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An approximation scheme for solution to the optimal investment problem in incomplete markets

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Abstract. We provide an approximation scheme for the maximal expected utility and optimal investment policies for the portfolio choice problem in an incomplete market. Incompleteness stems from the presence of a stochastic factor which affects the dynamics of the correlated stock price. The scheme is built on the Trotter-Kato approximation and is based on an intuitively pleasing splitting of the Hamilton-Jacobi-Bellman (HJB) equation in two sub-equations. The first is the HJB equation of a portfolio choice problem with a stochastic factor but in a complete market, while the other is a linear equation corresponding to the evolution of the orthogonal (non-traded) part of the stochastic factor. We establish convergence of the scheme to the unique viscosity solution of the marginal HJB equation, and, in turn, derive a computationally tractable representation of the maximal expected utility and construct an ε-optimal portfolio in a feedback form.

Key words. Merton's problem, Hamilton-Jacobi-Bellman equation, Trotter-Kato splitting, viscosity solutions

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1. Introduction.

This paper is a contribution to the analysis of stochastic optimization problems arising in models of optimal portfolio choice in incomplete markets. Incompleteness stems from a stochastic factor that affects the dynamics of the traded stock, with which it is imperfectly correlated. The stock price and the level of the stochastic factor are modeled as a two-dimensional diffusion process. The investor trades between the stock and a riskless security, and aims to maximize her expected utility of terminal wealth. Stochastic factors are ubiquitous modeling elements and have been widely used for the representation of time-varying stock returns, volatility of stocks as well as stochastic interest rates (see the review article [44] for extensive bibliography).

The model herein is the simplest and most direct extension of the log-normal one considered originally by Merton (see the seminal papers [31] and [32]), when the market becomes incomplete. However, little is known about the maximal expected utility as well as the form and properties of the optimal policies once the log-normality assumption is relaxed and imperfect correlation between the stock and the factor is introduced. This is despite the Markovian nature of the problem at hand, the advances in theories of fully nonlinear PDEs and stochastic control, and the computational tools that exist today. Specifically, results on the validity of the Dynamic Programming Principle, regularity of the value function, existence and verification of the optimal feedback controls, explicit representation of the value function and numerical approximations are still lacking. We highlight some of these issues next.

The Markovian assumptions on the stock price and the stochastic factor dynamics allow us to study the value function via the associated Hamilton-Jacobi-Bellman (HJB) equation,

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stated in (2.10) herein. Fundamental results in the theory of controlled diffusions yield that if the value function is smooth enough then it satisfies the HJB equation. Moreover, optimal policies may be constructed in a feedback form from the first-order conditions in the HJB equation, provided that the candidate feedback process is admissible and the wealth SDE has a strong solution when the candidate control is used. The latter usually requires further regularity on the value function. In the reverse direction, a smooth solution of the HJB equation, which also satisfies the appropriate terminal and boundary (or growth) conditions, may be identified with the value function, provided the solution is unique in the appropriate sense. These results are usually known as the "verification theorem" and we refer the reader to [12], [26] and [42] for a general exposition on the subject.

In maximal expected utility problems, it is rarely the case that the arguments in either direction of the verification theorem can be established. Indeed, it is difficult to show a priori regularity of the value function, with the main difficulties coming from the non-compactness of the set of admissible policies, state constraints and the unboundedness of the spatial domain. Similar reasons lead to possible degeneracies and singularities in the HJB equation, making it very difficult to establish existence, uniqueness and regularity of its solutions. As a matter of fact, in some cases, the strong regularity results that are needed for constructing optimal feedback controls might not even hold.

Partial results pertinent to the issues described above can be found, among others, in [5], where a "weak" form of the DPP has been established in a rather general setting, and in [22], [24] and [35], where, in particular, the existence and second order differentiability of the value function were established under very general assumptions on the model and the utility function. More recently, the authors of [4] have initiated a methodology, based on the stochastic solutions of Strook and Varadhan, which bypasses some of the above technical problems.

When the utility is homothetic (exponential, power and logarithmic) most of the above difficulties are bypassed because of convenient scaling properties of the problem. The HJB equation is reduced to a quasilinear one which, in some cases, can even be solved explicitly (see, among others, [6], [11], [21], [27], [28], [33], [43]). The analysis, then, simplifies considerably both from the analytic as well as the probabilistic points of view.

We stress that there is a very rich body of research for the analysis of the expected utility models which is based on duality techniques. This powerful approach is applicable to general market models, and yields elegant and universal results as well as various useful insights for the value function and the optimal wealth processes via the dual minimizers (see, among others, [18], [22], [23], [35] and [36]). When the market is complete, the duality results allow to solve the optimal investment problem in full generality. However, when incompleteness is introduced, the applicability of duality methods for studying the structure and properties of the value function and optimal portfolios is limited. Because of their volume as well as their different nature and focus, the duality results are not discussed herein.

In this paper, we provide a new way to study the optimal investment problem in a Markovian model. The new method, on the one hand, offers useful insights and, on the other, provides a mathematically rigorous approach to numerical approximations of the value function and the optimal policies. The main idea is to apply the Trotter-Kato approximation scheme to split the HJB equation in two simpler ones. The specific choice of the splitting is pivotal in the analysis: the first equation is non-linear, namely of HJB type, and corresponds to the portfolio choice problem in a complete market where the stochastic factor is perfectly correlated with the stock. The second equation is linear and related to the evolution of the part of stochastic factor that is orthogonal to the stock. From a conceptual point of view, the scheme provides a quite intuitive way to analyze the portfolio problem locally – and not globally, as it has been the case so far – via an infinitesimal decomposition of the solution into a controlled complete market component and a non-controlled component generated by the non-hedgeable part of the stochastic factor.

The analysis of the emerging partial differential equations associated with the original HJB equation presents various difficulties, for their generators are potentially degenerate and singular, due to the very nature of the underlying optimization problem. As a consequence, classical results for their analysis cannot be applied directly. In turn, results on the existence and uniqueness of their solutions as well as the convergence of the numerical schemes are not available.

Our contribution is multifold. We first analyze the marginal HJB equation, that is the equation satisfied by the x-derivative of the candidate value function. We build an approximation scheme associated with this equation and prove its convergence. We also show that the limit of this approximation is the unique viscosity solution to the marginal HJB equation. We, then, integrate this solution with respect to the spatial variable and prove that the result of the integration is the value function of the original optimization problem. We extend these results to a regularized version of the marginal HJB equation, showing, in addition, the smoothness of its solutions. Finally, we use these smooth solutions to produce ε-optimal portfolio processes in a feedback form. We compute the performance of these policies and show that they can approximate the maximal expected utility with arbitrary precision.

The overall contribution of this paper is the construction of computationally tractable solutions – both for the value function and the ε-optimal policies – in the presence of market incompleteness and under arbitrary risk preferences and general stochastic factor dynamics. The construction is based on an intuitive splitting of the investment problem and highlights in a transparent way the interplay between market incompleteness, and evolution of the stochastic factor and risk preferences.

The paper is organized as follows. In Section 2, we present the market model and the maximal expected utility problem. In Section 3, we introduce the approximation method and discuss the main ideas and intuition behind the splitting of the HJB equation. In Section 4, we focus on the marginal HJB equation, to which the splitting methodology is applied, and analyze the emerging auxiliary sub-problems. In Section 5, we establish the uniqueness of viscosity solutions and the convergence of the approximation scheme for the marginal HJB equation transformed by a change of variables. Section 6 contains the main results, specifically, the convergence of the approximation scheme to the solution of the marginal HJB equation, the representation of the value function via this solution and the construction of the feedback ε-optimal portfolios.

2. Optimal investment problem in the stochastic factor model. The market consists of two assets, a riskless and a risky one. The former is a bond earning constant interest rate, r,[1] while the latter is a stock whose price, $S=(S_t)_{t\geq 0}$, solves

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^1, \tag{2.1}$$

with $S_0>0$. The stochastic factor $Y=(Y_t)_{t\geq 0}$, satisfies

$$dY_t = b(Y_t)dt + a(Y_t)\left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \tag{2.2}$$

with $Y_0\in\mathbb{R}$. The process $W=\left(W_t^1,W_t^2\right)_{t\geq 0}$ is a standard Brownian motion on the filtered probability space $\left(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}\right)$, with its natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Choosing the bond as a numéraire, we introduce the market price of risk process, $(\lambda_t)_{t\geq 0}$,

$$\lambda_t = \lambda(Y_t) = \frac{\mu(Y_t)-r}{\sigma(Y_t)}. \tag{2.3}$$

Next, we introduce the following assumptions on the model coefficients.

Assumption 1. *It is assumed that:*

- *The functions $\mu:\mathbb{R}\to\mathbb{R}$ and $\sigma:\mathbb{R}\to(0,\infty)$ are continuous.*
- *The function $b:\mathbb{R}\to\mathbb{R}$ is continuously differentiable, while the functions $a:\mathbb{R}\to(0,\infty)$ and $\lambda:\mathbb{R}\to\mathbb{R}$ are twice continuously differentiable.*
- *The functions a, $1/a$, b, λ, a', b', λ', a'' and λ'' are absolutely bounded.*

It is easy to verify that the above assumptions imply that the system of SDE's (2.1) and (2.2) has a unique strong solution (see, for example, [19]).

An investor trades between the bond and the stock accounts in a finite (fixed) horizon $[0,T]$ generating a random payoff at the terminal time T. The investor's risk preferences at the end of the horizon, T, are modeled via a utility function, denoted by U_T, which is assumed to have the following properties.

Assumption 2. *The utility function $U_T:(0,\infty)\to\mathbb{R}$ is strictly increasing, concave and twice continuously differentiable. Moreover, U_T satisfies*

$$0<\inf_{x>0}\left(-\frac{xU_T{}''(x)}{U_T{}'(x)}\right)\leq\sup_{x>0}\left(-\frac{xU_T{}''(x)}{U_T{}'(x)}\right)<\infty, \tag{2.4}$$

$$0<\inf_{x>0}\left(x^{\gamma}U_T'(x)\right)\leq\sup_{x>0}\left(x^{\gamma}U_T'(x)\right)<\infty,\ \textit{for some } \gamma>0, \tag{2.5}$$

and $e^{(1+\gamma)z}U_T{}''\left(e^z\right)$ is a uniformly continuous function of $z\in\mathbb{R}$.

For example, the above assumption allows for any strictly concave twice continuously differentiable utility function on a positive half line, whose first two derivatives behave as power functions, asymptotically, at zero and infinity. This included the classical power utility $U_T(x)=x^{1-\gamma}$, as well as the logarithmic one $U_T(x)=\log x$, but it excludes any utility whose

[1] The results presented herein extend directly to the case of a stochastic interest rate as long as it is a deterministic function of Y_t.

derivative does not vanish at infinity or does not explode at zero, as well as any utility that is described by two power functions with different exponents, at zero and infinity, respectively.

Given an initial endowment $x>0$ at time $t\in[0,T)$, the investor's discounted allocations in the bond and the stock accounts at any time $s\in[t,T]$ are denoted, respectively, by π_s^0 and π_s. Then, his total discounted investment at time s, denoted by $X_s^{\pi,x,t}$, satisfies $X_s^{\pi,x,t}=\pi_s^0+\pi_s$. We will refer to $X_s^{\pi,x,t}$ as the discounted wealth. Given $\pi=(\pi_s)_{s\in[0,T]}$, the process $\pi^0=\left(\pi_s^0\right)_{s\in[0,T]}$ is uniquely determined by the self-financing condition. Hence, we will identify a trading strategy, or policy, with the process π. We easily derive that the process $\left(X_s^{\pi,x,t}\right)_{s\in[t,T]}$ satisfies

$$dX_s^{\pi,x,t}=\sigma\left(Y_s\right)\pi_s\left(\lambda\left(Y_s\right)ds+dW_s^1\right),\qquad X_t^{\pi,x,t}=x, \tag{2.6}$$

for any policy π from the set of admissible policies defined below.

Definition 2.1. *The set of admissible policies $\mathcal{A}$ consists of all locally square-integrable $\mathbb{F}$-progressively measurable stochastic processes $\pi=(\pi_s)_{s\in[0,T]}$ such that, for any initial condition $(x,t)\in(0,\infty)\times[0,T]$, the corresponding discounted wealth process $\left(X_s^{\pi,x,t}\right)_{s\in[t,T]}$, given by (2.6), stays strictly positive. In addition, if $\gamma\geq 1$, we require that*

$$\mathbb{E}\int_t^T\left(X_s^{\pi,x,t}\right)^{-p}\left(1+\pi_s^2\right)ds<\infty,\qquad \forall\, p\geq 0.$$

Of course, it is enough to check that the above inequality holds for $p=0$ and $p\to\infty$.

The investor aims at maximizing the expected utility of terminal wealth given today's information and over the admissible strategies. The object of interest is, then, the so called, *value function process.*

Definition 2.2. *Let U_T be the utility function and $\mathcal{A}$ as in Definition 2.1. The value function process $J\left(x,t\right)$ is defined for each $(x,t)\in(0,\infty)\times[0,T]$ as*

$$J\left(x,t\right)=esssup_{\pi\in\mathcal{A}}E\left(U_T\left(X_T^{\pi,x,t}\right)\middle|\,\mathcal{F}_t\right). \tag{2.7}$$

Remark 1. *The restriction of the set of admissible strategies $\mathcal{A}$ in the case $\gamma\geq 1$ is imposed merely for technical reasons, due to the fact that, in this case, the utility function may be unbounded from below, causing integrability problems when the wealth process approaches zero. We stress, however, that, even in the case $\gamma\geq 1$, the set $\mathcal{A}$ is still large enough and includes, for example, any square-integrable π which generates a wealth process that is bounded away from zero almost surely (by a constant which may depend upon π).*

In Markovian settings, as the one we consider herein, the value function process is typically associated with the HJB equation. Specifically, $J\left(x,t\right)$ is expected to have the functional representation

$$J\left(x,t\right)=U\left(x,Y_t,t\right), \tag{2.8}$$

where $U:\mathbb{D}\to\mathbb{R}$ is a deterministic function defined on the domain

$$\mathbb{D}=(0,\infty)\times\mathbb{R}\times[0,T]. \tag{2.9}$$

When such a function U exists, it is called the value function of the optimal investment problem and is expected to satisfy the HJB equation

$$U_t+\max_{\pi}\left(\frac{1}{2}\sigma^2(y)\pi^2U_{xx}+\pi\left(\sigma(y)\lambda(y)U_x+\rho\sigma(y)a(y)U_{xy}\right)\right)+\frac{1}{2}a^2(y)U_{yy}+b(y)U_y=0, \tag{2.10}$$

with terminal condition $U\left(x,y,T\right)=U_T\left(x\right)$ (cf. [44] and references therein).

Moreover, an optimal policy may be constructed in the so-called feedback form in terms of the partial derivatives of U, using the first order conditions in (2.10). Namely, let the function $\pi^*:\mathbb{D}\to\mathbb{R}$ be given by

$$\pi^*\left(x,y,t\right)=-\frac{\lambda(y)}{\sigma(y)}\frac{U_x\left(x,y,t\right)}{U_{xx}\left(x,y,t\right)}-\rho\frac{a\left(y\right)}{\sigma\left(y\right)}\frac{U_{xy}\left(x,y,t\right)}{U_{xx}\left(x,y,t\right)}. \tag{2.11}$$

Then, the optimal policy, denoted, with a slight abuse of notation, by $\left(\pi_s^*\right)_{s\in[t,T]}$, is given by

$$\pi_s^*=\pi^*(X_s^{\pi^*,x,t},Y_s,s), \tag{2.12}$$

where $X_s^{\pi^*,x,t}$ is the solution of (2.6) with the above policy being used. Since the correlation ρ controls the incompleteness of the market (when $|\rho|=1$, the market is complete), the representation (2.12), once established, can help quantify the effect of the incompleteness of the market on the optimal portfolio.

As mentioned in the introduction, the stochastic optimization problem (2.8) emerges in the simplest possible extension of the Merton problem when the market becomes incomplete. However, despite how ubiquitous this problem is in optimal portfolio management, very little is known about the validity of any of the above claims and results. Indeed, several key technical results are missing, namely, the existence and uniqueness of the candidate value function U as a solution to the HJB equation, the appropriate regularity and growth of the candidate value function U, the existence of the candidate feedback function π^*, and the existence and uniqueness of a strong solution to the wealth equation (2.6) when the policy π^* is implemented. These results are typically referred to as the verification theorem. To our knowledge, such complete results are lacking for the problem at hand. One of the main difficulties in analyzing (2.10) and establishing the verification theorem stems from the fact that the set of control policies is not compact and, thus, classical results in stochastic optimization of controlled diffusion processes cannot be directly applied. Another difficulty comes from the possibility of degeneracy of (2.10): e.g. when the value of π which attains the maximum in (2.10) is zero, the resulting quadratic form vanishes along the first axis, at that point. Both these difficulties are bypassed when the market is complete. In this case, the Fenchel-Legendre transform can be utilized to linearize the HJB equation and, in turn, obtain the dual solution via the Feynman-Kac formula (see, among others, [34], [18] and [8]). As we will see later on, this transformation is also crucial for the analysis herein when we isolate the "complete market" part of the original HJB equation.

Remark 2.*Notice that, in a typical application, the additional stochastic factor, which controls the volatility of S, may not be observed. As a result, its parameters a and b can only be approximated with certain precision. Establishing a representation of the optimal portfolio, in the spirit of (2.12), would, then, allow to investigate how such approximations affect the performance of the resulting, approximately optimal, trading strategy.*

3. Splitting the HJB equation . In this section we provide preliminary results on the approximation scheme for the value function and the optimal policy that we are going to construct. The scheme is built on the so-called *Trotter-Kato* approximation method. As discussed below, the method relies on the appropriate "splitting" of the HJB equation (cf. (2.10)). For technical reasons that we justify in detail later on, we do not apply the method directly to the original HJB equation but, rather, to a regularization of the corresponding equation satisfied by the x-marginal value function. See (4.2) for the marginal HJB equation and (4.3) for its regularized version. While most of the results we provide are for the marginal HJB equation, in order to build intuition and motivation about this specific choice of splitting, and how we build on existing results for the value function in complete markets, we explain the various elements of the scheme for the original HJB equation itself.

The Trotter-Kato approximation algorithm – also known as the dimensional spitting, operator splitting, Lie-Trotter-Kato formula, leapfrog, etc. – is a technique used to compute the solutions of a wide range of equations by splitting the original problem to more manageable ones. We describe the main ideas next. To this end, consider an initial value parabolic problem, say of the form

$$u_\tau = G\left(z, \tau, u, Du, D^2u\right), \tag{3.1}$$

where D and D^2 denote, respectively, the vector of the first order derivatives and the matrix of the second order derivatives with respect to the space variables. The operator G is, in general, nonlinear.

The *Trotter-Kato approximation* of (3.1) is defined via the "splitting" of the operator G into a sum of simpler operators, say G^i, $i = 1, ..., k$, with

$$G = \sum_{i=1}^{k} G^i, \tag{3.2}$$

so that each of the auxiliary initial value parabolic "G^i-problems",

$$\begin{cases} u_\tau = G^i\left(z, \tau, u, Du, D^2u\right), \\ \\ u(z,0) = F(z), \end{cases} \tag{3.3}$$

has an easy-to-compute, or even explicit, solution for an admissible initial value F.

For each time $\tau > 0$, we then denote by S^i_τ the individual solution operator that maps the initial condition F to the solution of the above G^i-equation at time τ. Then, the *n-th order Trotter-Kato approximation* of the true solution to (3.1), equipped with initial condition $u(.,0)$, is given by the Trotter's formula,

$$u^n(.,\tau) = \left(S^1_{\tau/n} \ldots S^k_{\tau/n}\right)^n u(.,0). \tag{3.4}$$

The first results on the corresponding operator product are given in [40], [7] and [20]. Some of the early applications for constructing the numerical solutions to partial differential equations of various types (linear or nonlinear, parabolic, elliptic or hyperbolic) can be found in [1], [16], [38] and [41]. The more recent references, containing a detailed overview of the existing results, include [15], [14], [29] and [30].

We stress that most of these works deal with the cases in which the existence and uniqueness of the solution to the limiting equation has been *already* established in the class of functions for which the approximation operators are defined. As mentioned earlier, such a result is still lacking for the HJB equation (2.10) due to the general form of the utility functions we consider as well as the unboundedness of both the spatial domain and the set of possible control values. As a matter of fact, none of the above results on the Trotter-Kato approximation can be directly applied to (2.10). As also mentioned above, the convergence is actually established for the approximation scheme of the *marginal HJB equation* (4.2). Hence, we show the existence and uniqueness of the solution to the latter equation, which can be viewed as a side result of this work.

Next, we describe how we construct the Trotter-Kato approximation scheme for (2.10). The first question is what is the appropriate way to split the HJB equation. To gain intuition, we first observe that the equation at hand has two components, namely, a nonlinear part which corresponds to the "controlled" part of the problem and a linear part which corresponds to the evolution of the stochastic factor.

Clearly, the latter part is easier to analyze, for solutions of linear parabolic problems can be conveniently represented via the Feynman-Kac formula and are, frequently, explicitly computed.

For the nonlinear part we observe the following. If the stochastic factor were perfectly correlated with the stock price process, the market would have been complete. Then, it would have been possible to solve the portfolio choice problem by using the Fenchel-Legendre transform. The latter is routinely used in complete market settings because it, not only, linearizes the equation but it, also, gives intuitively pleasing solutions in terms of the state-price density, the dual multiplier, etc. If, on the other hand, the stochastic factor is not perfectly correlated with the stock, the dual transformation fails to linearize (2.10).

The above observations motivate us to split the HJB equation in two parts, specifically,

$$U_t + \mathcal{H}\left(y, DU, D^2U\right) + \mathcal{L}\left(y, DU, D^2U\right) = 0 \tag{3.5}$$

where

$$\mathcal{H}\left(y, DU, D^2U\right) = \max_{\pi}\left(\frac{1}{2}\sigma^2(y)\pi^2 U_{xx} + \pi\left(\sigma(y)\lambda(y)U_x + \rho\sigma(y)a(y)U_{xy}\right)\right) + \frac{1}{2}\rho^2 a^2(y)U_{yy}$$

and

$$\mathcal{L}\left(y, DU, D^2U\right) = \frac{1}{2}\left(1-\rho^2\right)a^2(y)U_{yy} + b(y)U_y. \tag{3.6}$$

One, then, has to analyze the auxiliary problems:

$$U_t + \mathcal{H}\left(y, DU, D^2U\right) = 0 \tag{3.7}$$

and

$$U_t + \mathcal{L}\left(y, DU, D^2U\right) = 0, \tag{3.8}$$

equipped with terminal conditions of the form $U(x,y,T) = F(x,y)$.

It is now easy to see the intuition behind the above decomposition. The non-linear problem (3.7) can be interpreted as the one that would have emerged if the market were complete, with the stock dynamics affected by a perfectly correlated stochastic factor, say $\left(\hat{Y}_t\right)$, having zero drift and the volatility function $\rho a\left(y\right)$, and the investor endowed with terminal utility $F\left(x,y\right)$. Then, this equation can be readily analyzed using the aforementioned Fenchel-Legendre transform. The solution of the linearized equation can be represented via the Feynman-Kac formula, and the solution of the primal problem is, in turn, found by taking the dual transform of the latter solution.

The linear problem (3.8) does not correspond to any portfolio choice model. Rather, it is a linear parabolic problem, with terminal condition $F\left(x,y\right)$, and involves the generator of a stochastic factor, say $\left(\hat{Y}_t^{\perp}\right)$, which is driven by a Brownian motion that is orthogonal to the one driving the stock price, and has drift and volatility functions $b\left(y\right)$ and $\sqrt{1-\rho^2}a(y)$, respectively. Its solution can be, then, obtained using the Feynman-Kac formula.

4. The marginal HJB equation and its auxiliary problems. As mentioned earlier, the convergence of the scheme will be established for the x-spatial derivative, $V:\mathbb{D}\to(0,\infty)$,

$$V\left(x,y,t\right)=U_x\left(x,y,t\right), \tag{4.1}$$

and not for the solution U of the HJB equation itself. We choose to do this for several reasons. Firstly, one of the upcoming auxiliary problems (cf. (4.9)) will be analyzed using the aforementioned dual transformation which acts directly (as a spatial inverse) on the x-partial derivative of U. Moreover, as the feedback form (2.11) indicates, it is only the x-spatial, V, and its partial derivatives, V_x and V_y, that are needed for the construction of the optimal feedback portfolio function. Finally, a technical but very important advantage from focusing our analysis to V is the fact that the corresponding partial differential equation becomes *quasilinear* (linear in the second order derivatives), which, ultimately, makes it possible to develop a comparison principle for its viscosity solutions, even in the case of unbounded set of controls. Even though the marginal value function appears in various methods for analyzing the optimal investment problem (e.g. its x-inverse is the derivative of the dual value function), we are not aware of any methods that focus on the marginal value function as a primary subject of interest.

To this end, we assume for now that all involved functions are well defined and smooth enough, and that the second x-derivative is strictly negative, so that the "maximum" in the HJB equation (2.10) is well defined and attained. We then evaluate the maximum at this point and take the x-derivative of the resulting expression. This yields the *marginal HJB equation*,

$$\begin{aligned}V_t+\frac{1}{2}\left(\frac{\lambda(y)V+\rho a(y)V_y}{V_x}\right)^2V_{xx}-\frac{\lambda(y)V+\rho a(y)V_y}{V_x}\rho a(y)V_{xy}\\+\frac{1}{2}a^2(y)V_{yy}-\lambda^2(y)V-\rho a(y)\lambda(y)V_y+b(y)V_y=0.\end{aligned} \tag{4.2}$$

It is easy to notice that the above nonlinear equation is degenerate parabolic, and, hence, it may not have a classical solution. Due to this, as well as some other difficulties that we will explain in the sequel, we have to consider solutions of the above equation in the viscosity sense. However, having a viscosity solution is not always sufficient, especially when it comes to making a connection to the corresponding stochastic optimization problem. For this reason, we consider a regularized version of the above equation, with the additional time change $\tau = T - t$, namely,

$$\begin{cases} V^{\varepsilon}_{\tau} - \frac{1}{2}\left(\frac{\lambda(y)V^{\varepsilon}+\rho a(y)V^{\varepsilon}_y}{V^{\varepsilon}_x}\right)^2 V^{\varepsilon}_{xx} + \frac{\lambda(y)V^{\varepsilon}+\rho a(y)V^{\varepsilon}_y}{V^{\varepsilon}_x}\rho a(y)V^{\varepsilon}_{xy} - \frac{1}{2}a^2(y)V^{\varepsilon}_{yy} \\ -\varepsilon x^2 V^{\varepsilon}_{xx} - 2\varepsilon x V^{\varepsilon}_x + \lambda^2(y)V^{\varepsilon} + \rho a(y)\lambda(y)V^{\varepsilon}_y - b(y)V^{\varepsilon}_y = 0, \qquad (x,y,\tau)\in\mathbb{D}_0, \\ V^{\varepsilon}(x,y,0) = {U_T}'(x), \end{cases} \tag{4.3}$$

where $\varepsilon \in [0,1]$ and

$$\mathbb{D}_0 := (0,\infty)\times\mathbb{R}\times(0,T). \tag{4.4}$$

While the expected utility problem is naturally formulated at a terminal time, we chose to change the time variable in order to align the format of the upcoming initial value problems with the one appearing in the existing literature.

We will be working throughout with the *regularized marginal HJB equation* (4.3). Notice that, when $\varepsilon = 0$, the above initial value problem becomes the actual (non-regularized) marginal HJB equation. For the most part of this paper (namely, in Sections 4, 5 and in Subsection 6.1), we assume that $\varepsilon \in [0,1]$, thus, including the actual marginal HJB equation in consideration. However, in order to make connections to the optimal portfolio of the associated stochastic optimization problem, in Section 6, we will ultimately restrict ε to *strictly positive* values.

The first step in analyzing the above initial value problem is to specify the "correct" space of functions among which we will search for the solutions. To this end, we introduce the following spaces. The constant γ appearing below is the one introduced in Assumption 2.

Definition 4.1. *For a given $N > 0$, we define $\hat{\mathcal{D}}(N)$ as the space of functions $\hat{F} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ satisfying the following conditions:*

- *$-N \le \hat{F}(z,y,\tau) \le N$, for all $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$;*
- *if $\rho \neq 0$, then the function $\hat{F}(.,.,\tau)$ is globally Lipschitz with Lipschitz coefficient N, uniformly over $\tau\in[0,T]$; if $\rho = 0$, then the function $\hat{F}(.,y,\tau)$ is globally Lipschitz with Lipschitz coefficient N, uniformly over $(y,\tau)\in\mathbb{R}\times[0,T]$;*
- *the function $z \mapsto \hat{F}(z,y,\tau) - (\gamma - 1/N)\, z$ is nonincreasing, for each $(y,\tau)\in\mathbb{R}\times[0,T]$.*

Definition 4.2. *For a given $N > 0$, the function space $\mathcal{D}(N)$ is defined as the space of all functions $F : \mathbb{D} \to (0,\infty)$, with $\mathbb{D}$ given by (2.9), such that the associated function $\hat{F}$: $\mathbb{R}^2\times[0,T] \ni (z,y,\tau) \mapsto \log\left(F\left(e^z,y,\tau\right)\right) + \gamma z$ belongs to $\hat{\mathcal{D}}(N)$.*

Definition 4.3. *We define the function spaces $\hat{\mathcal{D}}$ and $\mathcal{D}$ as the corresponding union spaces:*

$$\hat{\mathcal{D}} = \bigcup_{N>0}\hat{\mathcal{D}}(N) \qquad \text{and} \qquad \mathcal{D} = \bigcup_{N>0}\mathcal{D}(N).$$

We will prove (see Theorem 6.1) that, for any regularization parameter $\varepsilon\in[0,1]$, the initial value problem (4.3) has a unique viscosity solution in $\mathcal{D}$ and will provide a numerical scheme that converges to this solution. We will, then, establish the regularity properties of this solution for strictly positive ε (see Theorem 6.2), and show that it generates investment strategies which approximate the value function of the corresponding stochastic optimization problem with precision of order ε (see Theorem 6.6).

It is worth mentioning that the existence and uniqueness of solution to (4.3) is closely related to the fact that it belongs to $\mathcal{D}$. In general, the coefficients in front of the second order derivatives in (4.3) are unbounded, which prevents us from developing the existence theory for this equation directly. One of the main contributions of the splitting scheme we propose here is that it allows us to construct a sequence of functions which approximate the potential solution and belong to $\mathcal{D}$. As a result, the coefficients of (4.3), evaluated on these functions, stay uniformly bounded and we are able to reduce the problem of convergence of this scheme to the one where the equation has bounded coefficients (or, equivalently, the set of control values in compact). The question of how to deduce the boundedness of the solution to (4.3) directly from the equation itself, to the best of our knowledge, is still open. We chose to prove it by splitting the equation and analyzing each part separately, since these parts can be reduced to linear equations and there is a much wider set of tools available for establishing the properties of their solutions. In addition, the proposed splitting scheme can be used to derive a numerical approximation for the solution of the marginal HJB equation.

Following the splitting of the original HJB, we set the regularized marginal HJB in the analogous to (3.5) form. We then introduce the *Trotter-Kato approximation scheme* for (4.3), namely,

$$V^{\varepsilon}_{\tau} = G^{1}\left(y, DV^{\varepsilon}, D^{2}V^{\varepsilon}\right) \tag{4.5}$$

and

$$V^{\varepsilon}_{\tau} = G^{2,\varepsilon}\left(y, DV^{\varepsilon}, D^{2}V^{\varepsilon}\right), \tag{4.6}$$

with initial conditions of the form $V^{\varepsilon}(x, y, 0) = F(x, y)$ and the generators G^1 and $G^{2,\varepsilon}$ given, respectively, by

$$G^{1}\left(y, DV^{\varepsilon}, D^{2}V^{\varepsilon}\right) = \frac{1}{2}\left(\frac{\lambda(y)V^{\varepsilon} + \rho a(y)V^{\varepsilon}_{y}}{V^{\varepsilon}_{x}}\right)^{2} V^{\varepsilon}_{xx} - \frac{\lambda(y)V^{\varepsilon} + \rho a(y)V^{\varepsilon}_{y}}{V^{\varepsilon}_{x}}\rho a(y)V^{\varepsilon}_{xy} \tag{4.7}$$

$$+\frac{1}{2}\rho^{2}a^{2}(y)V^{\varepsilon}_{yy} - \lambda^{2}(y)V^{\varepsilon} - \rho a(y)\lambda(y)V^{\varepsilon}_{y}$$

and

$$G^{2,\varepsilon}\left(y, DV^{\varepsilon}, D^{2}V^{\varepsilon}\right) = \varepsilon x^{2}V^{\varepsilon}_{xx} + 2\varepsilon xV^{\varepsilon}_{x} - \frac{1}{2}\left(1-\rho^{2}\right)a^{2}(y)V^{\varepsilon}_{yy} - b(y)V^{\varepsilon}_{y}. \tag{4.8}$$

In order to analyze the above problems, we, again, need to specify the "convenient" function space, such that, if the initial condition F of (4.5) or (4.6) belongs to this space, then so does the solution at any time level τ. We stress that this was *not* an obvious choice and that the correct specification of the solution space, presented below, was pivotal in our results.

Definition 4.4. *For each triplet $(\nu, \delta, \kappa) \in (0, 1)^3$, we define $\mathcal{D}_0(\nu, \delta, \kappa)$ as the space of continuous functions $F : (0, \infty) \times \mathbb{R} \to (0, \infty)$ satisfying the following conditions:*

- *$\delta \leq x^{\gamma}F(x, y) \leq 1/\delta$, for all $(x, y) \in (0, \infty) \times \mathbb{R}$;*

- *the function $F(x,y)$ has a continuous partial derivative with respect to x which satisfies*

$$\nu \leq -\frac{xF_x(x,y)}{F(x,y)} \leq \frac{1}{\nu}, \qquad \text{for all } (x,y) \in (0,\infty) \times \mathbb{R};$$

- *if $\rho \neq 0$, then the function $F(x,y)$ has a continuous partial derivative with respect to y which satisfies*

$$\left|\rho a(y) \frac{F_y(x,y)}{F(x,y)} + \lambda(y)\right| \leq \frac{1}{\kappa}, \qquad \text{for all } (x,y) \in (0,\infty) \times \mathbb{R}.$$

We define $\mathcal{D}_0$ as the union space: $\mathcal{D}_0 = \bigcup_{(\nu,\delta,\kappa)\in(0,1)^3} \mathcal{D}_0(\nu,\delta,\kappa)$.

In order to facilitate the subsequent analysis of the "nonlinear" auxiliary equation (4.5), we also need to introduce the "dual spaces".

Definition 4.5. *For each triplet $(\nu,\delta,\kappa) \in (0,1)^3$, we define $\mathcal{D}'_0(\nu,\delta,\kappa)$ as the space of continuous functions $f : (0,\infty) \times \mathbb{R} \to (0,\infty)$ satisfying the following conditions:*

- *$\delta \leq x^{1/\gamma} f(x,y) \leq 1/\delta$, for all $(x,y) \in (0,\infty) \times \mathbb{R}$;*
- *the function $f(x,y)$ has a continuous partial derivative with respect to x which satisfies*

$$\nu \leq -\frac{xf_x(x,y)}{f(x,y)} \leq \frac{1}{\nu}, \qquad \text{for all } (x,y) \in (0,\infty) \times \mathbb{R};$$

- *if $\rho \neq 0$, then the function $f(x,y)$ has a continuous partial derivative with respect to y which satisfies*

$$\left|\rho a(y) \frac{f_y(x,y)}{xf_x(x,y)} - \lambda(y)\right| \leq \frac{1}{\kappa}, \qquad \text{for all } (x,y) \in (0,\infty) \times \mathbb{R}.$$

We define $\mathcal{D}'_0$ as the union space: $\mathcal{D}'_0 = \bigcup_{(\nu,\delta,\kappa)\in(0,1)^3} \mathcal{D}'_0(\nu,\delta,\kappa)$.

The following lemma provides a useful connection between the spaces $\mathcal{D}_0$ and $\mathcal{D}'_0$.

Lemma 4.6. *If a function $F : (0,\infty) \times \mathbb{R} \to (0,\infty)$ belongs to $\mathcal{D}_0(\nu,\delta,\kappa)$, then its "x-inverse" $f : (0,\infty) \times \mathbb{R} \to (0,\infty)$, defined by $F(f(x,y),y) = x$, belongs to $\mathcal{D}'_0(\nu,\delta^{1/\gamma},\kappa)$. The inverse implication is also true, namely, if $f \in \mathcal{D}'_0(\nu,\delta,\kappa)$, then its x-inverse F belongs to $\mathcal{D}_0(\nu,\delta^\gamma,\kappa)$.*

Proof:

It is easy to see that, if $F \in \mathcal{D}_0(\nu,\delta,\kappa)$, then its x-inverse, f, is well defined and continuous. The inverse is also true. Then the assertion of the lemma follows easily from the standard relations between the derivatives of a function and its spatial inverse (see, for example, (4.11) below). ■

Next, we proceed with a detailed analysis of the auxiliary problems (4.5) and (4.6). For each problem, we establish that, if its initial condition F belongs to $\mathcal{D}_0$, then there exists a unique solution to the corresponding initial value problem which belongs to $\mathcal{D}_0(\nu,\delta,\kappa)$, for all $\tau \in [0,T]$, for some fixed triplet $(\nu,\delta,\kappa) \in (0,1)^3$.

4.1. The non-linear auxiliary problem. We start with the analysis of the first auxiliary problem (4.5). For the reader's convenience, we rewrite the relevant equation below. We are looking for a function $V^1 : \mathbb{D} \to \mathbb{R}$, which admits a triplet $(\nu, \delta, \kappa) \in (0,1)^3$, such that $V^1(.,.,\tau) \in \mathcal{D}_0(\nu,\delta,\kappa)$, for all $\tau \in [0,T]$, and solves

$$V^1_\tau = G^1\left(y, DV^1, D^2V^1\right), \tag{4.9}$$

with initial condition $V^1(x,y,0) = F(x,y)$, for $F \in \mathcal{D}_0$, where G^1 is defined in (4.7).

Before we provide a precise construction of the function V^1 (Definition 4.9), we need to specify what exactly we mean by a solution to the above equation. We stress that the latter is not at all obvious due to, on the one hand, the nonlinearity of equation (4.9) and, on the other, the requirement that its solution must belong to the space $\mathcal{D}_0(\nu,\delta,\kappa)$. As mentioned earlier, we are going to exploit the fact that equation (4.9) is similar to the marginal HJB equation of a complete market problem and, thus, we can linearize it. To this end, we introduce the function $v : \mathbb{D} \to (0,\infty)$ as

$$V^1\left(v(x,y,\tau), y, \tau\right) = x, \tag{4.10}$$

assuming that the function $V^1(x,y,\tau)$ is well defined and strictly monotone in x, for any $(y,\tau) \in \mathbb{R} \times [0,T]$. Then, all regularity properties of its derivatives hold for the derivatives of v and vice-versa. Assuming that the functions involved in (4.10), as well as the corresponding partial derivatives, do exist, we obtain the equalities

$$\begin{aligned} &V^1_x = \frac{1}{v_x}, \quad V^1_{xx} = -\frac{v_{xx}}{v_x^3}, \quad V^1_\tau = -\frac{v_\tau}{v_x}, \quad V^1_y = -\frac{v_y}{v_x}, \\ &V^1_{xy} = v_{xx}\frac{v_y}{v_x^3} - \frac{v_{xy}}{v_x^2}, \quad V^1_{yy} = -\frac{v_y^2 v_{xx}}{v_x^3} + 2\frac{v_y v_{xy}}{v_x^2} - \frac{v_{yy}}{v_x}, \end{aligned} \tag{4.11}$$

where the function V^1 and its partial derivatives are evaluated at $(v(x,y,\tau),y,\tau)$, while the function v and its derivatives are always considered at (x,y,τ). We then easily derive a corresponding partial differential equation for v

$$v_\tau - \frac{1}{2}x^2\lambda^2(y)v_{xx} + \rho x a(y)\lambda(y)v_{xy} - \frac{1}{2}\rho^2 a^2(y)v_{yy} - x\lambda^2(y)v_x + \rho a(y)\lambda(y)v_y = 0,$$

and notice that it is linear parabolic, with a possible degeneracy at $x \to 0$. In order to avoid this degeneracy, we make a change of variables, introducing the function $u : \mathbb{R}^2 \times [0,T] \to (0,\infty)$ defined as

$$u(z,y,\tau) = v(e^z, y, \tau). \tag{4.12}$$

Finally, using the relations in (4.11), after some tedious but routine calculations, we deduce that u is expected to satisfy the equation

$$\begin{aligned} u_\tau - \frac{1}{2}\lambda^2(y)u_{zz} + \rho a(y)\lambda(y)u_{zy} - \frac{1}{2}\rho^2 a^2(y)u_{yy} \\ -\frac{1}{2}\lambda^2(y)u_z + \rho a(y)\lambda(y)u_y = 0, \end{aligned} \tag{4.13}$$

with initial condition $u(z,y,0) = f(e^z,y,\tau)$, for $f \in \mathcal{D}'_0$.

We note that the above constructions are based on the assumption that function V^1 is a classical solution to (4.9), whose existence (and uniqueness), however, has not been verified. Therefore, in what follows, we will use the above equation as a starting point, and reverse-engineer the function V^1 directly from it. We, however, observe that (4.13) is still *degenerate* parabolic. Although we have eliminated one source of degeneracy (at $x = 0$), the quadratic form still vanishes, at each point (z, y), along the vector $(\lambda(y), \rho a(y))$. As a result, (4.13) may not have a classical solution. In addition, we note that the initial condition f does not necessarily have enough smoothness or integrability properties that could allow us to characterize u as a weak solution of the above equation using the standard techniques. Nevertheless, the Feynman-Kac representation may still be used, for it only relies on probabilistic methods.

Definition 4.7. *For any function $f \in \mathcal{D}_0'$ we define the associated functions $u : \mathbb{R}^2 \times [0, T] \to (0, \infty)$ and $v : \mathbb{D} \to (0, \infty)$ via:*

- *the expectation*

$$u(z, y, \tau) := \mathbb{E}\left(f\left(\exp\left(\hat{Z}_\tau^{z,y}\right), \hat{Y}_\tau^y\right)\right), \tag{4.14}$$

where, for each $(z, y) \in \mathbb{R}^2$, the stochastic processes $\hat{Z}^{z,y}$ and $\hat{Y}^y$ are given by the system of SDE's

$$\begin{cases} d\hat{Z}_\tau^{z,y} = \frac{1}{2}\lambda^2(\hat{Y}_\tau^y)d\tau + \lambda(\hat{Y}_\tau^y)dB_\tau, & \hat{Z}_0^{z,y} = z, \\ \\ d\hat{Y}_\tau^y = -\rho a(\hat{Y}_\tau^y)d\tau - \rho a(\hat{Y}_\tau^y)dB_\tau, & \hat{Y}_0^y = y, \end{cases} \tag{4.15}$$

with B being a standard Brownian motion;
- *and the change of variables*

$$v(x, y, \tau) := u(\log x, y, \tau). \tag{4.16}$$

The fact that, for any initial condition, the above system of SDE's has a unique strong solution follows immediately from the boundedness and Lipschitz properties of the coefficients a and λ. The weak uniqueness of the solution to (4.15) implies that the joint distribution of the corresponding stochastic processes does not depend upon the choice of the probability space and the driving Brownian motion. Hence, u is determined uniquely by f. The expectation is also well defined, since f is bounded by a power function.

The following lemma shows the additional regularity properties of the function u, defined in (4.14), and it is a key analytical result of this section. Its proof is given in Appendix A.

Lemma 4.8. *Fix an arbitrary $f \in \mathcal{D}_0'$, and consider the associated functions $u : \mathbb{R}^2 \times [0, T] \to (0, \infty)$ and $v : \mathbb{D} \to (0, \infty)$, given by Definition 4.7. Then, the following statements hold:*

(i) *The function v satisfies $v(.,.,\tau) \in \mathcal{D}_0'$, for any $\tau \in [0, T]$. Moreover, there exists a constant $\alpha' \geq 0$ and a continuous function $\beta' : (0, 1)^2 \to [0, \infty)$, independent of f and such that, for any $(\nu, \delta, \kappa) \in (0, 1)^3$, if $f \in \mathcal{D}_0'(\nu, \delta, \kappa)$, then, for all $\tau \in [0, T]$,*

$$v(.,.,\tau) \in \mathcal{D}_0'\left(\nu, \delta e^{-\alpha'\tau}, \left(1/\kappa + \tau \exp\left(\beta'(\nu, \delta)(1 + \tau/\kappa^2)\right)\right)^{-1}\right). \tag{4.17}$$

(ii) *If $\left((z, y) \mapsto e^{z/\gamma} f(e^z, y)\right) \in C_b^2\left(\mathbb{R}^2\right)$, then, there exists a constant $c \geq 0$, depending only upon the C^2-norm of $(z, y) \mapsto e^{z/\gamma} f(e^z, y)$ and upon λ, a ρ and γ, such that, for all*

$(z, y, \tau) \in \mathbb{R}^2 \times [0, T]$, *we have*

$$e^{z/\gamma} \left| u(z, y, \tau) - f(e^z, y) \right| \leq c\tau. \tag{4.18}$$

(iii) *If* $\left((z, y) \mapsto e^{z/\gamma} f(e^z, y)\right) \in C_b^\infty\left(\mathbb{R}^2\right)$, *then, the function* u *satisfies* $u \in C^{2,1}\left(\mathbb{R}^2 \times [0, T]\right)$, *and, moreover, it is a classical solution to (4.13), equipped with the initial condition* $u(z, y, 0) = f(e^z, y)$.

As discussed earlier, a classical solution to the equation (4.13) is not always well defined. It is then hard to expect that, in general, there would exist a classical solution to (4.9). However, we can still define V^1 directly via the function u, and, as a matter of fact, it turns out that such a definition is sufficient for our purposes.

Definition 4.9. *For any function* $F \in \mathcal{D}_0$, *denoting its* x*-inverse by* $f \in \mathcal{D}_0'$, *we define the associated function* $V^1 : \mathbb{D} \to (0, \infty)$ *as the* x*-inverse of the function* $v : \mathbb{D} \to (0, \infty)$, *which is associated with* f *according to Definition 4.7.*

Lemma 4.6 shows that the above definition is consistent, namely, the x-inverse of the function v is well defined.

The following proposition summarizes the most important properties of the function V^1, and it can be viewed as the main result of this subsection. Its proof is given in Appendix B.

Proposition 4.10. *Fix an arbitrary* $F \in \mathcal{D}_0$, *and consider the associated function* $V^1 : \mathbb{D} \to (0, \infty)$, *given by Definition 4.9. Then, the following statements hold:*

(i) *There exists a constant* $\alpha \geq 0$ *and a continuous function* $\beta : (0, 1)^2 \to [0, \infty)$, *independent of* F *and such that, for any* $(\nu, \delta, \kappa) \in (0, 1)^3$, *if* $F \in \mathcal{D}_0(\nu, \delta, \kappa)$, *then, for all* $\tau \in [0, T]$,

$$V^1(.,.,\tau) \in \mathcal{D}_0\left(\nu, \delta e^{-\alpha\tau}, \left(1/\kappa + \tau \exp\left(\beta(\nu, \delta)(1 + \tau/\kappa^2)\right)\right)^{-1}\right). \tag{4.19}$$

(ii) *If* $F \in \mathcal{D}_0(\nu, \delta, \kappa)$ *and* $((z, y) \mapsto e^{\gamma z} F(e^z, y)) \in C_b^2\left(\mathbb{R}^2\right)$, *then, there exists a constant* $c \geq 0$, *depending only upon the* C^2*-norm of* $(z, y) \mapsto e^{\gamma z} F(e^z, y)$, *the pair* (ν, δ) *and upon* λ, a, ρ *and* γ, *such that, for all* $(x, y, \tau) \in \mathbb{D}$,

$$\left| \log V^1(x, y, \tau) - \log F(x, y) \right| \leq c\tau. \tag{4.20}$$

(iii) *If* $((z, y) \mapsto e^{\gamma z} F(e^z, y)) \in C_b^\infty\left(\mathbb{R}^2\right)$, *then* $V^1 \in C^{2,1}\left(\mathbb{R}^2 \times [0, T]\right)$, *and, moreover, it is a classical solution to the equation (4.9), equipped with the initial condition* $V^1(x, y, 0) = F(x, y)$.

4.2. The linear auxiliary problem. We analyze the linear initial value problem (4.6). We are looking for a function $V^{\varepsilon,2} : \mathbb{D} \to \mathbb{R}$, which admits a triplet $(\nu, \delta, \kappa) \in (0, 1)^3$ such that $V^{\varepsilon,2}(.,.,\tau) \in \mathcal{D}_0(\nu, \delta, \kappa)$, for all $\tau \in [0, T]$, and $V^{\varepsilon,2}$ solves

$$V_\tau^{\varepsilon,2} = G^{2,\varepsilon}\left(y, DV^{\varepsilon,2}, D^2 V^{\varepsilon,2}\right), \tag{4.21}$$

with initial condition $V^{\varepsilon,2}(x, y, 0) = F(x, y)$, for $F \in \mathcal{D}_0$, where $G^{2,\varepsilon}$ is defined in (4.8). As in the previous subsection, before giving a rigorous definition of $V^{\varepsilon,2}$ (Definition 4.11), we discuss the difficulties associated with the above equation.

Notice that the second order differential operator in the above equation becomes degenerate as $x \to 0$. In order to resolve this issue, we work as before (see (4.12)) and make a change of variables, introducing the function $\tilde{V}^{\varepsilon,2} : \mathbb{R}^2 \times [0,T] \to (0,\infty)$ defined as

$$\tilde{V}^{\varepsilon,2}(z,y,\tau) = V^{\varepsilon,2}(e^z,y,\tau).$$

Then, $\tilde{V}^{\varepsilon,2}$ is expected to satisfy

$$\tilde{V}^{\varepsilon,2}_\tau - \frac{1}{2}\left(1-\rho^2\right)a^2(y)\tilde{V}^{\varepsilon,2}_{yy} - \varepsilon\tilde{V}^{\varepsilon,2}_{zz} - \varepsilon\tilde{V}^{\varepsilon,2}_{z} - b(y)\tilde{V}^{\varepsilon,2}_{y} = 0, \tag{4.22}$$

with initial condition $\tilde{V}^{\varepsilon,2}(z,y,0) = F(e^z,y)$, for $F \in \mathcal{D}_0$.

Working along the arguments we used in Subsection 4.1, we will, in fact, recover the solution $V^{\varepsilon,2}$ from the solution to the above auxiliary equation. Notice, however, that (4.22) is still degenerate if either $\rho^2 = 1$ or $\varepsilon = 0$. Since we would like to cover these cases as well, we cannot, in general, assume that there exists a classical solution to the above initial value problem. However, as in Subsection 4.1, the Feynman-Kac representation may still be used to define uniquely the function $\tilde{V}^{\varepsilon,2}$, and, in turn, $V^{\varepsilon,2}$.

Definition 4.11. *For any $\varepsilon \in [0,1]$ and any function $F \in \mathcal{D}_0$, we define the associated functions $\tilde{V}^{\varepsilon,2} : \mathbb{R}^2 \times [0,T] \to (0,\infty)$ and $V^{\varepsilon,2} : \mathbb{D} \to (0,\infty)$ via:*

- *the expectation*

$$\tilde{V}^{\varepsilon,2}(z,y,\tau) := \mathbb{E}\left(F\left(\exp\left(\tilde{Z}^{\varepsilon,z}_\tau\right), \tilde{Y}^y_\tau\right)\right), \tag{4.23}$$

  where, for each $(\varepsilon,z,y) \in [0,1] \times \mathbb{R}^2$, the stochastic processes $\tilde{Z}^{\varepsilon,z}$ and $\tilde{Y}^y$ are given by the system of SDE's

$$\begin{cases} d\tilde{Z}^{\varepsilon,z}_\tau = \varepsilon d\tau + \sqrt{2\varepsilon}dB_\tau, \quad \tilde{Z}^{\varepsilon,z}_0 = z, \\ d\tilde{Y}^y_\tau = b(\tilde{Y}^y_\tau)d\tau + \sqrt{1-\rho^2}a(\tilde{Y}^y_\tau)dW_\tau, \quad \tilde{Y}^y_0 = y, \end{cases} \tag{4.24}$$

  with B and W being two independent Brownian motions;
- *and the change of variables*

$$V^{\varepsilon,2}(x,y,\tau) := \tilde{V}^{\varepsilon,2}(\log x, y, \tau).$$

As in the previous subsection, it is easy to see that the above definition is consistent in that the corresponding stochastic processes are uniquely defined for any initial condition, their joint law does not depend upon the choice of the probability space or the driving Brownian motions, and that the corresponding expectation is finite.

The following proposition provides the most important properties of the function $V^{\varepsilon,2}$, defined by (4.23) above, and summarizes all the main results of this subsection. Its proof is given in Appendix C.

Proposition 4.12. *Fix arbitrary $\varepsilon \in [0,1]$ and $F \in \mathcal{D}_0$, and consider the associated function $V^{\varepsilon,2} : \mathbb{D} \to (0,\infty)$, given by Definition 4.11. Then, the following statements hold:*

(i) *There exists a constant $\theta \geq 0$ and a continuous function $\xi : (0,1) \to [0,\infty)$, independent of F and ε and such that, for any $(\nu,\delta,\kappa) \in (0,1)^3$, if $F \in \mathcal{D}_0(\nu,\delta,\kappa)$, then, for all $\tau \in [0,T]$,*

$$V^{\varepsilon,2}(.,.,\tau) \in \mathcal{D}_0\left(\nu, \delta e^{-\theta\tau}, \left(1/\kappa + \tau\exp\left(\xi(\delta)(1+\tau/\kappa^2)\right)\right)^{-1}\right). \tag{4.25}$$

(ii) *If $F \in \mathcal{D}_0(\nu,\delta,\kappa)$ and $((z,y) \mapsto e^{\gamma z}F(e^z,y)) \in C_b^2\left(\mathbb{R}^2\right)$, then, there exists a constant $c \geq 0$, depending only upon the C^2-norm of $(z,y) \mapsto e^{\gamma z}F(e^z,y)$, the pair (ν,δ) and upon λ a, b, ρ and γ, such that, for all $(x,y,\tau) \in \mathbb{D}$,*

$$\left|\log V^{\varepsilon,2}(x,y,\tau) - \log F(x,y)\right| \leq c\tau. \tag{4.26}$$

(iii) *If $((z,y) \mapsto e^{\gamma z}F(e^z,y)) \in C_b^\infty\left(\mathbb{R}^2\right)$, then $V^{\varepsilon,2} \in C^{2,1}(\mathbb{D})$, and, moreover, it is a classical solution to the equation (4.21), equipped with initial condition $V^{\varepsilon,2}(x,y,0) = F(x,y)$.*

5. The auxiliary approximation scheme. In this section, we make a change of variables in the *regularized marginal HJB equation* (4.3), obtaining a new parabolic equation which is somewhat easier to analyze. Using the results of the previous section, we construct a *Trotter-Kato approximation scheme* for the new equation and prove its convergence (see Theorem 5.8). This, in turn, will be used to establish the convergence of the Trotter-Kato scheme for (4.3) itself. The latter is one of the main contributions herein, and, for this reason, we choose to present it separately in the next section (see Theorem 6.1) together with the other main results of the paper.

To this end, assuming that V is a solution to (4.3), we, formally, introduce the auxiliary function $\hat{V}^\varepsilon : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ given by

$$\hat{V}^\varepsilon(z,y,\tau) = \log V(e^z,y,\tau) + \gamma z, \tag{5.1}$$

and derive the corresponding initial value problem

$$\begin{cases} \hat{V}^\varepsilon_\tau + G^\varepsilon(z,y,D\hat{V}^\varepsilon,D^2\hat{V}^\varepsilon) = 0, & (z,y,\tau) \in \mathbb{R}^2 \times (0,T) \\ \hat{V}^\varepsilon(z,y,0) = \log U_T'(e^z) + \gamma z, \end{cases} \tag{5.2}$$

with the function $G^\varepsilon : \mathbb{R}^2 \times (\mathbb{R} \setminus \{\gamma\}) \times \mathbb{R} \times \mathcal{S}(2) \to \mathbb{R}$ defined as

$$\begin{aligned} G^\varepsilon(z,y,(p_1,p_2)^T,X) &:= -\frac{1}{2}\left(\frac{\lambda + a\rho p_2}{p_1-\gamma}, -a\rho\right) X \left(\frac{\lambda + a\rho p_2}{p_1-\gamma}, -a\rho\right)^T - \frac{1-\rho^2}{2}(0,a)X(0,a)^T \\ -\varepsilon(1,0)X(1,0)^T &+ \frac{1}{2}\frac{(\lambda + a\rho p_2)^2}{p_1-\gamma} + \frac{1}{2}(\lambda + a\rho p_2)^2 - \frac{1}{2}a^2p_2^2 - \varepsilon p_1^2 - bp_2 - \varepsilon p_1, \end{aligned} \tag{5.3}$$

where $\mathcal{S}(2)$ is the space of symmetric 2×2 matrices and the superscript "T" denotes the transpose of a matrix. Notice that the initial condition in (5.2) is bounded and, moreover, the spatial domain does not have a boundary, which greatly simplifies the analysis of the problem and partially justifies the above transformation.

In order to define the Trotter-Kato approximation for (5.2), we need to introduce the function spaces corresponding to the above logarithmic change of variables.

Definition 5.1. *For any given triplet $(\nu,\delta,\kappa) \in (0,1)^3$, we define $\hat{\mathcal{D}}_0(\nu,\delta,\kappa)$ as the space of all functions $\hat{F} : \mathbb{R}^2 \to \mathbb{R}$ such that the associated function $F : (0,\infty) \times \mathbb{R} \ni (x,y) \mapsto x^{-\gamma}\exp\left(\hat{F}(\log x,y)\right)$ belongs to $\mathcal{D}_0(\nu,\delta,\kappa)$, given by Definition 4.4.*

The space $\hat{\mathcal{D}}_0$ is defined as the corresponding union space: $\hat{\mathcal{D}}_0 = \bigcup_{(\nu,\delta,\kappa)\in(0,1)^3} \hat{\mathcal{D}}_0(\nu,\delta,\kappa)$.

Next, we introduce the operators that are used to construct the auxiliary Trotter-Kato approximation.

Definition 5.2. *For any $\tau \in [0,T]$ and any $\varepsilon \in [0,1]$, the operators H_τ and L^ε_τ are defined as follows:*

- *The operator H_τ maps any function $F \in \mathcal{D}_0$ into $V^1(.,.,\tau) \in \mathcal{D}_0$, where the associated function V^1 is given by Definition 4.9.*
- *The operator L^ε_τ maps any function $F \in \mathcal{D}_0$ into $V^{\varepsilon,2}(.,.,\tau) \in \mathcal{D}_0$, where the associated function $V^{\varepsilon,2}$ is given by Definition 4.11.*

We have shown in Propositions 4.10 and 4.12 that the above semigroups are well defined, namely, the operators H_τ and L^ε_τ map $\mathcal{D}_0$ into itself. Let's define the analogues of these operators after the logarithmic change of variables (5.1).

Definition 5.3. *The operators $\hat{H}_\tau$, $\hat{L}^\varepsilon_\tau$ and $\hat{A}^\varepsilon_\tau$, acting on $\hat{F} \in \hat{\mathcal{D}}_0$, are defined as*

$$\hat{H}_\tau \hat{F}(z,y) = \log\left(H_\tau F(e^z,y)\right) + \gamma z, \qquad \hat{L}^\varepsilon_\tau \hat{F}(z,y) = \log\left(L^\varepsilon_\tau F(e^z,y)\right) + \gamma z$$

with the associated function $F : (0,\infty) \times \mathbb{R} \ni (x,y) \mapsto x^{-\gamma} \exp\left(\hat{F}(\log(x),y)\right)$, and

$$\hat{A}^\varepsilon_\tau \hat{F}(z,y) = \hat{L}^\varepsilon_\tau \hat{H}_\tau \hat{F}(z,y),$$

Notice that the definition of $\hat{A}^\varepsilon_\tau$ is consistent, because the fact that H_τ and L^ε_τ map $\mathcal{D}_0$ into itself implies that the operators $\hat{H}_\tau$ and $\hat{L}^\varepsilon_\tau$ map $\hat{\mathcal{D}}_0$ into itself.

Finally, we are ready to define the Trotter-Kato approximation for (5.2). Consider an arbitrary partition P of the interval $[0,T]$, given by

$$P = \{0 = \tau_0 < \cdots < \tau_N = T\}. \tag{5.4}$$

Denote $|P| := N$ and $\mathrm{mesh}(P) := \max\{\tau_i - \tau_{i-1}\}_{i=1}^N$.

Definition 5.4. *For any $\varepsilon \in [0,1]$ and any partition P, as in (5.4), the function $\hat{V}^{\varepsilon,P} : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ is defined as*

$$\hat{V}^{\varepsilon,P}(z,y,\tau) = \begin{cases} \left(\hat{A}^\varepsilon_{\tau-\tau_k} \Pi_{i=1}^k \hat{A}^\varepsilon_{\tau_{k-i+1}-\tau_{k-i}} \left(\log U'_T(\exp(.)) + (.)\gamma\right)\right)(z,y), & \tau \in (\tau_k, \tau_{k+1}], \\ \log U'_T(e^z) + \gamma z, \quad \tau = 0, \end{cases} \tag{5.5}$$

for each $k = 0,\dots,N-1$, and $\hat{V}^{\varepsilon,P}$ is the Trotter-Kato approximation associated with (5.2).

In this section we ultimately show that the auxiliary approximation $\hat{V}^{\varepsilon,P}$ has a limit, as the $\mathrm{mesh}(P) \to 0$, and this limit is the unique viscosity solution to the initial value problem (5.2).

5.1. Properties of the auxiliary scheme.

From the definition of the operator $\hat{A}^\varepsilon_\tau$, it follows that $\hat{V}^{\varepsilon,P}(.,.,\tau) \in \hat{\mathcal{D}}_0$, for all $\tau \in [0,T]$. As a matter of fact, we can prove the following stronger statement which turns out to be crucial for the proof of convergence of the approximation.

Proposition 5.5. *There exists a triplet $(\nu,\delta,\kappa) \in (0,1)^3$ and a constant $\varepsilon' > 0$, such that $\hat{V}^{\varepsilon,P}(.,.,\tau) \in \hat{\mathcal{D}}_0(\nu,\delta,\kappa)$, for all $\tau \in [0,T]$, all $\varepsilon \in [0,1]$ and all partitions P, with mesh$(P) < \varepsilon'$.*

Its proof is given in Appendix D.

To establish the convergence of the scheme (5.5), we follow the general approach of [3], which can be applied if the approximation operators $\left(\hat{A}^\varepsilon_\tau\right)_{\tau\in[0,T]}$ satisfy the following regularity conditions:

- Each operator $\hat{A}^\varepsilon_\tau$ is *monotone*: for any $u, v \in \hat{\mathcal{D}}_0$, satisfying $u(z,y) \le v(z,y)$, for all $(z,y) \in \mathbb{R}^2$, we have, for all $(z,y)\in\mathbb{R}^2$,
$$\hat{A}^\varepsilon_\tau u(z,y) \le \hat{A}^\varepsilon_\tau v(z,y). \tag{5.6}$$
- Each operator $\hat{A}^\varepsilon_\tau$ is *translation invariant*: for any constant $c\in\mathbb{R}$ and any function $u \in \hat{\mathcal{D}}_0$, we have
$$\hat{A}^\varepsilon_\tau\left(u+c\right) = \hat{A}^\varepsilon_\tau u + c. \tag{5.7}$$
- The family of operators $\left(\hat{A}^\varepsilon_\tau\right)_{\tau\in[0,T]}$ is *consistent*: for any $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C^\infty_b\left(\mathbb{R}^2\right)$ and any compact set $\mathcal{K}\subset\mathbb{R}^2$, we have, as $\tau\downarrow 0$,
$$\sup_{(z,y)\in\mathcal{K}}\left|\frac{1}{\tau}\left(\hat{A}^\varepsilon_\tau - I\right)\hat{\phi}(z,y) + G^\varepsilon\left(z,y,D\hat{\phi},D^2\hat{\phi}\right)\right| \to 0, \tag{5.8}$$
where I denotes the identity operator, and the above convergence is uniform over all $\hat{\phi}\in\hat{\mathcal{D}}_0(\nu,\delta,\kappa)\cap C^\infty_b\left(\mathbb{R}^2\right)$ with $\|\hat{\phi}\|_{C^3(\mathbb{R}^2)}\le N$, for any $(\nu,\delta,\kappa)\in(0,1)^3$ and $N>0$.

Proposition 5.6. *The semigroup of operators $\left(\hat{A}^\varepsilon_\tau\right)_{\tau\in[0,T]}$ satisfies the monotonicity, translation invariance and consistency properties, (5.6), (5.7) and (5.8), respectively.*

Its proof is provided in Appendix E.

5.2. Viscosity solutions and comparison principle. To complete the proof of the convergence of the auxiliary Trotter-Kato approximation scheme (5.5), we need to establish the *comparison principle* for the *viscosity solutions* of the corresponding initial value problem (5.2). For this, we first note that the equation appearing in (5.2) has, on the one hand, potentially unbounded coefficients in front of the second order derivatives, and, on the other, a possible singularity. To our knowledge, a comparison principle for such specific "irregular" equation is not available. We bypass this difficulty by "truncating" the coefficients of (5.2), so that existing results may be applied to establish the comparison principle for the "truncated" auxiliary equation. As a matter of fact, the analysis in the next subsection shows that, due to the a-priori properties of the approximating function $\hat{V}^{\varepsilon,P}$, established in the previous subsection (see Proposition 5.5), such a comparison principle is actually sufficient for our purposes.

To this end, we observe that the approximations $\hat{V}^{\varepsilon,P}$, given by Definition 5.4, satisfy
$$\left|\lambda(y) + a(y)\rho\hat{V}^{\varepsilon,P}_y(z,y,\tau)\right| \le \frac{1}{\kappa} \qquad \text{and} \qquad -\frac{1}{\nu} \le \hat{V}^{\varepsilon,P}_z(z,y,\tau) - \gamma \le -\nu,$$
for some $(\nu,\kappa)\in(0,1)^2$, uniformly over all $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$, all partitions P, with small enough mesh(P), and all $\varepsilon\in[0,1]$.

As previously mentioned, we expect the approximations $\hat{V}^{\varepsilon,P}$ to have a limit, $\hat{V}^{\varepsilon}$, which is the solution to (5.2). Therefore, the above inequalities are expected to hold for $\hat{V}^{\varepsilon}$ as well. We can, then, "truncate" the corresponding coefficients in (5.3), taking into account the above inequalities, and expecting that the limit of $\hat{V}^{\varepsilon,P}$, in fact, also solves the "truncated" equation. This argument is made precise in the proof of Theorem 5.8.

To this end, we introduce the "truncated" equation

$$\hat{V}^{\varepsilon}_{\tau} + G^{\varepsilon}_{K}(z, y, D\hat{V}^{\varepsilon}, D^2\hat{V}^{\varepsilon}) = 0, \tag{5.9}$$

with the corresponding function $G^{\varepsilon}_{K} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}(2) \to \mathbb{R}$ defined as

$$G^{\varepsilon}_{K}\left(z, y, (p_1, p_2)^T, X\right) := -\frac{1}{2}\left(\frac{\theta_K\left(p_1 - \gamma, \lambda + a\rho p_2\right)}{\eta_K\left(p_1 - \gamma, \lambda + a\rho p_2\right)}, -a\rho\right) X \left(\frac{\theta_K\left(p_1 - \gamma, \lambda + a\rho p_2\right)}{\eta_K\left(p_1 - \gamma, \lambda + a\rho p_2\right)}, -a\rho\right)^T$$

$$-\frac{1-\rho^2}{2}(0, a)X(0, a)^T - \varepsilon(1, 0)X(1, 0)^T \tag{5.10}$$

$$+\frac{1}{2}\frac{\theta^2_K\left(p_1 - \gamma, \lambda + a\rho p_2\right)}{\eta_K\left(p_1 - \gamma, \lambda + a\rho p_2\right)} + \frac{1}{2}\theta^2_K\left(p_1 - \gamma, \lambda + a\rho p_2\right) - \frac{1}{2}a^2 p_2^2 - \varepsilon p_1^2 - b p_2 - \varepsilon p_1,$$

where, for each "truncation" parameter $K > 1$, the functions $\eta_K, \theta_K \in C^{\infty}\left(\mathbb{R}^2\right)$ are chosen so that:

- $\eta_K(z, y) = z$ and $\theta_K(z, y) = y$, whenever $(z, y) \in [-K, -1/K] \times [-K, K]$,
- the first order partial derivatives of η_K and θ_K have compact support,
- $\eta_K(z, y) \in [-2K, 2K]$ and $\theta_K(z, y) \in [-2/K, -K/2]$, for all $(z, y) \in \mathbb{R}^2$.

The corresponding "truncated" initial value problem, then, becomes

$$\begin{cases} \hat{V}^{\varepsilon}_{\tau} + G^{\varepsilon}_{K}(z, y, D\hat{V}^{\varepsilon}, D^2\hat{V}^{\varepsilon}) = 0, \quad (z, y, \tau) \in \mathbb{R}^2 \times (0, T), \\ \hat{V}^{\varepsilon}(z, y, 0) = \log U'_T(e^z) + \gamma z. \end{cases} \tag{5.11}$$

In the remainder of this section, we study the viscosity solutions of (5.11). One can refer, for example, to [9] for the definition and key facts about the viscosity sub- and super-solutions to the initial value problems of the form (5.11). In particular, for any $u \in usc\left(\mathbb{R}^2 \times [0, T)\right)$, we denote by $\mathcal{P}^{2,+}_{\mathbb{R}^2\times(0,T)}u(z, y, \tau)$ the *super-jet* of u at an interior point $(z, y, \tau) \in \mathbb{R}^2 \times (0, T)$. Similarly, for any $v \in lsc\left(\mathbb{R}^2 \times [0, T)\right)$, we denote by $\mathcal{P}^{2,-}_{\mathbb{R}^2\times(0,T)}v(z, y, \tau)$ the *sub-jet* of v at an interior point $(z, y, \tau) \in \mathbb{R}^2 \times (0, T)$. For the definitions of the sub- and super-jets, we refer the reader to [9].

As discussed above, in order to prove the convergence of the approximate solutions $\hat{V}^{\varepsilon,P}$, we will need a comparison principle for the viscosity solutions of (5.11). Notice that the problem at hand is formulated in an unbounded domain, with the operator G^{ε}_{K} depending upon the space variable. Both these features bring the present setup outside the scope of the classical results, stated, for example, in [9]. However, the appropriate comparison principle has been developed in [13]. The following proposition is merely a corollary of the latter result, which, however, is sufficient for our analysis.

Proposition 5.7. *Fix arbitrary* $\varepsilon \in [0,1]$ *and* $K > 1$, *and let* $u \in usc\left(\mathbb{R}^2 \times [0,T)\right)$ *and* $v \in lsc\left(\mathbb{R}^2 \times [0,T)\right)$ *be, respectively, a sub- and a super-solution of (5.11), which are absolutely bounded on* $\mathbb{R}^2 \times [0,T)$. *Then,* $u(z,y,\tau) \leq v(z,y,\tau)$, *for all* $(z,y,\tau) \in \mathbb{R}^2 \times [0,T)$.

Proof:

By assumption, the functions u and v satisfy conditions $(A1)$-$(A3)$ in [13]. Moreover, the operator G^ε_K (cf. (5.10)) satisfies conditions $(F1)$, $(F3)$-$(F5)$, $(F6')$, $(F7)$, $(F9)$ and $(F10)$. Hence, we can apply Theorem 4.2 in [13] to conclude that the comparison principle holds. ■

5.3. Convergence of the auxiliary approximations. We are now ready to formulate a key convergence result which serves as a foundation for the main results of the paper presented in the next section. The following theorem shows that the approximation scheme (5.5) converges to the unique viscosity solution of both the initial value problems (5.2) and (5.11). The analogous result for the regularized marginal HJB equation (4.3) is formulated in the next section.

Theorem 5.8. *For each* $\varepsilon \in [0,1]$, *there exists a continuous function* $\hat{V}^\varepsilon : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$, *such that, as* $mesh(P) \to 0$,

$$\hat{V}^{\varepsilon,P}(z,y,\tau) \to \hat{V}^\varepsilon(z,y,\tau), \tag{5.12}$$

uniformly on all compacts in $(z,y,\tau) \in \mathbb{R}^2 \times [0,T]$. *In addition, for all* $\varepsilon \in [0,1]$, *we have:*

- $\hat{V}^\varepsilon$ *is a viscosity solution to the initial value problem (5.2),*
- *there exists* $K_0 > 1$, *independent of* ε, *such that* $\hat{V}^\varepsilon$ *is the unique bounded viscosity solution to (5.11), for all* $K \geq K_0$,
- *there exists* $N > 0$, *independent of* ε, *such that* $\hat{V}^\varepsilon \in \hat{\mathcal{D}}(N)$.

Proof:

The main ideas of the proof of this theorem stem from the results of [3] on the convergence of numerical approximations of viscosity solutions. As therein, we fix an arbitrary $\varepsilon \in [0,1]$, and introduce the functions

$$\hat{V}^{\varepsilon,*}(z,y,\tau) := \limsup_{\mathrm{mesh}(P)\to 0,\, (z',y',\tau')\to(z,y,\tau)} \hat{V}^{\varepsilon,P}(z',y',\tau')$$

and

$$\hat{V}^{\varepsilon}_*(z,y,\tau) := \liminf_{\mathrm{mesh}(P)\to 0,\, (z',y',\tau')\to(z,y,\tau)} \hat{V}^{\varepsilon,P}(z',y',\tau'),$$

for all $(z,y,\tau) \in \mathbb{R}^2 \times [0,T)$. It is easy to deduce that $\hat{V}^{\varepsilon,*} \in usc(\mathbb{R}^2 \times [0,T))$ and $\hat{V}^\varepsilon_* \in lsc(\mathbb{R}^2 \times [0,T))$.

We are going to show that $\hat{V}^{\varepsilon,*}$ and $\hat{V}^\varepsilon_*$ are, respectively, a sub- and a super-solution to the initial value problems (5.2) and (5.11). Then, we will apply the comparison principle to conclude that $\hat{V}^\varepsilon_*$ and $\hat{V}^{\varepsilon,*}$, in fact, coincide, and, therefore, yield the unique viscosity solution $\hat{V}^\varepsilon$. However, in the present case, few additional arguments need to be added to the method described in [3], because of the specifics of the problem at hand.

First, we recall that the consistency property of the approximation operators $\hat{A}_\tau$ was shown for the equation (5.2), while the comparison principle has been established for (5.11). These two equations, in principle, are different. Therefore, we need to ensure that the two generators G^ε and G^ε_K, defined in (5.10) and (5.3) respectively, coincide on the elements of the semi-jets of $\hat{V}^{\varepsilon}_*$ and $\hat{V}^{\varepsilon,*}$, for all large enough K. For this, we recall that, due to Proposition 5.5, there exists a triplet $(\nu,\delta,\kappa)\in(0,1)^3$, such that, for all small enough mesh(P), all $\tau\in[0,T]$ and all $\varepsilon\in[0,1]$, we have $\hat{V}^{\varepsilon,P}(.,.,\tau)\in\hat{\mathcal{D}}_0(\nu,\delta,\kappa)$. Thus, using the Definitions 4.1 and 5.1 and choosing large enough N, we conclude that $\hat{V}^{\varepsilon,P}\in\hat{\mathcal{D}}(N)$, for all partitions P with small enough mesh(P). Since $\hat{\mathcal{D}}(N)$ is closed with respect to the topology of pointwise convergence, we conclude that $\hat{V}^{\varepsilon}_*,\hat{V}^{\varepsilon,*}\in\hat{\mathcal{D}}(N)$. Using the Definition 4.1 again, we deduce that, for any $(\bar{z},\bar{y},\bar{\tau})\in\mathbb{R}^2\times(0,T)$ and a triplet

$$\left(a,(p_1,p_2)^T,X\right)\in\mathcal{P}^{2,+}_{\mathbb{R}^2\times(0,T)}\hat{V}^{\varepsilon,*}(\bar{z},\bar{y},\bar{\tau}),$$

we have that

$$-(N+\gamma)\le p_1-\gamma\le -1/N,\qquad\text{and}\qquad |p_2\rho|\le N|\rho|. \tag{5.13}$$

The same estimates hold for the elements of the sub-jet of $\hat{V}^{\varepsilon}_*$. It, then, follows from the definition of G^{ε}_K (cf. (5.10)), that for all $K\ge K_0$, with

$$K_0:=2\vee(N+\gamma)\vee|\rho N|\vee\sup_{y\in\mathbb{R}}\left(|\rho|Na(y)+|\lambda(y)|\right),$$

the generators of the equations (5.2) and (5.11), G^{ε} and G^{ε}_K respectively, coincide on the elements of the super-jets of $\hat{V}^{\varepsilon,*}$ as well as on the elements of the sub-jets of $\hat{V}^{\varepsilon}_*$.

Next, we need to show that $\hat{V}^{\varepsilon}_*$ and $\hat{V}^{\varepsilon,*}$ satisfy the corresponding initial condition (the same for (5.2) and (5.11). The only difficulty here is a possible lack of smoothness of the marginal utility function U'_T. Namely, the latter is assumed to be only once continuously differentiable, while we would like to make use of its second derivative. To resolve this issue we choose, for each $\varepsilon'>0$, a function, say $\hat{U}^{\varepsilon'}$, such that $\hat{U}^{\varepsilon'}\in C^2_b(\mathbb{R})$ and the inequality

$$\sup_{z\in\mathbb{R}}\left|\hat{U}^{\varepsilon'}(z)-\log U'_T(e^z)-\gamma z\right|<\varepsilon' \tag{5.14}$$

holds. Such function $\hat{U}^{\varepsilon'}$ exists due to Assumption 2. Since $U'_T\in\mathcal{D}_0$ and, therefore, $((z,y)\mapsto\log U'_T(e^z)+\gamma z)\in\hat{\mathcal{D}}_0$, it follows easily from the Definition 5.1 that there exists a triplet $(\nu,\delta,\kappa)\in(0,1)^3$ such that, for all small enough $\varepsilon'>0$, $\hat{U}^{\varepsilon'}\in\hat{\mathcal{D}}_0(\nu,\delta,\kappa)$ and, therefore, $\left((x,y)\mapsto x^{-\gamma}\exp\left(\hat{U}^{\varepsilon'}(\log x,y)\right)\right)\in\mathcal{D}_0(\nu,\delta,\kappa)$. Then, the second parts of Propositions 4.10 and 4.12 imply that there exists a constant $c_1=c_1(\varepsilon')>0$ such that

$$\sup_{(z,y)\in\mathbb{R}^2}\left(\left|\left(\hat{H}_\tau-\mathcal{I}\right)\hat{U}^{\varepsilon'}(z,y)\right|+\left|\left(\hat{L}^{\varepsilon}_\tau-\mathcal{I}\right)\hat{U}^{\varepsilon'}(z,y)\right|\right)\le c_1\tau,$$

for all $\tau\in[0,T]$. In turn, using the monotonicity and translation invariance of the operators $\hat{A}^{\varepsilon}_\tau$, $\hat{H}_\tau$ and $\hat{L}^{\varepsilon}_\tau$ (which follow from the monotonicity and scale invariance of H_τ and L^{ε}_τ, as shown in Appendix C), we obtain, for any partition $P=\{0=\tau_0<\ldots<\tau_N=T\}$, any $\tau\in(\tau_{k-1},\tau_k]$ and all $(z,y)\in\mathbb{R}^2$, that

$$-2c_1\tau+\hat{U}^{\varepsilon'}(z,y)\le\hat{A}^{\varepsilon}_{\tau-\tau_k}\hat{A}^{\varepsilon}_{\tau_k-\tau_{k-1}}\cdots\hat{A}^{\varepsilon}_{\tau_1}\hat{U}^{\varepsilon'}(z,y)\le\hat{U}^{\varepsilon'}(z,y)+2c_1\tau.$$

Notice that the choice of $\hat{U}^{\varepsilon'}$ (see (5.14)) and the definition of $\hat{V}^{\varepsilon,P}$ (see Definition 5.4), as well as the monotonicity and translation invariance of the operators $\hat{A}^{\varepsilon}_{\tau}$, imply that

$$\sup_{(z,y)\in\mathbb{R}^2}\left|\hat{A}^{\varepsilon}_{\tau-\tau_k}\hat{A}^{\varepsilon}_{\tau_k-\tau_{k-1}}\cdots\hat{A}^{\varepsilon}_{\tau_1}\hat{U}^{\varepsilon'}(z,y)-\hat{V}^{\varepsilon,P}(\tau,z,y)\right|\leq\varepsilon'.$$

Combining the above, we deduce that

$$\left|\hat{V}^{\varepsilon,P}(z,y,\tau)-\hat{U}^{\varepsilon'}(z)\right|\leq 2c_1\tau+\varepsilon', \tag{5.15}$$

for all partitions P and $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$. Passing to the limit, as $\tau\to 0$, we conclude that

$$\left|\hat{V}^{\varepsilon,*}(z,y,0)-\hat{U}^{\varepsilon'}(z)\right|\leq\varepsilon', \tag{5.16}$$

for any $\varepsilon'>0$. An analogous inequality can be obtained for $\hat{V}^{\varepsilon}_{*}$. Letting $\varepsilon'\to 0$ and recalling inequality (5.14), we recover the desired initial condition: $\hat{V}^{\varepsilon,*}(z,y,0)=\hat{V}^{\varepsilon}_{*}(z,y,0)=U_T'(e^z)+\gamma z$.

The rest of the proof proceeds along the lines of [3], and, therefore, we only highlight the main arguments. To this end, we show that $\hat{V}^{\varepsilon,*}$ is a subsolution to (5.11). Let $(\bar{z},\bar{y},\bar{\tau})\in\mathbb{R}^2\times(0,T)$ and consider a triplet $(a,p,X)\in\mathcal{P}^{2,+}_{\mathbb{R}^2\times(0,T)}\hat{V}^{\varepsilon,*}(\bar{z},\bar{y},\bar{\tau})$. Recall that there exists a function $\hat{\phi}:\mathbb{R}^2\times[0,T]\to\mathbb{R}$, satisfying: $\hat{\phi}\in C^{\infty}\left(\mathbb{R}^2\times(0,T)\right)$, $\hat{\phi}(\bar{z},\bar{y},\bar{\tau})=\hat{V}^{\varepsilon,*}(\bar{z},\bar{y},\bar{\tau})$, $D\hat{\phi}(\bar{z},\bar{y},\bar{\tau})=p$, $\hat{\phi}_{\tau}(\bar{z},\bar{y},\bar{\tau})=a$, $D^2\hat{\phi}(\bar{z},\bar{y},\bar{\tau})=X$ and $\hat{V}^{\varepsilon,*}-\hat{\phi}$ has a strict global maximum at $(\bar{z},\bar{y},\bar{\tau})$. Modifying, if necessary, the function $\hat{\phi}$ outside a neighborhood of $(\bar{z},\bar{y},\bar{\tau})$, we may additionally assume that there exists a triplet $(\nu',\delta',\kappa')\in(0,1)^3$ such that $\hat{\phi}(.,.,\tau)\in\hat{\mathcal{D}}_0(\nu',\delta',\kappa')$, for all $\tau\in(0,T)$, and, moreover, $\hat{\phi}(z,y,\tau)\geq -2N$, for all (z,y,τ) outside of a compact set.

Applying standard arguments (see, for example, [3]), we conclude that there exists a sequence of partitions $\{P_n\}$, with $\mathrm{mesh}(P_n)\to 0$, such that the function $\hat{V}^{\varepsilon,P_n}-\hat{\phi}$ attains its strict global maximum in $\mathbb{R}^2\times(0,T)$ at a point, say (z_n,y_n,τ_n), and such that $(z_n,y_n,\tau_n)\to(\bar{z},\bar{y},\bar{\tau})$. Next, we choose a sequence $\{\tau_n'\}$, satisfying $\tau_n'<\tau_n$ and $\tau_n'\to\bar{\tau}$, and notice that

$$\sup_{(z,y)\in\mathbb{R}^2}\left(\hat{V}^{\varepsilon,P_n}(z,y,\tau_n')-\hat{\phi}(z,y,\tau_n')\right)\leq\hat{V}^{\varepsilon,P_n}(z_n,y_n,\tau_n)-\hat{\phi}(z_n,y_n,\tau_n).$$

Using the above inequality and the monotonicity of the operators $\hat{A}^{\varepsilon}_{\tau}$, we obtain

$$\begin{aligned}0&=\hat{A}^{\varepsilon}_{\tau_n-\tau_n'}\left(\hat{V}^{\varepsilon,P_n}(.,.,\tau_n')-\hat{V}^{\varepsilon,P_n}(z_n,y_n,\tau_n)\right)(z_n,y_n)\\&\leq\hat{A}^{\varepsilon}_{\tau_n-\tau_n'}\left(\hat{\phi}(.,.,\tau_n')-\hat{\phi}(z_n,y_n,\tau_n)\right)(z_n,y_n).\end{aligned}$$

In turn, due to their translation invariance property, we have

$$\hat{\phi}(z_n,y_n,\tau_n)\leq\hat{A}^{\varepsilon}_{\tau_n-\tau_n'}\hat{\phi}(.,.,\tau_n')(z_n,y_n).$$

Subtracting $\hat{\phi}(z_n,y_n,\tau_n')$ from both sides of the above inequality, dividing by $\tau_n-\tau_n'$, and passing to the limit, as $n\to\infty$, we make use of the consistency of the semigroup $\left(\hat{A}^{\varepsilon}_{\tau}\right)$ to obtain

$$a+G^{\varepsilon}_{K}\left(\bar{z},\bar{y},p,X\right)=a+G^{\varepsilon}\left(\bar{z},\bar{y},p,X\right)\leq 0,$$

whenever $K \geq K_0$. Similarly, one can show that $\hat{V}^{\varepsilon}_*$ is a super-solution to (5.2) and (5.11), for all $K \geq K_0$.

Since $\hat{V}^{\varepsilon,*}$ and $\hat{V}^{\varepsilon}_*$ are absolutely bounded, we apply Proposition 5.7 to conclude that $\hat{V}^{\varepsilon,*} \leq \hat{V}^{\varepsilon}_*$. On the other hand, by construction, we know that the opposite inequality holds as well. Thus, we conclude that

$$\hat{V}^{\varepsilon}_*(z,y,\tau) = \hat{V}^{\varepsilon,*}(z,y,\tau) = \hat{V}^{\varepsilon}(z,y,\tau) := \lim_{\text{mesh}(P)\to 0} \hat{V}^{\varepsilon,P}(z,y,\tau).$$

It is easy to see that the above convergence holds for any $\varepsilon \in [0,1]$, uniformly on all compacts in $(z,y,\tau) \in \mathbb{R}^2 \times [0,T)$. To extend the function $\hat{V}^{\varepsilon}$, as well as the convergence result, to $\tau = T$, we simply notice that the above derivations can be repeated for $T + \varepsilon'$ in place of T. ■

5.4. Computational aspects. In the next section, we will show how Theorem 5.8 can be used to establish important theoretical results on the existence, uniqueness and approximation of the solution to the original optimization problem. However, one, naturally, may want to apply the above scheme to compute the numerical approximation of the solution of the marginal HJB equation, with a logarithmic change of variables, (5.2), and its truncated version, (5.11).

Recall that the operators $\hat{H}_\tau$, $\hat{L}^{\varepsilon}_\tau$, and, in turn, $\hat{A}^{\varepsilon}_\tau$, are defined via solutions to the linear partial differential equations (4.13) and (4.22). Therefore, to obtain the numerical approximations of the solution to (5.2), one needs to approximate the solutions of the aforementioned linear equations. Notice that these equations are standard linear parabolic, and there exist a variety of methods for solving such equations numerically: see, for example, [37], [17] and [10]. The only potential difficulty, in this case, is the degeneracy of the second order differential operators in (4.13) and (4.22). However, this difficulty can be resolved, for example, by approximating these equations with strictly parabolic ones (as it is done in the proofs of Lemma 4.8 and Proposition 4.12), and applying the "vanishing viscosity" method of Barles and Perthame (cf. [2]).

Alternatively, in view of the Definitions 4.7 and 4.11, one may construct a probabilistic approximation of the solutions to (4.13) and (4.22). An example of such method can be described as follows. Choose a sequence of uniform partitions P^N, with $|P| = N$ and $\text{mesh}(P) = \tau_N$. To approximate the functions in $\mathcal{D}_0$, we choose a uniform partition Ω_N of a compact domain $[-R_N, R_N]^2$ into squares with side h_N. Similarly, for functions in $\mathcal{D}'$, we choose a uniform partition Ω'_N of $[U'_T(\exp(R_N)), U'_T(\exp(-R_N))] \times [-R_N, R_N]$, with diameter h'_N. A function f, defined on Ω_N, is extended to a function on $\mathbb{R}^2$ via the linear interpolation inside the domain $[-R_N, R_N]^2$: first, in z, then, in y, along with the condition $f(z,y) = U'_T(\exp(z))$ outside of the domain. Similarly, we identify any function f on Ω'_N with a function on $\mathbb{R}^2$ by interpolating linearly inside $[U'_T(\exp(R_N)), U'_T(\exp(-R_N))] \times [-R_N, R_N]$: first in z, then, in y, and setting $f(z,y) = \log{(U'_T)}^{-1}(z)$ outside of the domain. The solution of (4.13) (or, (4.22)), with the initial condition f, is given by

$$\int f d\mu^N_{z,y},$$

defined for all $(z, y) \in \mathbb{R}^2$, with μ^N being the distribution of the associated diffusion at time τ_N, started at (z, y). Using the Monte Carlo methods, we can generate a sample whose empirical distribution $\bar{\mu}^N_{z,y}$ approximates $\mu^N_{z,y}$. Then

$$\bar{u}^f(z, y) = \int f d\bar{\mu}^N_{z,y},$$

defined for all (z, y) in Ω (or Ω'), is the numerical approximation of the true solution of (4.13) (or (4.22)). The precision of this approximation is measured by

$$\Delta_N = \sup \left\{ \int h(z', y') \left(e^{-z'\gamma} + e^{-z'/\gamma} \right) \left(\bar{\mu}^N_{z,y}(dz', dy') - \mu^N_{z,y}(dz', dy') \right) \,\middle|\, (z, y) \in \Omega_N \cup \Omega'_N, \right.$$
$$\left. h \text{ -- absolutely bounded by one, with Lipschitz coefficient not exceeding one} \right\}$$

To ensure that $\bar{u}^f$ satisfies the same boundary conditions as f, we define $\bar{\mu}^N_{z,y}$ to be a delta function at (z, y), whenever (z, y) is on the boundary. In addition, due to the form of the associated diffusion process, $\bar{u}^f(z, y)$ can be made strictly monotone in z, provided such is the function f itself. Interpolating linearly between the grid points, we can, then, compute the z-inverse of $\bar{u}^f(z, y)$, to pass from Ω to Ω', and back. Thus, we obtain the discrete-space approximations of the operators $\hat{H}_\tau$ and $\hat{L}^\varepsilon_\tau$. Applying them repeatedly, we construct a numerical approximation for the solution of (5.2). We, now, propose that, if R_N, h_N and $\bar{\mu}^N$ are chosen so that Δ_N/h_N vanishes fast enough, as $N \to \infty$, then, this discrete approximation converges to the true solution.

We, however, do not provide a rigorous proof of the above statement in this paper. As mentioned earlier, our interest in the splitting scheme for an HJB equation was motivated, mainly, by the theoretical insights it can provide. These insights are summarized in the next section. We leave the proof of the above conjecture, as well as the detailed analysis of the performance of the numerical approximation itself, for further research.

6. Main results. In this section, we present the main results of the paper on the convergence of the Trotter-Kato approximation scheme for the marginal HJB equation (cf. (4.2) and (4.3)), as well as the approximations of the value function of the original stochastic optimization problem and of the associated optimal policy.

In Subsection 6.1, we introduce the Trotter-Kato approximation $V^{\varepsilon,P}$ for the regularized marginal HJB equation (4.3) and show that, for each regularization parameter $\varepsilon \in [0, 1]$, it converges to the unique viscosity solution of (4.3). We denote this limit by V^ε and show that, for all strictly positive ε, the function V^ε is, in fact, a smooth solution of (4.3).

Recall that the regularized marginal HJB equation is the equation we obtain by differentiating, at the formal level, the original HJB equation (cf. (2.10)) with respect to the wealth variable x. In Subsection 6.2, we integrate V^ε with respect to x, producing the function U^ε, for each $\varepsilon \in [0, 1]$ and $\gamma \neq 1$ (see (6.7)). We, then, establish that, for all strictly positive ε, U^ε is a smooth solution to the regularized HJB equation (6.6).

In Subsection 6.3 we revert our attention to the optimal policies. We show that the results of Subsection 6.2 can be used to produce an approximate admissible feedback policy, denoted by π^ε. Specifically, we construct π^ε as an analogue of the feedback policy in (2.11), but now using the aforementioned auxiliary function U^ε in place of U. We calculate the expected utility payoff of this policy and, ultimately, show that it can be made arbitrarily close to the optimal value. We, therefore, characterize π^ε as an *ε-optimal policy*, even though the corresponding approximation inequalities (cf. (6.11)–(6.13)) do not correspond exactly to the traditional definition of such policies (see, for example, [42] and [26]), but rather to a logarithmic scale format.

We conclude in Subsection 6.4, where we establish a precise connection between the solutions to initial value problems constructed in preceding subsections and the value function of the original stochastic control problem. Specifically, we show that the value function process has the functional representation (2.8) with the value function given by U^0, constructed in (6.7) with $\varepsilon = 0$. To our knowledge, such an explicit construction is new.

6.1. Viscosity and classical solutions to the regularized marginal HJB equation. We define the Trotter-Kato approximation associated with the regularized marginal HJB equation (4.3) as

$$V^{\varepsilon,P} : (x, y, \tau) \mapsto x^{-\gamma} \exp\left(\hat{V}^{\varepsilon,P}(\log x, y, \tau)\right),$$

where $(x, y, \tau) \in \mathbb{D}$, with $\mathbb{D}$ as in (2.9) and $\hat{V}^{\varepsilon,P}$ as in Definition 5.4.

The following theorem shows that the Trotter-Kato approximation $V^{\varepsilon,P}$ converges to the unique viscosity solution of (4.3).

Theorem 6.1. *For each $\varepsilon \in [0, 1]$, we have:*

(i) *There exists a continuous function $V^\varepsilon : \mathbb{D} \to (0, \infty)$ such that, as mesh$(P) \to 0$,*

$$V^{\varepsilon,P}(x, y, \tau) \to V^\varepsilon(x, y, \tau), \tag{6.1}$$

uniformly on all compacts in $(x, y, \tau) \in \mathbb{D}$.

(ii) *The function V^ε is the unique viscosity solution of the regularized marginal HJB equation (4.3) in the class $\mathcal{D}$, where $\mathcal{D}$ is given in Definition 4.3.*

Proof:

(i) It follows immediately from Theorem 5.8 and the definition of $V^{\varepsilon,P}$ that, for each $\varepsilon \in [0, 1]$, the function $V^\varepsilon : \mathbb{D} \to (0, \infty)$, given by (6.1), is well defined and satisfies

$$V^\varepsilon(x, y, \tau) = x^\gamma \exp\left(\hat{V}^\varepsilon(\log x, y, \tau)\right), \tag{6.2}$$

with the function $\hat{V}^\varepsilon$ as in the statement of Theorem 5.8. Since $\hat{V}^\varepsilon$ is a viscosity solution to (5.2), we easily deduce that V^ε is a viscosity solution to (4.3). Moreover, because $\hat{V}^\varepsilon \in \hat{\mathcal{D}}(N)$ (see Theorem 5.8 and Definition 4.1), we obtain that $V^\varepsilon \in \mathcal{D}(N) \subset \mathcal{D}$.

(ii) We fix an arbitrary $\varepsilon \in [0, 1]$ and consider a function $V' \in \mathcal{D}$, which is a viscosity solution to (4.3). Then, the function $\hat{V}' : (z, y, \tau) \mapsto \log V'(e^z, y, \tau) + \gamma z$, with $(z, y, \tau) \in \mathbb{R}^2 \times [0, T]$, belongs to $\hat{\mathcal{D}}(N)$, for some $N > 0$, and is a viscosity solution to (5.2). In addition, for all large enough $K > 1$, the generators G^ε_K and G^ε coincide on the elements of the semi-jets of $\hat{V}'$ (see the proof of Theorem 5.8). Therefore, $\hat{V}'$ is a viscosity solution to (5.11). Since $\hat{V}' \in \hat{\mathcal{D}}(N)$, it is bounded and the comparison principle (Proposition 5.7) yields that $\hat{V}(z, y, \tau) = \log V^\varepsilon(e^z, y, \tau) + \gamma z$. Thus, V' coincides with V^ε. ■

The next result shows that the viscosity solution to the regularized marginal HJB equation constructed above is, in fact, a *classical solution* to this equation, if the regularization parameter ε is strictly positive. The degree of regularity of the solutions of (4.3) for $\varepsilon = 0$ is, to the best of our knowledge, an open problem.

Theorem 6.2. *For any $\varepsilon \in (0,1]$, the function V^ε defined in Theorem 6.1 is twice continuously differentiable in the spatial variables and once continuously differentiable in the time variable. Moreover, it is the unique classical solution to the regularized marginal HJB equation (4.3) in the class $\mathcal{D}$, and, in addition, it satisfies*

$$\sup_{(z,y,\tau)\in\mathbb{R}^2\times(\varepsilon',T)} \left(|\partial_\tau \log V^\varepsilon(e^z,y,\tau)| + \sum_{j+k=1}^{2} \left|\partial_z^j \partial_y^k \log V^\varepsilon(e^z,y,\tau)\right| \right) < \infty, \tag{6.3}$$

for any $\varepsilon' \in (0,T)$.

Proof:

First, we show that the initial value problem (5.11) has a classical solution. For this, we introduce the following normalization of an arbitrary function, say $u : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$:

$$v(z,y,\tau) = e^{-R\tau} u(z,y,\tau), \tag{6.4}$$

for some large constant $R > 0$. Next, we fix an arbitrary $\varepsilon \in (0,1]$ and $K > 1$, and notice that, if u satisfies the equation appearing in (5.11), then the corresponding equation for v is

$$v_\tau + G^\varepsilon_{K,R}(z,y,v,Dv,D^2v) = 0, \tag{6.5}$$

where $G^\varepsilon_{K,R}(z,y,v,Dv,D^2v) = G^\varepsilon_K(z,y,Dv,D^2v) + Rv$, with G^ε_K as in (5.10).

It is then easy to see that the above operator $\bar{G}^\varepsilon_K$ satisfies the conditions of Definition 1 in Section 5.5 and Lemma 3 in Section 6.1 of [25], and, therefore, in the notation of [25], $G^\varepsilon_{K,R}$ belongs to the class $\bar{\mathcal{F}}(\varepsilon', N, \mathbb{R}^2\times(0,T))$, for some $\varepsilon' \in (0,T)$ and $N > 0$.[2] Thus, we may apply Theorem 3 in Section 6.4 of [25] to conclude that equation (6.5), equipped with the same initial condition as (5.11), has a classical solution, which is of class $C^{2,1}\left(\mathbb{R}^2 \times (0,T)\right)$ and, moreover, it is absolutely bounded on $\mathbb{R}^2 \times [0,T]$. In addition, the second part of the latter theorem shows that the $C^{2,1}\left(\mathbb{R}^2 \times (\varepsilon',T)\right)$-norm of the solution is finite, for any $\varepsilon' > 0$. Undoing the normalization in (6.4), we deduce that the initial value problem (5.11) has an absolutely bounded classical solution. The uniqueness of a bounded viscosity solution to (5.11), then, implies that the aforementioned classical solution has to coincide with $\hat{V}^\varepsilon$, constructed in Theorem 5.8. Finally, Theorem 6.1 and the relation (6.2) imply that V^ε is the unique classical solution to (4.3) in the class $\mathcal{D}$, and the boundedness of the $C^{2,1}\left(\mathbb{R}^2 \times (\varepsilon',T)\right)$-norm of $\hat{V}^\varepsilon$ yields (6.3). ■

6.2. Classical solutions to the regularized HJB equation. So far, we have investigated solutions, classical and viscosity, of the regularized marginal HJB equation (4.3), including the case of zero regularization parameter ε. Next, we focus our attention on the HJB equation (2.10) itself. To our knowledge, the precise connection between the original optimization

[2] Alternatively, one can see this from Example 8 in Section 6.1 of [25], with the set Ω being a singleton.

problem and the solution to the HJB equation (2.10) is an open question in the case of a general utility function and an unbounded set of control values. Herein, we do not study the HJB equation directly, but rather consider its regularized version, whose solution, however, will be ultimately used to approximate the value function and the optimal policy of the original optimization problem with arbitrary precision. In this subsection, we show that, integrating the function V^ε, constructed in the previous subsection, with respect to the x-variable, we obtain a well defined function U^ε, for all $\varepsilon \in [0,1]$. Furthermore, we establish that for strictly positive ε, U^ε is a smooth solution to the regularized HJB equation

$$\begin{cases} U_t + \max_{\pi\in\mathbb{R}} \left(\frac{1}{2}\pi^2\sigma^2(y)U_{xx} + \pi\sigma(y)\left(\lambda(y)U_x + \rho a(y)U_{xy}\right)\right) \\ \qquad\qquad\qquad\qquad\qquad + \frac{1}{2}a^2(y)U_{yy} + \varepsilon x^2 U_{xx} + b(y)U_y = 0, \\ U(x,y,T) = U_T(x). \end{cases} \tag{6.6}$$

We stress that the existence and uniqueness of a classical solution to (6.6) is by no means immediate, as the domain and terminal condition of the problem are unbounded and, moreover, the equation may have singularities due to the possible range of the control variable. We also note that, in this subsection, and throughout the rest of the paper, we consider $\gamma \neq 1$, since, in this case, U_T' is absolutely integrable at either 0 or ∞ (see Assumption 2), and, therefore, the utility function has a finite limit at one of these points.

Theorem 6.3. *Let $\gamma \neq 1$, $\varepsilon \in [0,1]$ and $V^\varepsilon : \mathbb{D} \to (0,\infty)$ be as in Theorem 6.1. Introduce the function $U^\varepsilon(x,y,t) : \mathbb{D} \to \mathbb{R}$, defined as*

$$U^\varepsilon(x,y,t) := \begin{cases} U_T\left(0^+\right) + \int_0^x V^\varepsilon(z,y,T-t)dz, & \text{if } \gamma \in (0,1) \\ U_T\left(\infty\right) - \int_x^\infty V^\varepsilon(z,y,T-t)dz, & \text{if } \gamma \in (1,\infty), \end{cases} \tag{6.7}$$

where U_T is the utility function, and $U_T(0^+)$ and $U_T(\infty)$ are its right and left limits at 0 and ∞ respectively.

The above function U^ε is well defined for all $\varepsilon \in [0,1]$. Moreover, for all strictly positive ε, we have:

(i) *The function U^ε satisfies $U^\varepsilon \in C^{2,1}(\mathbb{D})$.*

(ii) *There exists $N > 0$, independent of ε, such that the inequalities*

$$\begin{aligned} &1/N \le x^\gamma U_x^\varepsilon(x,y,t) \le N, \qquad -N \le x^{1+\gamma}U_{xx}^\varepsilon(x,y,t) \le -1/N, \\ &\left|x^{-1+\gamma}U_y^\varepsilon(x,y,t)\right| \le N, \qquad \left|\frac{\lambda(y)U_x^\varepsilon(x,y,t) - \rho a(y)U_{xy}^\varepsilon(x,y,t)}{xU_{xx}^\varepsilon(x,y,t)}\right| \le N, \end{aligned} \tag{6.8}$$

hold for all $(x,y,t) \in \mathbb{D}_0$, with $\mathbb{D}_0$ as in (4.4).

(iii) *The function U^ε is a classical solution to the regularized HJB equation (6.6).*

Proof:

We only show the assertions for the case $\gamma \in (0,1)$, since the arguments for the case $\gamma \in (1,\infty)$ are similar. The fact that the function U^ε is well defined, then, follows immediately from the fact that $V^\varepsilon \in \mathcal{D}$ (see Definition 4.3).

(i) Theorem 6.2 implies that, for any $\varepsilon' > 0$, there exists $c_1(\varepsilon')$, such that

$$|V^\varepsilon(x,y,\tau)| + |V^\varepsilon_\tau(x,y,\tau)| + |xV^\varepsilon_x(x,y,\tau)| + \left|V^\varepsilon_y(x,y,\tau)\right|$$
$$+ \left|x^2V^\varepsilon_{xx}(x,y,\tau)\right| + \left|xV^\varepsilon_{xy}(x,y,\tau)\right| + \left|V^\varepsilon_{yy}(x,y,\tau)\right| \le c_1(\varepsilon')x^{-\gamma},$$

for all $(x,y,\tau) \in (0,\infty)\times\mathbb{R}\times(\varepsilon',T)$. Fubini's theorem, then, yields that $U^\varepsilon \in C^{2,1}(\mathbb{D})$.

(ii) Using (6.7) and the above estimates, we can interchange the differentiation and integration when computing the partial derivatives of U^ε. Recalling (6.2), we obtain

$$\begin{aligned}
U^\varepsilon_x(x,y,t) &= x^{-\gamma}\exp\left(\hat V^\varepsilon(\log x,y,T-t)\right),\\
U^\varepsilon_{xx}(x,y,t) &= \left(\hat V^\varepsilon_z(\log x,y,T-t)-\gamma\right)x^{-1-\gamma}\exp\left(\hat V^\varepsilon(\log x,y,T-t)\right),\\
U^\varepsilon_{xy}(x,y,t) &= x^{-\gamma}\hat V^\varepsilon_y(\log x,y,T-t)\exp\left(\hat V^\varepsilon(\log x,y,T-t)\right),\\
\frac{\lambda(y)U^\varepsilon_x(x,y,t)+\rho a(y)U^\varepsilon_{xy}(x,y,t)}{xU^\varepsilon_{xx}(x,y,t)} &= \frac{\lambda(y)V^\varepsilon(x,y,t)+\rho a(y)V^\varepsilon_y(x,y,t)}{xV^\varepsilon_x(x,y,t)}\\
&= \frac{\lambda(y)+\rho a(y)\hat V^\varepsilon_y(\log x,y,T-t)}{\hat V^\varepsilon_z(\log x,y,T-t)-\gamma}.
\end{aligned}$$

Using that $\hat V^\varepsilon \in \hat{\mathcal{D}}(N)$, uniformly over $\varepsilon\in[0,1]$ (see Theorem 5.8), we easily deduce the inequalities in (6.8).

(iii) Changing the "time" variable from τ back to $t = T-\tau$ and integrating (4.3) with respect to x, we apply Fubini's theorem to obtain

$$U^\varepsilon_t - \frac{1}{2}\left.\frac{\left(\lambda(y)U^\varepsilon_x+\rho a(y)U^\varepsilon_{xy}\right)^2}{U^\varepsilon_{xx}}\right|_0^x + \frac{1}{2}a^2(y)U^\varepsilon_{yy} + \varepsilon\left.\left(x^2U^\varepsilon_{xx}\right)\right|_0^x + b(y)U^\varepsilon_y = 0,$$

with terminal condition $U^\varepsilon(x,y,T) = U_T(x)$. Using the inequalities in (6.8), we conclude that the corresponding limits at $x=0$, in the above, vanish and, therefore, U^ε satisfies (6.6). ■

6.3. Construction of the "ε-optimal" portfolios. As mentioned earlier, the existence of an optimal portfolio in the feedback form (2.12) has not been established. Note that it is not even clear if such a result is valid for an arbitrary utility function, for the appropriate regularity and growth conditions of the value function and its derivatives might not actually hold. Herein, we construct an approximately optimal portfolio process, denoted by (π^ε_s), using a feedback structure analogous to the one in (2.11), but with the involved partial derivatives being the ones of the auxiliary function U^ε. Recall that, as shown in Theorem 6.3, these derivatives exist for all $\varepsilon\in(0,1]$.

To this end, we introduce the function $\pi^\varepsilon : \mathbb{D}\to\mathbb{R}$, defined by

$$\pi^\varepsilon(x,y,t) = \frac{\lambda(y)}{\sigma(y)}\frac{U^\varepsilon_x(x,y,t)}{U^\varepsilon_{xx}(x,y,t)} - \rho\frac{a(y)}{\sigma(y)}\frac{U^\varepsilon_{xy}(x,y,t)}{U^\varepsilon_{xx}(x,y,t)}, \tag{6.9}$$

with U^ε given in (6.7). We remind the reader that our standing assumptions throughout the rest of this section is $\gamma\neq 1$.

Theorem 6.3 yields that the function $\pi^\varepsilon(x,y,t)$ is globally Lipschitz and linearly bounded in x. From this, we easily deduce the following result.

Lemma 6.4. *Fix arbitrary $t \in [0,T)$ and consider the stochastic differential equation*

$$dX_s^{\varepsilon,x,t} = \sigma(Y_s)\pi^\varepsilon\left(X_s^{\varepsilon,x,t}, Y_s, s\right)\left(\lambda(Y_s)ds + dW_s^1\right), \tag{6.10}$$

with the stochastic factor process Y given by (2.2) and π^ε given in (6.9). Then, for any $x > 0$, (6.10) has a unique strong solution, $\left(X_s^{\varepsilon,x,t}\right)_{s\in[t,T]}$, satisfying $X_t^{\varepsilon,x,t} = x$.

The following result shows that the feedback policy defined via (6.9) is admissible.

Lemma 6.5.*Let $\mathcal{A}$ be the set of admissible policies given in Definition 2.1. Then, for any $(x,t) \in (0,\infty)\times[0,T)$, the policy $(\pi_s^\varepsilon)_{s\in[0,T]}$ defined as*

$$\pi_s^\varepsilon = \pi^\varepsilon\left(X_s^{\varepsilon,x,t}, Y_s, s\right)\mathbf{1}_{[t,T]}(s),$$

with π^ε given in (6.9) and the process $\left(X_s^{\varepsilon,x,t}\right)_{s\in[t,T]}$ given in Lemma 6.4, belongs to $\mathcal{A}$.

Proof:

As mentioned earlier, Theorem 6.3 yields that π_s^ε is absolutely bounded by a linear function of $X_s^{\varepsilon,x,t}$. Therefore, we consider the equation for $\log X_s^{\varepsilon,x,t}$ and deduce by standard arguments that the n-th moment of $X_s^{\varepsilon,x,t}$ is integrable in $s\in[0,T]$, for arbitrary (positive or negative) integer n. We easily conclude that $(\pi_s)\in\mathcal{A}$. ■

The following theorem is one of the main results of this paper. It shows that the original value function process J can be approximated with an arbitrary precision by the expected utility of the terminal wealth generated by (π_s^ε). This justifies the interpretation of (π_s^ε) as a ε-optimal portfolio.

Theorem 6.6. *Let Y and J be given, respectively, by equation (2.2) and Definition 2.2. Let also $X^{\pi,x,t}$, U^ε and $X^{\varepsilon,x,t}$ be given, respectively, by (2.6), (6.7) and (6.10). Then, there exists a constant $C\geq 0$, such that, for all $\varepsilon\in(0,1]$, $x>0$ and $t\in[0,T]$, the following inequalities hold almost surely:*

$$\text{If } \gamma\in(0,1)\cup(1,\infty), \text{ then } \quad U^\varepsilon(x,Y_t,t) \leq \mathbb{E}\left(U_T\left(X_T^{\varepsilon,x,t}\right)\Big|\mathcal{F}_t\right) \leq J(x,t). \tag{6.11}$$

$$\text{If } \gamma\in(0,1), \text{ then } \quad J(x,t) \leq e^{C\varepsilon(T-t)}U^\varepsilon(x,Y_t,t) + U_T\left(0^+\right)\left(1-e^{C\varepsilon(T-t)}\right). \tag{6.12}$$

$$\text{If } \gamma\in(1,\infty), \text{ then } \quad J(x,t) \leq e^{-C\varepsilon(T-t)}U^\varepsilon(x,Y_t,t) + U_T\left(\infty\right)\left(1-e^{-C\varepsilon(T-t)}\right). \tag{6.13}$$

Proof:

The second inequality in (6.11) is obviously satisfied because $(\pi_s^\varepsilon)\in\mathcal{A}$. Next, we show assertion (6.12) and the first inequality in (6.11). Consider an arbitrary $\gamma\in(0,1)$. Without loss of generality, we assume that $U_T(0^+)=0$. Since U^ε is smooth enough (see Theorem 6.3), we choose an arbitrary portfolio $\pi\in\mathcal{A}$ and a constant $c_1\geq 0$, and apply Itô's formula to obtain

$$d\left(e^{c_1\varepsilon(T-s)}U^\varepsilon\left(X_s^{\pi,x,t}, Y_s, s\right)\right) \tag{6.14}$$

$$= e^{c_1\varepsilon(T-s)}\left(\left(-c_1\varepsilon U^\varepsilon + U_t^\varepsilon + \frac{1}{2}\sigma^2\pi_s^2 U_{xx}^\varepsilon + \pi_s\left(\sigma\lambda U_x^\varepsilon + \rho\sigma a U_{xy}^\varepsilon\right) + \frac{1}{2}a^2 U_{yy}^\varepsilon + bU_y^\varepsilon\right) ds\right.$$
$$\left. + \left(U_x^\varepsilon \sigma\pi_s + \rho U_y^\varepsilon a\right) dW_s^1 + \sqrt{1-\rho^2}U_y^\varepsilon a dW_s^2\right).$$

On the other hand, since U^ε is a solution to the regularized HJB equation (6.6), we deduce that

$$\left(U_t^\varepsilon + \frac{1}{2}\sigma^2\pi_s^2 U_{xx}^\varepsilon + \pi_s\left(\sigma\lambda U_x^\varepsilon + \rho\sigma a U_{xy}^\varepsilon\right) + \frac{1}{2}a^2 U_{yy}^\varepsilon + bU_y^\varepsilon\right)\left(X_s^{\pi,x,t}, Y_s, s\right)$$
$$\leq -\varepsilon\left(X_s^{\pi,x,t}\right)^2 U_{xx}^\varepsilon\left(X_s^{\pi,x,t}, Y_s, s\right) \leq \varepsilon c_2\left(X_s^{\pi,x,t}\right)^{1-\gamma},$$

with a constant $c_2 > 0$, independent of ε. We used the second inequality in (6.8) to obtain the above. Next, we integrate the first inequality in (6.8) to obtain

$$U^\varepsilon\left(X_s^{\pi,x,t}, Y_s, s\right) \geq c_3\left(X_s^{\pi,x,t}\right)^{1-\gamma},$$

for some constant $c_3 > 0$, independent of ε. We, then, conclude that c_1 can be chosen large enough, so that the drift term in the right hand side of (6.14) is always non-positive. Since U^ε satisfies (6.6), applying the Itô's formula, we deduce that the drift of $U^\varepsilon\left(X_s^{\varepsilon,x,t}, Y_s, s\right)$ is positive. Hence, localizing the drifts and the local martingale terms, we conclude that there exists a sequence of stopping times, $\{\tau_n\}_{n=1}^\infty$, with values in $[t, T]$, such that $\tau_n \to T$, almost surely, and the processes

$$e^{c_1\varepsilon(T-t)}U^\varepsilon\left(X_{s\wedge\tau_n}^{\varepsilon,x,t}, Y_{s\wedge\tau_n}, s\wedge\tau_n\right) \qquad \text{and} \qquad e^{c_1\varepsilon(T-s\wedge\tau_n)}U^\varepsilon\left(X_{s\wedge\tau_n}^{\pi,x,t}, Y_{s\wedge\tau_n}, s\wedge\tau_n\right)$$

are a sub- and a supermartingale in $s \in [t, T]$, respectively. Moreover, at $s = t$, both processes are equal to $e^{c_1\varepsilon(T-t)}U^\varepsilon(x, Y_t, t)$. Combining the above, we deduce that the following inequalities hold almost surely, for all $t \in [0, T]$ and all $n \geq 1$,

$$\mathbb{E}\left(e^{c_1\varepsilon(T-T\wedge\tau_n)}U^\varepsilon\left(X_{T\wedge\tau_n}^{\pi,x,t}, Y_{T\wedge\tau_n}, T\wedge\tau_n\right)\middle|\mathcal{F}_t\right) \leq e^{c_1\varepsilon(T-t)}U^\varepsilon(x, Y_t, t)$$
$$\leq e^{c_1\varepsilon(T-t)}\mathbb{E}\left(U^\varepsilon\left(X_{T\wedge\tau_n}^{\varepsilon,x,t}, Y_{T\wedge\tau_n}, T\wedge\tau_n\right)\middle|\mathcal{F}_t\right). \tag{6.15}$$

Next, we notice that, almost surely:

$$\lim_{n\to\infty} e^{c_1\varepsilon(T-T\wedge\tau_n)}U^\varepsilon\left(X_{T\wedge\tau_n}^{\pi,x,t}, Y_{T\wedge\tau_n}, T\wedge\tau_n\right) = U_T(X_T^{\pi,x,t}),$$

$$\lim_{n\to\infty} U^\varepsilon\left(X_{T\wedge\tau_n}^{\varepsilon,x,t}, Y_{T\wedge\tau_n}, T\wedge\tau_n\right) = U_T(X_T^{\varepsilon,x,t}).$$

Finally, we recall that the random variables $\left(X_{T\wedge\tau_n}^{\varepsilon,x,t}\right)^{1-\gamma}$ are almost surely bounded by an integrable random variable, uniformly over $n \geq 1$. This follows from the explicit representation of $X^{\varepsilon,x,t}$ in (6.10) and the linear boundedness of π^ε, using, for example, Doob's maximal inequality. Passing to the limit in (6.15), as $n \to \infty$, we make use of Fatou's lemma and the dominated convergence theorem to conclude that inequality (6.12) and the first inequality in (6.11) are satisfied.

It remains to establish inequality (6.13) when $\gamma \in (1, \infty)$. To do this, we repeat the above derivations with "$-c_1$" in place of "c_1", without the use of localizing sequences, for the Fatou's lemma cannot be applied in this case. Hence, we obtain the sub- and super-martingale properties directly from the definition of the set $\mathcal{A}$. ■

6.4. Constructing the value function. We conclude by making a precise connection between the value function of the original optimization problem, defined by (2.8), and the auxiliary functions U^ε, constructed in Subsection 6.2. We, again, remind the reader that our standing assumption throughout this subsection is $\gamma \neq 1$.

Lemma 6.7. *For any $\varepsilon \in [0,1]$, let $U^\varepsilon : \mathbb{D} \to \mathbb{R}$ be as in Theorem 6.3. Then, as $\varepsilon \to 0$, $U^\varepsilon(x,y,t) \to U^0(x,y,t)$, uniformly on all compacts in $(x,y,t) \in \mathbb{D}$.*

Proof:

We first recall, using (6.7) and (6.2), that the function $\hat{V}^\varepsilon : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$, given in Theorem 5.8, satisfies

$$\hat{V}^\varepsilon(z,y,\tau) = \log U^\varepsilon_x(e^z, y, T-\tau) - \gamma z,$$

and, moreover, that it is a viscosity solution to (5.11), for each $\varepsilon \in [0,1]$. Applying the results of [2], we establish the stability of viscosity solutions to (5.11) with respect to the parameter ε. The arguments are well known, and, for this, we omit the details. We only note that one can repeat the proof of Proposition VII.4.1 in [12] – using the fact that the functions $\hat{V}^\varepsilon$ are bounded uniformly over $\varepsilon \in [0,1]$ in order to compensate for the unbounded domain in the present case – and obtain that the functions $\hat{V}^*$ and $\hat{V}_*$, defined as

$$\hat{V}^*(z,y,\tau) = \limsup_{(z',y',\tau',\varepsilon)\to(z,y,\tau,0)} \hat{V}^\varepsilon(z',y',\tau'),$$

$$\hat{V}_*(z,y,\tau) = \liminf_{(z',y',\tau',\varepsilon)\to(z,y,\tau,0)} \hat{V}^\varepsilon(z',y',\tau'),$$

are, respectively, a sub- and a super-solution of the equation in (5.11), with $\varepsilon = 0$. In order to show that $\hat{V}^*$ and $\hat{V}_*$ satisfy the initial condition in (5.11), we simply notice that the estimate (5.15) holds uniformly over $\varepsilon \in [0,1]$, as it follows from its derivation and the second statements of Propositions 4.10 and 4.12.

Since $\hat{V}^*$ and $\hat{V}_*$ are bounded, we apply the comparison principle and conclude that they coincide. On the other hand, they also form a viscosity solution to (5.11), with $\varepsilon = 0$, and, therefore, have to coincide with $\hat{V}^0$. Applying Fubini's theorem, we easily conclude that U^ε converges to U^0. ■

We now present one of the main results herein.

Theorem 6.8.*Let the value function process J be given by Definition 2.2, the stochastic factor process Y be given by (2.2) and the function U^0 be given by (6.7), with $\varepsilon = 0$. Then, the value function process J admits the functional representation*

$$J(x,t) = U^0(x, Y_t, t), \tag{6.16}$$

almost surely, for all $(x,t) \in (0,\infty) \times [0,T]$.

Proof:

The assertion follows by taking limit, as $\varepsilon \to 0$, in the inequalities in Theorem 6.6 and using Lemma 6.7. ■

Appendix A. Proof of Lemma 4.8. In view of (4.16), we will verify the corresponding properties of the function v by analyzing the function u instead. The main difficulty in studying the properties of u, is that, in general, it may not have enough smoothness to be a classical solution to (4.13). Thus, for arbitrary $\varepsilon' > 0$, we introduce the auxiliary function $u^{\varepsilon'}$ as the unique exponentially bounded classical solution to the regularized equation

$$u^{\varepsilon'}_\tau - \varepsilon' u^{\varepsilon'}_{zz} - \frac{1}{2}\lambda^2(y)u^{\varepsilon'}_{zz} - \frac{1}{2}\rho^2 a^2(y)u^{\varepsilon'}_{yy} + \rho a(y)\lambda(y)u^{\varepsilon'}_{zy} - \frac{1}{2}\lambda^2(y)u^{\varepsilon'}_z + \rho a(y)\lambda(y)u^{\varepsilon'}_y = 0, \quad (A.1)$$

with initial condition $u^{\varepsilon'}(z, y, 0) = f(e^z, y)$. The classical solution to the above problem is well defined due to Assumption 1. Notice that $u^{\varepsilon'} \in C^{2,1}(\mathbb{R}^2 \times (0, T))$, and, moreover, using the Feynman-Kac formula we deduce that it can be represented as

$$u^{\varepsilon'}(z, y, \tau) = \mathbb{E}\left(f\left(\exp\left(\hat{Z}^{z,y,\varepsilon'}_\tau\right), \hat{Y}^y_\tau\right)\right),$$

where $\hat{Y}^y$ and $\hat{Z}^{z,y,\varepsilon'}$ satisfy the system of SDE's, consisting of the second equation in (4.15) and the equation

$$d\hat{Z}^{z,y,\varepsilon'}_\tau = \frac{1}{2}\lambda^2(\hat{Y}^y_\tau)d\tau + \lambda(\hat{Y}^y_\tau)dB_\tau + \sqrt{2\varepsilon'}dW_t, \quad \hat{Z}^{z,y,\varepsilon'}_0 = z,$$

where W is a Brownian motion, independent of B appearing in (4.15). Clearly, we have: $\hat{Z}^{z,y,\varepsilon'}_\tau = \hat{Z}^{z,y}_\tau + \sqrt{2\varepsilon'}W_\tau$. Using this representation, the definition of u, together with the properties of f as an element of $\mathcal{D}'_0$, and applying standard probabilistic techniques, we easily deduce that

$$e^{z/\gamma}\left|u^{\varepsilon'}(z, y, \tau) - u(z, y, \tau)\right| \to 0, \quad (A.2)$$

as $\varepsilon' \to 0$, uniformly over $(z, y, \tau) \in \mathbb{R}^2 \times [0, T]$. Notice that, in particular, the above implies that the function u is continuous.

(i) We start with verifying the first condition of Definition 4.5. Specifically, we need to establish that

$$\delta e^{-\alpha'\tau} \le x^{1/\gamma} v(x, y, \tau) \le e^{\alpha'\tau}/\delta,$$

for some constant $\alpha' \ge 0$. Elementary arguments show that the above inequality is equivalent to

$$\delta e^{-\alpha'\tau} \le e^{z/\gamma}u(z, y, \tau) \le e^{\alpha'\tau}/\delta. \quad (A.3)$$

To show (A.3), we work as follows. We first notice that the function $e^{z/\gamma}u^{\varepsilon'}$ also solves an initial value problem. Consequently, it can be represented using the Feynman-Kac formula, namely,

$$e^{z/\gamma}u^{\varepsilon'}(z, y, \tau) = \mathbb{E}\left(f\left(\exp\left(\hat{Z}^{z,y,\varepsilon'}_\tau\right), \hat{Y}^y_\tau\right)\exp\left(\frac{1}{\gamma}\hat{Z}^{z,y,\varepsilon'}_\tau + \frac{1-\gamma}{2\gamma^2}\int_0^\tau \lambda^2(\hat{Y}^y_s)ds\right)\right).$$

From the above representation, as well as the properties of f as an element of $\in \mathcal{D}_0'(\nu,\delta,\kappa)$ and the boundedness of λ, we deduce that

$$e^{-\alpha'\tau}\delta \le e^{-\alpha'\tau} \inf_{(z,y)\in\mathbb{R}^2} e^{z/\gamma} f(e^z,y) \le \inf_{(z,y)\in\mathbb{R}^2} e^{z/\gamma} u^{\varepsilon'}(z,y,\tau) \tag{A.4}$$

$$\le \sup_{(z,y)\in\mathbb{R}^2} e^{z/\gamma} u^{\varepsilon'}(z,y,\tau) \le e^{\alpha'\tau} \sup_{(z,y)\in\mathbb{R}^2} e^{z/\gamma} f(e^z,y) \le e^{\alpha'\tau}/\delta,$$

for some constant $\alpha' \ge 0$, depending only upon the function λ. Recalling (A.2), we easily conclude.

Next, we establish the second condition of Definition 4.5. First, we rewrite it, for convenience, as

$$\nu \le -\frac{u_z(z,y,\tau)}{u(z,y,\tau)} \le \frac{1}{\nu}. \tag{A.5}$$

Notice that we first need to establish that u_z is well defined and continuous. As before, we start with analyzing the approximation $u^{\varepsilon'}$. It is easy to see that $u_z^{\varepsilon'}$ is well defined and continuous, and that, in fact, it satisfies the equation (A.1) with initial condition $u_z^{\varepsilon'}(z,y,0) = e^z f_x(e^z,y)$. Making use of the absolute boundedness of the function $e^{(1+1/\gamma)z} f_x(e^z,y)$, we use the Feynman-Kac formula to obtain

$$e^{z/\gamma} u_z^{\varepsilon'}(z,y,\tau) = \mathbb{E}\left(\exp\left((1+1/\gamma)\,\hat{Z}_\tau^{\varepsilon',z,y}\right) f_x\left(\exp\left(\hat{Z}_\tau^{\varepsilon',z,y}\right), \hat{Y}_\tau^y\right)\right) \tag{A.6}$$

$$= \frac{1}{\sqrt{2\pi}}\mathbb{E}\left(\int_{\mathbb{R}} \exp\left((1+1/\gamma)\left(\hat{Z}_\tau^{z,y} + s\sqrt{2\varepsilon'\tau}\right)\right) f_x\left(\exp\left(\hat{Z}_\tau^{z,y} + s\sqrt{2\varepsilon'\tau}\right), \hat{Y}_\tau^y\right) e^{-s^2/2} ds\right).$$

In turn, using the dominated convergence theorem, we conclude that

$$e^{z/\gamma} u_z^{\varepsilon'}(z,y,\tau) \to \mathbb{E}\left(\exp\left((1+1/\gamma)\,\hat{Z}_\tau^{z,y}\right) f_x\left(\exp\left(\hat{Z}_\tau^{z,y}\right), \hat{Y}_\tau^y\right)\right), \tag{A.7}$$

as $\varepsilon' \to 0$, uniformly in $(z,y,\tau) \in \mathbb{R}^2 \times [0,T]$. The uniform convergence is due to the continuity and absolute boundedness of $e^{(1+1/\gamma)z} f_x(e^z,y)$. The above, together with (A.2), implies that u is once continuously differentiable in z, and, moreover, that $u_z(z,y,\tau) = \lim_{\varepsilon'\to 0} u_z^{\varepsilon'}(z,y,\tau)$, for any $(z,y,\tau) \in \mathbb{R}^2 \times [0,T]$. Next, we establish (A.5) with $u^{\varepsilon'}$ in place of u. Notice that this is equivalent to verifying that

$$\partial_z\left(e^{z/\nu} u^{\varepsilon'}(z,y,\tau)\right) \ge 0 \qquad \text{and} \qquad \partial_z\left(e^{\nu z} u^{\varepsilon'}(z,y,\tau)\right) \le 0, \tag{A.8}$$

for all $(z,y,\tau) \in \mathbb{R}^2 \times [0,T]$. For this, we derive the corresponding linear equation for the function $\partial_z\left(e^{z/\nu} u^{\varepsilon'}\right)$, and observe that this equation preserves the non-negativity of its initial condition, due to the Feynman-Kac formula. Similarly, the equation for the function $\partial_z\left(e^{\nu z} u^{\varepsilon'}\right)$ preserves the non-positivity of its initial condition. Thus, we conclude that (A.5) holds for $u^{\varepsilon'}$ and, consequently, for the function u.

It remains to show that v satisfies the last condition of Definition 4.5. Assuming that $\rho \ne 0$, and using the notation $k = 1/\kappa$, the latter condition can be written as

$$\left|\rho a(y)\frac{u_y(z,y,\tau)}{u_z(z,y,\tau)} - \lambda(y)\right| \le k + \tau e^{\beta'(\nu,\delta)(1+\tau k^2)}, \tag{A.9}$$

for each $(z, y, \tau) \in \mathbb{R}^2 \times [0, T]$ and for some continuous function $\beta' : (0,1)^2 \to [0, \infty)$, provided, of course, that u_y is well defined. To show the latter, we work again with the approximation $u^{\varepsilon'}$. We, first, establish (A.9) for $u^{\varepsilon'}$ and then pass to the limit as $\varepsilon' \to 0$. In fact, in the present case, it is more convenient to introduce the change of variables

$$\hat{u}^{\varepsilon'}(z, y, \tau) = u^{\varepsilon'}\left(z - \int_0^y \frac{\lambda(s) - k}{\rho a(s)} ds, y, \tau\right). \tag{A.10}$$

We, then, easily deduce that $\hat{u}^{\varepsilon'}$ satisfies

$$\hat{u}^{\varepsilon'}_\tau - \frac{1}{2}\left((k^2 + \varepsilon')\hat{u}^{\varepsilon'}_{zz} - 2k\rho a \hat{u}^{\varepsilon'}_{zy} + \rho^2 a^2 \hat{u}^{\varepsilon'}_{yy}\right) + \frac{\lambda^2 - 2\lambda k - \rho a \lambda' + \rho a'(\lambda - k)}{2} \hat{u}^{\varepsilon'}_z + \rho a \lambda \hat{u}^{\varepsilon'}_y = 0, \tag{A.11}$$

with initial condition $\hat{u}^{\varepsilon'}(z, y, 0) = f\left(\exp\left(z - \int_0^y \frac{\lambda(s)-k}{\rho a(s)} ds\right), y\right)$. Next, we introduce the function $w^{\varepsilon'}$ via $w^{\varepsilon'}(z, y, \tau) = \hat{u}^{\varepsilon'}_y(z, y, \tau)$, and notice that it satisfies

$$w^{\varepsilon'}_\tau - \frac{1}{2}\left((k^2 + \varepsilon')w^{\varepsilon'}_{zz} - 2k\rho a w^{\varepsilon'}_{zy} + \rho^2 a^2 w^{\varepsilon'}_{yy}\right) + \frac{1}{2}\left(\lambda^2 - k\lambda - \rho a \lambda' + k\rho a' + \rho a' \lambda\right) w^{\varepsilon'}_z \tag{A.12}$$
$$+\rho a\left(\lambda - \rho a'\right) w^{\varepsilon'}_y + \rho (a\lambda)' w^{\varepsilon'} = \left(k\lambda' - \lambda\lambda' + \frac{1}{2}\rho\left(a\lambda' - a'\lambda\right)' + k\rho a''\right) \hat{u}^{\varepsilon'}_z.$$

Clearly, the above linear equation, equipped with initial condition $w^{\varepsilon'}(z, y, 0) = \bar{f}(z, y)$, where

$$\bar{f}(z, y) = f_y\left(\exp\left(z - \int_0^y \frac{\lambda(s) - k}{\rho a(s)} ds\right), y\right) \tag{A.13}$$
$$-\frac{\lambda(y) - k}{\rho a(y)} \exp\left(z - \int_0^y \frac{\lambda(s) - n}{\rho a(s)} ds\right) f_x\left(\exp\left(z - \int_0^y \frac{\lambda(s) - n}{\rho a(s)} ds\right), y\right),$$

has a unique classical solution. To see this, recall that, due to Assumption 1, the coefficients of the above equation are bounded, while the initial condition and the right hand side are exponentially bounded, due to the previously obtained exponential estimates of $u^{\varepsilon'}_z$, and, consequently, of $\hat{u}^{\varepsilon'}_z$. Clearly, the solution to the above initial value problem has to coincide with $w^{\varepsilon'}$. In order to obtain an absolutely bounded function, we consider the exponentially weighted transformation of $w^{\varepsilon'}$, denoted by $\tilde{w}^{\varepsilon'}$, and given by

$$\tilde{w}^{\varepsilon'}(z, y, \tau) := e^{\frac{1}{\gamma}\left(z - \int_0^y \frac{\lambda(s)-k}{\rho a(s)} ds\right)} w^{\varepsilon'}(z, y, \tau).$$

Then, $\tilde{w}^{\varepsilon'}$ satisfies the equation

$$\tilde{w}^{\varepsilon'}_\tau - \frac{1}{2}\left((k^2 + \varepsilon')\tilde{w}^{\varepsilon'}_{zz} - 2k\rho a \tilde{w}^{\varepsilon'}_{zy} + \rho^2 a^2 \tilde{w}^{\varepsilon'}_{yy}\right) + A(y)\tilde{w}^{\varepsilon'}_z + B(y)\tilde{w}^{\varepsilon'}_y + C(y)\tilde{w}^{\varepsilon'} = R^{\varepsilon'}(z, y, \tau), \tag{A.14}$$

with $A(y) = A_0(y) + kA_1(y) + k^2 A_2(y)$, $B(y) = B_0(y) + kB_1(y)$, $C(y) = C_0(y) + kC_1(y) + k^2 C_2(y)$,

$$R^{\varepsilon'}(z, y, \tau) = \exp\left(\frac{1}{\gamma}\left(z - \int_0^y \frac{\lambda(s) - k}{\rho a(s)} ds\right)\right) \hat{u}^{\varepsilon'}_z(z, y, \tau) \left(R_0(y) + kR_1(y)\right)$$

and initial condition

$$\tilde{w}^{\varepsilon'}(z,y,0) = \tilde{f}(z,y) := \exp\left(\frac{1}{\gamma}\left(z - \int_0^y \frac{\lambda(s)-k}{\rho a(s)}ds\right)\right)\bar{f}(z,y). \tag{A.15}$$

The above functions A_i, B_i, C_i and R_i are absolutely bounded and continuous, and they only depend upon λ, a, ρ and γ. As before, applying Assumption 1 and the estimate on the functions $e^{z/\gamma}|u_z^{\varepsilon'}(z,y,\tau)|$ and $e^{z/\gamma}|u_y^{\varepsilon'}(z,y,\tau)|$, we conclude that the coefficients, the initial condition, and the right hand side of (A.14), are absolutely bounded. Making use of the absolute boundedness of $\tilde{f}$, C and $R^{\varepsilon'}$, we use the Feynman-Kac formula to conclude that, for each $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$, we have

$$\tilde{w}^{\varepsilon'}(z,y,\tau) = \mathbb{E}\left(\tilde{f}\left(\bar{Z}_\tau^{z,y,\varepsilon'},\bar{Y}_\tau^y\right)\exp\left(-\int_0^\tau C\left(\bar{Y}_t^y\right)dt\right)\right. \tag{A.16}$$
$$\left. + \int_0^\tau R^{\varepsilon'}\left(\bar{Z}_s^{z,y,\varepsilon'},\bar{Y}_s^y,\tau-s\right)\exp\left(-\int_0^s C\left(\bar{Y}_t^y\right)dt\right)ds\right),$$

where $\left(\bar{Z}^{z,y,\varepsilon'},\bar{Y}^y\right)$ is the diffusion process given by the generator of (A.14) (we omit the precise definition, as it is analogous to the previous constructions). Making use of the dominated convergence theorem, we deduce that as $\varepsilon'\to 0$ the right hand side of (A.16) converges to

$$\mathbb{E}\left(\tilde{f}\left(\bar{Z}_\tau^{z,y,0},\bar{Y}_\tau^y\right)e^{-\int_0^\tau C\left(\bar{Y}_t^y\right)dt} + \int_0^\tau R\left(\bar{Z}_s^{z,y,0},\bar{Y}_s^y,\tau-s\right)e^{-\int_0^s C\left(\bar{Y}_t^y\right)dt}ds\right), \tag{A.17}$$

where the continuous function $R(z,y,\tau)$ is the limit of $R^{\varepsilon'}(z,y,\tau)$, as $\varepsilon'\to 0$, uniformly over $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$. Note that the latter limit exists due to (A.7). Applying the dominated convergence theorem once more, we see that the quantity in (A.17) is continuous in $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$. In particular, this implies that u is once continuously differentiable in y, and, moreover, that $u_y(z,y,\tau)=\lim_{\varepsilon'\to 0}u_y^{\varepsilon'}(z,y,\tau)$, for any $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$.

Next, we consider the case $\rho>0$. The fact that $f\in\mathcal{D}_0'(\nu,\delta,\kappa)$ yields that the function $\tilde{f}$, given by (A.15), is non-positive. Making use of the Feynman-Kac representation in (A.16) and using the straight forward estimates on the functions C and $R^{\varepsilon'}$, we conclude that

$$\tilde{w}^{\varepsilon'}(z,y,\tau) \le c_1\frac{\tau}{\nu\delta}\exp\left(c_2\tau(1+k^2)(1+1/\gamma^2)\right),$$

where, above and throughout the rest of the proof, c_i's stand for positive constants which depend only upon a, λ, ρ and γ, and are independent of $\varepsilon'\in(0,1]$ and $(\nu,\delta,\kappa)\in(0,1)^3$. Recalling the definition of $\tilde{w}^{\varepsilon'}$, we deduce that the above inequality implies

$$e^{z/\gamma}u_y^{\varepsilon'}(z,y,\tau) - \frac{\lambda(y)-k}{\rho a(y)}e^{z/\gamma}u_z^{\varepsilon'}(z,y,\tau) \le c_1\frac{\tau}{\nu\delta}\exp\left(c_2\tau(1+k^2)(1+1/\gamma^2)\right),$$

uniformly over $\varepsilon'\in(0,1]$. Since we have shown that $u_z^{\varepsilon'}$ and $u_y^{\varepsilon'}$ converge to u_z and u_y respectively, we conclude that the above inequality holds with u in place of $u^{\varepsilon'}$. Hence,

$$\frac{u_y(z,y,\tau)}{u_z(z,y,\tau)} - \frac{\lambda(y)}{\rho a(y)} \ge -\frac{1}{|\rho|a(y)}\left(k + c_3\frac{\tau}{\nu^2\delta^2}\exp\left(c_2\tau(1+k^2)(1+1/\gamma^2)\right)\right),$$

where we made use of Assumption 1 once more. When $\rho < 0$, we obtain an upper bound. In order to obtain the corresponding upper (lower) bound for positive (negative) correlation coefficient, we simply repeat the above derivations substituting "k" to "$-k$" in the definition of $\hat{u}^{\varepsilon'}$ in (A.10). Thus, recalling that $k = 1/\kappa$, we conclude that (4.17) holds with $\beta'(\nu,\delta) = c_2(1+1/\gamma^2)(1+T) + |\log(c_3 \vee 1) - 2\log(\nu\delta)|$.

(ii) To show (4.18), we first notice that if $\left((z,y) \mapsto e^{z/\gamma} f(e^z,y)\right) \in C_b^2\left(\mathbb{R}^2\right)$, then the function $(z,y,\tau) \mapsto e^{z/\gamma}\left(u^{\varepsilon'}(z,y,\tau) - f(e^z,y)\right)$ satisfies a linear parabolic equation with zero initial condition and with the coefficients, as well as the right hand side (the latter is a linear combination of the derivatives of f), being absolutely bounded by a constant depending only on the C^2-norm of $e^{z/\gamma} f(e^z,y)$, and on a, λ, ρ and γ. Using the Feynman-Kac formula once more, we easily obtain (4.18) with $u^{\varepsilon'}$ in place of u. Passing to the limit as $\varepsilon' \to 0$, we obtain (4.18).

(iii) We notice that, if $\left((z,y) \mapsto e^{z/\gamma} f(e^z,y)\right) \in C_b^{\infty}\left(\mathbb{R}^2\right)$, we can apply the Oleinik's estimate directly (see, for example, Corollary 2.4.5 in [39]) to show that there exists a unique bounded classical solution to the initial value problem (formally) corresponding to $e^{z/\gamma} u(z,y,\tau)$. This, in turn, implies that there exists a unique exponentially bounded classical solution to (4.13) equipped with the initial condition $f(e^z,y)$. Using the Feynman-Kac formula, we easily conclude that this solution has to coincide with u. ∎

Appendix B. Proof of Proposition 4.10. Part (i) follows directly from Lemmas 4.6 and 4.8. Indeed, we only need to notice that the constant α and the function β can be chosen as $\alpha := \gamma\alpha'$ and $\beta(\nu,\delta) := \beta'\left(\nu, \delta^{1/\gamma}\right)$, where α' and β' are given in Lemma 4.8.

(ii) Denote by $f \in \mathcal{D}_0'$ the x-inverse of F. Notice that, due to Lemma 4.6, we have that $f \in \mathcal{D}_0'\left(\nu, \delta^{1/\gamma}, \kappa\right)$ and, in particular, that the function $e^{z/\gamma} f(e^z,y)$ is absolutely bounded. Next, we need to show that, under the conditions of the second part of the proposition, the C^2-norm of $e^{z/\gamma} f(e^z,y)$ is finite. First, notice that, since $F \in \mathcal{D}_0(\nu,\delta,\kappa)$ and $((z,y) \mapsto e^{\gamma z} F(e^z,y)) \in C_b^2\left(\mathbb{R}^2\right)$, we have $((z,y) \mapsto \log(e^{\gamma z} F(e^z,y))) \in C^2(\mathbb{R}^2)$. This implies that

$$\sup_{(z,y)\in\mathbb{R}^2} \left|\partial_z^k \left(\log F(e^z,y)\right)\right| < \infty,$$

for $k = 1,2$. Next, we observe that the function $(z,y) \mapsto \log f(e^z,y)$ is the z-inverse of $(z,y) \mapsto \log F(e^z,y)$. Therefore, using the relations between the derivatives of a function and its inverse (see, for example, (4.11)), and the fact that $\partial_z^k \left(\log F(e^z,y)\right) \in [-1/\nu, -\nu]$, we obtain

$$\sup_{(z,y)\in\mathbb{R}^2} \left|\partial_z^k \left(\log\left(e^{z/\gamma} f(e^z,y)\right)\right)\right| < \infty,$$

for $k = 1,2$. Making use of the above inequality and the fact that $e^{z/\gamma} f(e^z,y) \geq \delta^{1/\gamma}$, we conclude that

$$\sup_{(z,y)\in\mathbb{R}^2} \left|\partial_z^k \left(e^{z/\gamma} f(e^z,y)\right)\right| < \infty, \tag{B.1}$$

for $k = 1,2$, and, hence, $e^{z/\gamma} f(e^z,y) \in C_b^2(\mathbb{R}^2)$. In turn, Lemma 4.8 yields that there exists a constant $c_1 > 0$, such that

$$e^{z/\gamma}\left|u(z,y,\tau) - f(e^z,y)\right| \leq c_1\tau$$

and

$$\left|u\left(z,y,\tau\right)-u\left(z',y,\tau\right)\right|\geq\nu\delta^{1/\gamma}e^{-\alpha\tau/\gamma}e^{-(z\vee z')/\gamma}|z-z'|,$$

for all $(z',z,y,\tau)\in\mathbb{R}^3\times[0,T]$. We adopt the convention that all constants c_i appearing in the proof may depend upon the C^2-norm of $e^{z/\gamma}f(e^z,y)\in C_b^2(\mathbb{R}^2)$ and a, λ and ρ; any additional dependence will be indicated explicitly. Collecting the above inequalities, and plugging "$\log V^1(x,y,\tau)$" and "$\log F(x,y)$" in place of "z" and "z'", respectively, we obtain

$$\left|\log V^1(x,y,\tau)-\log F(x,y)\right|\leq c_2(\nu,\delta)\tau\left(F(x,y)\right)^{-1/\gamma}\exp\left(\frac{1}{\gamma}\left(\log V^1(x,y,\tau)\vee\log F(x,y)\right)\right),$$

for some constant $c_2=c_2(\nu,\delta)>0$. Finally, making use of the inequality

$$\left|\log V^1(x,y,\tau)\vee\log F(x,y)\right|\leq\log F(x,y)-2\log\delta+\alpha\tau,$$

which holds because of (4.19), we conclude that

$$\left|\log V^1(x,y,\tau)-\log F\left(x,y\right)\right|\leq c_3(\nu,\delta)\tau,$$

for all $(x,y,\tau)\in\mathbb{D}$ and some constant $c_3=c_3(\nu,\delta)>0$.

iii) It suffices to notice that, in fact, (B.1) holds for arbitrary $k\geq1$, given that $e^{\gamma z}F(e^z,y)\in C_b^\infty\left(\mathbb{R}^2\right)$. Then, Lemma 4.8 and the equalities in (4.11) yield the desired result. ∎

Appendix C. Proof of Proposition 4.12. We will establish the desired properties of $V^{\varepsilon,2}$ by analyzing the auxiliary function $\tilde V^{\varepsilon,2}$, given in Definition 4.11. To this end, for any $\varepsilon'\in(0,1]$, we introduce the approximating function $\tilde V^{\varepsilon,2,\varepsilon'}:\mathbb{R}^2\times[0,T]\to(0,\infty)$ as a solution to the regularized equation

$$\tilde V_\tau^{\varepsilon,2,\varepsilon'}-\frac{1}{2}\left(1-\rho^2\right)a^2(y)\tilde V_{yy}^{\varepsilon,2,\varepsilon'}-\varepsilon'\tilde V_{yy}^{\varepsilon,2,\varepsilon'}-(\varepsilon+\varepsilon')\tilde V_{zz}^{\varepsilon,2,\varepsilon'}-\varepsilon\tilde V_z^{\varepsilon,2,\varepsilon'}-b(y)\tilde V_y^{\varepsilon,2,\varepsilon'}=0,\quad\text{(C.1)}$$

with initial condition $\tilde V^{\varepsilon,2,\varepsilon'}(z,y,0)=F(e^z,y)$. We easily deduce that $\tilde V^{\varepsilon,2,\varepsilon'}\in C^{2,1}(\mathbb{R}^2\times[0,T])$. Moreover, from the Feynman-Kac formula we obtain the stochastic representation

$$\tilde V^{\varepsilon,2,\varepsilon'}(z,y,\tau)=\mathbb{E}\left(F\left(\exp\left(\tilde Z_\tau^{\varepsilon,z,\varepsilon'}\right),\tilde Y_\tau^{y,\varepsilon'}\right)\right),$$

where $\left(\tilde Z^{\varepsilon,z,\varepsilon'},\tilde Y^{y,\varepsilon'}\right)$ is a diffusion given by the generator of (C.1). Following the proof of assertion (A.2), we obtain that

$$\lim_{\varepsilon'\to0}e^{\gamma z}\left|\tilde V^{\varepsilon,2,\varepsilon'}(z,y,\tau)-\tilde V^{\varepsilon,2}(z,y,\tau)\right|=0,\quad\text{(C.2)}$$

uniformly on $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$. In particular, the above implies that the function $\tilde V^{\varepsilon,2}$ is continuous.

(i) Let $F\in\mathcal{D}_0(\nu,\delta,\kappa)$, for some $(\nu,\delta,\kappa)\in(0,1)^3$. In order to verify the first condition of Definition 4.4, we proceed as in the proof of Lemma 4.8 (see (A.3)–(A.4)). To verify the second condition of Definition 4.4, we first notice that $\tilde V^{\varepsilon,2,\varepsilon'}(z,y,\tau)$ is continuously differentiable in

z. Repeating the argument used in the proof of Lemma 4.8 (see (A.5)–(A.7)), we easily deduce that $\tilde{V}^{\varepsilon,2}(z,y,\tau)$ is also continuously differentiable in z, and, moreover, that

$$\lim_{\varepsilon'\to 0} e^{\gamma z}\tilde{V}_z^{\varepsilon,2,\varepsilon'}(z,y,\tau) = e^{\gamma z}\tilde{V}_z^{\varepsilon,2}(z,y,\tau), \tag{C.3}$$

uniformly over $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$. In turn, using that the corresponding linear equations for $\partial_z\left(e^{z/\nu}\tilde{V}^{\varepsilon,2,\varepsilon'}\right)$ and $\partial_z\left(e^{\nu z}\tilde{V}^{\varepsilon,2,\varepsilon'}\right)$ preserve the sign of their respective initial conditions (see, for example, the argument following inequalities (A.8)), we similarly obtain that $\tilde{V}^{\varepsilon,2,\varepsilon'}$ satisfies the second condition of Definition 4.4. Due to (C.2) and (C.3), so does $\tilde{V}^{\varepsilon,2}$.

Next, we establish the last condition in Definition 4.4. To do this, we repeat the steps in the proof of Lemma 4.8, changing the variables by multiplying and dividing by an exponential and taking derivative with respect to y (see (A.10)–(A.14)). As a result, we obtain that the function $\bar{V}^{\varepsilon'}:\mathbb{R}^2\times[0,T]\to(0,\infty)$, defined as

$$\bar{V}^{\varepsilon'}(z,y,\tau) = e^{\gamma z}\left(\tilde{V}_y^{\varepsilon,2,\varepsilon'}(z,y,\tau) + \frac{\lambda(y)-k}{\rho a(y)}\tilde{V}^{\varepsilon,2,\varepsilon'}(z,y,\tau)\right), \tag{C.4}$$

satisfies the equation

$$\begin{aligned}\bar{V}_\tau^{\varepsilon'} - \frac{1}{2}a^2(y)\left(1-\rho^2\right)\bar{V}_{yy}^{\varepsilon'} - \varepsilon'\bar{V}_{yy}^{\varepsilon'} - (\varepsilon+\varepsilon')\bar{V}_{zz}^{\varepsilon'} - \left(\varepsilon(1-2\gamma)-2\varepsilon'\gamma\right)\bar{V}_z^{\varepsilon'}\\ +A(y)\hat{W}_y^{\varepsilon'} + B(y)\bar{V}^{\varepsilon'} = C(y)e^{\gamma z}\tilde{V}_y^{\varepsilon,2,\varepsilon'}(z,y,\tau)\end{aligned} \tag{C.5}$$

with

$$\begin{aligned}A(y) &= A_{0,0}(y) + A_{0,1}(y)k + \varepsilon'(A_{1,0}(y) + A_{1,1}(y)k),\\ B(y) &= B_{0,0}(y) + B_{0,1}(y)k + B_{0,2}(y)k^2 + \varepsilon'(B_{1,0}(y) + B_{1,1}(y)k + B_{1,2}(y)k^2),\\ C(y) &= C_{0,0}(y) + C_{0,1}(y)k + C_{0,2}(y)k^2 + \varepsilon'(C_{1,0}(y) + C_{1,1}(y)k + C_{1,2}(y)k^2),\end{aligned}$$

and initial condition

$$\bar{V}^{\varepsilon'}(z,y,0) = \hat{F}(z,y) = e^{\gamma z}\left(F_y(e^z,y) + \frac{\lambda(y)-k}{\rho a(y)}F(e^z,y)\right), \tag{C.6}$$

where $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are absolutely bounded continuous functions, which depend only upon λ, a, b, ρ and γ. As before, applying Assumption 1 and making use of the absolute boundedness of $e^{\gamma z}\tilde{V}^{\varepsilon,2,\varepsilon'}(z,y,\tau)$, we conclude that the coefficients, the initial condition and the right hand side of equation (C.5) are absolutely bounded. In addition, the limit in (C.2) implies that the right hand side of (C.5) converges, as $\varepsilon'\to 0$, uniformly in $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$. Then, applying the Feynman-Kac formula and the dominated convergence theorem (see the derivation of (A.17)), we conclude that $\bar{V}^{\varepsilon'}(z,y,\tau)$ has a limit, as $\varepsilon'\to 0$, and, therefore, in view of (C.2),

$$\tilde{V}_y^{\varepsilon,2}(z,y,\tau) = \lim_{\varepsilon'\to 0}\tilde{V}_y^{\varepsilon,2,\varepsilon'}(z,y,\tau),$$

for any $(z,y,\tau)\in\mathbb{R}^2\times[0,T]$.

Next, we consider the case $\rho < 0$. Notice that $F \in \mathcal{D}_0(\nu,\delta,\kappa)$ yields that the function $\hat{F}$, given in (C.6), is nonnegative. Using the Feynman-Kac representation of the solution to (C.5), and applying the straightforward estimates on the right hand side of (C.5), we conclude that

$$\bar{V}^{\varepsilon'}(z,y,\tau) \geq -c_1 \frac{\tau}{\delta} e^{c_2\tau(1+k^2)},$$

where, as before, c_i's stand for positive constants, which depend only upon λ, a, b, ρ and γ, and are independent of $\varepsilon' \in (0,1]$ and $(\nu,\delta,\kappa) \in (0,1)^3$. Using the definition of $\bar{V}^{\varepsilon'}$, we deduce from the above inequality that

$$e^{\gamma z}\tilde{V}^{\varepsilon,2,\varepsilon'}_y(z,y,\tau) + \frac{\lambda(y)-k}{\rho a(y)} e^{\gamma z}\tilde{V}^{\varepsilon,2,\varepsilon'}(z,y,\tau) \geq -c_1 \frac{\tau}{\delta} e^{c_2\tau(1+k^2)},$$

uniformly over $\varepsilon' \in (0,1]$. Recalling that $\tilde{V}^{\varepsilon,2,\varepsilon'}$ and $\tilde{V}^{\varepsilon,2,\varepsilon'}_y$ converge to $\tilde{V}^{\varepsilon,2}$ and $\tilde{V}^{\varepsilon,2}_y$, respectively, we conclude that the above inequality holds with $\tilde{V}^{\varepsilon,2}$ in place of $\tilde{V}^{\varepsilon,2,\varepsilon'}$. Therefore,

$$\frac{\tilde{V}^{\varepsilon,2}_y(z,y,\tau)}{\tilde{V}^{\varepsilon,2}(z,y,\tau)} + \frac{\lambda(y)}{\rho a(y)} \geq -\frac{1}{|\rho| a(y)}\left(k + c_3\frac{\tau}{\delta^2}e^{c_2\tau(1+k^2)}\right),$$

where we made use of Assumption 1 once more. In the case $\rho > 0$, we obtain an upper bound. In order to obtain the corresponding upper (lower) bound for negative (positive) correlation coefficient, we simply repeat the above derivations substituting "k" to "$-k$" in the definition of $\bar{V}^{\varepsilon'}$ in (C.4). Thus, we conclude that (4.25) holds with $\xi(\delta) := c_2(T+1) + |\log c_3 - 2\log\delta|$.

(ii) To show (4.26), we follow, again, the proof of Lemma 4.8. First, we notice that, if $((z,y) \mapsto e^{\gamma z}F(e^z,y)) \in C^2_b(\mathbb{R}^2)$, then, the function $(z,y,\tau) \mapsto e^{\gamma z}\left(\tilde{V}^{\varepsilon,2,\varepsilon'}(z,y,\tau) - F(e^z,y)\right)$ satisfies a linear parabolic equation with zero initial condition, and with the coefficients and the right hand side (which is a linear combination of the derivatives of F) being absolutely bounded by a constant, depending only upon the C^2-norm of $(z,y) \mapsto e^{\gamma z}F(e^z,y)$, and upon λ, a, b, ρ and γ. Using the Feynman-Kac formula and the uniform boundedness of $e^{\gamma z}\tilde{V}^{\varepsilon,2,\varepsilon'}(z,y,\tau)$ from below, we obtain (4.26) with $\tilde{V}^{\varepsilon,2,\varepsilon'}$ in place of $\tilde{V}^{\varepsilon,2}$. Passing to the limit, as $\varepsilon' \to 0$, we deduce (4.26).

(iii) If $((z,y) \mapsto e^{\gamma z}F(e^z,y)) \in C^\infty_b(\mathbb{R}^2)$, we repeat the last part of the proof of Lemma 4.8, making use of the Oleinik's result, to conclude that $\tilde{V}^{\varepsilon,2}$ is the unique exponentially bounded classical solution to (4.22) with initial condition $\tilde{V}^{\varepsilon,2}(z,y,0) = F(e^z,y)$. This yields that $V^{\varepsilon,2}$ is a classical solution to (4.21) with initial condition $V^{\varepsilon,2}(x,y,0) = F(x,y)$. ∎

Appendix D. Proof of Proposition 5.5. We start by noticing that

$$\hat{A}^\varepsilon_\tau \hat{F}(z,y) = \log\left(L^\varepsilon_\tau H_\tau F(e^z,y)\right) + \gamma z,$$

with $\hat{F}$ and F as in Definition 5.3. Thus, we need to analyze the properties of operators $A^\varepsilon_\tau = L^\varepsilon_\tau H_\tau$. The first parts of Propositions 4.10 and 4.12 imply that there exist a constant $\alpha \geq 0$ and a continuous function $\beta : (0,1)^2 \to [0,\infty)$ (possibly, different from the ones appearing in Proposition 4.10) such that, if $F \in D_0(\nu,\delta,\kappa)$, then

$$A^\varepsilon_\tau F \in \mathcal{D}_0\left(\nu, \delta e^{-\alpha\tau}, \left(1/\kappa + \tau\exp\left(\beta(\nu,\delta)(1+\tau/\kappa^2)\right)\right)^{-1}\right),$$

for any $\varepsilon \in [0,1]$. Then, since $U'_T \in \mathcal{D}_0(\nu,\delta,\kappa)$, for some $(\nu,\delta,\kappa) \in (0,1)^3$, we easily deduce that there exists a function $n(P,\tau) > 1$, such that, for any $\varepsilon \in [0,1]$ and any partition P, we have

$$\left((x,y) \mapsto x^{-\gamma}\hat{V}^{\varepsilon,P}(\log x, y, \tau)\right) = A^{\varepsilon}_{\tau-\tau_k} \cdots A^{\varepsilon}_{\tau_1-\tau_0} U'_T \in \mathcal{D}_0\left(\nu, \delta e^{-\alpha T}, 1/n(P,\tau)\right), \tag{D.1}$$

for all $k = 1,\ldots,N-1$ and $\tau \in (\tau_k, \tau_{k+1}]$. It only remains to show that we can choose the function $n(P,\tau)$ to be independent of $\tau \in (0,T]$ and P, provided that the mesh(P) is small enough.

To this end, we fix (ν,δ,κ) such that $U'_T \in \mathcal{D}_0(\nu,\delta,\kappa)$ and introduce the constant $\bar{\beta}$ given by

$$\bar{\beta} = \max\left(\beta(\nu,\delta), \beta\left(\nu, \delta e^{-\alpha T}\right), 1\right).$$

In turn, we fix a partition P and denote $\varepsilon' := \bar{\beta}\,\mathrm{mesh}(P)$. We, also, introduce the family of functions

$$I_\tau : (0,\infty) \ni x \mapsto x + \tau \exp\left(\bar{\beta} + \varepsilon' x^2\right) \in (0,\infty).$$

We deduce that (D.1) is satisfied if $n(P,\tau)$ is chosen to be not smaller than

$$I_{\tau-\tau_k} \circ I_{\tau_k-\tau_{k-1}} \circ \ldots \circ I_{\tau_1-\tau_0}(1/\kappa),$$

for $\tau \in (\tau_k, \tau_{k+1}]$, where "$\circ$" denotes the composition of two functions. We, also, notice that the above expression, as a function of $\tau \in [0,T]$, is bounded from above by the solution to the differential equation

$$\frac{d}{d\tau} g = \exp\left(\bar{\beta} + \varepsilon' g^2(\tau)\right),$$

with $g(0) = 1/\kappa$. It is easy to see that, for all small enough $\varepsilon' > 0$, the above equation has a non-exploding solution on $[0,T]$, and, therefore, the function g is well defined. This implies that the quantity $n(P,\tau)$ can be chosen as $g(T)$, independent of (P,τ). ∎

Appendix E. Proof of Proposition 5.6. To establish property (5.6), we first observe that the monotonicity of the operator $\hat{A}^{\varepsilon}_\tau$ follows from the monotonicity of L^{ε}_τ and H_τ. The operator L^{ε}_τ is, clearly, monotone, as its value is given by the Feynman-Kac formula corresponding to a linear partial differential equation (recall Definitions 4.11 and 5.2). On the other hand, the operator H_τ corresponds to a nonlinear equation. However, its value is defined as the x-inverse of a function given by the Feynman-Kac representation associated with a linear equation (recall Definitions 4.9 and 5.2), which is monotone with respect to the initial condition. It only remains to notice that such monotonicity is preserved under the "x-inversion". The translation invariance of the operator $\hat{A}^{\varepsilon}_\tau$ follows from the "scale invariance" of the operators L^{ε}_τ and H_τ, in the sense that $L^{\varepsilon}_\tau(cu) = cL^{\varepsilon}_\tau u$ and $H_\tau(cu) = cH_\tau u$. This, in turn, follows directly from the Definitions 5.2, 4.9 and 4.11.

To establish the consistency property (5.8), we work as follows. We first choose an arbitrary $\hat{\phi} \in \hat{\mathcal{D}}_0 \cap C^\infty_b\left(\mathbb{R}^2\right)$ and notice that

$$\hat{\phi}(z,y) = \log\phi(e^z, y) + \gamma z, \tag{E.1}$$

for some $\phi \in \mathcal{D}_0 \cap C^\infty((0,\infty)\times\mathbb{R})$ such that the function $(z,y) \mapsto \log\phi(\exp(z),y)$ has bounded derivatives of order one and higher. Therefore,

$$\frac{1}{\tau}\left(\hat{A}^{\varepsilon}_{\tau} - I\right)\hat{\phi}(z,y) = \frac{1}{\tau}\left(\log\left(H_\tau\phi\right) - \log\phi\right)(e^z,y) \tag{E.2}$$

$$+\frac{1}{\tau}\left(\log\left(L^{\varepsilon}_{\tau}\phi\right) - \log\phi\right)(e^z,y) + \frac{1}{\tau}\left(\log\left(L^{\varepsilon}_{\tau}H_\tau\phi\right) - \log\left(L^{\varepsilon}_{\tau}\phi\right) - \log\left(H_\tau\phi\right) + \log\phi\right)(e^z,y).$$

We study the above terms separately. We have

$$\frac{1}{\tau}\left(\log\left(H_\tau\phi\right) - \log\phi\right)(e^z,y) = \frac{1}{\tau}\log\frac{V^{\varepsilon,1}(e^z,y,\tau)}{\phi(e^z,y)} = \frac{1}{\tau}\log\left(1 + \frac{V^{\varepsilon,1}(e^z,y,\tau) - \phi(e^z,y)}{\phi(e^z,y)}\right),$$

where $V^{\varepsilon,1}$ is given by Definition 4.9, with ϕ in place of the initial condition F. We recall that $((z,y)\mapsto e^{\gamma z}\phi(e^z,y)) \in C^\infty_b\left(\mathbb{R}^2\right)$, and, therefore, all conditions of the last part of Proposition 4.10 are satisfied. Thus, $V^{\varepsilon,1} \in C^{2,1}(\mathbb{D})$ and, moreover, $V^{\varepsilon,1}$ is a classical solution to (4.9) with initial condition $V^{\varepsilon,1}(x,y,0) = \phi(x,y)$. Next, we introduce the auxiliary function $u : (z,y,\tau) \mapsto v(e^z,y,\tau) \in (0,\infty)$, with $(z,y,\tau) \in \mathbb{R}^2 \times [0,T]$ and v being the x-inverse of $V^{\varepsilon,1}(x,y,\tau)$. We also introduce the function φ as the x-inverse of ϕ and the function $\psi : (z,y) \mapsto \varphi(e^z,y) \in (0,\infty)$, with $(z,y)\in\mathbb{R}^2$. It is then easy to show, using Lemma 4.6 and the relations in (4.11), that $((x,y)\mapsto v(x,y,\tau)) \in \mathcal{D}'_0 \cap C^2((0,\infty)\times\mathbb{R})$, for all $\tau\in[0,T]$, and, in addition, that the function u is a classical solution to the linear equation (4.13), with initial condition $u(z,y,0) = \psi(z,y)$. Thus, we apply the mean value theorem to obtain

$$\frac{1}{\tau}\left(\left(\log H_\tau - \log\right)\phi\right)(e^z,y) = \frac{1}{\tau}\log\left(1 - \frac{V^{\varepsilon,1}(e^z,y,\tau) - V^{\varepsilon,1}\left(v(\phi(e^z,y),y,\tau),y,\tau\right)}{\phi(e^z,y)}\right) \tag{E.3}$$

$$= \frac{1}{\tau}\log\left(1 - \frac{e^{\eta(z,y,\tau)}}{\phi(e^z,y)}\frac{u\left(\log\phi(e^z,y),y,\tau\right) - \psi(\log\phi(e^z,y),y)}{u_z\left(\eta(\tau,z,y),y,\tau\right)}\right),$$

for some $\eta(z,y,\tau)$ between $\psi\left(\log\phi(e^z,y),y\right)$ and $u\left(\log\phi(e^z,y),y,\tau\right)$.

Next, using that $\phi\in\mathcal{D}_0$, we conclude that $\partial_z\log\phi(e^z,y)$ is bounded away from zero by a constant. In addition, we observe that all derivatives of the function $\log\phi(e^z,y)$, of order one and higher, are absolutely bounded. We also note that the function $\log\psi$ is the z-inverse of $(z,y)\mapsto\log\phi(e^z,y)$. Then, it can be easily verified (cf. (4.11)) that all derivatives of $\log\psi$, of order one and higher, are absolutely bounded as well. In addition, since $\hat{\phi}\in\hat{\mathcal{D}}_0(\nu,\delta,\kappa)$, for some $(\nu,\delta,\kappa)\in(0,1)^3$, we have $e^{z/\gamma}\psi(z,y)\le 1/\delta$, for all $(z,y)\in\mathbb{R}^2$. Therefore, for each pair of integers $i,j\ge1$, there exists a continuous function, say $g^{i+j}:[0,\infty)\to[0,\infty)$, independent of ψ and (ν,δ,κ), such that

$$\delta\sup_{(z,y)\in\mathbb{R}^2}\left|e^{z/\gamma}\partial^i_z\partial^j_y\psi(z,y)\right| \le g^{i+j}\left(\sum_{k+l=1}^{i+j}\sup_{(z,y)\in\mathbb{R}^2}\left|\partial^k_z\partial^l_y\log\psi(z,y)\right|\right). \tag{E.4}$$

For any given continuous function $C:[0,\infty)\to[0,\infty)$, any integer $n\ge1$ and any function ψ constructed as above, we denote by C^n_ψ the value of the right hand side of (E.4), with $i+j=n$

and C in place of g^{i+j}. Using the last statement in Lemma 4.8, we recall that u solves (4.13), and, using its Feynman-Kac representation and the Itô's formula, we obtain

$$u(z,y,\tau)=\psi(z,y)+\mathbb{E}\int_0^\tau\left(\frac{1}{2}\lambda^2\psi_{zz}-\rho a\lambda\psi_{zy}+\frac{1}{2}\rho^2a^2\psi_{yy}+\frac{1}{2}\lambda^2\psi_z-\rho a\lambda\psi_y\right)(\hat{Z}^{z,y}_s,\hat{Y}^y_s)ds,$$

where the stochastic processes $\hat{Z}^{z,y}$ and $\hat{Y}^y$ are defined in (4.15). Using Itô's formula once more, we expand further the partial derivatives of ψ in the above integral. Then, applying the estimate (E.4), we deduce that, for any compact set $\mathcal{K}\subset\mathbb{R}^2$, there exist continuous functions $C,R:[0,\infty)\to[0,\infty)$, with $R(0)=0$, such that the inequality

$$\begin{aligned}\left|u(z,y,\tau)-\psi(z,y)-\tau\left(\frac{\lambda^2}{2}\psi_{zz}+\frac{\rho^2}{2}a^2\psi_{yy}-\rho a\lambda\psi_{zy}+\frac{\lambda^2}{2}\psi_z-\rho a\lambda\psi_y\right)(z,y)\right|\\ \le\frac{\tau R(\tau)C^3_\psi}{\delta}\end{aligned}\tag{E.5}$$

holds for all $(z,y,\tau)\in\mathcal{K}\times[0,T]$ and all δ and ψ associated with some $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C^\infty_b\left(\mathbb{R}^2\right)$. Next, we notice that the function u_z satisfies the same equation as u and with initial condition $u_z(z,y,0)=\psi_z(z,y)$. We, then, similarly deduce that, for any compact set $\mathcal{K}\subset\mathbb{R}^2$, there exist continuous functions C and R as above, such that the inequality

$$\left|u_z(z,y,\tau)-\bar{\phi}_z(z,y)\right|\le R(\tau)C^3_\psi/(\nu\delta)\tag{E.6}$$

holds for all $(z,y,\tau)\in\mathcal{K}\times[0,T]$ and all (ν,δ) and ψ associated with some $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C^\infty_b\left(\mathbb{R}^2\right)$. We also note that the partial derivative u_z is bounded away from zero whenever $|z|$ is bounded. Thus, plugging (E.5) and (E.6) into (E.3), we conclude that, for any compact set $\mathcal{K}\subset\mathbb{R}^2$, there exists a continuous function C, as above, and continuous functions $\alpha:(0,1)^2\to\mathbb{R}$ and $\beta:(0,1)^2\times[0,\infty)\to\mathbb{R}$, such that

$$\begin{aligned}&\frac{1}{\tau}\left(\left(\log H_\tau-\log\right)\phi\right)(e^z,y)\\ &=-\frac{\left(\frac{1}{2}\lambda^2\psi_{zz}+\frac{1}{2}\rho^2a^2\psi_{yy}-\rho a\lambda\psi_{zy}+\frac{1}{2}\lambda^2\psi_z-\rho a\lambda\psi_y\right)\left(\log\phi(e^z,y),y\right)}{\psi_z\left(\log\phi(e^z,y),y\right)}+\alpha(\nu,\delta)C^3_\psi\overline{\overline{o}}(1)\\ &=\left(\left(\frac{\lambda\phi+\rho a\phi_y}{\phi_x}\right)^2\frac{\phi_{xx}}{2}-\left(\lambda\phi_x+\rho a\phi_{xy}\right)\frac{\lambda\phi+\rho a\phi_y}{\phi_x}+\frac{\rho^2}{2}a^2\phi_{yy}\right)(e^z,y)\\ &\quad+\beta\left(\nu,\delta,\left\|\hat{\phi}\right\|_{C^3(\mathbb{R}^2)}\right)\overline{\overline{o}}(1),\end{aligned}\tag{E.7}$$

where $\overline{\overline{o}}(1)$ vanishes, as $\tau\to0$, uniformly over all $(z,y)\in\mathcal{K}$, all (ν,δ) and all ψ associated with some $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C^\infty_b\left(\mathbb{R}^2\right)$. Above, we also made use of the relations $\psi(z,y)=\varphi(e^z,y)$ and $\varphi(\phi(x,y),y)=x$, and the corresponding relations between their derivatives.

The analysis of the terms involving the operator L^ε_τ is considerably simpler since L^ε_τ is linear. As $\tau\to0$,

$$\begin{aligned}\frac{1}{\tau}\left[\log\left(L^\varepsilon_\tau\phi\right)-\log\phi\right](e^z,y)&=\left(\frac{1}{2}a^2\left(1-\rho^2\right)\phi_{yy}-\varepsilon x^2\phi_{xx}-2\varepsilon x\phi_x+b\phi_y\right)(e^z,y)\\ &\quad+\gamma\left(\delta,\nu,\|\hat{\phi}\|_{C^3(\mathbb{R}^2)}\right)\overline{\overline{o}}\left(1\right),\end{aligned}\tag{E.8}$$

holds uniformly over all (z,y) changing on a compact and over all $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C_b^\infty\left(\mathbb{R}^2\right)$, with continuous $\gamma:(0,1)^2\times[0,\infty)\to\mathbb{R}$. For the last term in the right hand side of (E.2), we observe that

$$\frac{1}{\tau}\left[\log\left(L^\varepsilon_\tau H_\tau\phi\right)-\log\left(L^\varepsilon_\tau\phi\right)-\log\left(H_\tau\phi\right)+\log\phi\right](e^z,y)$$
$$=\frac{1}{\tau}\log\left(1+\frac{\phi(L^\varepsilon_\tau-I)(H_\tau-I)\phi-\left(L^\varepsilon_\tau-I\right)\left(\phi(H_\tau-I)\phi\right)}{L^\varepsilon_\tau\left(\phi H_\tau\phi\right)}\right)(e^z,y).$$

Next, we use the above results and the exponential boundedness of $(H_\tau-I)\,\phi(e^z,y)/\tau$ and $\phi\,(H_\tau-I)\,\phi(e^z,y)/\tau$, to apply the Feynman-Kac formula together with the dominated convergence theorem, and deduce that, for any compact set $\mathcal{K}\subset\mathbb{R}^2$, there exists a continuous function $\gamma':(0,1)^2\times[0,\infty)\times[0,T]\to[0,\infty)$, with $\gamma'(\nu,\delta,x,0)=0$, such that the inequality

$$|\phi(e^z,y)(L^\varepsilon_\tau-I)(H_\tau-I)\phi(e^z,y)|+|(L^\varepsilon_\tau-I)\left(\phi(H_\tau-I)\phi\right)(e^z,y)|\leq\tau\,\gamma'\left(\nu,\delta,\|\hat{\phi}\|_{C^3(\mathbb{R}^2)},\tau\right)$$

holds for all $(z,y,\tau)\in\mathcal{K}\times[0,T]$ and all $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C_b^\infty(\mathbb{R}^2)$. Finally, since $L^\varepsilon_\tau\left(\phi H_\tau\phi\right)(e^z,y)$ is bounded away from zero on any compact in (z,y), uniformly over $\tau\in[0,T]$, we deduce that, as $\tau\to0$,

$$\frac{1}{\tau}\left[\log\left(L^\varepsilon_\tau H_\tau\phi\right)-\log\left(L^\varepsilon_\tau\phi\right)-\log\left(H_\tau\phi\right)+\log\phi\right](e^z,y)=\gamma''\left(\nu,\delta,\|\hat{\phi}\|_{C^3(\mathbb{R}^2)}\right)\overline{\overline{o}}\,(1)$$

holds uniformly over all (z,y) changing on a compact and over all $\hat{\phi}\in\hat{\mathcal{D}}_0\cap C_b^\infty(\mathbb{R}^2)$, with a continuous function $\gamma'':(0,1)^2\times[0,\infty)\to\mathbb{R}$. We conclude by plugging (E.7) and (E.8) into (E.2) and using (E.1) to write the resulting expression in terms of $\hat{\phi}$. ∎

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