A Class of Homothetic Forward Investment Performance Processes with Non-zero Volatility

Sergey Nadtochiy and Thaleia Zariphopoulou

Abstract We study forward investment performance processes with non-zero forward volatility. We focus on the class of homothetic preferences in a single stochastic factor model. The forward performance process is represented in a closed-form via a deterministic function of the wealth and the stochastic factor. This function is, in turn, given as a distortion transformation of the solution to a linear ill-posed problem. We analyze the solutions of this problem in detail. We, also, provide two examples for specific dynamics of the stochastic factor, specifically, log-mean reverting and Heston-type dynamics.

Keywords Forward investment performance · Hamilton–Jacobi–Bellman equation · Distortion transformation · Widder theorem · Heston model

Mathematics Subject Classification (2010) 91G20

1 Introduction

This paper is a contribution to the recently developed approach of forward investment performance measurement (see [8] and [9]). This approach allows for dynamic update of the investor's performance criterion and offers an alternative to the classical maximal expected utility objective which is defined only at a single instant. The underlying object is a stochastic process, the so called forward investment performance process, which is defined for all times. Its key properties are the supermartingality at admissible self-financing policies and martingality at an optimum.

S. Nadtochiy (✉)
Department of Mathematics, University of Michigan, Ann Arbor MI 48109, USA
e-mail: sergeyn@umich.edu

T. Zariphopoulou
Oxford-Man Institute, University of Oxford, Oxford, UK
e-mail: thaleia.zariphopoulou@oxford-man.ox.ac.uk

T. Zariphopoulou
Departments of Mathematics and IROM, McCombs School of Business, The University of Texas at Austin, Austin, TX, USA

Y. Kabanov et al. (eds.), *Inspired by Finance*, DOI 10.1007/978-3-319-02069-3_22,

Constructing such a process is a formidable task, for the underlying stochastic optimization problem is formulated "forward in time" and might be ill-posed.

In [10], a stochastic partial differential equation was introduced which the forward performance process is expected to satisfy. In many aspects, this SPDE is the stochastic analogue of the deterministic Hamilton–Jacobi–Bellman equation for the classical (backward) case. There are several elements which make the study of the SPDE and the derivation of analogous verification results hard. Indeed, one has to specify the appropriate class of initial conditions and, also, address the ill-posedness and the possible degeneracy of the equation.

Besides these issues, one also has to specify the correct family of forward performance volatility processes. These processes are chosen by the investor and constitute one of the novel elements of the forward investment theory. They are exogenous inputs for the volatility term of the SPDE. Note that their classical analogue is uniquely determined due to the static nature of the utility criterion (see Remark 3 herein).

To date, existence and uniqueness of solutions to the forward SPDE have not been established and the related verification results are still lacking. General results have been produced only for the case of zero volatility (see [9]). Under this rather strong assumption, the performance process is monotone in time (decreasing) and can be represented as a compilation of a deterministic function and the market input (see (16)). This form, however, is not any more valid when the investor allows for volatility in his criterion.

Herein, we do not study general questions but only analyze a family of forward processes and construct specific examples. Moreover, we concentrate on the class of homothetic criteria. We are motivated to look at this family because it offers the closest analogue of the classical value function under power utilities.

The market consists of one riskless asset and a stock whose dynamics are affected by a stochastic factor, denoted by Y_t. The latter is imperfectly correlated with the stock which makes the market incomplete. Such a model arises frequently when one assumes predictability of returns and/or stochastic volatility.

The homotheticity assumption suggests a separable form for the candidate processes with one of the components depending exclusively on the stochastic factor. In turn, the assumptions on the model dynamics suggest that the latter component is a process, denoted by $V(Y_t, t)$, that can be represented as a function of the stochastic factor and time. Constructing the function $V(y, t)$ is the main goal of this paper together with, as mentioned earlier, the specification of the correct initial condition and the appropriate class of volatility processes.

A distortion transformation on $V(y, t)$ yields a linear equation with a potential term. The forward in time nature of the underlying stochastic optimization problem makes this linear equation ill-posed. Specifying its nonnegative solutions is, to our knowledge, an open problem. Indeed, the only known case for which necessary and sufficient conditions for nonnegative solutions of such problems have been established is when the potential term is absent. This is the celebrated Widder's theorem. Herein, we study the more general case and provide results in this direction. A special case of these results yields one part (sufficiency) of Widder's theorem.

Once the form of the function $V(y,t)$ is specified, we are able to construct an admissible volatility process. This process is, also, taken to be homothetic in the wealth argument. A solution to the forward performance SPDE is then readily obtained.

Finally, we provide two concrete examples. In the first example, the stochastic volatility is taken to be a mean reverting process satisfying linear SDE, while in the second it satisfies Heston-type dynamics. In both cases, we calculate explicitly the appropriate initial condition and the volatility process as well as the associated forward performance process. We, also, study the robustness of the latter when its volatility vanishes and we compare it with its zero-volatility counterpart.

The paper is organized as follows. In Sect. 2, we describe the model and recall the investment performance criterion. In Sect. 3, we focus on homothetic performance processes and provide some preliminary informal results for the form of candidate processes. In Sect. 4, we study the underlying linear equation. We conclude in Sect. 5 where we present the two examples.

2 The Stochastic Factor Model and Investment Performance Measurement

The market consists of a risky and a riskless asset. The risky asset is a stock whose price S_t, $t \ge 0$, is modeled as a diffusion process solving

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t^1, \tag{1}$$

with $S_0 > 0$. The stochastic factor Y_t, $t \ge 0$, satisfies

$$dY_t = b(Y_t)dt + d(Y_t)\big(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\big), \tag{2}$$

with $Y_0 = y$, $y \in \mathbb{R}$. The process $W_t = (W_t^1, W_t^2)$, $t \ge 0$, is a standard 2-dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The underlying filtration is $\mathscr{F}_t = \sigma(W_s : 0 \le s \le t)$. It is assumed that the correlation coefficient $\rho \in (-1, 1)$.

The coefficients μ, σ, b and d satisfy the appropriate continuity and Lipschitz conditions such that the above system of equations has a unique strong solution. It is, also, assumed that $\sigma(y) > 0$, $y \in \mathbb{R}$.

The riskless asset, the savings account, offers constant interest rate $r > 0$.

We introduce the process, frequently called the market price of risk,

$$\lambda(Y_t) = \frac{\mu(Y_t) - r}{\sigma(Y_t)}. \tag{3}$$

Starting with an initial endowment x, the investor invests at future times in the riskless and risky assets. The present value of the amounts allocated in the two accounts are denoted, respectively, by π_t^0 and π_t. The present value of her investment is, then, given by $X_t^\pi = \pi_t^0 + \pi_t$, $t > 0$. We will refer to X_t^π as the discounted wealth. Using (1) we easily deduce that it satisfies

$$dX_t^\pi = \sigma(Y_t)\pi_t\big(\lambda(Y_t)dt + dW_t^1\big). \tag{4}$$

The investment strategies will play the role of control processes and are taken to satisfy the standard assumption of being self-financing. Such a portfolio, π_t, is deemed admissible if, for $t > 0$, $\pi_t \in \mathscr{F}_t$, $E_{\mathbb{P}}(\int_0^t \sigma^2(Y_s)\pi_s^2 ds) < +\infty$ and the associated discounted wealth satisfies the state constraint $X_t^\pi \geq 0$, $t \geq 0$. We will denote the set of admissible strategies by $\mathscr{A}$.

Stochastic factors have been used in portfolio choice to model asset predictability and stochastic volatility. A detailed survey of asset allocation models with a single stochastic factor can be found in [16] and we refer the reader therein for a complete bibliography.

2.1 Forward Investment Performance Process

The performance of implemented investment strategies is typically measured in terms of optimizing an expected criterion of the generated wealth. In the academic literature, this criterion is predominantly given by the investor's utility function (see, for example, the seminal papers [6] and [7]). One, then, chooses an investment horizon, say T, and a utility function at this time, $U_T(x)$, and maximizes, over all admissible self-financing strategies, the expected utility of terminal wealth. Such problems have been widely studied under rather general assumptions on market coefficients and constitute one of the cornerstones in modern mathematical portfolio management theory.

There is, however, a limitation in this setting. Indeed, the performance criterion is not dynamic in the sense that, from one hand, it cannot be revised at any previous investment time, $t < T$, and, from the other, it cannot be extended at any time $t > T$. One could say that the terminal utility criterion corresponds to a static objective. This does not mean that the associated value function is time independent, an obviously wrong conclusion. Rather, we state that it is the criterion per se that is static, for it is (pre)specified for only one time instant, T.

Recently, one of the authors and M. Musiela introduced an alternative approach which bypasses these shortcomings. The associated criterion is developed in terms of a family of stochastic processes defined on $[0, +\infty)$ and indexed by the wealth argument. It is called the forward investment performance process. Its key properties are its martingality at an optimum and its supermartingality away from it. These are in accordance with the analogous properties of the value function process which stem out from the Dynamic Programming Principle. However, in contrast to the existing framework, the risk preferences are specified for today and not at a (possibly remote) future time. As we will see in the upcoming analysis, one of the fundamental questions in this approach is the correct specification of the initial conditions in order for the relevant stochastic optimization problem to be well posed (see, for example, Propositions 6 and 8 herein and [9]).

For completeness, we provide the definition of the forward investment process below but we refer the reader to [8] and [9] for details. We recall that $\mathscr{F}_t$, $t \geq 0$, is the filtration generated by $W_t = (W_t^1, W_t^2)$, $t \geq 0$, and $\mathscr{A}$ the set of admissible policies.

Definition 1 An $\mathscr{F}_t$-progressively measurable process $U(x,t)$ is a forward investment performance if for $t \geq 0$ and $x \geq 0$:

(i) the mapping $x \to U(x,t)$ is concave and increasing,
(ii) for each portfolio process $\pi \in \mathscr{A}$, $E_{\mathbb{P}}(U(X_t^{\pi}, t))^+ < \infty$, and

$$E_{\mathbb{P}}\big(U\big(X_s^{\pi}, s\big)|\mathscr{F}_t\big) \leq U\big(X_t^{\pi}, t\big), \quad s \geq t, \tag{5}$$

(iii) there exists a portfolio process $\pi^* \in \mathscr{A}$, for which

$$E_{\mathbb{P}}\big(U\big(X_s^{\pi^*}, s\big)|\mathscr{F}_t\big) = U\big(X_t^{\pi^*}, t\big), \quad s \geq t. \tag{6}$$

While the above definition might appear like a pedantic rephrase of the Dynamic Programming Principle it is actually not. Indeed, it gives rise to a forward in time stochastic optimization problem which belongs to the family of the so called "ill-posed" problems. Such problems are notoriously difficult with regards to their well-posedeness, stability and finiteness of solutions. Herein, we do not address this question but, rather, construct specific examples. Specifying forward processes that satisfy the above definition is an open problem and is currently under investigation by the authors and others (see, for example, [1, 3, 10], and [17]).

2.2 *The Forward Performance SPDE*

Recently, it was shown in [10] and [16] that a sufficient condition for a (sufficiently smooth) process $U(x,t)$ to be a forward performance is that it satisfies a stochastic partial differential equation (see (7) below). For the single stochastic factor model we examine herein, Proposition 2 in [16] takes the following form.

Proposition 1 (i) *Let $U(x,t)$ be an $\mathscr{F}_t$-progressively measurable process such that the mapping $x \to U(x,t)$ is strictly increasing and concave. Let, also, $U(x,t)$ be a solution to the stochastic partial differential equation*

$$dU(x,t) = \frac{1}{2}\frac{(\lambda(Y_t)U_x(x,t) + a_x^1(x,t))^2}{U_{xx}(x,t)}dt + \big(a(x,t)\cdot dW_t\big), \tag{7}$$

where $a(x,t) = (a^1(x,t), a^2(x,t))$ is an $\mathscr{F}_t$-progressively measurable process. Then $U(x,t)$ is a forward investment performance process.

(ii) *Consider the process* π_t^*, $t \geq 0$, *given by*

$$\pi_t^* = -\frac{\lambda(Y_t)U_x(X_t^*, t) + a_x^1(X_t^*, t)}{\sigma(Y_t)U_{xx}(X_t^*, t)}, \tag{8}$$

where X_t^*, $t \geq 0$, *solves*

$$dX_t^* = \sigma(Y_t)\pi_t^*\big(\lambda(Y_t)dt + dW_t^1\big), \tag{9}$$

with $X_0^* = x$, $x \geq 0$. *If* $\pi_t^* \in \mathscr{A}$ *and* (9) *has a strong solution, then* π_t^* *and* X_t^* *are optimal.*

As it is shown in [10], the same stochastic partial differential equation emerges in the classical formulation of the optimal portfolio choice problem. Indeed, fix an investment horizon, say T, and recall the traditional value function process, denoted by $V(x, t; T)$ and defined as

$$V(x, t; T) = \sup_{\mathscr{A}_T} E_{\mathbb{P}}\big(U(X_T)|\mathscr{F}_t, X_t = x\big),$$

with $\mathscr{A}_T$ being the direct analogue of $\mathscr{A}$ in $[0, T]$. Let us now assume that there exists a smooth enough function, say $v(x, y, t)$ such that the representation

$$V(x, t; T) = v(x, Y_t, t) \tag{10}$$

holds. We note that the existence and regularity of such a function has not been established to date, expect for special utilities.

The associated Hamilton-Jacobi-Bellman (HJB) equation is then given (informally) by

$$\begin{aligned} v_t + \max_\pi\bigg(\frac{1}{2}\sigma^2(x)\pi^2 v_{xx} + \pi\big(\mu(y)v_x + \rho\alpha(y)\sigma(y)v_{xy}\big)\bigg) \\ + \frac{1}{2}d^2(y)v_{yy} + b(y)v_y, \end{aligned} \tag{11}$$

with $v(x, y, T) = U(x)$.

Using the representation (10) and expanding the process $v(x, Y_t, t)$ yield,

$$\begin{aligned} dv(x, Y_t, t) = \bigg(v_t(x, Y_t, t) + \frac{1}{2}d^2(Y_t)v_{yy}(x, Y_t, t) + b(Y_t)v_y(x, Y_t, t)\bigg)dt \\ + \rho d(Y_t)v_y(x, Y_t, t)dW_t^1 + \sqrt{1-\rho^2}d(Y_t)v_y(x, Y_t, t)dW_t^2. \end{aligned}$$

Using that $v(x, y, t)$ satisfies (11) and rearranging terms, we deduce that

$$dv(x, Y_t, t) = \frac{1}{2}\frac{(\lambda(Y_t)v_x(x, Y_t, t) + \rho d(Y_t)v_{xy}(x, Y_t, t))^2}{v_{xx}(x, Y_t, t)}dt$$
$$+ \rho d(Y_t)v_y(x, Y_t, t)dW_t^1 + \sqrt{1-\rho^2}d(Y_t)v_y(x, Y_t, t)dW_t^2.$$

From (10) we, then, deduce that the value function process, which now plays the role of the (backward) investment performance, satisfies the same SPDE as in (7). Specifically, for $0 \le t < T$, the process $V(x, t; T)$ satisfies the equation

$$dV(x, t; T) = \frac{1}{2}\frac{(\lambda(Y_t)V_x(x, t; T) + a_x^1(x, t; T))^2}{V_{xx}(x, t; T)}dt + \left(a(x, t; T) \cdot dW_t\right)$$

with terminal condition $V(x, T; T) = U(x)$ and the components of volatility process given by

$$a^1(x, t; T) = \rho d(Y_t)v_y(x, Y_t, t) \quad \text{and} \quad a^2(x, t; T) = \sqrt{1-\rho^2}d(Y_t)v_y(x, Y_t, t). \tag{12}$$

Its is worth noticing that the terminal data suggest that $\lim_{t\to T} a^i(x, t; T) = 0$.

Remark 1 It is important to notice three fundamental differences between the classical (backward) and the forward cases. Firstly, in the backward optimal investment model, we are given a terminal condition while in the forward an initial one. Secondly, in the former case, the performance process satisfies $V(x, T) \in \mathscr{F}_0$ while in the latter, $U(x, t) \in \mathscr{F}_t$. Finally, in the backward case, there is *no* flexibility in choosing the volatility coefficients, for they are *uniquely* obtained from the Itô decomposition of the value function process while in the forward case, the volatility process is up to the investor to choose. How the investor should make this choice is one of the main challenges in the new approach.

2.3 The Zero Volatility Case

An important class of forward performance processes are the ones that correspond to the choice of zero volatility, $a(x, t) \equiv 0$, $t \ge 0$. We easily see, using the concavity of the forward process and (7), that these processes are decreasing in time. Despite the strong assumption on the volatility, these processes yield a rich family of performance criteria which compile in an intuitively pleasing way the dynamic risk profile of the investor and the information coming from the evolution of the investment opportunity set, as (16) below indicates. They also provide an important benchmark when volatility is not zero, as it is discussed in Propositions 7 and 9 herein. They are extensively studied in [8] and [9], and we refer the reader therein for the proofs of the results that follow. Herein, we only state the main result and discuss some insights about the admissibility of the candidate initial conditions. Because all involved functions are smooth, we will not refer to their specific regularity (see [9]).

Theorem 1 *Let $u_0 : \mathbb{R}^+ \to \mathbb{R}$ be strictly increasing and concave and such that the function $h_0 : \mathbb{R} \to \mathbb{R}^+$ defined by*

$$u_0'\big(h_0(x)\big) = e^{-x} \tag{13}$$

can be represented as the Laplace transform of a finite positive Borel measure, denoted by ν, namely,

$$h_0(x) = \int_0^\infty e^{xy}\,\nu(dy), \tag{14}$$

such that $h_0(x) < \infty$, for all $x \in \mathbb{R}$. Let, also, $u : \mathbb{R}^+ \times (0,\infty) \to \mathbb{R}$ be a strictly concave and increasing in the spatial argument function satisfying

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}, \tag{15}$$

and $u(x,0) = u_0(x)$. Then, with $\lambda(Y_t)$, $t \geq 0$, as in (3), the process

$$U(x,t) = u\left(x, \int_0^t \lambda^2(Y_s)ds\right) \tag{16}$$

is a forward investment performance.

Relations (13) and (14) demonstrate the admissibility condition for a candidate initial condition, $u_0(x)$. Specifically, the inverse of its first derivative must be represented via a Laplace transform as in (14).

In [9] (see, also [1]) the following is shown. Let $h : \mathbb{R}\times[0,\infty) \to \mathbb{R}^+$ be given by the "dynamic" analogue of (17), namely,

$$h(x,t) = \int_0^\infty e^{xy - \frac{1}{2}y^2 t}\,\nu(dy). \tag{17}$$

Then, the solution $u(x,t)$ of (15) satisfies

$$u_x\big(h(x,t),t\big) = e^{-x+\frac{t}{2}}, \tag{18}$$

while $h(x,t)$ solves the backward heat equation

$$h_t + \frac{1}{2}h_{xx} = 0.$$

The reader is invited to compare (13) and its "dynamic" analogue (18) as well as the role of the measure ν as the essential defining element in generating solutions for positive times. Generalizations of some of these results is one of the main contributions herein (see Sect. 4).

3 Homothetic Forward Investment Performance Processes

We concentrate on forward investment performance processes which are homothetic in the spatial variable. We are motivated to do so for two reasons. Firstly, these processes are the natural analogues of the popular power utilities. Secondly, as the analysis will indicate, the homogeneity assumption allows for significant tractability and closed form solutions.

To this end, we are looking for initial conditions and volatility processes which produce well defined solutions, $U(x,t)$, to (7) that have the property

$$U(kx,t) = k^{\gamma} U(x,t), \tag{19}$$

for all $t \geq 0$ and $k \in \mathbb{R}^+$, with $0 < \gamma < 1$. We easily deduce that the forward processes must be of the multiplicative form

$$U(x,t) = \frac{x^{\gamma}}{\gamma} K_t, \tag{20}$$

where the multiplicative process K_t, $t \geq 0$, is to be determined[1] but does not depend on the spatial variable x. In the sequel, we will further restrict the class of solutions by looking at factors that depend functionally on time and the current state of the stochastic factor (see (24)).

Note that (20) tells us that the only admissible initial conditions are of the form

$$u_0(x) = \frac{x^{\gamma}}{\gamma} K_0. \tag{21}$$

3.1 *The Zero-Volatility Homothetic Case*

We recall the homothetic time-monotone performance process. We will revert to this case later in the analysis when we investigate their robustness of the forward process for vanishing volatilities (see Propositions 7 and 9).

Proposition 2 *Assume that* $a(x,t) \equiv 0$, $t \geq 0$, *in* (7) *and let the initial condition be as in* (21). *Then, the forward performance process is given by*

$$U(x,t) = \frac{x^{\gamma}}{\gamma} K_0 \exp\left(\int_0^t \frac{1}{2} \frac{\gamma}{\gamma - 1} \lambda^2(Y_s) ds \right), \tag{22}$$

for $x \geq 0$ *and* Y_t, $t \geq 0$, *solving* (2).

[1]For convenience, we introduce the factor $1/\gamma$. Moreover, we do not consider the case $\gamma < 0$, which can be analyzed with similar, albeit more tedious computationally arguments.

Proof The claim follows from (16) and the fact that the function

$$u(x,t) = \frac{x^\gamma}{\gamma} e^{\frac{1}{2}\frac{\gamma}{\gamma-1}t} K_0, \quad x \ge 0,$$

solves the nonlinear equation (15) with initial condition $u(x,0) = \frac{x^\gamma}{\gamma} K_0$. □

3.2 Non-zero Volatility Homothetic Case

We now focus our attention to the case of non-zero volatility coefficients, which is the main topic herein. As mentioned earlier, the underlying problem is to specify an initial condition, $u_0(x)$, a (non-zero) volatility process, $a(x,t)$, and a process $U(x,t)$, such that the latter solves (7) with $U(x,0) = u_0(x)$. Moreover, we will be looking at processes with the Markovian structure

$$U(x,t) = \frac{x^\gamma}{\gamma} K(Y_t, t), \tag{23}$$

which corresponds to the factor in (20) to be of the functional form

$$K_t = K(Y_t, t), \tag{24}$$

for an appropriately chosen function $K : \mathbb{R} \times [0,\infty) \to \mathbb{R}^+$. Such processes constitute the simplest extension of their zero volatility counterparts.

We start with an informal analysis. To this end, let us make the distortion transformation[2]

$$K(y,t) = \big(v(y,t)\big)^\delta \tag{25}$$

with the power δ given by

$$\delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}. \tag{26}$$

Combining (23) and (25) , and plugging in (7) yields that the process in (23), indeed, satisfies (7), provided that, from one hand, the function $v : \mathbb{R} \times [0,\infty) \to \mathbb{R}^+$ solves the linear problem

$$v_t + \frac{1}{2} d^2(y) v_{yy} + \left(b(y) + \rho \frac{\gamma}{1-\gamma} \lambda(y) d(y) \right) v_y + \frac{1}{2\delta} \frac{\gamma}{1-\gamma} \lambda^2(y) v = 0, \tag{27}$$

with initial condition

$$v(y,0) = \big(K(y,0)\big)^{1/\delta}, \tag{28}$$

[2] Solutions of similar structure were produced for the traditional value function in [15].

and, from the other, the volatility process is set to be $a(x,t) = (a^1(x,t), a^2(x,t))$ with

$$a^1(x,t) = \rho\delta \frac{x^\gamma}{\gamma} d(Y_t) v_y(Y_t,t)\big(v(Y_t,t)\big)^{\delta-1} \tag{29}$$

and

$$a^2(x,t) = \sqrt{1-\rho^2}\delta \frac{x^\gamma}{\gamma} d(Y_t) v_y(Y_t,t)\big(v_y(Y_t,t)\big)^{\delta-1}. \tag{30}$$

The calculations are routine but tedious and are, thus, omitted.

What the above shows is that, in order to construct a solution to (7), it suffices to construct a well defined solution to the initial problem (27) and for the appropriate initial condition (28). This is the subject of investigation in the next section.

4 Non-negative Solutions to an Ill-Posed Heat Equation with a Potential

We consider the backward linear Cauchy problem

$$H_t + \frac{1}{2} a_1^2(x) H_{xx} + a_2(x) H_x + a_3(x) H = 0, \tag{31}$$

for $(t,x) \in (0,+\infty) \times \mathbb{R}$, and initial condition $H(x,0) = H_0(x)$.

The coefficients, a_1, a_2 and a_3 satisfy the following conditions: $a_1(x) > 0$ and is twice continuously differentiable, $a_2(x)$ is continuously differentiable, and $a_3(x)$ is continuous.

We are interested in characterizing the set of non-negative solutions, $H(x,t)$, to the above equation as well as the set of initial conditions, $H_0(x)$, for which (31) has a well-defined solution.

The first step in the analysis of solutions of (31) is to put the equation in the so-called *canonical form*. To this end, consider the change of variables (see, for example, Sect. 4.3 of [14]) $Z : \mathbb{R} \to \mathbb{R}$, given by

$$Z(x) = \sqrt{2} \int_\zeta^x \frac{dz}{a_1(z)}, \tag{32}$$

for some fixed $\zeta \in \mathbb{R}$. In turn, introduce the function $F : \mathbb{R}\times[0,\infty) \to \mathbb{R}^+$ defined as

$$F(z,t) = H\big(X(z),t\big) e^{\frac{1}{2}\int_\zeta^z b(z')dz'} \tag{33}$$

where

$$b(z) = \sqrt{2}\frac{a_2(X(z))}{a_1(X(z))} - \frac{1}{\sqrt{2}} a_1'\big(X(z)\big), \tag{34}$$

with $X : \mathbb{R} \to \mathbb{R}$ given by $X(z) = Z^{-1}(z)$.

In the new variables, Eq. (31) takes the canonical form

$$F_t + F_{zz} + q(z)F = 0,$$

where $q : \mathbb{R} \to \mathbb{R}$ is a continuous function given by

$$q(z) = -\frac{1}{4}b^2(z) - \frac{1}{2}b'(z) + a_3\big(X(z)\big), \tag{35}$$

with $b(z)$ as in (34).

The aim is, then, to specify the class of admissible initial conditions, $F_0 : \mathbb{R} \to \mathbb{R}^+$, and the associated nonnegative solutions $F : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^+$, for the initial value problem

$$(IV) \begin{cases} F_t + F_{zz} + q(z)F = 0, \\ F(z, 0) = F_0(z). \end{cases} \tag{36}$$

A common approach in analyzing the set of solutions to linear time-homogeneous parabolic pdes is to consider the associated *Sturm-Liouville* problem.

In the context of the problems of financial mathematics, the use of Sturm-Liouville theory is, for example, demonstrated in [2] and [11].

Denoting by $f(z, .)$ the Laplace transform of $F(z, .)$, we obtain

$$f_{zz}(z, \lambda) + \big(\lambda + q(z)\big) f(z, \lambda) = f_0(z). \tag{37}$$

We remind the reader that the calculations that follow are, for the moment, formal.

The homogeneous version of (37) is (with a slight abuse of notation),

$$f_{zz}(z, \lambda) + \big(\lambda + q(z)\big) f(z, \lambda) = 0. \tag{38}$$

The following result shows how to generate solutions to (36) using (38). This result is, in many aspects, similar to Widder's theorem (see [18]) which holds for the case $q(z) \equiv 0$ and provides necessary and sufficient conditions for constructing positive solutions to (36). We recall this theorem and provide some comments in the sequel (see Sect. 4.1).

We note that, to our knowledge, an extension to Widder's theorem for non-zero potentials, as the case we study herein, is still lacking. The result below offers only a sufficient condition for constructing positive solutions to (36) but not a necessary one. A further study in this direction can be found in [12].

Proposition 3 *Let us assume that* $\{\psi(., p, \lambda)\}_{(p,\lambda)\in\mathscr{P}\times\Lambda}$ *is a family of solutions to the homogeneous equation* (38), *parameterized by* (p, λ), *where* $\Lambda \subset \mathbb{R}$ *is a Borel set and* $\mathscr{P}$ *is an abstract measurable space. Let us, also, assume that, for each* $z \in \mathbb{R}$, *the function* $\psi(z, ., .)$ *is a nonnegative measurable function on* $\mathscr{P} \times \Lambda$ *and that* ξ *is a measure on* $\mathscr{P} \times \Lambda$, *such that*

$$\sup_{(z,t)\in\mathscr{K}} \left(\int_{\mathscr{P}\times\Lambda} \big(1 + \lambda^2\big) e^{t\lambda} \psi(z, p, \lambda) \xi(dp, d\lambda) \right) < \infty, \tag{39}$$

for any compact set $\mathscr{K} \subset \mathbb{R} \times [0, \infty)$.

Let $F_0 : \mathbb{R} \rightarrow \mathbb{R}^+$ *be defined by*

$$F_0(z) = \int_{\mathscr{P} \times \Lambda} \psi(z, p, \lambda) \xi(dp, d\lambda). \tag{40}$$

Then, *Eq*. (36), *equipped with the above initial condition has a nonnegative solution*, $F(z, t)$, *given by*

$$F(z, t) = \int_{\mathscr{P} \times \Lambda} \psi(z, p, \lambda) e^{t\lambda} \xi(dp, d\lambda). \tag{41}$$

Proof It can be verified by direct computation that the function $F(z, t)$ satisfies (36). Therefore, we only need to show that F and its derivatives exist and are continuous, and that we can interchange the differentiation and integration in (41). These statements will follow from repeated applications of Fubini's theorem.

To this end, we first observe that $F(z, t)$ is well defined, for the corresponding integral converges absolutely due to the integrability assumption (39).

Using (38), we have, for $z \in \mathbb{R}$, that

$$\begin{aligned} &\int_0^z \int_0^{z'} \int_{\mathscr{P} \times \Lambda} \left|\psi_{xx}(x, p, \lambda)\right| e^{t\lambda} \xi(dp, d\lambda) dx dz' \\ &\quad = \int_0^z \int_0^{z'} \int_{\mathscr{P} \times \Lambda} \left|\lambda + q(x)\right| \left|\psi(x, p, \lambda)\right| e^{t\lambda} \xi(dp, d\lambda) dx dz' < \infty, \end{aligned}$$

as it follows from (39) and the continuity of the potential coefficient $q(z)$. Thus, we can interchange the order of integration to obtain

$$\begin{aligned} &\int_0^z \int_0^{z'} \int_{\mathscr{P} \times \Lambda} \psi_{xx}(x, p, \lambda) e^{t\lambda} \xi(dp, d\lambda) dx \\ &\quad = \int_{\mathscr{P} \times \Lambda} \big(\psi(z, p, \lambda) - \psi(0, p, \lambda) - z\psi_z(0, p, \lambda)\big) e^{t\lambda} \xi(dp, d\lambda). \end{aligned}$$

Notice that the integral in the right hand side above is absolutely convergent, because such is the integral in the left hand side. In addition, because of (39), the integral $\int_{\mathscr{P} \times \Lambda} (\psi(z, p, \lambda) - \psi(0, p, \lambda)) e^{t\lambda} \xi(dp, d\lambda)$ also converges absolutely. Therefore, the function $e^{t\lambda} \psi_z(0, p, \lambda)$ is absolutely integrable with respect to $\xi(dp, d\lambda)$. We, easily, deduce that, for some constant c_1,

$$\int_0^z \int_0^{z'} \int_{\mathscr{P} \times \Lambda} \psi_{xx}(x, p, \lambda) e^{t\lambda} \xi(dp, d\lambda) dx dz' = F(z, t) - F(0, t) - c_1 z,$$

for all $(z, t) \in \mathbb{R} \times [0, \infty)$.

Let $\phi(z, t)$ be given by

$$\phi(z, t) = \int_0^z \int_0^{z'} \int_{\mathscr{P} \times \Lambda} \psi_{xx}(x, p, \lambda) e^{t\lambda} \xi(dp, d\lambda) dx dz'.$$

Then,

$$\phi(z,t) = F(z,t) - F(0,t) - c_1 z$$

and, by construction, it is continuously differentiable in z, with absolutely continuous derivative. Therefore, the same holds for $F(z,t)$, and, for almost all $z \in \mathbb{R}$, we have

$$F_z(z,t) = c_1 + \phi_z(z,t)$$

and

$$F_{zz}(z,t) = \phi_{zz}(z,t) = \int_{\mathscr{P}\times\Lambda} \psi_{zz}(z,p,\lambda)e^{t\lambda}\xi(dp,d\lambda).$$

Following similar arguments, we can show that, for any fixed $z \in \mathbb{R}$, the function $F(z,.)$ is absolutely continuous on $[0,\infty)$, and, in turn,

$$F_t(z,t) = \int_{\mathscr{P}\times\Lambda} \lambda\psi(z,p,\lambda)e^{t\lambda}\xi(dp,d\lambda),$$

for (almost all) $t \geq 0$.

It remains to show that the partial derivatives are continuous in $(z,t) \in \mathbb{R}\times[0,\infty)$. We start with $F_t(z,t)$. Let $(z,t), (z',t') \in \mathbb{R}\times[0,\infty)$. Then,

$$\begin{aligned}
&\left|\int_{\mathscr{P}\times\Lambda} \lambda\big(\psi(z,p,\lambda)e^{t\lambda} - \psi\big(z',p,\lambda\big)e^{t'\lambda}\big)\xi(dp,d\lambda)\right| \\
&\quad \leq \int_{\mathscr{P}\times\Lambda} |\lambda|e^{t\lambda}\psi(z,p,\lambda)\big|1 - e^{\lambda(t'-t)}\big|\xi(dp,d\lambda) \\
&\qquad + \int_{\mathscr{P}\times\Lambda} |\lambda|e^{t'\lambda}\big|\psi(z,p,\lambda) - \psi\big(z',p,\lambda\big)\big|\xi(dp,d\lambda). \qquad (42)
\end{aligned}$$

We estimate the above integrals separately. We first observe that, for some constant c_2, the first integral satisfies

$$\begin{aligned}
&\int_{\mathscr{P}\times\Lambda} |\lambda|e^{t\lambda}\psi(z,p,\lambda)\big|1 - e^{\lambda(t'-t)}\big|\xi(dp,d\lambda) \\
&\quad \leq c_2\big|t - t'\big|\int_{\mathscr{P}\times\Lambda} \lambda^2 e^{t\lambda}\psi(z,p,\lambda)\xi(dp,d\lambda).
\end{aligned}$$

The expression in the right hand side above converges to zero as $t' \to t$, since the integral therein is finite, due to (39). For the second integral in (42) we have

$$\begin{aligned}
&|\lambda|e^{t'\lambda}\big|\psi(z,p,\lambda) - \psi\big(z',p,\lambda\big)\big| \\
&\quad = |\lambda|e^{t'\lambda}\left|\int_z^{z'}\int_0^{x'} \psi_{xx}(x,p,\lambda)dx dx' + \big(z' - z\big)\psi_z(0,p,\lambda)\right|.
\end{aligned}$$

We readily deduce that the left hand side above is absolutely integrable with respect to ξ, uniformly over t' changing on a compact set in $[0,\infty)$. Therefore, the right hand side has the same property. On the other hand, (39) yields that

$$\int_{\mathscr{P}\times\Lambda}\int_z^{z'}\int_0^{x'}|\lambda|e^{t'\lambda}\big|\psi_{xx}(x,p,\lambda)\big|dxdx'\xi(dp,d\lambda)$$
$$=\int_{\mathscr{P}\times\Lambda}\int_z^{z'}\int_0^{x'}|\lambda|e^{t'\lambda}\big|\lambda+q(x)\big|\psi(x,p,\lambda)dxdx'\xi(dp,d\lambda)$$

is bounded, uniformly on t' changing on a compact set. Therefore, the function $\lambda e^{t'\lambda}\psi_z(0,p,\lambda)$ is absolutely integrable with respect to $\xi(dp,d\lambda)$, uniformly over t' varying on a compact set. We, then, deduce that

$$\int_{\mathscr{P}\times\Lambda}|\lambda|e^{t'\lambda}\big|\psi(z,p,\lambda)-\psi\big(z',p,\lambda\big)\big|\xi(dp,d\lambda)$$
$$\le c_3\int_{\mathscr{P}\times\Lambda}\int_z^{z'}\int_0^{x'}\big(1+\lambda^2\big)e^{t'\lambda}\psi(x,p,\lambda)dxdx'\xi(dp,d\lambda)$$
$$+\big|z'-z\big|\int_{\mathscr{P}\times\Lambda}|\lambda|e^{t'\lambda}\big|\psi_z(0,p,\lambda)\big|\xi(dp,d\lambda)$$
$$\le c_3\big|z-z'\big|\big(|z|+|z'|\big)\sup_{x\in[z,z']}\int_{\mathscr{P}\times\Lambda}\big(1+\lambda^2\big)e^{t'\lambda}\psi(x,p,\lambda)\xi(dp,d\lambda)$$
$$+\big|z'-z\big|\int_{\mathscr{P}\times\Lambda}|\lambda|e^{t'\lambda}\big|\psi_z(0,p,\lambda)\big|\xi(dp,d\lambda).$$

The above integrals are bounded uniformly over t' changing on a compact set, and, therefore, the above right hand side converges to zero, as $(z',t')\to(z,t)$.

Working along similar arguments, we obtain the continuity in (z,t) of the function $\int_{\mathscr{P}\times\Lambda}\psi(z,p,\lambda)e^{t\lambda}\xi(dp,d\lambda)$. We easily conclude. □

The above result shows how one can construct solutions to Eq. (36) directly from the appropriate initial condition. It is not, however, always clear how to actually construct a nonnegative solution to (38). This is what we explore next.

For the rest of the analysis, we focus on the class of coefficients $q(z)$ which are bounded from above. We remind the reader that the term $q(z)$ represents the negative of a potential term, as the latter appears in the literature. A natural assumption for potentials is that they are bounded from below: notice, for example, that the assumption of nonnegativity of the "killing rate" in [4] is another way of saying that the corresponding potential is nonnegative.

Proposition 4 *Let us assume that there exists* $\bar{\lambda}\in\mathbb{R}$, *such that the potential term in* (36) *satisfies* $q(z)\le\bar{\lambda}$, $z\in\mathbb{R}$, *and denote* $\mathscr{D}=(-\infty,-\bar{\lambda})$. *Then, the following statements hold*:

(i) *Assume that there exists* $L_1 \in \mathbb{R}$, *such that*

$$\int_0^\infty \left|q(z) - L_1\right| dz < \infty.$$

Then, for any $\lambda \in \mathscr{D}$, *there exists a unique solution of* (38), *denoted by* $\psi^{(1)}(.,\lambda)$, *which is square integrable over* $(0,\infty)$ *and satisfies* $\psi^{(1)}(0,\lambda) = 1$. *Moreover, for each* $z \in \mathbb{R}$, *the function* $\psi^{(1)}(z,.)$ *is nonnegative and continuous on* $\mathscr{D}$.

Let, also, μ_1 *be a Borel measure on* $\mathscr{D}$, *satisfying*

$$\sup_{(t,z)\in\mathscr{K}} \left(\int_{\mathbb{R}} \left(1+\lambda^2\right) e^{t\lambda} \psi^{(1)}(z,\lambda) \mu_1(d\lambda) \right) < \infty,$$

for any compact set $\mathscr{K} \subset [0,\infty) \times \mathbb{R}$, *and define the function* $F_0^{(1)} : \mathbb{R} \to \mathbb{R}^+$ *by*

$$F_0^{(1)}(z) = \int_{\mathbb{R}} \psi^{(1)}(z,\lambda) \mu_1(d\lambda). \tag{43}$$

Then, Eq. (36) *has a nonnegative classical solution, say* $F^{(1)}(z,t)$, *given by*

$$F^{(1)}(z,t) = \int_{\mathbb{R}} \psi^{(1)}(z,\lambda) e^{t\lambda} \mu_1(d\lambda), \tag{44}$$

satisfying $F^{(1)}(z,0) = F_0^{(1)}(z)$.

(ii) *Assume that there exists* $L_2 \in \mathbb{R}$, *such that*

$$\int_{-\infty}^0 \left|q(z) - L_2\right| dy < \infty. \tag{45}$$

Then, for any $\lambda \in \mathscr{D}$, *there exists a unique solution of* (38), *denoted by* $\psi^{(2)}(.,\lambda)$, *which is square integrable over* $(-\infty,0)$ *and satisfies* $\psi^{(2)}(0,\lambda) = 1$. *Moreover, for each* $z \in \mathbb{R}$, *the function* $\psi^{(2)}(z,.)$ *is nonnegative and continuous on* $\mathscr{D}$.

Let, also, μ_2 *be a Borel measure on* $\mathscr{D}$, *satisfying*

$$\sup_{(t,z)\in\mathscr{K}} \left(\int_{\mathbb{R}} \left(1+\lambda^2\right) e^{t\lambda} \psi^{(2)}(z,\lambda) \mu_2(d\lambda) \right) < \infty,$$

for any compact set $\mathscr{K} \subset [0,\infty) \times \mathbb{R}$, *and define the function* $F_0^{(2)} : \mathbb{R} \to \mathbb{R}^+$ *given by*

$$F_0^{(2)}(z) = \int_{\mathbb{R}} \psi^{(2)}(z,\lambda) \mu_2(d\lambda). \tag{46}$$

Then, Eq. (36) *has a nonnegative classical solution, say* $F^{(2)}(z,t)$, *given by*

$$F^{(2)}(z,t) = \int_{\mathbb{R}} \psi^{(2)}(z,\lambda) e^{t\lambda} \mu_2(d\lambda), \tag{47}$$

satisfying $F^{(2)}(z,0) = F_0^{(2)}(z)$.

(iii) *Let the above assumptions hold in both* (i) *and* (ii). *Then, problem* (36), *equipped with the initial condition* $F_0(z) = F_0^{(1)}(z) + F_0^{(2)}(z)$, *with* $F_0^{(1)}(z)$ *and* $F_0^{(2)}(z)$ *given, respectively, by* (43) *and* (46), *has a nonnegative classical solution, say* $F(z,t)$, *given by*

$$F(z,t) = F^{(1)}(z,t) + F^{(2)}(z,t),$$

with $F^{(1)}(z,t)$ *and* $F^{(2)}(z,t)$ *as in* (44) *and* (47), *respectively.*

Proof We only establish part (i), for part (ii) follows along the same arguments using a change of variables "$z \mapsto -z$" and part (iii) follows trivially from parts (i) and (ii).

We start with some elementary transformations which will facilitate the upcoming analysis. To this end, fix $\delta > 0$, and consider all (possibly complex) numbers λ, satisfying $\mathfrak{Re}(\lambda) < -\delta - \bar{\lambda}$. Let $\varepsilon \in (0, \frac{1}{2}\delta(1 - e^{-2\sqrt{\delta}}))$ and $N \geq 0$ satisfying $\int_N^\infty |q(z) - L_1| dy < \varepsilon$, and introduce the change of variables

$$\tilde{\lambda} = \lambda + L_1 \quad \text{and} \quad \tilde{q}(z) = q(z+N) - L_1.$$

It, then, follows that a function $f(z,\lambda)$, is a solution to (38), if and only if the function $g(z,\tilde{\lambda})$, defined by

$$g(z,\tilde{\lambda}) = f(z+N, \tilde{\lambda} - L_1)$$

satisfies the homogeneous problem

$$g_{zz}(z,\tilde{\lambda}) + \big(\tilde{\lambda} + \tilde{q}(z)\big) g(z,\tilde{\lambda}) = 0. \tag{48}$$

Let H_δ be the set

$$H_\delta = \big\{ z \in \mathbb{C} \,|\, \mathfrak{Re}(z) < -\delta - \bar{\lambda} + L_1 \big\}.$$

It is clear that $L_1 \leq \bar{\lambda}$ and, therefore, $-\delta - \bar{\lambda} + L_1 < 0$.

We proceed as follows. We first establish that for any $\tilde{\lambda} \in H_\delta$, there exists a square integrable solution, say $\chi(z,\tilde{\lambda})$, to the above equation (48), for $z \in [0,\infty)$. Then, we show that this solution can be extended to the entire set $\mathbb{R}$ and that it is the unique (up to a multiplicative factor) such solution that is square integrable. We conclude showing that $\chi(z,\tilde{\lambda})$ does not change its sign.

To this end, let $\tilde{\lambda} \in H_\delta$ and consider the following integral equation for functions of $z \in [0,+\infty)$,

$$\begin{aligned}\chi(z,\tilde{\lambda}) = e^{iz\sqrt{\tilde{\lambda}}} &- \frac{1}{2i\sqrt{\tilde{\lambda}}} \int_0^z e^{i(z-x)\sqrt{\tilde{\lambda}}} \tilde{q}(x) \chi(x,\tilde{\lambda}) dx \\ &- \frac{1}{2i\sqrt{\tilde{\lambda}}} \int_z^\infty e^{i(x-z)\sqrt{\tilde{\lambda}}} \tilde{q}(x) \chi(x,\tilde{\lambda}) dx.\end{aligned} \tag{49}$$

Herein, we choose a version of the "square root" which generates a continuous mapping from $\mathbb{C} \setminus [0, \infty)$ to the upper half plane.

It is, then, easy to see that if the above equation has a solution $\chi(., \tilde{\lambda})$, then, it is twice continuously differentiable and solves (48).

On the other hand, it is shown in Sect. 6.2 (p. 119) of [14] that the iterative scheme

$$\chi_1(z, \tilde{\lambda}) = e^{iz\sqrt{\tilde{\lambda}}}$$

and

$$\chi_{n+1}(z, \tilde{\lambda}) = e^{iz\sqrt{\tilde{\lambda}}} - \frac{1}{2i\sqrt{\tilde{\lambda}}} \int_0^z e^{i(z-x)\sqrt{\tilde{\lambda}}} \tilde{q}(x)\chi_n(x, \tilde{\lambda})dx$$
$$- \frac{1}{2i\sqrt{\tilde{\lambda}}} \int_z^\infty e^{i(x-z)\sqrt{\tilde{\lambda}}} \tilde{q}(x)\chi_n(x, \tilde{\lambda})dx, \quad \text{for } n \geq 1,$$

converges to the solution of (49), $\chi(., \tilde{\lambda})$.

In particular, it is shown in formulas (6.2.5) and (6.2.6) therein that the approximating terms satisfy

$$\left|\chi_{n+1}(z, \tilde{\lambda}) - \chi_n(z, \tilde{\lambda})\right| \leq \left(\frac{\varepsilon}{2\delta}\right)^n \left|e^{iz\sqrt{\tilde{\lambda}}}\right|,$$

and, hence, the convergence is uniform in $\tilde{\lambda}$ changing on any compact set in H_δ. This, in turn, yields that the function $\chi(z, .)$ is holomorphic in H_δ, for any $z \in [0, \infty)$. Moreover, the following estimate holds

$$\left|\chi(z, \tilde{\lambda})\right| \leq \frac{|e^{iz\sqrt{\tilde{\lambda}}}|}{1 - \varepsilon/(2\delta)}.$$

We easily deduce that $\chi(., \tilde{\lambda})$ solves (48) and that it is square integrable on $[0, +\infty)$.

Next, we extend $\chi(., \tilde{\lambda})$ to the entire set $\mathbb{R}$. To this end, notice that any solution of (48) can be uniquely represented as a linear combination of two solutions, say, $\varphi(z, \tilde{\lambda})$ and $\theta(z, \tilde{\lambda})$, satisfying

$$\varphi(0, \tilde{\lambda}) = 0 \quad \text{and} \quad \varphi_z(0, \tilde{\lambda}) = -1,$$

and

$$\theta(0, \tilde{\lambda}) = 1 \quad \text{and} \quad \theta_z(0, \tilde{\lambda}) = 0.$$

Therefore, one obtains the representation

$$\chi(z, \tilde{\lambda}) = K_1(\tilde{\lambda})\theta(z, \tilde{\lambda}) + K_2(\tilde{\lambda})\varphi(z, \tilde{\lambda}),$$

for some functions K_1 and K_2. On the other hand, differentiating (49) and applying the dominated convergence theorem yield that $\chi_z(z,.)$ is continuous in H_δ, for any $z \in [0,\infty)$. Notice, also, that

$$K_1(\tilde{\lambda}) = \chi(0,\tilde{\lambda}) \quad \text{and} \quad K_2(\tilde{\lambda}) = -\chi_z(0,\tilde{\lambda}),$$

and, hence, the functions K_1 and K_2 are continuous in H_δ. It also follows—see for example Theorem 1.5 in Sect. 1.5 of [14]—that $\varphi(z,.)$ and $\theta(z,.)$ are entire functions (holomorphic in $\mathbb{C}$), for any $z \in \mathbb{R}$. Combining the above, we conclude that $\chi(z,.)$ is continuous in H_δ.

Next, we establish that $\chi(z,\tilde{\lambda})$ is the unique (up to a multiplicative factor) square integrable solution to (48). We argue by contradiction. To this end, assume that, for some $\tilde{\lambda} \in H_\delta$, there exists a solution to (48), which is square integrable over $(0,\infty)$ and linearly independent of $\chi(.,\tilde{\lambda})$. Then, this solution, together with $\chi(.,\tilde{\lambda})$, will span the space of all solutions to (48). Hence, every solution is square integrable over $(0,+\infty)$. However, from Eq. (5.3.1) in Sect. 5.3 of [14], we obtain the following representation of φ,

$$\varphi(z,\tilde{\lambda}) = \frac{e^{-iz\sqrt{\tilde{\lambda}}}}{2i\sqrt{\tilde{\lambda}}}\Bigg(-1 + e^{2iz\sqrt{\tilde{\lambda}}} - \int_0^z e^{i(z-x)\sqrt{\tilde{\lambda}}} e^{iz\sqrt{\tilde{\lambda}}}\varphi(x,\tilde{\lambda})\tilde{q}(x)dx \\ + \int_0^z e^{ix\sqrt{\tilde{\lambda}}}\varphi(x,\tilde{\lambda})\tilde{q}(x)dx\Bigg).$$

Using the above representation and Lemma 5.2 in Sect. 5.2 of [14], we obtain the estimate (given in the last equation on page 98 in Sect. 5.3 therein),

$$\left|\varphi(z,\tilde{\lambda})\right| \leq \frac{1}{\delta}\exp\left(\frac{\varepsilon}{\delta}\right)\left|e^{-iz\sqrt{\tilde{\lambda}}}\right|.$$

Using the above estimate we obtain, for $z \geq 1$, that

$$\left|e^{2iz\sqrt{\tilde{\lambda}}} - \int_0^z e^{i(z-x)\sqrt{\tilde{\lambda}}} e^{iz\sqrt{\tilde{\lambda}}}\varphi(x,\tilde{\lambda})\tilde{q}(x)dx + \int_0^z e^{ix\sqrt{\tilde{\lambda}}}\varphi(x,\tilde{\lambda})\tilde{q}(x)dx\right| \\ \leq e^{-2\sqrt{\delta}} + 2\frac{\varepsilon}{\delta}\exp\left(\frac{\varepsilon}{\delta}\right) < 1,$$

where the last inequality follows from the choice of ε as in the beginning of the proof.

Thus, from the above representation of φ, we conclude that, for all $z \geq 1$,

$$\left|\varphi(z,\tilde{\lambda})\right| \geq \frac{c_1}{2i\sqrt{\tilde{\lambda}}}\left|e^{-iz\sqrt{\tilde{\lambda}}}\right|.$$

However, sending $z \to \infty$, we have $\lim_{z\to\infty}|e^{-iz\sqrt{\tilde{\lambda}}}| = \infty$, which contradicts the square integrability of $\varphi(z,\tilde{\lambda})$ over $(0,\infty)$, and we easily conclude that $\chi(.,\tilde{\lambda})$ is the unique solution to (48) that is square integrable (up to a multiplicative constant).

Next, we show that $\chi(., \tilde{\lambda})$ does not change the sign. Indeed, notice that because $\tilde{\lambda} + \tilde{q}(z) < 0$, for all $\tilde{\lambda} \in H_\delta$ and $z \in \mathbb{R}$, any solution to (48) is concave on the intervals on which it is negative, and it is convex on the intervals on which it is positive.

Fix, now, some $\tilde{\lambda} \in H_\delta$ and assume that there is $z_0 \in \mathbb{R}$, such that $\chi(z_0, \tilde{\lambda}) = 0$. We know that, if $\chi_z(z_0, \tilde{\lambda}) = 0$, then, due to the uniqueness of a solution to the Eq. (48) with a given pair of initial conditions, the function $\chi(., \tilde{\lambda})$ has to be identically zero. This, however, is not possible since the function identically equal to zero does not satisfy (49). Therefore, without loss of generality, we assume that $\chi_z(z_0, \tilde{\lambda}) < 0$. Then, there exist $\varepsilon' > 0$ and $z' > z_0$, such that $\chi(z', \tilde{\lambda}) = -\varepsilon' < 0$ and $\chi(., \tilde{\lambda}) < -\varepsilon'$, in some right neighborhood of z'. This, in turn, implies that $\chi(z, \tilde{\lambda}) < -\varepsilon'$ for all $z > z'$, since, otherwise the concavity of the function $\chi(., \tilde{\lambda})$ in the interval $[z', \inf(z > z' \mid \chi(z, \tilde{\lambda}) = -\varepsilon')]$ will be violated. On the other hand, if $\chi(z, \tilde{\lambda}) < -\varepsilon'$, for all $z > z'$, we then obtain a contradiction to the square integrability of $\chi(., \tilde{\lambda})$, for $z \in (0, \infty)$. Similarly, we arrive to a contradiction if we assume that $\chi_z(z_0, \tilde{\lambda}) > 0$.

Combining the above we deduce that the function $\chi(., \tilde{\lambda})$ does not change its sign on $\mathbb{R}$. Therefore, the function $\psi^{(1)}(z, \lambda)$, defined as

$$\psi^{(1)}(z, \lambda) = \frac{\chi(\lambda + L_1, z - N)}{\chi(\lambda + L_1, -N)},$$

is well defined for all $\lambda \in (-\infty, -\delta - \bar{\lambda})$ and $z \in \mathbb{R}$. Moreover, it is uniquely characterized as a solution to (48), which is square integrable over $(0, +\infty)$ and satisfies $\psi^{(1)}(0, \lambda) = 1$. We have, also, shown that $\psi^{(1)}(z, \lambda) > 0$ and, moreover, it is continuous as a function of λ, changing on $(-\infty, -\delta - \bar{\lambda})$, for any $z \in \mathbb{R}$. Notice that, since $\delta > 0$ is arbitrary, the above properties extend to all $\lambda \in (-\infty, -\bar{\lambda})$.

Finally, we apply Proposition 3 to conclude that (44) is well defined and it is a solution to (36) with the initial condition (43). □

4.1 The Backward Heat Equation

When $a_1 \equiv \frac{1}{2}$, $a_2 \equiv 0$, and $a_3 \equiv 0$, the Eq. (31) reduces to the well-known backward heat equation, presented earlier in Sect. 2.3 and rewritten below to ease the presentation (we also denote the solution by F to preserve the above notation). As mentioned earlier, its non-negative solutions are completely characterized by the celebrated Widder's theorem given, for completeness, below. Its proof can be found in Chap. XI in [18].

Theorem 2 (Widder's) *Consider the heat equation*

$$F_t + \frac{1}{2} F_{xx} = 0 \tag{50}$$

for $(x, t) \in \mathbb{R}\times(0, \infty)$. *A function* $F : \mathbb{R}\times(0, \infty) \to \mathbb{R}^+$ *is a solution to the above if and only if it can be represented as*

$$F(x, t) = \int_{\mathbb{R}} e^{zx - \frac{1}{2}z^2 t}\nu(dz) \tag{51}$$

where ν *is a positive finite Borel measure.*

As the above theorem shows, the *only* functions that can serve as initial conditions to (50) are given by a bilateral Laplace transform of the underlying measure ν, namely,

$$F(x, 0) = \int_{\mathbb{R}} e^{xz}\nu(dz), \tag{52}$$

given that the above integral converges for any $x \in \mathbb{R}$. We next show how the results proved herein can be used to obtain one direction of the above theorem. Specifically, we show how formula (51) can be obtained[3] by using the construction approach provided in Proposition 4.

Proposition 5 *Let* $F : \mathbb{R}\times[0, \infty) \to \mathbb{R}^+$ *be given by*

$$F(x, t) = \int_{\mathbb{R}} e^{xy - \frac{1}{2}y^2 t}\nu(dy),$$

where ν *is a positive Borel measure, such that the above integral is finite for* $t = 0$ *and all* $x \in \mathbb{R}$. *Then,* F *is a nonnegative solution of* (50), *satisfying initial condition* (52).

Proof Rewrite equation (50) for $G(x, t) = F(x, 2t)$. Then, we obtain Eq. (36) with $q \equiv 0$. Applying Proposition 4 with $L_1 = L_2 = \bar{\lambda} = 0$, we conclude that the corresponding solutions $\psi^{(1)}$ and $\psi^{(2)}$ are given, respectively, by

$$\psi^{(1)}(x, \lambda) = e^{-ix\sqrt{\lambda}} \quad \text{and} \quad \psi^{(2)}(x, \lambda) = e^{ix\sqrt{\lambda}}.$$

Then, Eq. (36) has a nonnegative solution, say $G(x, t)$, for any initial condition of the form

$$G_0(x) = \int_{-\infty}^{0} e^{-ix\sqrt{\lambda}}\mu_1(d\lambda) + \int_{-\infty}^{0} e^{ix\sqrt{\lambda}}\mu_2(d\lambda),$$

where μ_1 and μ_2 are Borel measures on $(-\infty, 0)$, satisfying the integrability conditions in parts (i) and (ii) of Proposition 4, respectively.

[3] Of course, one can easily verify that (51) indeed solves (50). The aim is, however, to develop a general approach for equations of the general form (36).

Notice that we, then, have

$$\begin{aligned}G(x,t) &= \int_{-\infty}^{0} e^{-ix\sqrt{\lambda}+t\lambda}\mu_1(d\lambda) + \int_{-\infty}^{0} e^{ix\sqrt{\lambda}+t\lambda}\mu_2(d\lambda)\\ &= \int_{0}^{+\infty} e^{xs-ts^2}\tilde{\mu}_1(ds) + \int_{0}^{+\infty} e^{-xs-ts^2}\tilde{\mu}_2(ds)\\ &= \int_{\mathbb{R}} e^{xs-ts^2}\big(\tilde{\mu}_1(ds)\mathbf{1}_{\mathbb{R}_+}(s) + \tilde{\mu}_2\big(d(-s)\big)\mathbf{1}_{\mathbb{R}_-}(s)\big),\end{aligned}$$

where

$$\tilde{\mu}_1 = \mu_1 \circ m^{-1} \quad \text{and} \quad \tilde{\mu}_2 = \mu_2 \circ m^{-1},$$

with $m(s) = \sqrt{-s}$.

It is easy to see that μ_1 and μ_2 satisfy the corresponding integrability conditions if and only if the above integral is finite for $t = 0$ and all $x \in \mathbb{R}$.

Reverting to the original variables, we obtain $F(x,t) = G(x,t/2)$, and note that we have proved the statement of the proposition for all measures ν, which satisfy the appropriate integrability conditions and have no mass at zero. Finally, we notice that if ν is a Dirac delta-function at zero, then the resulting function F is identically equal to one, and, therefore, solves (50). Using the linearity of (50), we conclude the proof. □

5 Examples

In this section we present two examples of processes satisfying the forward SPDE (7). For this, we apply the methodology developed in the previous section and the form of the candidate solutions. We do not, however, derive or study the associated optimal policy and optimal wealth processes. Such questions will be presented in a future paper in which a more general class of solutions will be considered (see [13]).

5.1 Mean Reverting Stochastic Volatility

We assume that the coefficients in (1) and (2) take, respectively, the form

$$\mu(y) = \mu \quad \text{and} \quad \sigma(y) = (\mu - r)e^{-y} \tag{53}$$

and

$$b(y) = c_1 e^{y} + c_2 \quad \text{and} \quad d(y) = d, \tag{54}$$

for $y \in \mathbb{R}$, and c_1, c_2, d, μ and r constants with $d > 0$ and $c_1 < 0$. An extra assumption on the ratio $|c_1|/d$ will be imposed in the sequel. For the other constants, we assume, without loss of generality, that $\mu > r > 0$ and $c_2 \geq 0$.

Under (53) and (54), Eqs. (1) and (2) become

$$dS_t = S_t \mu dt + S_t(\mu - r)e^{-Y_t} dW_t^1 \tag{55}$$

and

$$dY_t = \big(c_1 e^{Y_t} + c_2\big)dt + d\big(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\big), \tag{56}$$

with $S_0 > 0$ and $Y_0 \in \mathbb{R}$. The above choice of the stochastic factor corresponds to a stock volatility

$$N_t = (\mu - r)e^{-Y_t} \tag{57}$$

which satisfies

$$dN_t = \left(|c_1|(\mu - r) + \left(\frac{d^2}{2} - c_2\right)N_t\right)dt - dN_t dW_t, \tag{58}$$

and, hence, if c_2 is large enough, exhibits mean reverting behavior. One can easily show that the above equation, and, consequently, the system consisting of (55) and (56), has a unique strong solution.

Next, we use the change of variables introduced at the beginning of Sect. 4, in order to derive a canonical form of Eq. (27). Recall that in this case, we have

$$a_1(y) = \frac{1}{2}d^2, \qquad a_2(y) = e^y\left(c_1 + \rho d \frac{\gamma}{1-\gamma}\right) + c_2, \qquad a_3(y) = \frac{1}{2\delta}\frac{\gamma}{1-\gamma}e^{2y}.$$

To this end, rescaling time, from t to $d^2 t/2$, and applying the change of variables described at the beginning of Sect. 4, we get that the function $g : \mathbb{R}\times[0,\infty) \to \mathbb{R}^+$ defined by

$$g(y,t) = v\left(y, \frac{2}{d^2}t\right)\exp\left(C_2 + \frac{c_2}{d^2}y - C_2 e^y\right),$$

with v introduced in Sect. 3.2 and the constants C_1 and C_2 as

$$C_1 = \frac{1}{d^2}\left(\frac{c_1^2}{d^2} + \rho\frac{2c_1}{d}\frac{\gamma}{1-\gamma} - \frac{\gamma}{1-\gamma}\right) \quad \text{and} \quad C_2 = \frac{1}{d}\left(\frac{|c_1|}{d} - \rho\frac{\gamma}{1-\gamma}\right), \tag{59}$$

needs to satisfy the linear equation

$$g_t + g_{yy} + q(y)g = 0, \tag{60}$$

with initial condition

$$g(y,0) = \exp\left(C_2 + \frac{c_2}{d^2}y - C_2 e^y\right)\big(K(y)\big)^{1/\delta}, \tag{61}$$

where the distortion power δ is as in (26) and the potential term is given by

$$q(y) = -C_1 e^{2y} + C_2\left(1 + \frac{2c_2}{d^2}\right)e^y - \frac{c_2^2}{d^4}. \tag{62}$$

It is further assumed that $|c_1|/d$ is large enough, so that both constants $C_1, C_2 > 0$.

We recall that, according to Proposition 3, one needs to represent the above initial condition as an integral over λ's of the nonnegative solutions to the corresponding Sturm-Liouville equation

$$\psi_{yy}(y,\lambda) + \big(\lambda + q(y)\big)\psi(\lambda, y) = 0, \tag{63}$$

with $q(y)$ given in (62). We, also, remind the reader that, herein, we are not looking for the entire class of solutions, but we seek to construct merely one solution. To this end, we first observe that the function

$$\varphi(y) = \exp\left(C_2 + \frac{c_2}{d^2}y - \sqrt{C_1}e^y\right),$$

satisfies (63) with $\lambda = 0$. Applying Proposition 3 with $\mathscr{P}$ being a singleton and $\Lambda = \{0\}$, we easily obtain that the same function is a solution for $t > 0$, i.e. the function $g : \mathbb{R}\times[0,\infty) \to \mathbb{R}^+$ given by

$$g(y,t) = \exp\left(C_2 + \frac{c_2}{d^2}y - \sqrt{C_1}e^y\right)$$

solves (60).

Therefore, if we choose the factor $K(y)$ to be

$$K(y) = \exp\big(\delta(C_2 - \sqrt{C_1})e^y\big),$$

we deduce that $g(y,0) = \varphi(y)$. Hence,

$$v(y,t) = \exp\big((C_2 - \sqrt{C_1})e^y\big),$$

and we easily conclude.

We summarize the above findings below.

Proposition 6 *Assume that the stock and the stochastic factor solve* (55) *and* (56). *Also, assume that the aforementioned assumptions on the involved coefficients hold and that the distortion power δ is as in* (26).

Define the process $a(x,t)$ by

$$a(x,t) = \left(\frac{x^\gamma}{\gamma}\rho Z_t, \frac{x^\gamma}{\gamma}\sqrt{1-\rho^2}Z_t\right) \tag{64}$$

where

$$Z_t = d\delta(C_2 - \sqrt{C_1})\exp\big(Y_t + \delta(C_2 - \sqrt{C_1})\big(e^{Y_t} - e^{Y_0}\big)\big) \tag{65}$$

and the constants C_1 and C_2 are as in (59).

Moreover, let the initial condition $u_0 : \mathbb{R}^+ \to \mathbb{R}^+$ *be*

$$u_0(x) = \frac{x^\gamma}{\gamma}.$$

Then, the process $U(x,t)$ *given by*

$$U(x,t) = \frac{x^\gamma}{\gamma} \exp\big(\delta(C_2 - \sqrt{C_1})\big(e^{Y_t} - e^{Y_0}\big)\big) \tag{66}$$

solves the forward performance SPDE (7) *with the above performance volatility process* $a(x,t)$ *and initial condition* $U(x,0) = u_0(x)$.

Next, we study the behavior of the forward investment performance process as the forward volatility vanishes. This occurs when the coefficient $d \to 0$.

Proposition 7 *Let* $U^{(d)}(x,t)$ *be the forward investment performance process given in* (66). *Then, for each* $t > 0$,

(i) *the performance volatility process* $a(x,t)$ (*cf.* (64)) *satisfies a.s. for all* $x \geq 0$,

$$\lim_{d \to 0} a(x,t) = 0, \tag{67}$$

and

(ii) *the forward investment performance process satisfies a.s. for all* $x \geq 0$,

$$\lim_{d \to 0} U^{(d)}(x,t) = \frac{x^\gamma}{\gamma} \exp\left(-\frac{\gamma}{2c_1(1-\gamma)}\left(e^{Y_t^{(0)}} - e^{Y_0}\right)\right), \tag{68}$$

where $Y_t^{(0)}$ *is the solution to the deterministic problem*

$$dY_t^{(0)} = \big(c_1 e^{Y_t^{(0)}} + c_2\big)dt$$

with $Y_0^{(0)} = Y_0$.

Proof We first observe, using (59), that

$$\delta(C_2 - \sqrt{C_1}) = \frac{\gamma}{1-\gamma}\left(-c_1 - \rho\frac{d\gamma}{1-\gamma} + \sqrt{c_1^2 + 2\rho\frac{c_1 d\gamma}{(1-\gamma)} - \frac{\gamma d^2}{1-\gamma}}\right)^{-1}$$

and, in turn,

$$\lim_{d \to 0} \delta(C_2 - \sqrt{C_1}) = -\frac{\gamma}{2c_1(1-\gamma)} > 0.$$

Next, we recall that the process $N_t^{(d)}$, $t \geq 0$, defined in (57) solves the affine SDE (58), with $N_0^{(d)} = (\mu - r)e^{-Y_0}$. On the other hand, the solution of this equation can be represented explicitly (see, for example, Sect. 5.6 in [5]). From this explicit representation, it is easy to deduce that almost surely, for all $t > 0$,

$$\lim_{d\to 0} N_t^{(d)} = N_t^{(0)} = (\mu - r)e^{-Y_t^{(0)}}.$$

We easily obtain that $\lim_{d\to 0} Z_t = 0$, and using (64) and passing to the limit we obtain (67). Assertion (68) follows easily. □

5.2 *Heston-Type Stochastic Volatility*

We choose the model coefficients

$$\mu(y) = \mu \quad \text{and} \quad \sigma(y) = (\mu - r)\sqrt{y}$$

and

$$b(y) = c_1 y + c_2 \quad \text{and} \quad d(y) = d\sqrt{y},$$

for $y \in \mathbb{R}^+$. It is assumed that c_1, c_2, d, μ and r are constants, such that $r \geq 0$ and $c_2, d > 0$. In addition, without loss of generality, we assume that $\mu > r$. In order to prevent the process Y_t, $t \geq 0$, from hitting zero, we also assume that $d^2 < 2c_2$. An additional assumption on c_2/d will be made in the sequel.

Under the above assumptions, the stock and the stochastic factor processes (cf. (1) and (2)) satisfy

$$dS_t = S_t \mu dt + S_t(\mu - r)\sqrt{Y_t} dW_t^1 \tag{69}$$

and

$$dY_t = (c_1 Y_y + c_2)dt + d\sqrt{Y_t}\big(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\big), \tag{70}$$

with $S_0, Y_0 > 0$. It is well known that the above system has a unique strong solution.

According to the methodology developed in Sect. 4, we perform the following change of variables in order to bring Eq. (27) in its canonical form. Specifically, in the notation of Sect. 4, we obtain

$$Z(y) = \frac{2\sqrt{2}}{d}\sqrt{y} \quad \text{and} \quad X(z) = Z^{-1}(z) = \frac{d^2}{8} z^2, \tag{71}$$

and introduce the function $g : \mathbb{R}^+ \times (0,\infty) \to \mathbb{R}^+$ given by

$$g(t,y) = \frac{1}{\sqrt{y}} \exp\left(\frac{c_1}{d^2}\left(y^2 \frac{d^2}{8} - 1\right) + C_2 \log\left(y^2 \frac{d^2}{8}\right)\right) v\left(\frac{\sigma^2}{8} y^2, t\right), \tag{72}$$

where v is as in (27), and the constants C_1 and C_2 are given by

$$C_1 = \left(\frac{c_2^2}{2d^2} + \frac{c_2 \rho\gamma}{d(1-\gamma)} + \frac{3d^2}{32} - \frac{c_2}{2} - \frac{\gamma}{2(1-\gamma)}(1 + d\rho)\right)\frac{8}{d^2}$$

and

$$C_2 = \frac{c_2}{d^2} + \frac{\rho\gamma}{d(1-\gamma)}.$$

We, also, conclude that g has to solve

$$g_t + g_{yy} + q(y)g = 0 \tag{73}$$

with initial condition

$$g(y,0) = \frac{1}{\sqrt{y}} \exp\left(\frac{c_1}{d^2}\left(y^2\frac{d^2}{8} - 1\right) + C_2 \log\left(y^2\frac{d^2}{8}\right)\right)\left(K\left(\frac{d^2}{8}y^2\right)\right)^{1/\delta}, \tag{74}$$

where the coefficient $q(y)$ is given by

$$q(y) = -\frac{c_1^2}{16}y^2 - C_1\frac{1}{y^2} - c_1 C_2,$$

We assume that c_2/d is large enough, so that $C_1 > -1/4$.

Elementary calculations yield that the functions $\psi^{(i)} : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, given by

$$\psi^{(1)}(y) = e^{y^2 c_1/8} y^{1/2+\sqrt{C_1+1/4}} \quad \text{and} \quad \psi^{(2)}(y) = e^{y^2 c_1/8} y^{1/2-\sqrt{C_1+1/4}}$$

satisfy the corresponding Sturm–Liouville equation,

$$\frac{\partial}{\partial y^2}\psi(\lambda, y) + \big(\lambda + q(y)\big)\psi(\lambda, y) = 0, \tag{75}$$

with respective values λ_1 and λ_2 given by

$$\lambda_1 = \frac{c_1 c_2}{d^2} + \frac{c_1\rho\gamma}{d(1-\gamma)} - \frac{c_1(1+\sqrt{C_1+1/4})}{2}$$

and

$$\lambda_2 = \frac{c_1 c_2}{d^2} + \frac{c_1\rho\gamma}{d(1-\gamma)} - \frac{c_1(1-\sqrt{C_1+1/4})}{2}.$$

Next, we choose the factor $K : (0,\infty) \to (0,\infty)$ as

$$\begin{aligned} K(y) = {} & \left(\frac{2\sqrt{2}}{d}\right)^{\delta} y^{\delta(\frac{1}{2}-C_2)} \exp\left(\frac{c_1\delta}{d^2}\right) \\ & \times \left(k_1\left(\frac{2\sqrt{2}}{d}\right)^{\sqrt{C_1+1/4}} y^{\sqrt{C_1+1/4}/2} \right. \\ & \left. + k_2\left(\frac{2\sqrt{2}}{d}\right)^{-\sqrt{C_1+1/4}} y^{-\sqrt{C_1+1/4}/2}\right)^{\delta}, \end{aligned} \tag{76}$$

for any constants $k_1, k_2 \in [0, \infty)$. Then, the solution to the linear equation (73) is given by

$$g(y,t) = \sqrt{y}e^{y^2 c_1/8}\big(k_1 y^{\sqrt{C_1+1/4}}e^{\lambda_1 t} + k_2 y^{-\sqrt{C_1+1/4}}e^{\lambda_2 t}\big).$$

Consequently, we deduce that v is given by

$$\begin{aligned} v(y,t) &= \frac{2\sqrt{2}}{d}\exp\left(\frac{c_1}{d^2}\right)y^{1/2-C_2} \\ &\quad \times \left(k_1\left(\frac{2\sqrt{2}}{d}\right)^{\sqrt{C_1+1/4}} y^{\sqrt{C_1+1/4}/2}e^{\lambda_1 t} \right. \\ &\quad \left. + k_2\left(\frac{2\sqrt{2}}{d}\right)^{-\sqrt{C_1+1/4}} y^{-\sqrt{C_1+1/4}/2}e^{\lambda_2 t}\right). \end{aligned} \tag{77}$$

Summarizing the above, we have the following result.

Proposition 8 *Assume that the stock and the stochastic factor solve* (69) *and* (70). *Also, assume that the aforementioned assumptions on the involved coefficients hold and that the distortion power δ is as in* (26).

Define the process $a(x,t)$ by

$$a(x,t) = \left(\frac{x^\gamma}{\gamma}\rho Z_t, \frac{x^\gamma}{\gamma}\sqrt{1-\rho^2}Z_t\right) \tag{78}$$

where

$$Z_t = d\delta\sqrt{Y_t}\frac{v_y(Y_t,t)}{v(Y_0,0)}\left(\frac{v(Y_t,t)}{v(Y_0,0)}\right)^{\delta-1} \tag{79}$$

with $v : (0,+\infty)\times[0,+\infty) \to \mathbb{R}^+$ given by (77) *above.*

Moreover, consider the initial condition $u_0 : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$u_0(x) = \frac{x^\gamma}{\gamma}.$$

Then, the process

$$U_t(x) = \frac{x^\gamma}{\gamma}\left(\frac{v(Y_t,t)}{v(Y_0,0)}\right)^\delta, \tag{80}$$

satisfies the SPDE (7) *with the above performance volatility process $a(x,t)$ and initial condition $U(x,0) = u_0(x)$.*

Next, we study the behavior of the forward investment performance process in (80) as its volatility process $a(x,t)$ vanishes. For this, we will send the parameter $d \to 0$. Notice, however, that in the present case, if none of k_1 or k_2 is equal to zero, the particular choice of their values will affect the forward performance process. Therefore, for the sake of simplicity we assume that $k_2 = 0$.

Proposition 9 *Let $U^{(d)}(x,t)$ be the forward investment performance process given in (80), with $k_2=0$. Then, for each $t>0$,*

(i) *the performance volatility process $a(x,t)$ (cf. (78)) satisfies a.s for all $x\geq 0$,*

$$\lim_{d\to 0} a(x,t)=0,$$

(ii) *the forward performance process satisfies a.s. for all $x\geq 0$*

$$\lim_{d\to 0} U^{(d)}(x,t)=\frac{x^\gamma}{\gamma}\left(\frac{Y_t^{(0)}}{Y_0}e^{-c_1 t}\right)^{\delta(1-\frac{1}{2c_2}(\frac{\rho\gamma}{1-\gamma})^2+\frac{1}{2c_2}\frac{\gamma}{1-\gamma})},$$

where $Y_t^{(0)}$ is the solution to the deterministic problem

$$dY_t^{(0)}=\left(c_1 Y_t^{(0)}+c_2\right)dt,$$

with $Y_0^{(0)}=Y_0$.

Proof First, we make use of the assumption $c_2>0$ to obtain that for small enough $d>0$ the following calculations are valid:

$$\begin{aligned}
A(d)&=\frac{1}{2}\sqrt{C_1+1/4}-C_2\\
&=C_2\left(\sqrt{1-\frac{(\frac{\rho\gamma}{1-\gamma})^2+\frac{d^2}{4}-c_2-\frac{\gamma}{1-\gamma}\rho d}{d^2C_2^2}}-1\right)\\
&=-\frac{(\frac{\rho\gamma}{1-\gamma})^2+\frac{d^2}{4}-c_2-\frac{\gamma(1+\rho d)}{1-\gamma}}{(c_2+\frac{\gamma}{1-\gamma}\rho d)\sqrt{1-\frac{(\frac{\rho\gamma}{1-\gamma})^2+\frac{d^2}{4}-c_2-\frac{\gamma}{1-\gamma}\rho d}{(\frac{c_2}{d}+\frac{\rho\gamma}{1-\gamma})^2}}}.
\end{aligned}$$

We, then, easily deduce that

$$\lim_{d\to 0} A(d)=\frac{1}{2}-\frac{1}{2c_2}\left(\frac{\rho\gamma}{1-\gamma}\right)^2+\frac{1}{2c_2}\frac{\gamma}{1-\gamma}.$$

Finally, we note that because $c_1>0$, we have

$$\lim_{d\to 0}\lambda_1(d)=-c_1+\frac{c_1}{2c_2}\left(\frac{\rho\gamma}{1-\gamma}\right)^2-\frac{c_1}{2c_2}\frac{\gamma}{1-\gamma},$$

and therefore,

$$\lim\frac{v^{(d)}(y,t)}{v^{(d)}(Y_0,0)}=\left(\frac{y}{Y_0}e^{-c_1 t}\right)^{1-\frac{1}{2c_2}(\frac{\rho\gamma}{1-\gamma})^2+\frac{1}{2c_2}\frac{\gamma}{1-\gamma}}.$$

Using standard results for the CIR process, we deduce that there exists a modification of the family of processes $\{(Y_t^{(d)})_{t\geq 0}\}$, solving (70) for each $d > 0$, such that a.s for any $t \geq 0$,

$$\lim_{d\to 0} Y_t^{(d)} = Y_t^{(0)} = \left(\frac{c_2}{c_1} + Y_0\right)e^{c_1 t} - \frac{c_2}{c_1}.$$

We easily conclude. □

References

1. Barrier, F., Rogers, L.C., Tehranchi, M.: 2009, A characterization of forward utility functions. Preprint. http://www.statslab.cam.ac.uk/~mike/papers/forward-utilities.pdf
2. Carr, P., Nadtochiy, S.: Static hedging under time-homogeneous diffusions. SIAM J. Financ. Math. **2**(1), 794–838 (2011)
3. El Karoui, N., M'Rad, M.: 2010, Stochastic utilities with a given optimal portfolio: approach by stochastic flows. Preprint. arXiv:1004.5192
4. Itô, K., McKean, H.P. Jr: Diffusion Processes and Their Sample Paths (Classics in Mathematics), 2nd edn. Springer, Berlin (1974)
5. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, Berlin (1998)
6. Merton, R.: Lifetime portfolio selection under uncertainty: the continuous-time case. Rev. Econ. Stat. **51**, 247–257 (1969)
7. Merton, R.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **3**, 373–413 (1971)
8. Musiela, M., Zariphopoulou, T.: Portfolio choice under dynamic investment performance criteria. Quant. Finance **9**, 161–170 (2009)
9. Musiela, M., Zariphopoulou, T.: Portfolio choice under space-time monotone performance criteria. SIAM J. Financ. Math. **1**, 326–365 (2010)
10. Musiela, M., Zariphopoulou, T.: Stochastic partial differential equations in portfolio choice. In: Chiarella, C., Novikov, A. (eds.) Contemporary Quantitative Finance, pp. 195–216. Springer, Berlin (2010)
11. Linetski, V., Davydov, D.: Pricing options on scalar diffusions: an eigenfunction expansion approach. Oper. Res. **51**(2), 185–209 (2003)
12. Nadtochiy, S., Tehranchi, M.: Optimal investment for all time horizons and Martin boundary of space-time diffusions (2013). arXiv:1308.2254
13. Nadtochiy, S., Zariphopoulou, T.: The SPDE for the forward investment performance process (2010). Work in progress
14. Titchmarsh, E.C.: In: Eigenfunction Expansions Associated with Second-Order Differential Equations, Clarendon, Oxford (1946)
15. Zariphopoulou, T.: A solution approach to valuation of unhedgeable risks. Finance Stoch. **5**, 61–82 (2001)
16. Zariphopoulou, T.: Optimal asset allocation in a stochastic factor model—an overview and open problems. Adv. Financ. Model. Radon Ser. Comput. Appl. Math. **8**, 427–453 (2009)
17. Zitkovic, G.: A dual characterization of self-generation and exponential forward performances. Ann. Appl. Probab. **19**(6), 2176–2210 (2008)
18. Widder, D.V.: The Heat Equation. Academic Press, San Diego (1975)