T], \tag{8}$$

where V_t and U_t are given in (7) and (6), respectively.

The forward indifference price is useful for characterizing the option holder's optimal exercise time τ^*. Under appropriate integrability conditions ([21], Theorem D.12), the optimal stopping time is the first time the value process reaches the reward process. From (7) and (8), we have

$$\begin{aligned}\tau_t^* &= \inf\{t \leq s \leq T : V_s(X_s) = U_s(X_s + g_s)\} \\ &= \inf\{t \leq s \leq T : U_s(X_s + p_s) = U_s(X_s + g_s)\} \\ &= \inf\{t \leq s \leq T : p_s = g_s\}.\end{aligned} \tag{9}$$

The representation (9) implies that the option holder will exercise the American option as soon as the forward indifference price reaches (from above) the option pay-off. It allows us to analyse the holder's optimal exercise policy through his forward indifference price.

In Sections 3 and 4, we will focus our study on two specific financial applications: (i) the valuation of an American option written on a stock S with stochastic volatility under

forward performance criterion of *exponential* type [to be defined in (23)] and (ii) modelling early exercises of ESOs for criteria beyond the exponential forward performance.

2.2.   *Forward performance of generalized constant absolute risk aversion/constant relative risk aversion type*

Henceforth, we will focus our attention on a special class of forward performance processes introduced by Musiela and Zariphopoulou [29], namely, the time-monotone forward performance processes. These processes are represented by the compilation of a deterministic function $u(x,t)$ which models the investor's dynamic risk preferences, and a stochastic time-change $(A_t)_{t\geq0}$ that solely depends on the market. Recent studies [4] and [30] addressed various properties and alternative characterizations of this family of forward performances.

THEOREM 3. Define the stochastic process ([29], Theorem 4])

$$A_t = \int_0^t \lambda_s^2 \, ds, \quad t \geq 0. \tag{10}$$

Let $u : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}$ be $\mathcal{C}^{3,1}$, strictly concave, and increasing in its spatial argument. Assume that it satisfies the nonlinear partial differential equation

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}} \tag{11}$$

with initial condition $u(x,0) = u_0(x)$, where $u_0 \in \mathcal{C}^3(\mathbb{R})$. Then, the process $U_t(x)$, defined by

$$U_t(x) = u(x, A_t), \quad t \geq 0, \tag{12}$$

is a forward performance process. Moreover, the trading strategy π^* given by

$$\pi_t^* = -\frac{\lambda_t}{\sigma_t} \frac{u_x(X_t^*, A_t)}{u_{xx}(X_t^*, A_t)}, \quad t \geq 0, \tag{13}$$

where $X^* = X^{\pi^*}$ is the associated wealth process following (3), is optimal.

By its definition in (10), A is an increasing stochastic process that depends on the Sharpe ratio of the traded asset S. Also, it is commonly called the *mean-variance trade-off* process (see [34] and references therein). In constructing the forward performance process in (12), A acts as a stochastic time change to the deterministic preference function $u(x,t)$.

We stress that because Equation (11) is ill-posed, one needs to specify the class of initial conditions that yields a well-defined solution for all times. This is not a trivial matter and was investigated in detail in [30]. A related problem, which was also studied there, is to determine for which initial conditions the policies specified by (12) are admissible. Because the related arguments for both the aforementioned questions are quite lengthy, we provide the key results in the Appendix. The time-monotone forward performance criteria used in Sections 3 and 4 belong to the admissible class.

A quantity that plays a crucial role in the description of the optimal wealth and portfolio processes (X^*, π^*) is the so-called *local risk tolerance* function $R : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$, defined by

$$R(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)} \tag{14}$$

with u solving (11). Using (13) and (14), the dynamics of the optimal wealth X^* can be expressed as

$$dX_t^* = R(X_t^*, A_t)\lambda_t(\lambda_t \, dt + dW_t). \tag{15}$$

Furthermore, by applying differentiation to (14), one can show that R is the solution of an equation of fast diffusion type, namely

$$R_t + \frac{1}{2}R^2 R_{xx} = 0. \tag{16}$$

The above *autonomous* equation for $R(x, t)$ suggests that one could first model the local risk tolerance directly, and in turn recover the dynamic risk preference function $u(x, t)$ from (14). This provides an alternative way to construct forward performance criteria. This idea was further developed in [37] which proposed the following two-parameter family of risk tolerance functions:

$$R(x, t; \alpha, \beta) := \sqrt{\alpha x^2 + \beta e^{-\alpha t}}, \quad x \in \mathbb{R}, \ t \geq 0, \alpha, \beta > 0. \tag{17}$$

We illustrate an example of this risk tolerance in Figure 1.

There are several reasons to work with this family of risk tolerance. First, it yields, in the limits as α or β goes to zero, the risk tolerance functions that resemble those related to the three most popular cases, specifically, the *exponential*, *power* and *logarithmic*. We summarize from [37] the limiting cases leading to risk tolerance functions and the corresponding utilities as follows:

$$\lim_{\alpha \to 0} R(x, t; \alpha, \beta) = \sqrt{\beta}, \quad u(x, t) = -e^{-\frac{x}{\sqrt{\beta}} + \frac{t}{2}}, \ x \in \mathbb{R} \text{ (exponential)}, \tag{18}$$

$$\lim_{\beta \to 0} R(x, t; \alpha, \beta) = \sqrt{\alpha} x, \quad u(x, t) = \frac{x^\delta}{\delta} e^{-\frac{\delta}{2(1-\delta)}t}, \ x \geq 0, \ \alpha \neq 1 \text{ (power)}, \tag{19}$$

$$\lim_{\beta \to 0} R(x, t; 1, \beta) = x, \quad u(x, t) = \log x - \frac{t}{2}, \ x > 0 \text{ (logarithmic)}, \tag{20}$$
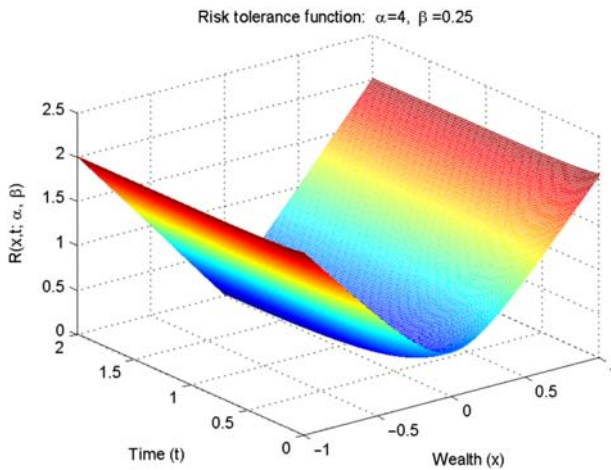
where $\delta := (\sqrt{\alpha} - 1)/\sqrt{\alpha}$.

Figure 1. The risk tolerance function $R(x, t; \alpha, \beta)$ in (17) with $\alpha = 4$, and $\beta = 0.25$. For any fixed wealth x, $R(x, \cdot; \alpha, \beta)$ decreases with time, while for any fixed time t, $R(\cdot, t; \alpha, \beta)$ increases as wealth decreases or increases away from zero.
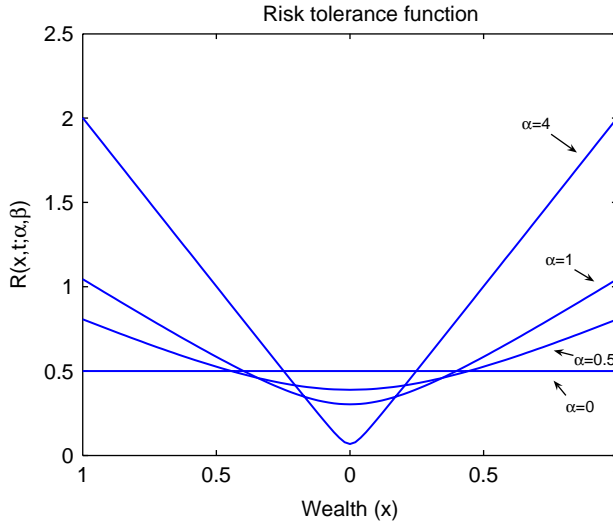
Figure 2. As α decreases from 4 to 0, with $\beta = 0.25$ and $t = 1$, the risk tolerance function $R(x, t; \alpha, \beta)$ converges to the constant level $\sqrt{\beta} = 0.5$, as predicted by the limit in (18).

According to (19) and (20), in the limit $\beta \downarrow 0$, $R(x, t; \alpha, \beta)$ is defined only over a positive/strictly positive wealth domain. In Figure 2, we illustrate the limit in (18) where the risk tolerance function converges to the constant $\sqrt{\beta}$ as $\alpha \downarrow 0$. In Section 3, we will work with the exponential forward performance which corresponds to constant risk tolerance in (18). In view of the limits in (18) and (19), we may call $\sqrt{\alpha}$ the *power risk tolerance* and $\sqrt{\beta}$ the *exponential risk tolerance*. Hence, the risk tolerance $R(x, t; \alpha, \beta)$ for $\alpha, \beta > 0$ can be viewed as a combination/interpolation of the power and exponential extremes.

For the general case with $\alpha, \beta > 0$, Zariphopoulou and Zhou [37] compute, via integration of (14), the dynamic risk preference function $u(x, t; \alpha, \beta)$ associated with $R(x, t; \alpha, \beta)$ in (17):

PROPOSITION 4. The dynamic risk preference function $u(x, t; \alpha, \beta)$ associated with $R(x, t; \alpha, \beta)$ in (17) for $\alpha, \beta > 0$ is given by ([37], Proposition 3.2)

$$u(x, t; \alpha, \beta) = m \frac{\kappa^{1+(1/\kappa)}}{\alpha - 1} e^{((1-\kappa)/2)t} \frac{(\beta/\kappa)e^{-\alpha t} + (1 + \kappa)x\left(\kappa x + \sqrt{\alpha x^2 + \beta e^{-\alpha t}}\right)}{\left(\kappa x + \sqrt{\alpha x^2 + \beta e^{-\alpha t}}\right)^{1+(1/\kappa)}} + n, \quad \alpha \neq 1,$$

(21)

$$u(x, t; 1, \beta) = \frac{m}{2}\left(\log\left(x + \sqrt{x^2 + \beta e^{-t}}\right) - \frac{e^{-t}}{\beta}x\left(x - \sqrt{x^2 + \beta e^{-t}}\right) - \frac{t}{2}\right) + n, \quad \alpha = 1,$$

where $\kappa = \sqrt{\alpha}$, and $m > 0, n \in \mathbb{R}$ are integration constants.

As mentioned earlier, in the context of the domain of the local risk tolerance, the function $u(x, t; \alpha, \beta)$ is also well defined for all $x \in \mathbb{R}$, except in the limit case $\beta \downarrow 0$. This property is particularly useful in indifference valuation, for it eliminates the non-negativity constraints on the investor's wealth (with and without the claim at hand).

3. American options under stochastic volatility

In this section, we study the forward indifference valuation of an American option in a stochastic volatility model. We work with the exponential forward performance, which, as mentioned in the previous section, corresponds to the parameter choice $\alpha = 0$. A comparative analysis with the classical exponential utility indifference pricing is provided in Section 3.3.

The discounted stock price S is modelled as a diffusion process satisfying

$$dS_t = S_t \sigma(Y_t)(\lambda(Y_t) \, dt + dW_t). \tag{22}$$

The Sharpe ratio $\lambda(Y_t)$ and volatility coefficient $\sigma(Y_t)$ are driven by a *non-traded* stochastic factor process $(Y_t)_{t \geq 0}$ which evolves according to

$$dY_t = b(Y_t) \, dt + c(Y_t)\left(\rho \, dW_t + \sqrt{1 - \rho^2} \, d\hat{W}_t\right). \tag{23}$$

The processes W and $\hat{W}$ are two independent Brownian motions defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $\mathcal{F}_t$ is taken to be the augmented σ-algebra generated by $((W_u, \hat{W}_u); 0 \leq u \leq t)$. The coefficient $\rho \in (-1, 1)$ accounts for the correlation between S and Y. The volatility function $\sigma(\cdot)$ and the diffusion coefficient $c(\cdot)$ are smooth, positive and bounded. The Sharpe ratio $\lambda(\cdot)$ is bounded continuous, and $b(\cdot)$ is Lipschitz continuous on $\mathbb{R}$. Similar conditions can be found in [36] and, as therein, our model excludes the Heston model whose volatility function is not bounded. For indifference pricing under the Heston model, we refer to [15].

The American option yields pay-off $g(S_\tau, Y_\tau, \tau)$ at any exercise time $\tau \in [0, T]$, where $g(\cdot, \cdot, \cdot)$ is a smooth and bounded function. The holder of the American option dynamically trades between the stock and money market account, and his discounted trading wealth follows:

$$dX_t^\pi = \pi_t \sigma(Y_t)(\lambda(Y_t) \, dt + dW_t), \tag{24}$$

where $(\pi_t)_{t \geq 0}$ is the discounted cash amount invested in stock (cf. (19)).

3.1. Exponential forward indifference price

We model the American option holder's risk preferences by the exponential forward performance process. This corresponds to the limiting case in (18) where the risk tolerance becomes a constant $\sqrt{\beta}$ (see also Figure 2). As seen in (18), the function $u(x, t)$ is given by

$$u(x, t) = -e^{-\gamma x + (t/2)}, \tag{25}$$

where $\gamma := 1/\sqrt{\beta}$ can be considered as the investor's *local risk aversion* parameter. In turn, applying Theorem 3, we obtain the exponential forward performance process

$$U_t^e(x) = -e^{-\gamma x + ((1/2) \int_0^t \lambda(Y_s)^2 \, ds)}, \quad t \geq 0. \tag{26}$$

As defined in (7), the option holder's value process based on the exponential forward performance is given by

$$\begin{aligned}
V_t^e(X_t) &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{ -e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{(1/2) \int_0^\tau \lambda(Y_s)^2 \, ds} \big| \mathcal{F}_t \right\} \\
&= e^{(1/2) \int_0^t \lambda(Y_s)^2 \, ds} \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{ -e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{(1/2) \int_t^\tau \lambda(Y_s)^2 \, ds} \big| \mathcal{F}_t \right\}. \quad (27)
\end{aligned}$$

We observe that the second term in (27) is the value of a combined stochastic control and optimal stopping problem. Working under the Markovian stochastic volatility market (22) and (23), we look for a candidate optimal $\mathcal{F}_t$-adapted Markovian strategy by studying the associated HJB variational inequality.

To facilitate notation, we introduce the following differential operators and Hamiltonian:

$$
\begin{aligned}
\mathcal{L}_{SY}v &= \frac{1}{2}\sigma(y)^2s^2v_{ss} + \rho c(y)\sigma(y)sv_{sy} + \frac{1}{2}c(y)^2v_{yy} + \lambda(y)\sigma(y)sv_s + b(y)v_y, \\
\mathcal{L}_{SY}^0v &= \frac{1}{2}\sigma(y)^2s^2v_{ss} + \rho c(y)\sigma(y)sv_{sy} + \frac{1}{2}c(y)^2v_{yy} + (b(y) - \rho c(y)\lambda(y))v_y,
\end{aligned}
\tag{28}
$$

and

$$
\mathcal{H}(v_{xx}, v_{xy}, v_{xs}, v_x) = \max_\pi \left(\frac{\pi^2\sigma(y)^2}{2}v_{xx} + \pi\big(\rho\sigma(y)c(y)v_{xy} + \sigma(y)^2sv_{xs} + \lambda(y)\sigma(y)v_x\big) \right).
$$

Note that $\mathcal{L}_{SY}$ and $\mathcal{L}_{SY}^0$ are, respectively, the infinitesimal generators of the Markov process $(S_t, Y_t)_{t\geq 0}$ under the historical measure $\mathbb{P}$ and the MMM Q^0. The latter measure is defined in (35).

Next, we consider the HJB variational inequality:

$$
\begin{cases}
V_t + \mathcal{L}_{SY}V + \mathcal{H}\big(V_{xx}, V_{xy}, V_{xs}, V_x\big) + \frac{\lambda(y)^2}{2}V \leq 0, \\
V(x, s, y, t) \geq -\mathrm{e}^{-\gamma(x+g(s,y,t))}, \\
\left(V_t + \mathcal{L}_{SY}V + \mathcal{H}(V_{xx}, V_{xy}, V_{xs}, V_x) + \frac{\lambda(y)^2}{2}V\right) \cdot \left(-\mathrm{e}^{-\gamma(x+g(s,y,t))} - V(x, s, y, t)\right) = 0, \\
V(x, s, y, T) = -\mathrm{e}^{-\gamma(x+g(s,y,T))},
\end{cases}
\tag{29}
$$

for $(x, s, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [0, T]$. Given a solution $V(x, s, y, t)$ to (29) that is $\mathcal{C}^{2,2,2,1}$, except across a lower dimensional optimal exercise boundary, one can show by standard verification arguments (see, for example, Theorem 4.2 of [33]) that V is the value function for the combined optimal control/stopping problem in (27). Therefore, we can write

$$
V_t^e(X_t) = \mathrm{e}^{(1/2)\int_0^t \lambda(Y_s)^2\,\mathrm{d}s}V(X_t, S_t, Y_t, t).
\tag{30}
$$

As is common in classical indifference pricing of American options, the existence of a solution (in an appropriate regularity class) to the HJB equation or variational inequality is a non-trivial and technical issue. In the classical exponential utility indifference pricing for American options, Oberman and Zariphopoulou [32] show the existence of a unique viscosity solution of the HJB variational inequality for the value function. In fact, our variational inequality (29) differs from that in [32] only by the term $((\lambda(y)^2)/2)V$. For our analysis in this section, we assume the existence of a unique solution $V(x, s, y, t)$ to the variational inequality (29) with the regularity needed for the verification arguments.

Assumption 5. We assume that there exists a unique smooth solution $V(x, s, y, t)$ to the variational inequality (29) so that (30) holds.

Applying (26) and (30) to Definition 2, the option holder's exponential forward indifference price function $p(x, s, y, t)$ is given by

$$p(x, s, y, t) = -\frac{1}{\gamma}\log(-V(x, s, y, t)) - x. \tag{31}$$

Substituting (31) into the variational inequality (29), we derive the variational equality for $p(x, s, y, t)$. It turns out that the indifference price is independent of the wealth argument x and solves the free boundary problem

$$\begin{cases} p_t + \mathcal{L}_{SY}^0 p - \frac{1}{2}\gamma(1 - \rho^2)c(y)^2 p_y^2 \leq 0, \\ p(s, y, t) \geq g(s, y, t), \\ \left(p_t + \mathcal{L}_{SY}^0 p - \frac{1}{2}\gamma(1 - \rho^2)c(y)^2 p_y^2\right) \cdot \left(g(s, y, t) - p(s, y, t)\right) = 0, \\ p(s, y, T) = g(s, y, T), \end{cases} \tag{32}$$

for $(s, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times [0, T]$.

By the first-order condition in (29) and the formula (31), the optimal hedging strategy $(\tilde{\pi}_t^*)_{0 \leq t \leq T}$ can be expressed in terms of the partial derivatives of the forward indifference price, namely

$$\tilde{\pi}_t^* = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{S_t}{\gamma}p_s(S_t, Y_t, t) + \frac{\rho c(Y_t)}{\gamma\sigma(Y_t)}p_y(S_t, Y_t, t).$$

The first term in this expression is the optimal strategy in (13) when there is no claim. The second and third parts of the strategy $\tilde{\pi}_t^*$ account for the sensitivity of the indifference price with respect to the traded and non-traded assets S and Y, respectively.

The optimal exercise time is the first time that the indifference price reaches the option pay-off:

$$\tau_t^* = \inf\{t \leq u \leq T : p(S_u, Y_u, u) = g(S_u, Y_u, u)\}. \tag{33}$$

In practice, one can numerically solve the variational inequality (32) to obtain the optimal exercise boundary which represents the critical levels of S and Y at which the option should be exercised. We remark that the indifference price, the optimal hedging and exercising strategies are all wealth independent. The same phenomenon occurs in the classical indifference valuation with exponential utility.

3.2. Dual representation

The option holder's forward performance maximization in (27) can be considered as the primal optimization problem, and it yields the first expression for the forward indifference price in (31). In this subsection, our objective is to derive a dual representation for the forward indifference price, which turns out to be related to pricing the American option with entropic penalty. This result will allow us to express the price in a way analogous to the classical exponential indifference price. We carry out this comparison in Section 3.3.

First, we denote by $\mathcal{M}(\mathbb{P})$ the set of equivalent local martingale measures with respect to $\mathbb{P}$ on $\mathcal{F}_T$. As is well known (see, for example, [11]), these measures are characterized by their respective density process with respect to $\mathbb{P}$, which is given by the stochastic

exponential

$$Z_t^{\phi} = \frac{dQ^{\phi}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \lambda(Y_s)^2 + \phi_s^2 \, ds - \int_0^t \lambda(Y_s) \, dW_s - \int_0^t \phi_s \, d\hat{W}_s\right), \qquad (34)$$

where $(\phi_t)_{0 \leq t \leq T}$ is a $\mathcal{F}_t$-progressively measurable process satisfying $\int_0^T \phi_s^2 \, ds < \infty$, $\mathbb{P}$-a.s., and $\mathbb{E}\{Z_T^{\phi}\} = 1$.

By Girsanov's Theorem, it follows that the two processes $W_t^{\phi} = W_t + \int_0^t \lambda(Y_s) \, ds$ and $\hat{W}_t^{\phi} = \hat{W}_t + \int_0^t \phi_s \, ds$ are independent Q^{ϕ}-Brownian motions. The process ϕ is commonly referred to as the *volatility risk premium* for the second Brownian motion $\hat{W}$. When $\phi = 0$, the resulting measure Q^0 is the well-known MMM, whose Radon–Nikodym derivative is

$$\frac{dQ^0}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \lambda(Y_s)^2 \, ds - \int_0^T \lambda(Y_s) \, dW_s\right), \qquad (35)$$

see [10].

Next, we define the conditional relative entropy of Q^{ϕ} with respect to $\mathbb{P}$ over the interval $[t, \tau]$, with $\tau \in \mathcal{T}_{t,T}$, as

$$H_t^{\tau}(Q^{\phi}|\mathbb{P}) := \mathbb{E}^{Q^{\phi}}\left\{\log\frac{Z_{\tau}^{\phi}}{Z_t^{\phi}}\bigg|\mathcal{F}_t\right\}. \qquad (36)$$

Direct computation from (34) shows that this relative entropy is, in fact, a quadratic penalization on the risk premia λ and ϕ. In other words,

$$H_t^{\tau}(Q^{\phi}|\mathbb{P}) = \frac{1}{2}\mathbb{E}^{Q^{\phi}}\left\{\int_t^{\tau} \lambda(Y_s)^2 + \phi_s^2 \, ds|\mathcal{F}_t\right\}. \qquad (37)$$

We denote the set of equivalent local martingale measures with finite relative entropy (with respect to $\mathbb{P}$) as

$$\mathcal{M}_f := \left\{Q^{\phi} \in \mathcal{M}(\mathbb{P}) : H_0^T(Q^{\phi}|\mathbb{P}) < \infty\right\}.$$

The probability measure that yields the minimum relative entropy with respect to $\mathbb{P}$ is called the MEMM and is defined by

$$Q^E := \underset{Q^{\phi} \in \mathcal{M}(\mathbb{P})}{\arg\min} H_t^T(Q^{\phi}|\mathbb{P}). \qquad (38)$$

Key results on the MEMM in a general semimartingale market framework can be found in [12,13]. This measure also arises in hedging and indifference valuation under exponential utility; see [8,35], among others.

Remark 6. If the Sharpe ratio is constant, i.e. $\lambda(y) = \lambda$, then the conditional relative entropy simplifies to

$$H_t^T(Q^{\phi}|\mathbb{P}) = \frac{\lambda^2}{2}(T - t) + \mathbb{E}^{Q^{\phi}}\left\{\int_t^T \phi_s^2 \, ds|\mathcal{F}_t\right\}.$$

As a result, setting $\phi = 0$ minimizes $H_t^T(Q^\phi|\mathbb{P})$. This is a well-known example in which the MEMM Q^E coincides with the MMM Q^0.

We may also express any measure Q^ϕ in terms of Q^0 via the Radon–Nikodym derivative, namely

$$\frac{\mathrm{d}Q^\phi}{\mathrm{d}Q^0} = \frac{\mathrm{d}Q^\phi}{\mathrm{d}\mathbb{P}} \Big/ \frac{\mathrm{d}Q^0}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \phi_s^2 \,\mathrm{d}s - \int_0^T \phi_s \,\mathrm{d}\hat{W}_s^0 \right). \tag{39}$$

We denote the density process of Q^ϕ with respect to Q^0 by $Z_t^{\phi,0} = \mathbb{E}^{Q^0}\{(\mathrm{d}Q^\phi)/(\mathrm{d}Q^0)|\mathcal{F}_t\}$.

Treating Q^0 as the prior risk-neutral measure, we can define the conditional relative entropy $H_t^\tau(Q^\phi|Q^0)$ of Q^ϕ with respect to Q^0 over the interval $[t,\tau]$ as

$$H_t^\tau(Q^\phi|Q^0) = \mathbb{E}^{Q^\phi}\left\{ \log\frac{Z_\tau^{\phi,0}}{Z_t^{\phi,0}} |\mathcal{F}_t \right\} = \frac{1}{2}\mathbb{E}^{Q^\phi}\left\{ \int_t^\tau \phi_s^2 \,\mathrm{d}s|\mathcal{F}_t \right\}. \tag{40}$$

With these notations, we are now ready to state the duality formula for the exponential forward indifference price.

PROPOSITION 7. The American option holder's exponential forward indifference price $p(s, y, t)$ is the solution of the combined stochastic control and optimal stopping problem:

$$p(S_t, Y_t, t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi \in \mathcal{M}_f} \left(\mathbb{E}^{Q^\phi}\{g(S_\tau, Y_\tau, \tau)|\mathcal{F}_t\} + \frac{1}{\gamma}H_t^\tau(Q^\phi|Q^0) \right). \tag{41}$$

Before giving the proof in the next subsection, let us first discuss the intuitive interpretation of the forward indifference price according to the duality formula (41). In essence, the holder tries to value the American option over a set of equivalent local martingale measures, and his selection criterion for the optimal pricing measure is based on relative entropic penalization (scaled by risk aversion γ). Indeed, the second term in (41) is the relative entropy of a candidate measure Q^ϕ with respect to the MMM Q^0 up to the exercise time. Therefore, the holder assigns the corresponding optimal risk premium ϕ^* according to (42). Due to the entropic penalty, we observe from (41) that the exponential forward indifference pricing rule is nonlinear in terms of the number of options held.

There are two ways to establish Proposition 7. The first approach is to apply the variational inequalities in Section 3.1. One can check that the variational inequality (32) for the indifference price $p(s, y, t)$ in (31) is identical to the one for the stochastic control/stopping problem on the RHS of (41). Using this approach, the associated optimal control ϕ^* must satisfy

$$\phi_t^* = -\gamma c(Y_t)\sqrt{1 - \rho^2}p_y(S_t, Y_t, t), \quad 0 \le t \le T, \tag{42}$$

subject to integrability condition so that $Q^{\phi^*} \in \mathcal{M}_f$. This approach requires a number of regularity conditions for the nonlinear variational inequalities (32) and (41) and for the candidate optimal control ϕ^*.

Hence, in the next subsection, we will prove Proposition 7 via an alternative approach which does not involve the variational inequalities. The key idea is to derive the dual representation for the forward value function in (30) using an analogous duality formula

from the classical exponential indifference pricing for American options [25]; see Theorem 8 below. Before we present the proof, we first need to recall and discuss the classical exponential utility indifference price.

3.3. *Comparison with the classical exponential utility indifference price*

In this section, we first summarize the duality results from the classical exponential utility indifference pricing, and then apply them to derive the forward indifference formula (41). Moreover, we also provide a comparative analysis between the classical and forward indifference valuation approaches.

We start with a brief review of the classical indifference pricing with exponential utility under stochastic volatility models. We refer the reader to, for example, [3,15,36] for European-style derivatives, as well as [25,32] for American options.

In the classical setting, the investor's risk preferences at time T are modelled by the exponential utility function $-e^{-\gamma x}$, with risk aversion parameter $\gamma > 0$. In the stochastic volatility model described in (22) and (23), the value function of the Merton problem (cf. (4)) is

$$M(X_t, Y_t, t) = \sup_{\pi \in \mathcal{Z}_{t,T}} \mathbb{E}\{-e^{-\gamma X_T^\pi}|\mathcal{F}_t\} \tag{43}$$

with $(X_t^\pi)_{t \geq 0}$ given by (24).

As is well known, see for example, [8,35], the Merton value function admits a dual representation in terms of relative entropy minimization, namely

$$M(X_t, Y_t, t) = -\exp\left(-\gamma X_t - H_t^T(Q^E|\mathbb{P})\right), \tag{44}$$

where $H_t^T(Q^E|\mathbb{P})$ is the conditional relative entropy of Q^E with respect to $\mathbb{P}$ over $[t,T]$.

If the American option g is held, then the investor seeks the optimal trading strategy and exercise time to maximize the expected utility of wealth from both his dynamic portfolio and the option's pay-off at exercise. Upon exercise of the option, the investor will reinvest the contract proceeds, if any, to his trading portfolio, and continue to trade up to time T. As a consequence, the holder faces the optimization problem

$$\hat{V}(X_t, S_t, Y_t, t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\{M(X_\tau^\pi + g(S_\tau, Y_\tau, \tau), Y_\tau, \tau)|\mathcal{F}_t\} \tag{45}$$

$$= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\{-e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau))} e^{-H_\tau^T(Q^E|\mathbb{P})}|\mathcal{F}_t\}, \tag{46}$$

where M is defined in (43). The classical indifference price $\hat{p}$ of the American option is then determined from the equation

$$M(x, y, t) = \hat{V}(x - \hat{p}(x, s, y, t), s, y, t). \tag{47}$$

Under a general semimartingale framework, Leung and Sircar [25] have derived a duality formula for the optimization problem (45) and the exponential indifference price $\hat{p}$. Herein, we summarize the results as written for our stochastic volatility market setting. We use the shorthand notation $\mathbb{E}_t^{Q^\phi}\{\cdot\} \equiv \mathbb{E}^{Q^\phi}\{\cdot|\mathcal{F}_t\}$.

THEOREM 8 ([25], PROPOSITIONS 2.4 AND 2.8). *The classical value function in (45) for holding the American option* g *is given by*

$$\hat{V}(X_t, S_t, Y_t, t)$$

$$= -e^{-\gamma X_t} \exp\left(-\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi \in \mathcal{M}_f} \left(\mathbb{E}_t^{Q^\phi} \{\gamma g(S_\tau, Y_\tau, \tau)\} + H_t^\tau(Q^\phi | \mathbb{P}) + \mathbb{E}_t^{Q^\phi} \{H_\tau^T(Q^E | \mathbb{P})\}\right)\right).$$

(48)

The classical exponential indifference price is given by

$$\hat{p}(S_t, Y_t, t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi \in \mathcal{M}_f} \left(\mathbb{E}_t^{Q^\phi} \{g(S_\tau, Y_\tau, \tau)\} + \frac{1}{\gamma} H_t^\tau(Q^\phi | Q^E) \right).$$

(49)

Now, we apply Theorem 8 to establish Proposition 7, namely, the duality formula (41) for the *forward* exponential indifference price.

Proof of Proposition 7. We begin by writing the function $V(X_t, S_t, Y_t, t)$ in (30) as

$$V(X_t, S_t, Y_t, t) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{-e^{-\gamma(X_\tau^\pi + g(S_\tau, Y_\tau, \tau) - (1/(2\gamma)) \int_t^\tau \lambda(Y_s)^2 \, ds)} | \mathcal{F}_t\right\}$$

$$= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,sup}_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E}\left\{-e^{-\gamma(X_\tau^\pi + \tilde{g}(S_\tau, Y_\tau, \tau))} e^{-H_\tau^T(Q^E | \mathbb{P})} | \mathcal{F}_t\right\},$$

where

$$\tilde{g}(S_\tau, Y_\tau, \tau) = g(S_\tau, Y_\tau, \tau) - \frac{1}{2\gamma} \int_t^\tau \lambda(Y_s)^2 \, ds - \frac{1}{\gamma} H_\tau^T(Q^E | \mathbb{P}).$$

In other words, the optimization problem $V(X_t, S_t, Y_t, t)$ has the same form as $\hat{V}(X_t, S_t, Y_t, t)$ in (48), but with a new option pay-off $\tilde{g}(S_\tau, Y_\tau, \tau)$, instead of $g(S_\tau, Y_\tau, \tau)$, at any exercise time $\tau \in \mathcal{T}_{t,T}$.

Therefore, substituting the pay-off $\tilde{g}$ for g in Theorem 8 yields

$$V(X_t, S_t, Y_t, t)$$

$$= -e^{-\gamma X_t} \exp\left(-\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi \in \mathcal{M}_f} \left(\mathbb{E}_t^{Q^\phi} \{\gamma \tilde{g}(S_\tau, Y_\tau, \tau)\} + H_t^\tau(Q^\phi | \mathbb{P}) + \mathbb{E}_t^{Q^\phi} \{H_\tau^T(Q^E | \mathbb{P})\}\right)\right)$$

$$= -e^{-\gamma X_t} \exp\left(-\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi \in \mathcal{M}_f} \left(\mathbb{E}_t^{Q^\phi} \left\{ \gamma g(S_\tau, Y_\tau, \tau) - \frac{1}{2} \int_t^\tau \lambda(Y_s)^2 \, ds \right\} + H_t^\tau(Q^\phi | \mathbb{P}) \right)\right)$$

$$= -e^{-\gamma X_t} \exp\left(-\operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \operatorname*{ess\,inf}_{Q^\phi \in \mathcal{M}_f} \left(\mathbb{E}_t^{Q^\phi} \{\gamma g(S_\tau, Y_\tau, \tau)\} + H_t^\tau(Q^\phi | Q^0) \right)\right),$$

(50)

where the last equality follows from (36) and (40). This is an alternative representation for V in (30). Finally, applying the duality formula (31) to (50) yields the forward exponential indifference price formula (41). □

The classical and forward exponential indifference prices in Theorem 8 and Proposition 7 bear a striking similarity, except that the relative entropy term in (49) is computed with respect to Q^E, but in (41) it is computed with respect to Q^0. To highlight this, we shall compare the variational inequality of the forward indifference price in (32) with its classical analogue.

As is well known, the classical Merton function M admits a separation of variables due to the choice of exponential utility.

PROPOSITION 9. *The value function $M(x, y, t)$ is given by*

$$M(x, y, t) = -e^{-\gamma x} f(y, t)^{(1/(1-\rho^2))}, \tag{51}$$

where ρ is the correlation coefficient in (23), and f solves

$$f_t + \mathcal{L}_Y^0 f = \frac{1}{2}(1 - \rho^2)\lambda(y)^2 f, \tag{52}$$

for $(x, t) \in \mathbb{R} \times [0, T)$, with $f(y, T) = 1$, for $y \in \mathbb{R}$. The operator $\mathcal{L}_Y^0$ is the infinitesimal generator of Y under the MMM Q^0, and is given by

$$\mathcal{L}_Y^0 f = \frac{1}{2}c(y)^2 f_{yy} + \big(b(y) - \rho c(y)\lambda(y)\big)f_y.$$

Details can be found, for example, in Theorem 2.2 of [36].

Using (51) and (47), we obtain the formula

$$\hat{V}(x, s, y, t) = -e^{-\gamma(x+\hat{p}(x,s,y,t))} f(y, t)^{(1/(1-\rho^2))}. \tag{53}$$

To derive the variational inequality for the indifference price, one can use the variational inequality for V and then apply the transformation (53). Again, the choice of exponential utility yields wealth-independent indifference prices, i.e. $\hat{p}(x, s, y, t) = \hat{p}(s, y, t)$. We obtain

$$\begin{cases} \hat{p}_t + \mathcal{L}_{SY}^E \hat{p} - \frac{1}{2}\gamma(1 - \rho^2)c(y)^2\hat{p}_y^2 \leq 0, \\ \hat{p}(s, y, t) \geq g(s, y, t), \\ \left(\hat{p}_t + \mathcal{L}_{SY}^E \hat{p} - \frac{1}{2}\gamma(1 - \rho^2)c(y)^2\hat{p}_y^2\right) \cdot \big(g(s, y, t) - \hat{p}(s, y, t)\big) = 0, \\ \hat{p}(s, y, T) = g(s, y, T), \end{cases} \tag{54}$$

for $(s, y, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [0, T]$. Here, $\mathcal{L}_{SY}^E$ is the infinitesimal generator of (S, Y) under the MEMM Q^E, namely

$$\mathcal{L}_{SY}^E w = \mathcal{L}_{SY}^0 w + l(y, t)c(y)\sqrt{1 - \rho^2}w_y, \tag{55}$$

where $\mathcal{L}_{SY}^0$ is given in (28) and

$$l(y, t) = \frac{1}{\sqrt{1 - \rho^2}} c(y)\frac{f_y(y, t)}{f(y, t)}. \tag{56}$$

As shown in ([36], Section 2), the function $l(y,t)$ is smooth and bounded, and is the risk premium corresponding to the MEMM Q^E, namely

$$\frac{\mathrm{d}Q^E}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \left(\lambda(Y_s)^2 + l(Y_s,s)^2\right)\mathrm{d}s - \int_0^T \lambda(Y_s)\,\mathrm{d}W_s + \int_0^T l(Y_s,s)\,\mathrm{d}\hat{W}_s\right). \quad (57)$$

Therefore, the operator $\mathcal{L}_{SY}^E$ is the infinitesimal generator of (S,Y) under Q^E.

It is important to notice that the fundamental difference between the variational inequalities (32) and (54) lies in the operators $\mathcal{L}_{SY}^0$ in (28) and $\mathcal{L}_{SY}^E$ in (55). Indeed, these two variational inequalities reflect, respectively, the special roles of the MMM in the forward indifference setting and the MEMM in the classical model.

Note that the classical indifference price $\hat{p}$ in (54) involves the MEMM operator $\mathcal{L}_{SY}^E$ which in turn depends on $f(t,y)$. Therefore, the computation of $\hat{p}$ requires first solving the partial differential equation (PDE) (52) followed by solving the variational inequality (54). However, in the forward indifference valuation, the indifference price can be obtained by solving only one variational inequality (32). Hence, under the forward exponential performance, the forward indifference formulation allows for more efficient computation than in the classical framework.

Remark 10. If the claim is written on Y only, say with pay-off function $g(y,t)$, then the indifference price does not depend on S. Applying a logarithmic transformation to the variational inequality (32), the nonlinear variational inequality can be linearized. Then, under Assumption 5, the forward indifference price admits the probabilistic representation:

$$p(y,t) = -\frac{1}{\gamma(1-\rho^2)}\log\inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^{Q^0}\left\{e^{-\gamma(1-\rho^2)g(Y_\tau,\tau)}|Y_t = y\right\}. \quad (58)$$

In contrast, the classical exponential utility indifference price of an American option with the same pay-off function $g(y,t)$ can be found in [32] and is given by

$$\hat{p}(y,t) = -\frac{1}{\gamma(1-\rho^2)}\log\inf_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^{Q^E}\left\{e^{-\gamma(1-\rho^2)g(Y_\tau,\tau)}|Y_t = y\right\} \quad (59)$$

with Q^E given in (57). Again, we see that the measure Q^0 in the forward performance framework plays a similar role as Q^E in the classical setting.

Remark 11. If the Sharpe ratio λ is constant, then the measures Q^0 and Q^E coincide by Remark 6. In fact, direct substitution shows that the function $f(t) := e^{-(1-\rho^2)(\lambda^2/2)(T-t)}$ solves the PDE (52). This implies that $l(y,t) = 0$ and $Q^E = Q^0$. As a result, the classical and forward indifference prices in (58) and (59) above, where the claim is written on Y only, are in fact identical.

3.4. Risk aversion and volume asymptotics

Proposition 7 provides a convenient representation for analysing the exponential forward indifference price's sensitivity with respect to risk aversion and the number of options held. Next, we further elaborate on these dependencies.

First, let us consider a risk-averse investor with local risk aversion γ who holds $a > 0$ units of American options, and suppose that all a units are constrained to be exercised simultaneously. In this case, the holder's indifference price $p(s, y, t; \gamma, a)$ is again given by (41) but with the pay-off $g(S_\tau, Y_\tau, \tau)$ replaced by $ag(S_\tau, Y_\tau, \tau)$. The optimal exercise time $\tau^*(a, \gamma)$ is the first time that the forward indifference price reaches the pay-off from exercising all a units:

$$\tau^*(a, \gamma) = \inf\{t \le u \le T : p(S_u, Y_u, u; \gamma, a) = ag(S_u, Y_u, u)\}. \tag{60}$$

PROPOSITION 12. Fix $a > 0$ and $t \in [0, T]$. If $\gamma_2 \ge \gamma_1 > 0$, then

$$p(s, y, t; \gamma_2, a) \le p(s, y, t; \gamma_1, a)$$

and

$$\tau^*(a, \gamma_2) \le \tau^*(a, \gamma_1), \quad \text{almost surely.}$$

Proof. For $\gamma_2 \ge \gamma_1 > 0$, it follows from (41) that $p(s, y, t; \gamma_2, a) \le p(s, y, t; \gamma_1, a)$. Therefore, as γ increases, $p(s, y, t; \gamma, a)$ decreases, while the pay-off $ag(s, y, t)$ does not depend on γ. By (60), this leads to a shorter exercise time (almost surely). $\qquad \square$

Furthermore, we deduce formally the risk-aversion limits of the indifference price. For the technical details, we refer the reader to Leung and Sircar [25] who have shown these asymptotic results for the traditional exponential indifference price of American options in a general semimartingale framework, and their proofs can be easily adapted here.

First, as γ increases to infinity, the penalty term in the indifference price representation (41) vanishes. Consequently, we deduce the following limit:

$$\lim_{\gamma \to \infty} p(s, y, t; \gamma, a) = a \cdot \sup_{\tau \in \mathcal{T}_{t,T}} \inf_{Q^\phi \in \mathcal{M}_f} \mathbb{E}^{Q^\phi}\{g(S_\tau, Y_\tau, \tau)|S_t = s, Y_t = y\} =: a \cdot \underline{c}(s, y, t). \tag{61}$$

This limiting price $\underline{c}(s, y, t)$ is commonly referred to as the sub-hedging price of the American options (see, for example, [20]). Interestingly, the classical indifference price also converges to the same limit as $\gamma \to \infty$ (see [25], Proposition 2.17).

On the other hand, as the holder's risk aversion γ decreases to zero, one can deduce from (41) that it is optimal not to deviate from the prior measure Q^0 (i.e. $\phi = 0$), yielding zero entropic penalty. This leads to valuing the American options under the MMM Q^0, namely

$$\lim_{\gamma \to 0} p(s, y, t; \gamma, a) = a \cdot \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{Q^0}\{g(S_\tau, Y_\tau, \tau)|S_t = s, Y_t = y\}. \tag{62}$$

In contrast, the classical indifference price converges to the risk-neutral price of the American options under the MEMM Q^E instead of Q^0.

Finally, the forward indifference price satisfies the volume-scaling property:

$$\frac{p(s, y, t; \gamma, a)}{a} = p(s, y, t; a\gamma, 1).$$

As the number of options held increases, the average indifference price $p(s, y, t; \gamma, a)/a$ will decrease, and by (60) the options will be exercised earlier. The classical indifference price for American options also possesses the same volume-scaling property and exercise phenomenon.

Moreover, the risk-aversion limits in (61) and (62) lead to the large volume limit:

$$\lim_{a \to \infty} \frac{p(s, y, t; \gamma, a)}{a} = \underline{c}(s, y, t),$$

which is the sub-hedging price, and the small volume limit:

$$\lim_{a \to 0} \frac{p(s, y, t; \gamma, a)}{a} = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{Q^0}\{g(S_\tau, Y_\tau, \tau) | S_t = s, Y_t = y\}.$$

To summarize, in all these limiting cases, both the classical and forward indifference pricing rules become *linear* with respect to quantity. In the large risk-aversion and large volume limits, the classical and forward indifference prices will both converge to the sub-hedging price. However, in the zero risk-aversion and zero volume limits, the classical and forward indifference prices, respectively, converge to the risk-neutral prices under the MEMM Q^E and the MMM Q^0. As pointed out in Remark 6, when the Sharpe ratio λ is constant, the MEMM and MMM coincide, so the corresponding zero risk-aversion and zero volume limits of the classical and forward indifference prices are in fact the same.

4. Modelling early exercises of ESOs

Now, we consider the problem of exercising ESOs under a time-monotone forward performance criterion with the risk tolerance function $R(x, t; \alpha, \beta)$ in (17). These options are American calls granted by a company to its employees as a form of compensation. A typical ESO contract prohibits the employee from selling the option and from hedging by short selling the firm's stock. The sale and hedging restrictions may induce the employee to exercise the ESO early and invest the option proceeds elsewhere. Modelling the employee's exercise timing is crucial to the accurate valuation of ESOs.

Empirical studies (see, for example, [5]) show that employees tend to exercise their ESOs very early. Recent studies, including [16] and [24], apply classical indifference pricing to ESO valuation. In those papers, the employee was assumed to have a classical exponential utility specified at the expiration date T of the options. Here, we assume a forward performance criterion for the employee, which is not anchored to a specific future time, and then numerically solve for the optimal exercise strategies under different scenarios.

We assume that the employee trades dynamically in a liquid correlated market index and a riskless money market account in order to partially hedge against his ESO position. Alternative hedging strategies for ESOs have also been proposed. For instance, Leung and Sircar [25] considered combining static hedges with market-traded European or American puts with the dynamic investment in the market index.

We focus our study on the case of a single ESO. Typically, ESOs have a vesting period during which they cannot be exercised early. The incorporation of a vesting period amounts to lifting the employee's pre-vesting exercise boundary to infinity to prevent exercise, but leaving the post-vesting policy unchanged. The case with multiple ESOs can be studied as a straightforward extension to our model though the numerical computations will be more complex and time-consuming; see [14] for the case of multiple perpetual

ESOs with exponential utility. Our main objective is to examine the non-trivial effects of forward investment performance criterion on the employee's optimal exercise timing.

4.1. The employee's optimal forward performance with an ESO

We assume that the money market account yields a constant interest rate $r \geq 0$. The *discounted* prices of the market index and the firm's stock are modelled as correlated log-normal processes, namely

$$dS_t = S_t \sigma (\lambda \, dt + dW_t) \quad \text{(traded)}, \tag{63}$$

$$dY_t = bY_t \, dt + cY_t \left(\rho \, dW_t + \sqrt{1 - \rho^2} \, d\hat{W}_t \right) \quad \text{(non-traded)}, \tag{64}$$

where λ, σ, b, c are constant parameters. The ESO studied here has a discounted capped American pay-off given by

$$g(Y_\tau, \tau) = (Y_\tau - K \, e^{-r\tau})^+ \wedge L_0, \quad \text{for } \tau \in \mathcal{T}_{0,T},$$

where T is the expiration date and L_0 is a large upper bound to be used in our numerical method (see Section 4.2).

This market set-up is nested in the Itô diffusion market described in Section 2. Here, the Sharpe ratio λ of the index S is now a constant, and the option pay-off is independent of S. The employee trades dynamically in the index S and the money market account, so his discounted wealth process satisfies

$$dX_t^\pi = \pi_t \sigma (\lambda \, dt + dW_t). \tag{65}$$

We proceed with the employee's forward performance criterion $U_t(x)$. First, we adopt the risk tolerance function in (17), namely, $R(x,t) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}$, and the corresponding dynamic risk preference function $u(x,t)$ given in Proposition 4. Then, we apply Theorem 3 to obtain the employee's forward performance $U_t(x) = u(x, \lambda^2 t)$. In turn, the employee's maximal forward performance in the presence of the ESO is given by

$$V(x, y, t) = \sup_{\tau \in \mathcal{T}_{t,T}} \sup_{\pi \in \mathcal{Z}_{t,\tau}} \mathbb{E} \{ u(X_\tau^\pi + g(Y_\tau, \tau), \lambda^2 \tau) | X_t = x, Y_t = y \}. \tag{66}$$

In contrast to the stochastic volatility problem in Section 3, the option pay-off depends on Y only, and the state variable S disappears from the value function V.

To solve for the employee's value function, we look for a solution to the following HJB variational inequality:

$$\begin{cases} V_t + \mathcal{L}_Y V - \frac{(\rho c y V_{xy} + \lambda V_x)^2}{2 V_{xx}} \leq 0, \\ V(x, y, t) \geq u(x + g(y, t), \lambda^2 t), \\ \left(V_t + \mathcal{L}_Y V - \frac{(\rho c y V_{xy} + \lambda V_x)^2}{2 V_{xx}} \right) \cdot (u(x + g(y, t), \lambda^2 t) - V(x, y, t)) = 0, \\ V(x, y, T) = u(x + g(y, T), \lambda^2 T), \end{cases} \tag{67}$$

for $(x, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times [0, T]$, with $\mathcal{L}_Y V = (1/2)c(y)^2 V_{yy} + b(y) V_y$.

We remark that the variational inequality (67) is highly nonlinear, and it can be simplified only for very special local utility functions. In the exponential forward performance case, this can be viewed as a special case under the stochastic volatility model discussed in Section 3. In the perpetual case with exponential utility, Henderson [17] derives an explicit solution for the value function. Recent ESO valuation models, including [14,16,24], are also designed with the classical exponential utility. As for the general case, we do not attempt to address the related existence, uniqueness and regularity questions.

4.2. Numerical solutions

We apply a *fully explicit* finite-difference scheme to numerically solve (67) for the employee's optimal exercising strategy. First, we restrict the domain $\mathbb{R} \times \mathbb{R}_+ \times [0, T]$ to a finite domain $\mathcal{D} = \{(x, y, t) : -L_1 \leq x \leq L_2, 0 \leq y \leq L_0, 0 \leq t \leq T\}$, where L_k, $k = 0, 1, 2$, are chosen to be sufficiently large to preserve the accuracy of the numerical solutions.

Next, a number of boundary conditions are imposed. Along $y = 0$, the firm's stock price, and thus the ESO, become worthless. Therefore, we set $V(x, 0, t) = u(x, \lambda^2 t)$. When Y hits the high-level L_0, we assume that the ESO will be exercised there, implying the condition

$$V(x, L_0, t) = u(x + g(L_0, t), \lambda^2 t).$$

Along $x = -L_1$ and $x = L_2$, we adopt the Dirichlet boundary conditions

$$V(-L_1, y, t) = u(-L_1 + g(y, t), \lambda^2 t) \quad \text{and} \quad V(L_2, y, t) = u(L_2 + g(y, t), \lambda^2 t),$$

which imply that the employee will exercise the ESO at these boundaries. Over a uniform grid, we apply an explicit finite-difference approximations and solve for V iteratively backward in time starting at T.

At each time step, the inequality constraint $V(x, y, t) \geq u(x + g(y, t), \lambda^2 t)$ is enforced. By comparing the value function and the obstacle term, we identify the *continuation region* $\mathcal{C}$ where the ESO is not exercised, and the *exercise region* $\mathcal{E}$ where it is exercised, namely

$$\mathcal{C} = \{(x, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times [0, T] : V(x, y, t) > u(x + g(y, t), \lambda^2 t)\}, \tag{68}$$

$$\mathcal{E} = \{(x, y, t) \in \mathbb{R} \times \mathbb{R}_+ \times [0, T] : V(x, y, t) = u(x + g(y, t), \lambda^2 t)\}. \tag{69}$$

From the numerical example in Figure 3, we observe that the value function dominates the obstacle term. At any time t and wealth x, we locate the optimal stock price level $y^*(x, t)$ that separates the two regions $\mathcal{C}$ and $\mathcal{E}$. As a result, the employee will exercise the ESO as soon as Y_t hits the threshold $y^*(X_t, t)$:

$$\tau^* = \inf\{0 \leq t \leq T : Y_t = y^*(X_t, t)\}. \tag{70}$$

In the case of call options, the boundary lies above the strike K. Figure 4 shows an example of the optimal exercise boundary for the ESO.

4.3. Behaviour of the optimal exercise policy

We illustrate the employee's optimal exercise boundary in Figure 4. Not surprisingly, the exercise boundary $y^*(x, t)$ decreases with respect to time, which implies that the employee is willing to exercise the ESO at a lower stock price as it gets closer to expiry.
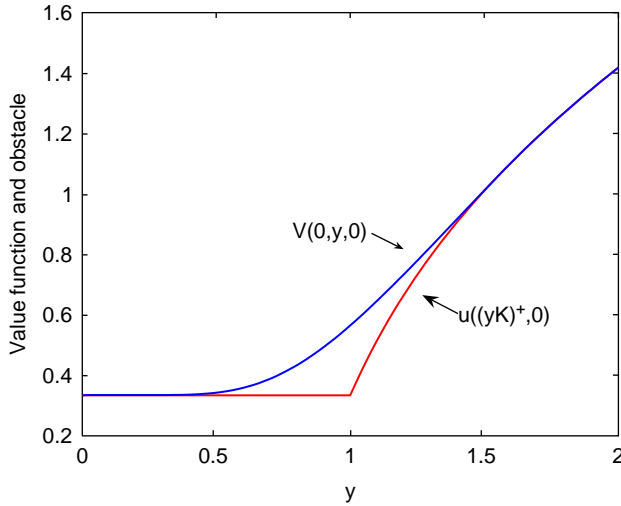
Figure 3. The value function $V(x, y, t)$ dominates the obstacle term $u(x + g(y, t), \lambda^2 t)$. The parameters are $\lambda = 33\%$, $\sigma = 35\%$, $b = 6\%$, $c = 40\%$, $\rho = 50\%$, $r = 1\%$, $K = 1$, $T = 1$, $\alpha = 4$ and $\beta = 0.25$. At $t = 0$ and $x = 0$, the critical stock price $y^*(0, 0) = 1.58$ is the point at which the value function touches the obstacle term (above the strike).

From Figure 5, we observe that the exercise boundary is wealth dependent. The employee tends to delay exercising the ESO when his wealth deviates away from zero. We can gain some intuition from our choice of risk tolerance function $R(x, t; \alpha, \beta)$. As wealth approaches zero, the employee's risk tolerance decreases (recall Figure 1) or, equivalently, risk aversion increases. Higher risk aversion influences the employee to exercise earlier to secure small gains rather than waiting for future uncertain pay-offs.
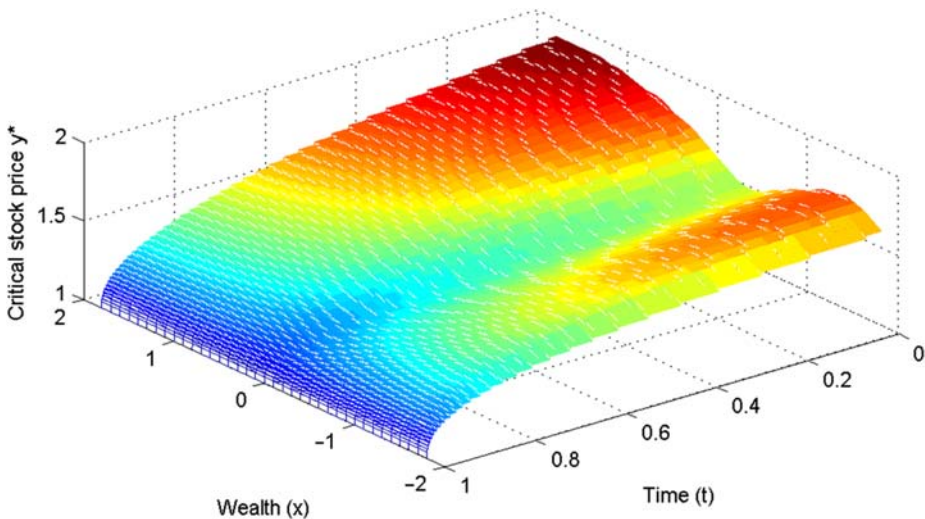

Figure 4. The optimal exercise policy is characterized by the critical stock price $y^*(x, t)$ as a function of wealth x and time t. It decreases as time approaches maturity. In addition, it tends to shift lower as wealth is near zero.
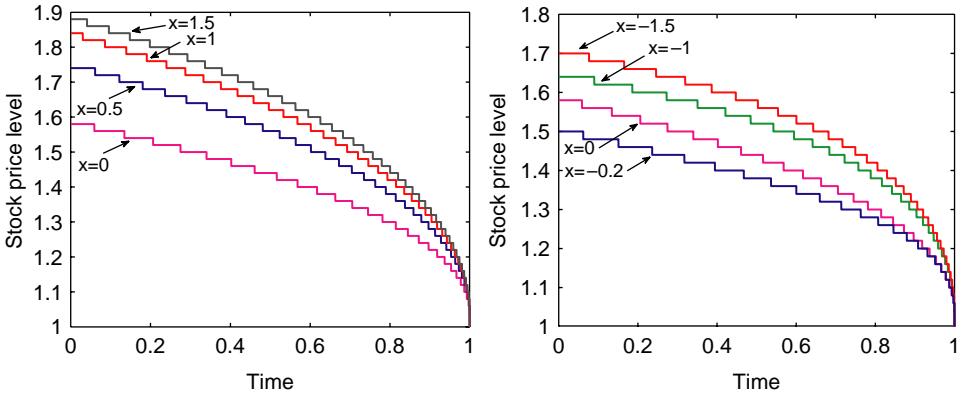
Figure 5. The optimal exercise boundary represents the critical stock price level at which the ESO is exercised, and varies for different wealth level x. Left: the exercise boundary shifts upward as wealth x increases from 0 to 1.5. Right: the exercise boundary is the lowest when wealth $x = -0.2$. As wealth decreases from -1 to -1.5, the exercise boundary rises again above the boundary with $x = 0$. The parameters here are the same as in Figure 3.

Finally, we show in Figure 6 that the exercise boundary tends to shift upward for higher values of α and β, given the initial wealth $x = 0$. The effect of β is intuitive because the risk tolerance function is increasing with respect to β. Therefore, the option holder with a higher β is effectively less risk averse and may be willing to hold on to the ESO longer.

5. Marginal forward indifference price of American options

In this section, we introduce the marginal forward indifference price of American options. A related concept in the classical utility framework is the *marginal utility price* introduced by Davis [7], which is useful as an approximation for pricing a small number of claims. For completeness and the upcoming comparison with the forward analogue, we provide a brief review of the marginal utility price in the diffusion market.
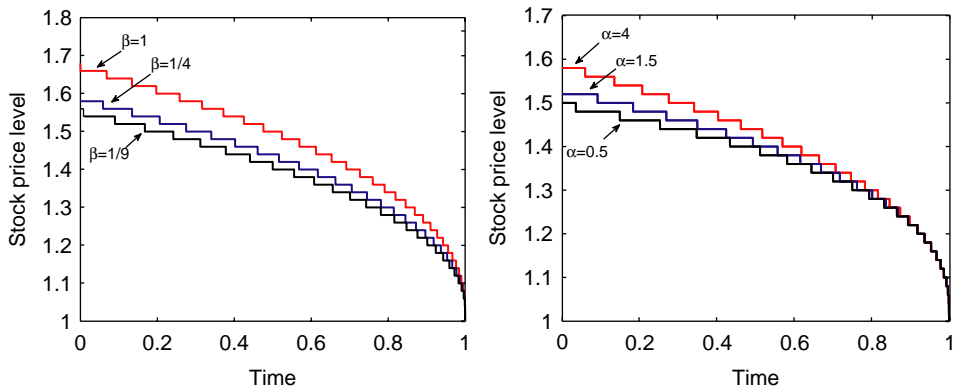

Figure 6. With initial wealth $x = 0$, the optimal exercise boundary varies for different values of β and α. Left: a higher value of β leads to a higher exercise boundary. Right: a higher value of α shifts the exercise boundary upward. The parameters here are taken to be same as those in Figure 3, except for α and β specified in the figures above.

5.1. *The classical marginal utility price*

In traditional utility maximization, the investor's risk aversion is modelled by a deterministic utility function, say $\hat{U}(x)$, defined at time T. In the Itô diffusion market introduced in Section 2, the investor trades dynamically between the money market and stock S, and solves the Merton portfolio optimization problem in (4).

Next, suppose that the investor decides to buy δ units of a European claim, each offering pay-off $C_T \in \mathcal{F}_T$. The marginal utility price is defined as the per-unit price that the investor is willing to pay for an infinitesimal position ($\delta \approx 0$) in the claim. This concept is introduced by Davis [7]. He shows by a formal small δ expansion that the investor's marginal utility price at time t is given by

$$\hat{h}_t = \frac{\mathbb{E}\left\{\hat{U}'\left(\hat{X}_T^*\right) C_T | \mathcal{F}_t\right\}}{M_t'(X_t)}, \quad t \in [0, T], \tag{71}$$

where $\hat{X}_T^*$ is the optimal terminal wealth for the Merton problem $M_t(X_t)$ defined in (4), and $\hat{U}'$ and M_t' are the derivatives with respect to the wealth argument. Kramkov and Sirbu [23] directly adopt (71) as the definition of the marginal utility price for European claims, which we also adapt to the case of American options.

DEFINITION 13. The marginal utility price process $(h_t)_{0 \leq t \leq T}$ for an American option with pay-off process $(g_t)_{0 \leq t \leq T}$ is defined as

$$h_t = \frac{\underset{\tau \in \mathcal{T}_{t,T}}{\text{ess sup}}\, \mathbb{E}\left\{M_\tau'\left(\hat{X}_\tau^*\right) g_\tau | \mathcal{F}_t\right\}}{M_t'(X_t)}, \tag{72}$$

where $M_t(X_t)$ is given in (4).

Among others, one important question is under what conditions will the marginal utility price be independent of the investor's wealth. In the classical setting for options without early exercise, wealth independence of marginal utility prices is very rare. In fact, Kramkov and Sirbu [23] show that only exponential and power utilities yield wealth-independent marginal utility prices for any pay-off and in any financial market.

5.2. *The marginal forward indifference price formula*

Following the definition of the classical marginal indifference price, we introduce the marginal forward indifference price for our model. Henceforth, we will give the definitions and results based on the Itô diffusion market settings described in Section 2, where the discounted stock price S follows (2) and the option holder's trading wealth X_t follows (3).

DEFINITION 14. Let $U_t(x) = u(x, A_t)$, with $A_t = \int_0^t \lambda_s^2\, ds$, be the investor's forward performance process, and assume X^* is the optimal wealth process in (15) (cf. Theorem 3). The marginal forward indifference price process $(\tilde{p}_t)_{0 \leq t \leq T}$ for an American option with an $\mathcal{F}_t$-adapted bounded pay-off process $(g_t)_{0 \leq t \leq T}$ is defined as

$$\tilde{p}_t = \frac{\underset{\tau \in \mathcal{T}_{t,T}}{\text{ess sup}}\, \mathbb{E}\left\{u_x\left(X_\tau^*, A_\tau\right) g_\tau | \mathcal{F}_t\right\}}{u_x\left(X_t^*, A_t\right)}. \tag{73}$$

At first glance, the marginal forward indifference price in (73) might depend on the holder's risk preferences and wealth. However, as the next result shows, under a time-monotone forward performance the marginal forward indifference price is *independent* of both of these inputs, and is simply given as the expected discounted pay-off under the MMM, regardless of the investor's forward performance criterion.

THEOREM 15. The marginal forward indifference price of an American option with pay-off process $(g_t)_{0 \leq t \leq T}$ is given by

$$\tilde{p}_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{Q^0}\{g_\tau | \mathcal{F}_t\}, \tag{74}$$

where Q^0 is the MMM. Consequently, $\tilde{p}_t$ is independent of both the holder's wealth and his forward performance criterion.

Proof. Comparing (73) and (74), we observe that it is sufficient to show that

$$\frac{u_x(X_\tau^*, A_\tau)}{u_x(X_t^*, A_t)} = \exp\left(-\frac{1}{2}\int_t^\tau \lambda_s^2 \, ds - \int_t^\tau \lambda_s \, dW_s\right), \quad \tau \in \mathcal{T}_{t,T}. \tag{75}$$

Indeed, since λ is bounded, this leads to the desired measure change from the historical measure $\mathbb{P}$ to the MMM Q^0.

Applying Itô's formula to $u_x(X_t^*, A_t)$ and using the SDE (15) for X^* gives

$$du_x(X_t^*, A_t) = \lambda_t^2\left(u_{xt}(X_t^*, A_t) + R(X_t^*, A_t)u_{xx}(X_t^*, A_t) + \frac{R(X_t^*, A_t)^2}{2}u_{xxx}(X_t^*, A_t)\right) dt$$
$$+ \lambda_t R(X_t^*, A_t)u_{xx}(X_t^*, A_t) \, dW_t. \tag{76}$$

Next, we show that the drift vanishes. First, it follows from differentiating $u(x, t)$ in (11) that

$$u_{xt} = u_x - \frac{u_x^2 u_{xxx}}{2u_{xx}^2}.$$

Using this and the fact that $R(x, t) = -u_x(x, t)/u_{xx}(x, t)$ to (76), we see that the drift in (76) becomes zero. As a result, the SDE (76) simplifies to

$$du_x(X_t^*, A_t) = \lambda_t R(X_t^*, A_t)u_{xx}(X_t^*, A_t) \, dW_t$$
$$= -\lambda_t u_x(X_t^*, A_t) \, dW_t.$$

This implies that the process $(u_x(X_t^*, A_t))_{t \geq 0}$ is given by the stochastic exponential representation in (75). Hence, by a change of measure, formula (74) follows. □

Theorem 15 illustrates a crucial feature of the forward indifference pricing mechanism. If we consider that, in a general Itô diffusion market, different investors adopt different forward performances according to Theorem 3, then their marginal forward indifference prices for an American claim will necessarily be the same, regardless of their wealth and choices of forward performance. In particular, this is true for the

stochastic volatility model in (22) and (23) and the basis risk model in (63) and (64). In contrast, the classical marginal utility price for a general utility function is typically wealth and utility dependent ([23], Theorem 7). In the basis risk model as a special case, Kramkov and Sirbu [23] show that the marginal utility price is also found from pricing under the MMM, thus coinciding with the forward counterpart, even though they are derived from very different performance mechanisms.

6. Conclusions and extensions

In summary, we have discussed the forward indifference valuation for American options in an incomplete model with a stochastic factor. We have applied it to value American options under stochastic volatility and model the early exercises of ESOs. The option holder's optimal hedging and exercising strategies are found from solving the underlying variational inequalities.

The forward indifference valuation mechanism is profoundly different from the mechanism in the classical approach. This is best illustrated in Section 3, in which the exponential forward indifference price is expressed in terms of relative entropy minimization with respect to the MMM, rather than with respect to the MEMM, as is the case in the traditional setting. Lastly, we also introduced the marginal forward indifference price. In contrast to the classical marginal utility price, the marginal forward indifference price based on any time-monotone forward performance is independent of both the investor's wealth and the particular form of time-monotone forward performance, and is given as the risk-neutral expectation under the MMM.

Several major challenges and interesting problems remain for future investigation. These include the existence and regularity results for the variational inequalities associated with the optimal forward performance and the forward indifference price. The nonlinearity of the variational inequalities also requires the development of efficient numerical schemes. Moreover, even though we have focused on the valuation of American options, it is important to examine its impact in the host of other applications where traditional utility valuation has been used, for example, credit derivatives [19,26], volatility derivatives [15], insurance products [2] and order book modelling [1]. In all of these, the exponential utility is chosen for its convenient analytic properties. The forward performance criterion provides a convenient tool to (i) move away from exponential utility and (ii) remove the horizon dependence.

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Notes

1. Email: sircar@princeton.edu
2. Email: thaleia.zariphopoulou@oxford-man.ox.ac.uk

References

[1] M. Avellaneda and S. Stoikov, *High-frequency trading in a limit order book*, Quant. Finance 8 (2008), pp. 217–224.
[2] E. Bayraktar and M. Ludkovski, *Relative hedging of systematic mortality risk*, North Am. Actuar. J. 13 (2009), pp. 106–140.
[3] F. Benth and K. Karlsen, *A PDE representation of the density of the minimal entropy martingale measure in stochastic volatility markets*, Stoch. Stoch. Rep. 77 (2005), pp. 109–137.

[4] F. Berrier, L. Rogers, and M. Tehranchi, *A characterization of forward utility functions*, preprint, University of Cambridge, 2009.

[5] J.C. Bettis, J.M. Bizjak, and M.L. Lemmon, *Exercise behaviors, valuation, and the incentive effects of employee stock options*, J. Financ. Econ. 76 (2005), pp. 445–470.

[6] P. Cheridito and M. Kupper, *Recusiveness of indifference prices and translation-invariant preferences*, Mat. Financ. Econ. 2 (2009), pp. 173–188.

[7] M. Davis, *Option pricing in incomplete markets*, *Mathematics of Derivatives Securities*, Cambridge University Press, Cambridge, 1997, pp. 227–254.

[8] F. Delbaen, P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker, *Exponential hedging and entropic penalties*, Mat. Financ. 12 (2002), pp. 99–123.

[9] N. El Karoui and M. M'Rad, *Stochastic utilities with a given optimal portfolio: Approach by stochastic flows*, Technical Report, 2010.

[10] H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, in *Applied Stochastic Analysis, Stochastics Monographs*, M. Davis and R. Elliot, eds., Vol. 5, Gordon and Breach, London/New York, 1990, pp. 389–414.

[11] R. Frey, *Derivative asset analysis in models with level-dependent and stochastic volatility*, CWI Quart. 10 (1997), pp. 1–34.

[12] M. Fritelli, *The minimal entropy martingale measure and the valuation problem in incomplete markets*, Mat. Finance 10 (2000), pp. 39–52.

[13] P. Grandits and T. Rheinländer, *On the minimal entropy martingale measure*, Ann. Probab. 30 (2002), pp. 1003–1038.

[14] M. Grasselli and V. Henderson, *Risk aversion and block exercise of executive stock options*, J. Econ. Dyn. Control 33 (2009), pp. 109–127.

[15] M. Grasselli and T. Hurd, *Indifference pricing and hedging for volatility derivatives*, Appl. Mat. Finance 14 (2008), pp. 303–317.

[16] V. Henderson, *The impact of the market portfolio on the valuation, incentives and optimality of executive stock options*, Quant. Finance 5 (2005), pp. 1–13.

[17] V. Henderson, *Valuing the option to invest in an incomplete market*, Mat. Financ. Econ. 1 (2007), pp. 103–128.

[18] V. Henderson and D. Hobson, *Horizon-unbiased utility functions*, Stoch. Process. Appl. 117 (2007), pp. 1621–1641.

[19] S. Jaimungal and G. Sigloch, *Incorporating risk and ambiguity aversion into a hybrid model of default*, Mat. Finance 22 (2012), pp. 57–81.

[20] I. Karatzas and S. Kou, *Hedging American contingent claims with constrained portfolios*, Finance Stoch. 2 (1998), pp. 215–258.

[21] I. Karatzas and S. Shreve, *Methods of Mathematical Finance*, Springer, New York, 1998.

[22] S. Klöppel and M. Schweizer, *Dynamic indifference valuation via convex risk measures*, Mat. Finance 17 (2006), pp. 599–627.

[23] D. Kramkov and M. Sirbu, *Sensitivity analysis of utility based prices and risk-tolerance wealth processes*, Ann. Appl. Probab. 16 (2006), pp. 2140–2194.

[24] T. Leung and R. Sircar, *Accounting for risk aversion, vesting, job termination risk and multiple exercises in valuation of employee stock options*, Mat. Finance 19 (2009), pp. 99–128.

[25] T. Leung and R. Sircar, *Exponential hedging with optimal stopping and application to ESO valuation*, SIAM J. Control Optim. 48 (2009), pp. 1422–1451.

[26] T. Leung, R. Sircar, and T. Zariphopoulou, *Credit derivatives and risk aversion*, in *Advances in Econometrics*, T. Fomby, J.P. Fouque, and K. Solna, eds., Vol. 22, Elsevier Science, 2008, DOI: http://dx.doi.org/10.1016/S0731-9053(08)22011-6.

[27] M. Mania and M. Schweizer, *Dynamic exponential utility indifference valutation*, Ann. Appl. Probab. 15 (2005), pp. 2113–2143.

[28] R. Merton, *Lifetime portfolio selection under uncertainty: The continuous time model*, Rev. Econ. Stud. 51 (1969), pp. 247–257.

[29] M. Musiela and T. Zariphopoulou, *Portfolio choice under dynamic investment performance criteria*, Quant. Finance 9 (2008), pp. 161–170.

[30] M. Musiela and T. Zariphopoulou, *Portfolio choice under space–time monotone performance criteria*, SIAM J. Financ. Mat. 1 (2010), pp. 326–365.

[31] M. Musiela and T. Zariphopoulou, *Stochastic partial differential equations and portfolio choice*, in *Contemporary Quantitative Finance*, C. Chiarella and A. Novikov, eds., Springer, Berlin Heidelberg, 2010, pp. 195–216.

[32] A. Oberman and T. Zariphopoulou, *Pricing early exercise contracts in incomplete markets*, Comput. Manag. Sci. 1 (2003), pp. 75–107.

[33] B. Oksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, Springer, Berlin Heidelberg, 2005.

[34] H. Pham, T. Rheinländer, and M. Schweizer, *Mean-variance hedging for continuous processes: New proofs and examples*, Finance Stoch. 2 (1998), pp. 173–198.

[35] R. Rouge and N. El Karoui, *Pricing via utility maximization and entropy*, Mat. Finance 10 (2000), pp. 259–276.

[36] R. Sircar and T. Zariphopoulou, *Bounds and asymptotic approximations for utility prices when volatility is random*, SIAM J. Control Optim. 43 (2005), pp. 1328–1353.

[37] T. Zariphopoulou and T. Zhou, *Investment performance measurement under asymptotically linear local risk tolerance*, in *Handbook of Numerical Analysis*, P. Ciarlet, A. Bensoussan, and Q. Zhang, eds., North-Holland, Amsterdam, 2009, pp. 227–254.

[38] G. Žitković, *A dual characterization of self-generation and exponential forward performances*, Ann. Appl. Probab. 19 (2009), pp. 2176–2210.

Appendix: Admissibility and characterization of time-monotone forward performances

Musiela and Zariphopoulou [30] have shown that there exists a class of *admissible* initial conditions, $u_0(x)$, for which the time-monotone performance is well defined and the associated optimal portfolio process can be explicitly constructed. In this appendix, we highlight some of the main results relevant to our study.

The class of admissible initial conditions is given via a positive, finite Borel measure which is, in turn, linked with a space–time harmonic function [see (78) and (80) below]. As in [30], we define the set of measures $\mathcal{B}^+(\mathbb{R})$ by

$$\mathcal{B}^+(\mathbb{R}) = \left\{ \nu \in \mathcal{B}(\mathbb{R}) : \forall B \in \mathcal{B}, \nu(B) \geq 0 \text{ and } \int_{\mathbb{R}} e^{yx} \nu(dy) < \infty, \ x \in \mathbb{R} \right\}. \quad (77)$$

PROPOSITION 16 ([30], PROPOSITION 3). (i) *Let $\nu \in \mathcal{B}^+(\mathbb{R})$. Then, the function h defined, for $(x, t) \in \mathbb{R} \times [0, +\infty)$, by*

$$h(x, t) = \int_{\mathbb{R}} \frac{e^{yx - ((1/2)y^2 t)} - 1}{y} \nu(dy) + C, \quad (78)$$

is a strictly increasing solution to the PDE:

$$h_t + \frac{1}{2} h_{xx} = 0. \quad (79)$$

(ii) *Assume that h above is of full range for each $t \geq 0$, and let $h^{(-1)} : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ be its spatial inverse. Then, the function u defined by*

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s) + (s/2)} h_x\left(h^{(-1)}(x, s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz, \quad (80)$$

for $(x, t) \in \mathbb{R} \times [0, +\infty)$, is an increasing and strictly concave solution of the PDE (11).

The above result yields a class of admissible initial data for a forward performance process. Precisely, a function $u_0 : \mathbb{R} \to \mathbb{R}$ is admissible, if it can be represented as

$$u_0(x) = \int_0^x e^{-h^{(-1)}(z,0)} \, dz, \quad x \in \mathbb{R}, \tag{81}$$

where $h^{(-1)}$ is the spatial inverse of h defined in (78). Moreover, once the measure ν in (78) is defined, the function h yields directly a dynamic preference function u satisfying (11).

The following result, taken from [30], provides the explicit construction of the optimal portfolio and the optimal wealth process. In establishing this result, they rigorously proved the admissibility of the optimal portfolio π^* under an integrability condition on the measure ν given in (82) below.

THEOREM 17 ([30], THEOREM 4). (i) Let h be a strictly increasing solution to (79), for $(x, t) \in \mathbb{R} \times [0, +\infty)$, and assume that the associated measure ν satisfies

$$\int_{\mathbb{R}} e^{yx + ((1/2)y^2 \hat{t})} \nu(dy) < +\infty. \tag{82}$$

Let also A_t be as in (10) and introduce m_t, $t \geq 0$, as

$$m_t = \int_0^t \lambda_s \, dW_s.$$

Define the processes X_t^* and π_t^* by

$$X_t^* = h\big(h^{(-1)}(x, 0) + A_t + m_t, A_t\big) \tag{83}$$

and

$$\pi_t^* = h_x\big(h^{(-1)}(X_t^*, A_t), A_t\big) \frac{\lambda_t}{\sigma_t}, \tag{84}$$

for $t \geq 0$, $x \in \mathbb{R}$ with h as above and $h^{(-1)}$ standing for its spatial inverse. Then, the portfolio π_t^* is *admissible* and generates X_t^*, i.e.

$$X_t^* = x + \int_0^t \sigma_s \pi_s^* (\lambda_s \, ds + dW_s). \tag{85}$$

(ii) Let u be associated with h increasing and strictly concave solution to (11). Then, the process $u(X_t^*, A_t)$, $t \geq 0$, satisfies the SDE

$$du\big(X_t^*, A_t\big) = u_x\big(X_t^*, A_t\big) \sigma_t \pi_t^* \, dW_t \tag{86}$$

with X_t^* and π_t^* as in (83) and (84).

(iii) Let $U_t(x)$, $t \geq 0$, $x \in \mathbb{R}$ be given by (12) with u_0 being an admissible initial condition. Then, the processes X_t^* and π_t^* are optimal.

From (14) and (80), it can be shown that the local risk tolerance function is given by

$$R(x, t) = h_x\big(h^{-1}(x, t), t\big) \tag{87}$$

with h as in (78). Since both $h(x, t)$ and $u(x, t)$ are completely characterized by the measure ν, the same holds for the local risk tolerance function $R(x, t)$ in (87). In Example 12 of [30], it was shown that the measure linked to the parametric risk tolerance function $R(x, t; \alpha, \beta)$ in (17) is given by

$$\nu(\mathrm{d}y) = \frac{\sqrt{\beta}}{2}\left(\delta_{\sqrt{\alpha}} + \delta_{-\sqrt{\alpha}}\right),$$

with $\delta_{\pm\sqrt{\alpha}}$ are Dirac measures at $\pm\sqrt{\alpha}$. Hence, it is clear that this measure satisfies the integrability condition (82) in Theorem 17. Finally, in view of (78), the associated space–time harmonic function is given by

$$h(x, t) = \sqrt{\frac{\beta}{\alpha}}\mathrm{e}^{-((1/2)\alpha t)}\sin h\left(\sqrt{\alpha}x\right).$$

Using this, the optimal portfolio and wealth processes are in turn explicitly constructed as in (84) and (85).