Stochastic modeling and methods in optimal portfolio construction

Dedicated to my father George Zariphopoulos (1930-2014)

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Abstract. Optimal portfolio construction is one of the most fundamental problems in financial mathematics. The foundations of investment theory are discussed together with modeling issues and various methods for the analysis of the associated stochastic optimization problems. Among others, the classical expected utility and its robust extension are presented as well as the recently developed approach of forward investment performance. The mathematical tools come from stochastic optimization for controlled diffusions, duality and stochastic partial differential equations. Connections between the academic research and the investment practice are also discussed and, in particular, the challenges of reconciling normative and descriptive approaches.

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1. Introduction

Financial mathematics is a burgeoning area of research on the crossroads of stochastic processes, stochastic analysis, optimization, partial differential equations, finance, econometrics, statistics and financial economics. There are two main directions in the field related, respectively, to the so-called *sell* and *buy* sides of financial markets. The former deals with derivative valuation, hedging and risk management while the latter with investments and fund management.

Derivatives are financial contracts written on primary financial assets. Their development started in the late 1970s with the revolutionary idea of Black, Merton and Scholes of pricing via "perfect replication" of the derivatives' payoffs. Subsequently, the universal theory of arbitrage-free valuation, developed by Kreps, and Harrison and Pliska, was built on a surprising fit between stochastic calculus and quantitative needs. It revolutionized the derivatives industry, but its impact did not stop there. Because the theory provided a model-free approach to price and manage risks, the option pricing methodology has been applied in an array of applications, among others, corporate and non-corporate agreements, pension funds, government loan guarantees and insurance plans. In a different direction, applications of the theory resulted in a substantial growth of the fields of real options and decision analysis. Complex issues related, for example, to operational efficiency, financial flexibility,

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contracting, and initiation and execution of R&D projects were revisited and analyzed using derivative valuation arguments. For the last three decades, the theoretical developments, technological advances, modeling innovations and creation of new derivatives products have been developing at a remarkable rate. The recent financial crisis cast a lot of blame upon derivatives and quantitative methods and, more generally, upon financial mathematics. Despite the heated debate on what went really wrong, the theory of derivatives remains one of the best examples of a perfect match among mathematical innovation, technological sophistication and direct real world applicability.

In the complementary side of finance practice, investments deal with capital allocation under market uncertainty. The objective is not to eliminate the inherent market risks - as it is the case with derivatives - but to exploit optimally the market opportunities while undertaking such risks. The overall goal is to assess the trade-off between risks and payoffs. For this, one needs to have, from the one hand, models that predict satisfactorily future asset prices and, from the other, mechanisms that measure in a practically meaningful way the performance of investment strategies. There are great challenges in both these directions. Estimating the drift of stock prices is a notoriously difficult problem. Moreover, building appropriate investment criteria that reflect the investors' attitude is extremely complex, for these criteria need to capture an array of human sentiments like risk aversion, loss aversion, ambiguity, prudence, impatience, etc.. There is extensive academic work, based on the foundational concept of expected utility, that examines such issues. However, there is still a considerable gap between academic developments and investment practice, and between normative and descriptive investment theories. In many ways, we have not yet experienced the unprecedented progress that took place in the 1980s and 1990s when academia and the derivatives industry challenged and worked by each other, leading to outstanding scientific progress in financial mathematics and quantitative finance.

The aim of this paper is to describe the main academic developments in portfolio management, discuss modeling issues, present various methods and expose some of the current challenges that the investment research faces.

2. Model certainty and investment management

Models of optimal investment management give rise to stochastic optimization problems with expected payoffs. There are three main ingredients in their specification: the model for the stochastic market environment, the investment horizon and the optimality criterion.

The market consists of assets whose prices are modelled as stochastic processes in an underlying probability space. The associated measure is known as the real, or historical, measure $\mathbb{P}$. Popular paradigms of prices are diffusion processes (2.2), (2.3), Itô processes (2.11) and, more generally, semimartingales (sections 3.1 and 3.2). When the price model is known we say that there is no model uncertainty.

The trading horizon is the time during which trading takes place, typically taken to have deterministic finite length. Depending on the application, the horizon can be infinitesimal (high frequency trading), short (hedge funds), medium (mutual funds) or long (pension funds). Models of infinite horizon have been also considered, especially when intermediate consumption is incorporated or when the criterion is asymptotic, like optimal long-term growth, risk-sensitive payoff and others.

The optimality criterion is built upon the utility function, a concept measuring risk and uncertainty that dates back to D. Bernoulli (1738). He was the first to argue that utility should not be proportional to wealth but, rather, have decreasing marginal returns, thus, alluding for the first time to its concavity property. Bernoulli's pioneering ideas were rejected at that time and it took close to two centuries (with the exception of the work of Gossen) to be recognized. In 1936, Alt and few years later von Neumann and Morgenstern proposed the axiomatic foundation of expected utility and argued that the behavior of a rational investor must coincide with that of an individual who values random payoffs using an expected utility criterion. This normative work was further developed by Friedman and Savage, Pratt and Arrow. In the latter works, the quantification of individual aversion to risk - via the so called risk aversion coefficient - was proposed and few years later, Markowitz developed the influential "mean-variance" portfolio theory. In 1969, Merton built a continuous-time portfolio management model of expected utility for log-normal stock prices, and since then the academic literature in this area has seen substantial growth. We refer the reader to the review article [70] for further details and references.

The expected utility criterion enables us to quantify and rank the outcomes of investments policies π by mapping the wealth X_T^π they generate to its expected utility,

$$X_T^\pi \to E_{\mathbb{P}}\left(U\left(X_T^\pi\right)\right), \tag{2.1}$$

where $\mathbb{P}$ is the aforementioned historical measure and U a deterministic function that is smooth, strictly increasing and strictly concave, and satisfies appropriate asymptotic properties. The objective is then to *maximize* $E_{\mathbb{P}}\left(U\left(X_T^\pi\right)\right)$ over all admissible portfolios. The portfolios are the amounts (or proportions of current wealth) that are dynamically allocated to the different accounts. They are stochastic processes on their own and might satisfy (control) constraints, as it is discussed below.

There are two main directions in studying optimal portfolio problems. Under Markovian assumptions for the asset price processes, the value function is analyzed via PDE and stochastic control arguments applied to the associated Hamilton-Jacobi-Bellman (HJB) equation. We discuss this direction in detail next. For more general market settings, the powerful theory of duality is used. This approach yields elegant results for the value function and the optimal wealth. The optimal portfolios can be then characterized via martingale representation results for the optimal wealth process (see, among others, [27, 28, 30, 31, 55, 56]). We discuss the duality approach in sections 3.1 and 3.2 herein.

2.1. A diffusion market model and its classical (backward) expected utility criterion.

We consider the popular paradigm in which trading takes place between a riskless asset (bond) and a risky one (stock). The stock price is modelled as a diffusion process whose coefficients depend on a correlated stochastic factor. Stochastic factors have been used in a number of academic papers to model the time-varying predictability of stock returns, the volatility of stocks as well as stochastic interest rates (for an extended bibliography, see the review article [67]).

From the technical point of view, a stochastic factor model is the simplest and most direct extension of the celebrated Merton model in which stock dynamics are taken to be log-normal (see [40]). However, as it is discussed herein, relatively little is known about the regularity of the value function, and the form and properties of the optimal policies once the log-normality assumption is relaxed and correlation between the stock and the factor is introduced. This is despite the Markovian nature of the problem at hand, the advances in the theories of fully nonlinear PDE and stochastic optimization of controlled diffusion processes, as well as the available computational tools.

Specifically, complete results on the validity of the Dynamic Programming Principle, smoothness of the value function, existence and verification of optimal feedback controls, representation of the value function and numerical approximations are still lacking. The only cases that have been extensively analyzed are the ones of homothetic utilities (exponential, power and logarithmic). In these cases, convenient scaling properties reduce the HJB equation to a quasilinear one (even linear, see (2.9)). The analysis, then, simplifies considerably both from the analytic as well as the probabilistic points of view (see, for example, [52] and [66]).

The lack of rigorous results for the regularity and other properties of the value function, when the utility function is general, limits our understanding of the structure of the optimal policies. Informally speaking, the first-order conditions in the HJB equation yield that the optimal feedback portfolio consists of two components (see (2.7)). The first is the so-called *myopic portfolio* and has the same functional form as the one in the classical Merton problem. The second component, usually referred to as the *excess hedging demand*, is generated by the stochastic factor. Conceptually, very little is understood about this term. In addition, the sum of the two components may become zero which implies that it is optimal for a risk averse investor not to invest in a risky asset with positive risk premium. A satisfactory explanation for this counter intuitive phenomenon - related to the so-called market participation puzzle - is also lacking.

We continue with the description of the market model. The stock price S_t, $t \geq 0$, is modelled as a diffusion process solving

$$dS_t = \mu\left(Y_t\right) S_t dt + \sigma\left(Y_t\right) S_t dW_t^1, \tag{2.2}$$

with $S_0 > 0$. The stochastic factor Y_t, $t \geq 0$, satisfies

$$dY_t = b\left(Y_t\right) dt + d\left(Y_t\right)\left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right), \tag{2.3}$$

with $Y_0 = y$, $y \in \mathbb{R}$. The process $W_t = \left(W_t^1, W_t^2\right)$, $t \geq 0$, is a standard $2-$dim Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The underlying filtration is $\mathcal{F}_t = \sigma\left(W_s :\ 0 \leq s \leq t\right)$, and it is assumed that $\rho \in (-1, 1)$. The market coefficients $f = \mu, \sigma, b$ and d satisfy global Lipschitz and linear growth conditions and the non-degeneracy condition $\sigma\left(y\right) \geq l > 0$, $y \in \mathbb{R}$. The riskless asset offers constant interest rate $r > 0$.

Starting with an initial endowment x, at time $t \in [0, T)$, the investor invests at future times $s \in (t, T]$ in the riskless and risky assets. The present value of the amounts allocated in the two accounts are denoted, respectively, by π_s^0 and π_s. The investor's (discounted) wealth is, then, given by $X_s^\pi = \pi_s^0 + \pi_s$. It follows that it satisfies $dX_s^\pi = \sigma\left(Y_s\right) \pi_s \left(\lambda\left(Y_s\right) ds + dW_s^1\right)$, where $\lambda\left(Y_s\right) = \frac{\mu(Y_s) - r}{\sigma(Y_s)}$.

A portfolio, π_s, is admissible if it is self-financing, $\mathcal{F}_s$-adapted, $E_{\mathbb{P}}\left(\int_t^T \sigma^2\left(Y_s\right) \pi_s^2 ds\right) < \infty$ and the associated discounted wealth satisfies the state constraint $X_s^\pi \geq 0$, $\mathbb{P}-$a.s. We denote the set of admissible strategies by $\mathcal{A}$.

Frequently, portfolio constraints are also present which further complicate the analysis. Notable cases are the so-called *drawdown* constraints, for which $X_t^\pi \geq \alpha \max_{0 \leq s \leq t} X_s^\pi$ with $\alpha \in (0, 1)$, *leverage* constraints, when $|\pi_t| \leq g\left(X_t^\pi\right)$ for an admissible function g, and *stochastic target* constraints, for which $X_T^\pi \geq Z_T$ for a random level Z_T.

The objective, known as the *value function* (or indirect utility), is formulated as

$$V(x,y,t;T)=\sup_{\mathcal{A}} E_{\mathbb{P}}\left(U\left(X_T^{\pi}\right)\middle|\,\mathcal{F}_t, X_t^{\pi}=x, Y_t=y\right), \tag{2.4}$$

for $(x,y,t)\in\mathbb{R}_+\times\mathbb{R}\times[0,T]$. The utility function $U:\mathbb{R}_+\to\mathbb{R}$ is $C^4(\mathbb{R}_+)$, strictly increasing and strictly concave, and satisfies certain asymptotic properties (see, among others, [55] and [56]). As solution of a stochastic optimization problem, the value function is expected to satisfy the Dynamic Programming Principle (DPP), namely,

$$V(x,y,t)=\sup_{\mathcal{A}} E_{\mathbb{P}}\left(V\left(X_s^{\pi},Y_s,s\right)\middle|\,\mathcal{F}_t, X_t^{\pi}=x, Y_t=y\right), \tag{2.5}$$

for $s\in[t,T]$. This is a fundamental result in optimal control and has been proved for a wide class of optimization problems. For a detailed discussion on the validity (and strongest forms) of the DPP in problems with controlled diffusions, we refer the reader to [18] (see, also [6, 8, 14, 33, 35, 65]). Key issues are the measurability and continuity of the value function process as well as the compactness of the set of admissible controls. A weak version of the DPP was proposed in [9] where conditions related to measurable selection and boundness of controls are relaxed. Related results for the case of bounded payoffs can be found in [3] and, more recently, new results appeared in [71].

Besides its technical challenges, the DPP exhibits two important properties of the value function process. Specifically, the process $V(X_s^{\pi},Y_s,s)$, $s\in[t,T]$, is a *supermartingale* for an arbitrary admissible investment strategy and becomes a *martingale* at an optimum (provided certain integrability conditions hold). Moreover, observe that the DPP yields a backward in time algorithm for the computation of the maximal expected utility, starting at expiration with U and using the martingality property to compute the solution conditionally for earlier times. For this, we occasionally refer to the classical problem as the *backward* one.

The Markovian assumptions on the stock price and stochastic factor dynamics allow us to study the value function via the associated HJB equation, stated in (2.6) below. Fundamental results in the theory of controlled diffusions yield that if the value function is smooth enough then it satisfies the HJB equation. Moreover, optimal policies may be constructed in a feedback form from the first-order conditions in the HJB equation, provided that the candidate feedback process is admissible and the wealth SDE has a strong solution when the candidate control is used. The latter usually requires further regularity on the value function. In the reverse direction, a smooth solution of the HJB equation that satisfies the appropriate terminal and boundary (or growth) conditions may be identified with the value function, provided the solution is unique in the appropriate sense. These results are usually known as the "verification theorem" and we refer the reader to [6, 8, 14, 33, 35, 65] for a general exposition on the subject.

In maximal expected utility problems, it is rarely the case that the arguments in either direction of the verification theorem can be established. Indeed, it is difficult to show a priori regularity of the value function, with the main difficulties coming from the lack of global Lipschitz regularity of the coefficients of the controlled process with respect to the controls and from the non-compactness of the set of admissible policies. It is, also, difficult to establish existence, uniqueness and regularity of the solutions to the HJB equation. This is caused primarily by the presence of the control policy in the volatility of the controlled wealth process which makes the classical assumptions of global Lipschitz conditions of the equation with regards to the non linearities to fail. Additional difficulties come from state constraints and the non-compactness of the set of admissible portfolios.

Regularity results for the value function (2.4) for general utility functions have not been obtained to date except, as mentioned earlier, for the special cases of homothetic preferences. The most general result in this direction, and in a much more general market model, was obtained using duality methods in [32] where it is shown that the value function is twice differentiable in the spatial argument but without establishing the continuity of the derivative. Because of lack of general rigorous results, we proceed with an informal discussion about the optimal feedback policies. For the model at hand, the associated HJB equation is

$$V_t + \max_{\pi}\left(\frac{1}{2}\sigma^2\left(y\right)\pi^2 V_{xx} + \pi\left(\mu\left(y\right)V_x + \rho\sigma\left(y\right)d\left(y\right)V_{xy}\right)\right) \tag{2.6}$$
$$+\frac{1}{2}d^2\left(y\right)V_{yy} + b\left(y\right)V_y = 0,$$

with $V\left(x,y,T\right)=U\left(x\right),\ \left(x,y,t\right)\in\mathbb{R}_+\times\mathbb{R}\times\left[0,T\right]$. The verification results would yield that under appropriate regularity and growth conditions, the feedback policy $\pi_s^* = \pi^*\left(X_s^*, Y_s, s\right)$, $s\in\left(t,T\right]$, with

$$\pi^*\left(x,y,t\right) = -\frac{\lambda\left(y\right)}{\sigma\left(y\right)}\frac{V_x\left(x,y,t\right)}{V_{xx}\left(x,y,t\right)} - \rho\frac{d\left(y\right)}{\sigma\left(y\right)}\frac{V_{xy}\left(x,y,t\right)}{V_{xx}\left(x,y,t\right)}, \tag{2.7}$$

and $X_s^{\pi^*}$ solving $dX_s^{\pi^*} = \sigma\left(Y_s\right)\pi\left(X_s^{\pi^*}, Y_s, s\right)\left(\lambda\left(Y_s\right)ds + dW_s^1\right)$, is admissible and optimal.

Some answers to the questions related to the characterization of the solutions to the HJB equation may be given if one relaxes the requirement to have classical solutions. An appropriate class of weak solutions turns out to be the so called *viscosity solutions* ([11, 35, 36, 61]). Results related to the value function being the unique viscosity solution of (2.6) are rather limited. Recently, it was shown in [50] that the partial $V_x\left(x,y,t\right)$ is the unique viscosity solution of the marginal HJB equation. Other results, applicable for non-compact controls but for bounded payoffs, can be found in [3].

A key property of viscosity solutions is their robustness (see [36]). If the HJB has a unique viscosity solution (in the appropriate class), robustness is used to establish convergence of numerical schemes for the value function and the optimal feedback laws. Such numerical studies have been carried out successfully for a number of applications. However, for the model at hand, no such studies are available. Numerical results using Monte Carlo techniques have been obtained in [12] for a model more general than the one herein. More recently, the authors in [50] proposed a Trotter-Kato approximation scheme for the value function and an algorithm on how to construct $\varepsilon-$optimal portfolio policies.

Important questions arise on the dependence, sensitivity and robustness of the optimal feedback portfolio, especially of the excess hedging demand term, in terms of the market parameters, the wealth, the level of the stochastic factor and the risk preferences. Such questions are central in financial economics and have been studied, primarily in simpler models in which intermediate consumption is also incorporated. Recent results for more general models can be found, for example, in [34]. For diffusion models with a perfectly correlated stochastic factor, qualitative results can be found, among others, in [29] and [62] and for log-normal models in [7, 25, 42, 64]. However, a qualitative study for general utility functions and/or arbitrary factor dynamics has not been carried out to date. Another open question, which is more closely related to applications, is how one could infer the investor's risk preferences from her investment targets. This is a difficult inverse problem and has been partially addressed in [41] and [45].

Example 2.1. A commonly used utility function is the homothetic $U(x) = \frac{x^\gamma}{\gamma}$, $x \geq 0$, $\gamma \in (0,1)$. In this case, the value function is given by (see [66])

$$V(x,y,t) = \frac{x^\gamma}{\gamma}\left(F(y,t)\right)^\delta \tag{2.8}$$

where $\delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$ and F solves the linear equation

$$F_t + \frac{1}{2}d^2(y)F_{yy} + \left(b(y) + \rho\frac{\gamma}{1-\gamma}\lambda(y)a(y)\right)F_y + \frac{1}{2}\frac{\gamma}{(1-\gamma)\delta}\lambda^2(y)F = 0, \tag{2.9}$$

with $F(y,T) = 1$. The Feynman-Kac formula then yields the probabilistic representation

$$V(x,y,t) = \frac{x^\gamma}{\gamma}\left(E_{\bar{\mathbb{P}}}\left(e^{\int_t^T \frac{1}{2}\frac{\gamma}{(1-\gamma)\delta}\lambda^2\left(\bar{Y}_s\right)ds}\middle|\,\bar{Y}_t = y\right)\right)^\delta \tag{2.10}$$

where $\bar{Y}_t$, $t \in [0,T]$, solves $d\bar{Y}_t = (b\left(\bar{Y}_t\right) + \rho\frac{\gamma}{1-\gamma}\lambda\left(\bar{Y}_t\right)a\left(\bar{Y}_t\right))dt + d\left(\bar{Y}_t\right)dW_t^{\bar{\mathbb{P}}}$, with $W^{\bar{\mathbb{P}}}$ being a standard Brownian motion under a measure $\bar{\mathbb{P}}$.

2.2. An Itô market model and its forward performance criterion. Besides the difficulties discussed earlier, there are other issues that limit the development of a flexible enough optimal investment theory in complex market environments. One of them is the "static" choice of the utility function at the specific investment horizon. Indeed, once the utility function is a priori specified, no revision of risk preferences is possible at any intermediate trading time. In addition, once the horizon is chosen, no investment performance criteria can be formulated for horizons longer than the initial one. As a result, extending the investment horizon (due to new incoming investment opportunities, change of risk attitude, unpredictable price shocks, etc.) is not possible.

Addressing these limitations has been the subject of a number of studies and various approaches have been proposed. With regards to the horizon length, the most popular alternative has been the formulation of the investment problem in $[0,\infty)$ and either incorporating intermediate consumption or optimizing the investor's long-term optimal behavior. Investment modes with random horizon have been also considered, and the revision of risk preferences has been partially addressed by recursive utilities (see, for example, [13] and [59]).

An alternative approach which addresses both shortcomings of the expected utility approach has been proposed recently by the author and Musiela (see, [43–45]). The associated criterion, the so called *forward performance process,* is developed in terms of a family of utility fields defined on $[0,\infty)$ and indexed by the wealth argument. Its key properties are the (local) martingality at an optimum and (local) supermartingality away from it. These are in accordance with the analogous properties of the classical value function process, we discussed earlier, which stem out from the Dynamic Programming Principle (cf. (2.5)). Intuitively, the average value of an optimal strategy at any future date, conditional on today's information, preserves the performance of this strategy up until today. Any strategy that fails to maintain the average performance over time is, then, sub-optimal. We refer the reader to

[44] and [45] for further discussion on this new concept and its connection with the classical expected utility theory.

Next, we introduce the definition of the forward performance process and present old and more recent results. The market environment consists of one riskless security and k stocks. For $i = 1, ..., k$, the stock price S_t^i, $t > 0$, is an Itô process solving

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t^j \right) \tag{2.11}$$

with $S_0^i > 0$. The process $W_t = \left(W_t^1, ..., W_t^k \right)$ is a standard $d-$dim Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}_t = \sigma\left(W_s : 0 \le s \le t \right)$. The coefficients μ_t^i and σ_t^i, $i = 1, ..., k$, are $\mathcal{F}_t-$adapted processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. For brevity, we denote by σ_t the volatility matrix, i.e., the $d \times k$ random matrix $\left(\sigma_t^{ji} \right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. The riskless asset has the price process B satisfying $dB_t = r_t B_t dt$ with $B_0 = 1$, and a nonnegative $\mathcal{F}_t-$adapted interest rate process r_t. Also, we denote by μ_t the $k \times 1$ vector with coordinates μ_t^i. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions and it is further assumed that $\left(\mu_t - r_t \mathbf{1} \right) \in Lin \left(\sigma_t^T \right)$.

The market price of risk is given by the vector $\lambda_t = \left(\sigma_t^T \right)^+ \left(\mu_t - r_t \mathbf{1} \right)$, where $\left(\sigma_t^T \right)^+$ is the Moore-Penrose pseudo-inverse of σ_t^T. It is assumed that, for all $t > 0$, $E_{\mathbb{P}} \int_0^t \left| \lambda_s \right|^2 ds < \infty$.

Starting at $t = 0$ with an initial endowment $x \in \mathbb{D}$, $\mathbb{D} \subseteq [-\infty, \infty]$, the investor invests dynamically among the assets. The (discounted) value of the amounts invested are denoted by π_t^0 and π_t^i, $i = 1, ..., k$, respectively. The (discounted) wealth process is, then, given by $X_t^\pi = \sum_{i=0}^k \pi_t^i$, and satisfies

$$dX_t^\pi = \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot \left(\lambda_t dt + dW_t \right) = \sigma_t \pi_t \cdot \left(\lambda_t dt + dW_t \right), \tag{2.12}$$

where the (column) vector, $\pi_t = \left(\pi_t^i;\ i = 1, ..., k \right)$. The admissibility set, $\mathcal{A}$, consists of self-financing $\mathcal{F}_t-$adapted processes π_t such that $E_{\mathbb{P}} \int_0^t \left| \sigma_s \pi_s \right|^2 ds < \infty$, and $X_t^\pi \in \mathbb{D}$, for $t \ge 0$.

The initial datum is taken to be a strictly concave and strictly increasing function of wealth, $u_0 : \mathbb{D} \to \mathbb{R}$ with $u_0 \in \mathcal{C}^4(\mathbb{D})$. The specification of admissible initial conditions deserves special attention and is discussed later (see (2.20)).

Next, we present the definition of the forward performance process. The one below is a relaxed version of the original definition, given in [44], where stronger integrability conditions were needed.

Definition 2.2. An $\mathcal{F}_t-$adapted process $U(x, t)$ is a local forward performance process if for $t \ge 0$ and $x \in \mathbb{D}$:

i) the mapping $x \to U(x, t)$ is strictly concave and strictly increasing,

ii) for each $\pi \in \mathcal{A}$, the process $U\left(X_t^\pi, t \right)$ is a local supermartingale, and

iii) there exists $\pi^* \in \mathcal{A}$ such that the process $U\left(X_t^{\pi^*}, t \right)$ is a local martingale.

Variations of the above definition have appeared, among others, in [15] and [49]. In [69], the alternative terminology "*self-generating*"was introduced, for the forward performance satisfies, for *all* $0 \le t \le s$,

$$U(x,t) = ess \sup_{\mathcal{A}} E_{\mathbb{P}} \left(U\left(X_s^\pi, s\right) \middle| \mathcal{F}_t,\ X_t^\pi = x \right). \tag{2.13}$$

Note that in the classical (backward) case $(0 \le t \le s \le T)$ the above property is a direct *consequence* of the DPP. In the forward framework, however, it *defines* the forward performance process. Clearly, if for the backward problem with finite horizon T one uses as terminal utility $U_T(x) = U(x,T)$, the backward and the forward problems coincide on $[0,T]$.

The axiomatic construction of forward performance is an *open* problem, and results have been derived only for the exponential case (see [69]). More recently, the authors in [49] proposed a class of forward performances processes that are deterministic functions of underlying stochastic factors (see, for example, (2.24) herein).

2.2.1. Stochastic PDE for the forward performance process. In [46] a stochastic PDE was derived as a sufficient condition for a process to be a forward performance. In many aspects, the forward SPDE is the analogue of the HJB equation that appears in the classical theory of stochastic optimization.

Proposition 2.3.

i) *Let $U(x,t)$, $(x,t) \in \mathbb{D} \times [0,\infty)$, be an $\mathcal{F}_t$–adapted process such that the mapping $x \to U(x,t)$ is strictly concave, strictly increasing and smooth enough so that the Itô-Ventzell formula can be applied to $U(X_t^\pi, t)$, for any strategy $\pi \in \mathcal{A}$. Let us, also, assume that the process $U(x,t)$ satisfies*

$$dU(x,t) = \frac{1}{2} \frac{\left| U_x(x,t)\lambda_t + \sigma_t \sigma_t^+ a_x(x,t) \right|^2}{U_{xx}(x,t)} dt + a(x,t) \cdot dW_t, \tag{2.14}$$

where the volatility $a(x,t)$ is an $\mathcal{F}_t$–adapted, d–dimensional and continuously differentiable in the spatial argument process. Then, $U(X_t^\pi, t)$ is a local supermartingale for every admissible portfolio strategy π.

ii) *Assume that the stochastic differential equation*

$$dX_t = -\frac{U_x(X_t,t)\lambda_t + \sigma_t \sigma_t^+ a_x(X_t,t)}{U_{xx}(X_t,t)} \cdot (\lambda_t dt + dW_t)$$

has a solution X_t, with $X_0 = x$, and $X_t \in \mathbb{D}$, $t \ge 0$, and that the strategy π_t^*, $t \ge 0$, defined by

$$\pi_t^* = -\sigma_t^+ \frac{U_x(X_t,t)\lambda_t + a_x(X_t,t)}{U_{xx}(X_t,t)}$$

is admissible. Then, X_t corresponds to the wealth generated by this investment strategy, that is $X_t = X_t^{\pi^*}$, $t > 0$. The process $U\left(X_t^{\pi^*}, t\right)$ is a local martingale and, hence, $U(x,t)$ is a local forward performance value process. The process π_t^* is optimal.

An important ingredient of the forward SPDE is the *forward volatility process* $a(x,t)$. This is a novel model input that is up to the investor to choose, in contrast to the classical value function process whose volatility process is uniquely determined from its Itô decomposition. In general, the forward volatility may depend explicitly on t, x, U and its derivatives, as it is, for instance, shown in the examples below. More general dependencies and admissible volatility representations have been proposed in [15].

The initial condition $u_0(x)$ is an additional model input. In contrast to the classical framework where the class of admissible (terminal) utilities is rather large, the family of admissible forward initial data can be rather restricted.

The analysis of the forward performance SPDE (2.14) is a formidable task. The reasons are threefold. Firstly, it is not only degenerate and fully nonlinear but is, also, formulated forward in time, which might lead to "ill-posed" behavior. Secondly, one needs to specify the appropriate class of admissible volatility processes, namely, volatility inputs that generate strictly concave and strictly increasing solutions of (2.14). The volatility specification is quite difficult both from the modelling and the technical points of view. Thirdly, as mentioned earlier, one also needs to specify the appropriate class of initial conditions $u_0(x)$. As it has been shown in [45] and discussed in the sequel, even the simple case of zero volatility poses a number of challenges.

Addressing these issues is an ongoing research effort of several authors; see, among others, in [4, 15, 16, 46, 49] and [51].

2.2.2. The time-monotone case and its variants. A fundamental class of forward performance processes are the ones that correspond to non-volatile criteria, $a(x,t) \equiv 0$, $t \geq 0$. The forward performance SPDE (2.14) simplifies to

$$dU(x,t) = \frac{1}{2}|\lambda_t|^2 \frac{U_x^2(x,t)}{U_{xx}(x,t)}dt, \tag{2.15}$$

and, thus, its solutions are processes of finite variation. In particular, they are decreasing in time, as it follows from the strict concavity requirement. The analysis of these processes was carried out in [45], and we highlight the main results next.

There are three functions that play pivotal role in the construction of the forward performance process, as well as of the optimal wealth and optimal portfolio processes. The first function is $u : \mathbb{D}\times[0,\infty) \to \mathbb{R}$, with $u \in C^{4,1}(\mathbb{D}\times[0,\infty))$, solving the HJB type equation

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}, \tag{2.16}$$

and satisfying an admissible initial condition, $U(x,0) = u_0(x)$ (see (2.20)).

The second function is the so-called local absolute risk tolerance $r : \mathbb{D}\times[0,\infty) \to \mathbb{R}_+$, defined by $r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$. It solves an autonomous fast-diffusion type equation, $r_t + \frac{1}{2}r^2 r_{xx} = 0$, with $r(x,0) = -\frac{u_0'(x)}{u_0''(x,t)}$.

The third is an increasing space-time harmonic function, $h : \mathbb{R}\times[0,\infty) \to \mathbb{D}$, defined via a Legendre-Fenchel type transformation

$$u_x(h(x,t),t) = e^{-x+\frac{1}{2}t}. \tag{2.17}$$

It solves the (backward) heat equation

$$h_t + \frac{1}{2}h_{xx} = 0, \tag{2.18}$$

with initial condition $h(x,0) = \left(u_0'\right)^{(-1)}(e^{-x})$.

Using the classical results of Widder (see [63]) for the representation of positive solutions[1] of (2.18), it follows that $h(x,t)$ must be given in the integral form

$$h(x,t) = \int_S \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy), \tag{2.19}$$

where ν is a positive, finite, Borel measure with support $S \in [-\infty, \infty]$. Detailed analysis on the interplay among the support S, the range of h, the structure and the asymptotic properties of u can be found in [45]. It was also shown therein that there is a one-to-one correspondence between such solutions of (2.18) to strictly increasing and strictly concave solutions of (2.16) (see, Propositions 9, 13 and 14).

One then sees that the measure ν becomes the defining element in the entire construction, for it determines the function h and, in turn, u and r. How this measure could be extracted from various distributional investment targets is an interesting question and has been discussed in [41] and [45].

We also see that the definition (cf. (2.17)) of the auxiliary function h and its structural representation (2.19) dictate that the initial utility $u_0(x)$, $x \in \mathbb{D}$, is given by

$$(u_0')^{(-1)}(x) = \int_S \frac{e^{-y \ln x} - 1}{y} \nu(dy). \tag{2.20}$$

In other words, only utilities whose inverse marginals have the above form can serve as initial conditions. Characterizing the set of *admissible initial data* $u_0(x)$ for general volatile performance criteria and, moreover, provide an intuitively meaningful financial interpretation for them is an interesting open question.

We summarize the general results next. As (2.21) and (2.22) below show, one obtains rather *explicit* stochastic representations of the optimal wealth and portfolio policies, despite the ill-posedness of the underlying problem, the complexity of the price dynamics, and the path-dependence nature of all quantities involved.

Proposition 2.4. *Let $u : \mathbb{D} \times [0,\infty) \to \mathbb{R}$ be a strictly increasing and strictly concave solution of (2.16), satisfying an admissible initial condition $u(x,0) = u_0(x)$, and $r(x,t)$ be its local absolute risk tolerance function. Let also $h : \mathbb{R} \times [0,\infty) \to \mathbb{D}$ be the associated harmonic function (cf. (2.17)). Define the market-input processes A_t and M_t, $t \geq 0$, as*

$$M_t = \int_0^t \lambda_s \cdot dW_s \quad \text{and} \quad A_t = \langle M \rangle_t = \int_0^t |\lambda_s|^2 \, ds.$$

Then, the process $U(x,t) = u(x, A_t)$, $t \geq 0$, is a forward performance. Moreover, the optimal portfolio process is given by

$$\pi_t^{*,x} = r\left(X_t^{\pi^*}, A_t\right) \sigma_t^+ \lambda_t = h_x\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right) \sigma^+ \lambda. \tag{2.21}$$

The optimal wealth process $X_t^{\pi^*}$ solves $dX_t^{\pi^*} = \sigma_t \sigma_t^+ \lambda_t r\left(X_t^{\pi^*}, A_t\right) \cdot (\lambda_t dt + dW_t)$ with $X_0^{\pi^*} = x$, and is given by

$$X_t^{\pi^*} = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right). \tag{2.22}$$

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[1] Widder's results are not applied to $h(x,t)$ directly, for it might not be positive, but to its space derivative $h_x(x,t)$.

Representations (2.21) and (2.22) enable us to study the optimal processes in more detail. Among others, one can draw analogies between option prices and their sensitivities (gamma, delta and other "greeks") and study analogous quantities for the optimal investments. Moreover, one can study the distribution of hitting times of the optimal wealth, calculate its moments, running maximum, Value at Risk, expected shortfall and other investment performance markers.

Example 2.5.

i) Let $\mathbb{D} = \mathbb{R}$ and $\nu = \delta_0$, where δ_0 is a Dirac measure at 0. Then, $h(x,t) = x$ and $u(x,t) = 1 - e^{-x+\frac{t}{2}}$. The forward performance process is, for $t \geq 0$, $U(x,t) = 1 - e^{-x+\frac{A_t}{2}}$ (see [43] and [69]).

ii) Let $\mathbb{D} = \mathbb{R}_+$ and $\nu = \delta_\gamma$, $\gamma > 1$. Then $h(x,t) = \frac{1}{\gamma} e^{\gamma x - \frac{1}{2}\gamma^2 t}$. Since $\nu((0,1]) = 0$, it turns out that $u(x,t) = kx^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}t}$, $k = \frac{1}{\gamma-1}\gamma^{\frac{\gamma-1}{\gamma}}$. The forward performance process is, for $t \geq 0$,

$$U(x,t) = kx^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{2}A_t}. \tag{2.23}$$

There exist two interesting *variants* of the time-monotone forward performance process, which correspond to *non-zero* volatility processes. To this end, consider the auxiliary processes Y_t, Z_t, $t \geq 0$, solving

$$dY_t = Y_t \delta_t \cdot (\lambda_t dt + dW_t) \quad \text{and} \quad dZ_t = Z_t \varphi_t \cdot dW_t,$$

with $Y_0 = Z_0 = 1$ and the coefficients δ_t and φ_t being $\mathcal{F}_t-$adapted and bounded (by a deterministic constant) processes. We further assume that $\delta_t, \varphi_t \in Lin(\sigma_t)$.

- *The benchmark case:* $a(x,t) = -xU_x(x,t)\,\delta_t$. Then, $U(x,t) = u\left(\frac{x}{Y_t}, A_t^{(\delta)}\right)$ with $A_t^{(\delta)} = \int_0^t |\lambda_s - \delta_s|^2 ds$ is a forward performance process. The factor Y_t normalizes the wealth argument and, thus, can be thought as a *benchmark* (or a numeraire) in relation to which one might wish to measure the performance of investment strategies.
- *The market-view case*: $a(x,t) = U(x,t)\,\varphi_t$. Then, $U(x,t) = u\left(x, A_t^{(\varphi)}\right) Z_t$ with $A_t^{(\varphi)} = \int_0^t |\lambda_s + \varphi_s|^2 ds$ is a forward performance process. The factor Z_t can be thought as a device offering flexibility to the forward solutions in terms of the asset returns. This might be needed if the investor has different *views* about the future market movements or faces trading constraints. In such cases, the returns need to be modified which essentially points to a change of measure, away from the historical one. This is naturally done through an exponential martingale.

2.2.3. The stochastic factor case and its forward volatility process. We now revert to the stochastic factor example with dynamics (2.2) and (2.3), studied earlier under the classical (backward) formulation, and we examine its forward analogue. To this end, consider a process $U(x,t)$, $t \geq 0$, given by

$$U(x,t) = v(x, Y_t, t), \tag{2.24}$$

for a deterministic function $v : \mathbb{R}_+ \times \mathbb{R} \times [0,\infty)$. Then, the SPDE (2.14) takes the form

$$dU(x,t) = \frac{1}{2} \frac{\left(\lambda(Y_t)\, v_x(x,Y_t,t) + \rho d(Y_t)\, v_{xy}(x,Y_t,t)\right)^2}{v_{xx}(x,Y_t,t)} dt$$

$$+\rho d\left(Y_t\right) v_y\left(x, Y_t, t\right) dW_t^1+\sqrt{1-\rho^2} d\left(Y_t\right) v_y\left(x, Y_t, t\right) dW_t^2,$$

with the forward volatility given by $a\left(x,t\right) = \left(\rho, \sqrt{1-\rho^2}\right) d\left(Y_t\right) v_y\left(x, Y_t, t\right)$. One then sees that if v satisfies (2.6) but now with an admissible initial (and not terminal) condition, say $v\left(x,y,0\right) = u_0\left(x\right)$, the process given in (2.24) is a forward performance. Solving (2.6) with an initial condition is an open problem because it not only inherits the difficulties discussed in the previous section but, now, one needs to deal with the ill-posedness of the HJB equation.

The homothetic case $u_0\left(x\right) = \frac{x^\gamma}{\gamma}$, $\gamma \in (0,1)$, has been extensively studied in [51]. Therein, it is shown that the forward performance process is given by an analogous to (2.8) formula, namely,

$$U\left(x,t\right) = \frac{1}{\gamma} x^\gamma \left(f\left(Y_t, t\right)\right)^\delta \tag{2.25}$$

provided that $f\left(y,t\right)$ satisfies the linear equation (2.9) with initial (and not terminal) condition $f\left(x,0\right) = 1$. This problem is more general than (2.18) due to the form of its coefficients, and, thus, more involved arguments needed to be developed. The multi-dimensional analogue of (2.25) was recently analyzed in [49]. Therein, $f\left(y,t\right)$ solves a multi-dimensional ill-posed linear problem with state-dependent coefficients. For such problems, there is no standard existence theory. The authors addressed this by developing a generalized version of the classical Widder's theorem.

Forward versus backward homothetic utilities. It is worth commenting on the different features of the three homothetic performance processes (2.10), (2.23) and (2.25). Tthe traditional value function (2.10) requires, for each $s \in [t,T)$ *forecasting* of the market price of risk in the *remaining trading horizon* $[s,T)$. In contrast, both (2.23) and (2.25) are constructed *path-by-path, given the information for the market price of risk up to today, in* $[0,s]$. The process (2.23) is decreasing in time, while (2.25) is not.

3. Model uncertainty and investment management

In the previous section, a prevailing assumption was that the historical measure $\mathbb{P}$ is a priori known. This, however, has been challenged by a number of scholars and gradually led to the development of selection criteria under *model uncertainty,* otherwise known as *ambiguity* or *Knightian uncertainty.* Pathbreaking work was done by Gilboa and Schmeidler in [22] and [58] who built an axiomatic approach for preferences not only towards risk - as it was done by von Neumann and Morgenstern for (2.1) - but also towards model ambiguity. They argued that such preferences can be numerically represented by a *"coherent" robust* utility functional of the form

$$X_T^\pi \to \inf_{Q \in \mathcal{Q}} E_Q\left(U\left(X_T^\pi\right)\right), \tag{3.1}$$

where U is a classical utility function and $\mathcal{Q}$ a family of probability measures. These measures can be thought as corresponding to different "prior" market models and the above infimum serves as the "worst-case scenario" in model misspecification.

A standard criticism for the above criterion, however, is that it allows for very limited, if at all, differentiation of models with respect to their plausibility. As discussed in [57], if, for instance, the family of prior models is generated from a confidence set in statistical estimation, models with higher plausibility must receive a higher weight than models in the boundary of the confidence set. Furthermore, one should be able to incorporate information from certain stress test models and observed discrepancies with outcomes of models of possible priors. Such shortcomings of criterion (3.1) stem primarily from the axiom of certainty independence in [22]. Maccheroni et al. [37] relaxed this axiom and proposed a numerical representation of the form

$$X_T^\pi \rightarrow \inf_{Q\in\mathcal{Q}} \left(E_Q\left(U\left(X_T^\pi\right)\right) + \gamma\left(Q\right)\right), \tag{3.2}$$

where U is a classical utility function and the functional $\gamma(Q)$ serves as a *penalization* weight to each Q-market model.

The specification and representation of robust preferences and their penalty functionals have recently attracted considerable attention. It turns out that there is a deep connection between them, monetary utility functionals and risk measures. The latter, denoted by $\varphi(X)$ and $\rho(X)$, respectively, are mappings on financial positions X, represented as random variables on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X \in L^\infty$. They are related as $\varphi(X) = -\rho(X)$.

Coherent risk measures were first introduced in [1] and were later extended to their convex analogues by [19, 21, 23]. Risk measures constitute one of the most active areas in financial mathematics with a substantial volume of results involving several areas in mathematics spanning from capacity theory and Choquet integration to BSDE, nonlinear expectations and stochastic differential games.

The (minimal) penalty function associated with a convex risk measure and its associated concave monetary utility functional, is defined, for probability measures $Q \ll \mathbb{P}$, by

$$\gamma(Q) = \sup_{X\in L^\infty} \left(E_Q(-X) - \rho(X)\right) = \sup_{X\in L^\infty} \left(\varphi(X) - E_Q(X)\right). \tag{3.3}$$

Extending criterion (3.1) to (3.2) is in direct analogy to generalizing the coherent risk measures to their convex counterparts. There is a substantial body of work on representation results for (3.3) which is, however, beyond the scope of this article.

Recent generalizations to (3.2) include the case

$$X_T^\pi \longrightarrow \inf_{Q\ll\mathbb{P}} G\left(Q, E_Q\left(U\left(X_T^\pi\right)\right)\right), \tag{3.4}$$

where G is the dual function in the robust representation of a *quasi-concave* utility functional.

In the sequel, we provide representative results on portfolio selection under the classical robust criterion (3.2) and its recently developed robust forward analogue.

3.1. Classical robust portfolio selection. The problem of portfolio selection in a finite horizon $[0, T]$ with the coherent robust utility (3.1) was studied by [53], [60] and others. Its generalization, corresponding to the convex analogue (3.2), was analyzed, among others, in [57] and we present below some of the results therein.

For an extensive overview of robust preferences and robust portfolio choice we refer the reader to the review paper [20].

The market model in [57] is similar to the standard semimartingale model in [30] and [31]. There is one riskless and d risky assets available for trading in $[0, T]$, $T < \infty$. The discounted price processes are modelled by a $d-$dim semimartingale $S_t = \left(S_t^1, ..., S_t^d\right)$,

$t \in [0,T]$, on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$. For $t \in [0,T]$, the control policies $\alpha_t = \left(\alpha_t^1, ..., \alpha_t^d\right)$ are self-financing, predictable and $S-$integrable processes. The associated discounted wealth process, X_t^α, is then given by $X_t^\alpha = x + \int_0^t \alpha_s \cdot dS_s$, and needs to satisfy $X_t^\alpha \geq 0$, $t \in [0,T]$. This formulation is slightly different than the ones in sections 2.1 and 2.2 in that the controls α_t now denote the number of shares (and not the discounted amounts) held at time t in the stock accounts.

For $x > 0$, $\mathcal{X}(x)$ stands for the set of all discounted wealth processes satisfying $X_0 \leq x$ and $X_t \geq 0$, $t \in (0,T]$. The classical (absence of arbitrage) model assumption is that $\mathcal{M} \neq \varnothing$, where $\mathcal{M}$ denotes the measures equivalent to $\mathbb{P}$ under which each $X_t \in \mathcal{X}(1)$, $t \in (0,T]$, is a local martingale (see [30]).

The value function of the *robust portfolio selection* problem is then defined, for $x \geq 0$, as

$$v(x) = \sup_{X\in\mathcal{X}(x)} \inf_{Q\in\mathcal{Q}} \left(E_Q\left(U\left(X_T\right)\right) + \gamma(Q)\right), \tag{3.5}$$

where γ is a minimal penalty function as in (3.3) and $\mathcal{Q} = \{Q \ll \mathbb{P} | \gamma(Q) < \infty\}$.

Because of the semimartingale assumption for the stock prices, classical stochastic optimization arguments do not apply and the *duality approach* comes in full force. As mentioned in the previous section, this approach has been extensively applied to portfolio choice problems and provides general characterization results of the value function and optimal policies through the dual problem, which is in general easier to analyze. There is a rich body of work in this area and we refer the reader, among others, to the classical references [28, 30, 31].

In the presence of model ambiguity, there is an extra advantage in using the duality approach because the dual problem simply involves the minimization of a convex functional while the primal one requires to find a saddle point of a functional which is concave in one argument and convex in the other.

We now describe the main notions and results in [57]. We stress, however, that for the ease of presentation we abstract from a number of detailed modeling assumptions and technical conditions.

We recall that the convex conjugate of the utility function U is defined, for $y > 0$, as $\tilde{U}(y) = \sup_{x>0}(U(x) - xy)$. Then, for every measure Q, $u_Q(x) = \sup_{X\in\mathcal{X}(x)} E_Q(U(X_T))$ is a traditional value function as in (2.4). It was established in [30] that, for $Q \sim \mathbb{P}$ with finite primal value function $u_Q(x)$, the bidual relationships $u_Q(x) = \inf_{y>0}(\tilde{u}_Q(y) + xy)$ and $\tilde{u}_Q(y) = \sup_{x>0}(u_Q(x) - xy)$ hold, where the *dual value function* $\tilde{u}_Q(y)$ is given by $\tilde{u}_Q(y) = \inf_{Y\in\mathcal{Y}_Q(y)} E_Q\left(\tilde{U}(Y_T)\right)$, for $Q \in \mathcal{Q}$. The space $\mathcal{Y}_Q(y)$ is the set of all positive $Q-$supermartingales such that $Y_0 = y$ and the product XY is a $Q-$supermartingale for all $X \in \mathcal{X}(1)$.

In analogy, one then defines in [57] the *dual function of the robust portfolio* problem by

$$\tilde{u}(y) = \inf_{Q\in\mathcal{Q}} \left(\tilde{u}_Q(y) + \gamma(Q)\right) = \inf_{Q\in\mathcal{Q}} \inf_{Y\in\mathcal{Y}_Q(y)} \left(E_Q\left(\tilde{U}(Y_T)\right) + \gamma(Q)\right).$$

Then, for $y > 0$ such that $\tilde{u}(y) < \infty$, a pair (Q, Y) is a solution to the dual convex robust problem if $Q \in \mathcal{Q}$, $Y \in \mathcal{Y}_Q(y)$ and $\tilde{u}(y) = E_Q(\tilde{U}(Y_T)) + \gamma(Q)$. Let also $\mathcal{Q}^e = \{Q \in \mathcal{Q} | Q \sim \mathbb{P}\}$.

Theorems 2.4 and 2.6 in [57] provide characterization results for the primal and dual robust value functions, as well as for the optimal policies. In the next two propositions, we highlight some of their main results.

Proposition 3.1. *Assume that for some $x > 0$ and $Q_0 \in \mathcal{Q}^e$, $u_{Q_0}(x) < \infty$ and that $\tilde{u}(y) < \infty$ implies that, for some $Q_1 \in \mathcal{Q}^e$, $\tilde{u}_{Q_1}(y) < \infty$. Then, the robust value function $u(x)$ is concave and finite, and satisfies*

$$u(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \left(E_Q \left(U \left(X_T \right) \right) + \gamma(Q) \right) = \inf_{Q \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \left(E_Q \left(U \left(X_T \right) \right) + \gamma(Q) \right).$$

Moreover, the primal and the dual robust value functions u and $\tilde{u}$ satisfy

$$u(x) = \inf_{y>0} \left(\tilde{u}(y) + xy \right) \quad \textit{and} \quad \tilde{u}(y) = \sup_{x>0} \left(u(x) - xy \right).$$

If $\tilde{u}(y) < \infty$, then the dual problem admits a solution, say (Q^*, Y^*) that is maximal, in that any other solution (Q, Y) satisfies $Q \ll Q^*$ and $Y_T = Y_T^*$, $Q-$a.s.

Note that the optimal measure Q^* might not be equivalent to $\mathbb{P}$ (see, for instance, example 3.2 in [57]). In such cases, one can show that the Q^*-market may admit arbitrage opportunities.

The existence of optimal policies requires the additional assumption that for all $y > 0$ and each $Q \in \mathcal{Q}^e$ the dual robust value function satisfies $\tilde{u}_Q(y) < \infty$.

Proposition 3.2. *For any $x > 0$, there exists an optimal strategy $X^* \in \mathcal{X}(x)$ for the robust portfolio selection problem. If $y > 0$ is such that $\tilde{u}'(y) = -x$ and (Q^*, Y^*) is a solution of the dual problem, then $X_T^* = I(Y_T^*)$, Q^*-a.s. for $I(x) = -\tilde{U}'(x)$, and (Q^*, Y^*) is a saddle point for the primal robust problem,*

$$u(x) = \inf_{Q \in \mathcal{Q}} \left(E_Q \left(U \left(X_T^* \right) \right) + \gamma(Q) \right) = E_{Q^*} \left(U \left(X_T^* \right) \right) + \gamma(Q^*) = u_{Q^*}(x) + \gamma(Q^*).$$

Furthermore, the product $X_t^* Y_t^* Z_t^*$ is a martingale under $\mathbb{P}$, where Z_t^*, $t \in [0, T]$, is the density process of Q^* with respect to $\mathbb{P}$.

Example 3.3. Examples of penalty functionals

- *Coherent penalties:* γ takes the values 0 or ∞. Then, (3.2) reduces to (3.1).
- *Entropic penalties:* $\gamma(Q) = H(Q|\,\mathbb{P})$, where the entropy function H is defined, for $Q \ll \mathbb{P}$, as

$$H(Q|\,\mathbb{P}) = \int \frac{dQ}{d\mathbb{P}} \ln\left(\frac{dQ}{d\mathbb{P}} \right) d\mathbb{P} = \sup_{Y \in L^\infty} \left(E_Q(Y) - \ln E_{\mathbb{P}}\left(e^Y \right) \right). \tag{3.6}$$

  In this case, $\inf_{Q \in \mathcal{Q}} \left(E_Q \left(U \left(X_T \right) \right) + \gamma(Q) \right) = \ln E_{\mathbb{P}} \left(e^{-U(X_T)} \right)$ and the robust portfolio problem (3.5) reduces to the standard one of maximizing $E_{\mathbb{P}} \left(e^{-U(X_T)} \right)$.

- *Dynamically consistent penalties:* $\gamma_t(Q) = E_Q \left(\int_t^T h(\eta_s)\, ds \middle|\, \mathcal{F}_t \right)$, $t \in [0, T)$, where the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is generated by a $d-$dim Brownian motion. Then, for every measure $Q \ll \mathbb{P}$, there exists a d-dim predictable process η_t with $\int_0^T |\eta_t|^2\, dt < \infty$, Q-a.s. and $\frac{dQ}{d\mathbb{P}} = \mathcal{E}\left(\int_0 \eta_t \cdot dW_t \right)_T$ where $\mathcal{E}(M)_t = \exp\left(M_t - \langle M \rangle_t \right)$ for a continuous semimartingale M_t. The function h satisfies appropriate regularity and growth conditions (see example 3.4 in [57]). The specific choice $h(x) = \frac{1}{2}|x|^2$ corresponds to (3.6).

- *Shortfall risk penalties:* $\gamma(Q) = \inf_{\lambda>0}\left(\lambda x + \lambda E_{\mathbb{P}}\left(f^*\left(\frac{1}{\lambda}dQ/d\mathbb{P}\right)\right)\right)$, for $Q \ll \mathbb{P}$, and where $f : \mathbb{R} \to \mathbb{R}$ is convex and increasing and x is in the interior of $f(\mathbb{R})$, and f^* denotes its Legendre-Fenchel transform. The associated risk measure is given by $\rho(Y) = \inf\{m \in \mathbb{R} | E_{\mathbb{P}}(f(-Y-m)) \leq x\}$, $Y \in L^\infty$, and is the well known shortfall risk measure introduced by Föllmer and Schied. Its dynamic version is weakly dynamically consistent but fails to be dynamically consistent.
- *Penalties associated with statistical distance functions:* $\gamma(Q) = E_{\mathbb{P}}(g(dQ/d\mathbb{P}))$, for $Q \ll \mathbb{P}$ and suitable functions g.

3.2. Forward robust portfolio selection. We consider the model as in [69] with $d+1$ securities whose prices, $(S^0; S) = (S_t^0, S_t^1, ..., S_t^d)$, $t \geq 0$, with $S_0 = 1$ (the numeraire) and S_t, $t \geq 0$, is a d-dim càdlàg locally bounded semimartingale on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$. The wealth process is given by $X_t^\alpha = x + \int_0^t \alpha_s \cdot dS_s$, $t \geq 0$. The set $\mathcal{A}$ of admissible policies consists of weight portfolios α_t that are predictable and, for each $T > 0$ and $t \in [0,T]$, are S-integrable and $X_t^\alpha > -c$, $c > 0$. We denote the set of probability measures that are equivalent to $\mathbb{P}$ by $\mathcal{Q}$. For further details and all technical assumptions, see [69] and [26].

Definition 3.4.

i) A random field is a mapping $U : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ which is measurable with respect to the product of the optional σ-algebras on $\Omega \times [0,\infty)$ and $\mathcal{B}(\mathbb{R})$.

ii) A utility field is a random field such that, for $t \geq 0$ and $\omega \in \Omega$, the mapping $x \to U(\omega, x, t)$ is $\mathbb{P}$-a.s. a strictly concave and strictly increasing $C^1(\mathbb{R})$ function, and satisfies the Inada conditions $\lim_{x\to-\infty} \frac{\partial}{\partial x}U(\omega,x,t) = \infty$ and $\lim_{x\to\infty} \frac{\partial}{\partial x}U(\omega,x,t) = 0$. Moreover, for each $x \in \mathbb{R}$ and $\omega \in \Omega$, the mapping $t \to U(\omega,x,t)$ is càdlàg on $[0,\infty)$, and for each $x \in \mathbb{R}$ and $T \in [0,\infty)$, $U(\cdot,x,T) \in L^1(\mathbb{P})$.

For simplicity, the $\omega-$notation is suppressed in $U(x,t)$. Next, the concept of an admissible penalty function is introduced.

Definition 3.5.

i) Let $T > 0$ and $t \in [0,T]$, and $\mathcal{Q}_T = \{Q \in \mathcal{Q} : Q|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}\}$. Then, a mapping $\gamma_{t,T} : \Omega \times \mathcal{Q}_T \to \mathbb{R}_+ \cup \{\infty\}$, is a penalty function if $\gamma_{t,T}$ is $\mathcal{F}_t$-adapted, $Q \to \gamma_{t,T}(Q)$ is convex a.s., for $Q \in \mathcal{Q}_T$, and for $\kappa \in L_+^\infty(\mathcal{F}_t)$, $Q \to E_Q(\kappa\gamma_{t,T}(Q))$ is weakly lower semi-continuous on $\mathcal{Q}_T$.

ii) For a given utility random field $U(x,t)$, $\gamma_{t,T}$ is an admissible penalty function if, for each $T > 0$ and $x \in \mathbb{R}$, $E_Q(U(x,T)) < \infty$ for all $Q \in \mathcal{Q}_{t,T}$, with $\mathcal{Q}_{t,T} = \{Q \in \mathcal{Q}_T : \gamma_{t,T}(Q) < \infty, \text{a.s.}\}$.

Using the above notions, the following definition of the *robust forward performance process* was proposed in [26]. Because of the presence of the penalty term in (3.7) below, it is more convenient to formulate this concept in terms of the self-generation property (cf. (2.13)).

Definition 3.6. Let, for $t \geq 0$, $U(x,t)$ be a utility field and, for $T > 0$ and $t \in [0,T]$, $\gamma_{t,T}$ be an admissible family of penalty functions. Define the associated value field as a family of mappings $u(\cdot;t,T): L^{\infty} \to L^{0}(\mathcal{F}_t;\mathbb{R}\cup\{\infty\})$, given by

$$u(\xi;t,T) = ess \sup_{\pi\in\mathcal{A}_{bd}} ess \inf_{Q\in\mathcal{Q}_{t,T}} \left(E_Q \left(U(\xi + \int_t^T \alpha_s \cdot dS_s, T) \middle| \mathcal{F}_t \right) + \gamma_{t,T}(Q) \right), \tag{3.7}$$

with $\xi \in L^{\infty}(\mathcal{F}_t)$ and $\mathcal{A}_{bd}$ being the set of admissible policies in $\mathcal{A}$ that yield bounded wealth processes. Then, the pair $(U, \gamma_{t,T})$ is a forward robust criterion if, for $T > 0$ and $t \in [0,T]$, $U(\xi,t)$ is self-generating, that is $U(\xi,t) = u(\xi;t,T)$, a.s..

Preliminary results for the dual characterization of forward robust preferences were recently derived in [26]. The dual of the utility field $U(x,t)$ is defined, for $(y,t) \in \mathbb{R}_+ \times [0,\infty)$, as $\tilde{U}(y,t) = \sup_{x\in\mathbb{R}}(U(x,t) - xy)$. One, then, defines the dual value field, for $T > 0$ and $t \in [0,T]$, as the mapping $\tilde{u}(\cdot,t,T) : L^0_+(\mathcal{F}_t) \longrightarrow L^0(\mathcal{F}_t, \mathbb{R}\cup\{\infty\})$ given by

$$\tilde{u}(\eta;t,T) = ess \inf_{Q\in\mathcal{Q}_{t,T}} ess \inf_{Z\in\mathcal{Z}^a_T} \left(E_Q \left(\tilde{U}\left(\eta Z_{t,T}/Z_{t,T}{}^Q, T\right) \middle| \mathcal{F}_t \right) + \gamma_{t,T}(Q) \right). \tag{3.8}$$

Herein, $Z_{t,T} = Z_T/Z_t$ (resp. $Z^Q_{t,T} = Z^Q_T/Z^Q_t$), where Z_s (resp. Z^Q_s), $s = t, T$, is the well known density process for the absolutely continuous local martingale measures (resp. Q) (for further details, see [69]).

In turn, the pair $\left(\tilde{U}, \gamma_{t,T}\right)$, for an admissible family of penalty functions $\gamma_{t,T}$, is said to be self-generating if $\tilde{U}(\eta,t) = \tilde{u}(\eta;t,T)$, for all $\eta \in L^0_+(\mathcal{F}_t)$. Under additional assumptions, it was shown in [26] that the primal and the dual value fields satisfy, for all $T > 0$ and $t \in [0,T]$, the bidual relationships $u(\xi;t,T) = essinf_{\eta\in L^0_+(\mathcal{F}_t)}(\tilde{u}(\eta;t,T) + \xi\eta)$ and $\tilde{u}(\eta;t,T) = ess\sup_{\xi\in L^{\infty}(\mathcal{F}_t)}(u(\xi;t,T) - \xi\eta)$, for $\xi \in L^{\infty}(\mathcal{F}_t)$ and $\eta \in L^0_+(\mathcal{F}_t)$. It was also shown that the primal criterion $(U, \gamma_{t,T})$ is self-generating, and thus a forward robust criterion, if and only if its dual counterpart $\left(\tilde{U}, \gamma_{t,T}\right)$ is self-generating.

There are several open questions for the characterization and construction of the robust forward performance process. For example, there are certain assumptions on $\mathcal{Q}_{t,T}$ in Definition 3.5 (see Assumption 4.5 in [26]) which might be difficut to remove. Another issue is whether the penalty functions need to be themselves dynamically consistent, in that whether they need to satisfy $\gamma_{t,T}(Q) = \gamma_{t,s}(Q) + E_Q(\gamma_{s,T}(Q)|\mathcal{F}_t)$, for $T > 0$ and $t \in [0,T]$. As Definition 3.5 stands, this property is not needed as long as the pair $(U(x,t), \gamma_{t,T})$ is self-generating. However, examples (either for the primal or the dual forward robust criterion) for non dynamically consistent penalty functions have not been constructed to date. We remind the reader that classical robust utilities are well defined even if the associated penalties are not time-consistent, with notable example being the penalty associated with the shortfall risk measure. It is not clear, however, if in the forward setting such cases are indeed viable.

Because of the model ambiguity and the semimartingale nature of the asset prices, it is not immediate how to obtain the robust analogue of the forward performance SPDE (2.14). Some cases have been analyzed in [26]. Among others, it is shown that when asset prices follow Itô processes and the forward robust criterion is time-monotone, then its dual $\tilde{U}(x,t)$ solves a fully non-linear ill-posed PDE with random coefficients.

The time-monotone case with logarithmic initial datum, $U(x,0) = \ln x$, and time-consistent quadratic penalties can be explicitly solved. The optimal policy turns out to be a *fractional Kelly strategy,* which is widely used in investment practice. The fund manager invests in the growth optimal (Kelly) portfolio corresponding to her best estimate of the market price of risk. However, she is not fully invested but, instead, allocates in stock a fraction α_t^* of her optimal wealth that depends on her "trust" in this estimate. Her "trust" is modelled by a process δ_t that appears in the quadratic penalty. As $\delta_t \uparrow \infty$ (infinite trust in the estimation), α_t^* converges to the classical Kelly strategy associated with the most likely model while if $\delta_t \downarrow 0$ (no trust in the estimation), α_t^* converges to zero and deleveraging becomes optimal.

4. Concluding remarks

Despite the numerous advances on the theoretical development and analysis of portfolio management models and their associated stochastic optimization problems, there is relatively little intersection between investment practice and academic research. As mentioned in the introduction, the two main reasons for this are the fundamental difficulties in estimating the parameters for the price processes and the lack of practically relevant investment performance criteria.

While estimating the volatility of stock prices is a problem extensively analyzed (see, for example, [2] and [47]), estimating their drift is notoriously difficult (see, among others, [17] and [39]). Note that drift estimation is not an issue in derivative valuation, for pricing and hedging do not require knowledge of the historical measure but, rather, of the martingale one(s). As a result, there is no need to estimate the drift of the underlying assets. Recently, a line of research initiated by S. Ross ([54]) on the so called *Recovery Theorem* investigates if the historical measure can be recovered from its martingale counterpart(s) (see also [10]).

The lack of a realistic investment performance criterion poses equally challenging questions. There are two issues here: the form of the criterion per se, and its dynamic and time-consistent nature. A standard criticism from practitioners is that utility functions are elusive and inapplicable concepts. Such observations date back to 1968 in the old note of F. Black ([5]). Indeed, in portfolio practice, managers and investors have investment targets (expected return, volatility limits, etc.) and companies have constraints on their reserves and risk limits, and it is quite difficult, if possible at all, to map these inputs to a classical utility function. The only criterion that bridges part of this gap is the celebrated *mean-variance* one, developed by H. Markowitz ([38]), which corresponds to a quadratic utility with coefficients reflecting the desired variance and associated optimal mean. However, this widely used criterion is essentially a single-period one. In a multi-period setting, it becomes time-inconsistent, in contrast to criteria used in derivative pricing which are by nature dynamically consistent. It is not known to date how to construct genuinely dynamic and time-consistent mean-variance or other practically relevant investment criteria. Some attempts towards this direction can be found in the recent works [48] and [68].

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