Forward exponential indifference valuation in an incomplete binomial model*

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Abstract

We introduce and construct indifference prices under exponential forward performance criteria in an incomplete binomial model. We propose a pricing algorithm, which is iterative and yields the price in two sub-steps, locally in time. At the beginning of each period, an intermediate payoff is produced which is non-linear and replicable, and, in turn, it is priced by arbitrage, in the second sub-step. The indifference price is thus constructed via an iterative non-linear pricing operator, which also involves a martingale measure. The latter turns out to minimize the reverse relative entropy. Properties of the forward prices are discussed as well as differences with their classical counterparts.

1 Introduction

We introduce, construct and study indifference prices in an incomplete binomial model under forward performance criteria. Such criteria, proposed by two of the authors (see, among others, [12] and [15]), complement the traditional expected utility ones by allowing for dynamic adaptation of risk preferences as the market evolves. We refer the reader to, among others, [15], [16], [17] and [19] for an overview on the forward performance approach.

The binomial model we consider is more general than the ones studied in the traditional exponential indifference valuation literature, for it includes a non-traded stochastic factor that affects not only the claim's payoff (as it is the case, among others, in [1], [4], [10], [11], [23] and [24]) but, also, the transition probability and/or the values of the traded stock. This extension is crucial in incorporating models with stochastic investment opportunity sets. Binomial

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models of this kind were analyzed in the classical setting in [9] and [18] for power and exponential utilities, respectively.

We first construct a forward performance process for the incomplete model herein and, in turn, analyze the associated indifference prices. We focus on a criterion of exponential type (cf. (13)) since exponential risk preferences have been predominantly used in indifference valuation.

The main contribution is the construction of a valuation algorithm for the forward indifference prices. We show that, for a claim written at time 0 and maturing at t, its price $\nu_s(C_t)$, $s = 0, 1, ..., t$, satisfies

$$\nu_s(C_t) = \mathcal{E}^{(s.s+1)}_{\mathbb{Q}^*}(\nu_{s+1}(C_t)) := E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma\nu_{s+1}(C_t)}\middle|\mathcal{F}_s \vee \mathcal{F}^S_{s+1}\right)\middle|\mathcal{F}_s\right),$$

where $\mathcal{F}_s$ and $\mathcal{F}^S_s$ are the filtrations generated by both the stock and the stochastic factor, and the stock, respectively, and $\mathbb{Q}^*$ an appropriately chosen martingale measure.

Therefore, the price is constructed iteratively,

$$\nu_s(C_t) = \mathcal{E}^{(s,t)}_{\mathbb{Q}^*}(C_t) := \mathcal{E}^{(s,s+1)}_{\mathbb{Q}^*}\left(\mathcal{E}^{(s+1,s+2)}_{\mathbb{Q}^*}...\mathcal{E}^{(t-1,t)}_{\mathbb{Q}^*}(C_t)\right).$$

Each price iteration has two sub-steps. In the first, the intermediate payoff

$$\mathcal{C}^{(s,s+1)}(\nu_{s+1}(C_t)) := \frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma\nu_{s+1}(C_t)}\middle|\mathcal{F}_s \vee \mathcal{F}^S_{s+1}\right) \tag{1}$$

is produced, which is non-linear and replicable. In turn, its arbitrage-free price yields, in the second sub-step, the indifference price,

$$\nu_s(C_t) = E_{\mathbb{Q}^*}\left(\mathcal{C}^{(s,s+1)}(\nu_{s+1}(C_t))\middle|\mathcal{F}_s\right). \tag{2}$$

Central role plays the emerging pricing measure $\mathbb{Q}^*$, which turns out to be a martingale one that minimizes the reverse relative entropy (see Proposition 7). Moreover, it has the property that the conditional distribution of the stochastic factor, given the information on the traded stock, remains the same as the one under the historical measure (see (36)), in that, for $s = 1, 2, ..., t$,

$$\mathbb{Q}^*\left(Y_s\left|\mathcal{F}_{s-1} \vee \mathcal{F}^S_s\right.\right) = \mathbb{P}\left(Y_s\left|\mathcal{F}_{s-1} \vee \mathcal{F}^S_s\right.\right).$$

The forward indifference prices have intuitively pleasing properties. Among others, we show that the above intermediate payoff $\mathcal{C}^{(s,s+1)}(\nu_{s+1}(C_t))$ provides a direct analogue of the traditional certainty equivalent (cf. (51)). Namely, consider the nonlinear payoff of certainty equivalent type

$$CE^{(s,s+1)}(\nu_{s+1}(C_t)) := -U^{(-1)}_{s+1}\left(E_{\mathbb{P}}\left(U_{s+1}(-\nu_{s+1}(C_t))\middle|\mathcal{F}_s \vee \mathcal{F}^S_{s+1}\right)\right), \tag{3}$$

where $\mathbb{P}$ is the historical measure, U_{s+1} the forward performance process and $U^{(-1)}_{s+1}$ its spatial inverse. We establish that it coincides with the above payoff,

$$CE^{(s,s+1)}(\nu_{s+1}(C_t)) = \mathcal{C}^{(s,s+1)}(\nu_{s+1}(C_t)).$$

As a result, the forward indifference price can be represented as the arbitrage-free price of an appropriately chosen conditional certainty equivalent for each valuation period.

We also show that the single-period conditional distribution of the pricing measure $\mathbb{Q}^*$ depends exclusively on the associated single-period conditional risk neutral and historical probabilities (see (32),(33)). This, together with the form of $\mathcal{E}_{\mathbb{Q}^*}^{(s.s+1)}$ above, highlight the essential features of the indifference valuation under forward exponential criteria: the price is constructed by "single-period" operations - both in terms of the pricing functional and the involved pricing measure - which are repeated from one-period to the next with single-period adjustments of the conditional risk neutral and historical probabilities. Furthermore, all three pricing ingredients, $\mathbb{Q}^*$, $\mathcal{E}_{\mathbb{Q}^*}^{(s.s+1)}$ and $\mathcal{E}_{\mathbb{Q}^*}^{(s.t)}$, are independent of the maturity of the claim. Finally, because forward performance criteria are defined for all times, sequentially from one period to the next as the market moves (cf. (17)), one can price claims that arrive at later times with arbitrary maturities (see discussion below Corollary 13).

Note that most of these properties fail in the classical setting, where prices are defined in terms of expected utility from terminal wealth in a chosen horizon, say $[0, T]$. Indeed, while the forms of the corresponding single- and multi-step pricing functionals $\mathcal{E}_{\mathbb{Q}_T^{me}}^{(s.s+1)}$ and $\mathcal{E}_{\mathbb{Q}_T^{me}}^{(s.t)}$ are similar to the ones herein (see, [23], [24], [10] for complete markets and a claim written only on the nontraded asset, and [18] for a model like the one herein), the choice of the horizon strongly affects the pricing measure $\mathbb{Q}_T^{me}$, which is the minimal relative entropy one ([2], [5], [21]). Moreover, its conditional distribution does not have the aforementioned local features that $\mathbb{Q}^*$ has. From the indifference valuation perspective, once the investment horizon is (pre)chosen, no new claim arriving at a future time, that was not known a priori when the original investment horizon was set up, and maturing beyond T can be priced.

The paper is organized as follows. In section 2, we present the model and its forward investment performance process, and propose an example of exponential type. In section 3, we introduce the forward indifference price and in section 4 we construct the associated pricing algorithm. In section 5, we present various properties of the prices and discuss differences with their classical counterparts.

2 The model and its forward performance criteria

We start with the probabilistic setup of the incomplete multi-period binomial model. There are two traded assets, a riskless bond and a stock. The bond is assumed to offer zero interest rate.

The values of the stock are denoted by S_t, $t = 1, 2, ...$ with $S_0 > 0$. We define the random variables

$$\xi_t = \frac{S_t}{S_{t-1}}, \quad \xi_t = \xi_t^d,\ \xi_t^u \quad \text{with} \quad 0 < \xi_t^d < 1 < \xi_t^u. \tag{4}$$

Incompleteness comes from a non-traded stochastic factor. Its levels, denoted by Y_t, $t = 0, 1, ...$, satisfy $Y_t \neq 0$. We introduce the random variables

$$\eta_t = \frac{Y_t}{Y_{t-1}},\ \eta_t = \eta_t^d,\ \eta_t^u \quad \text{with} \quad 0 < \eta_t^d < \eta_t^u. \tag{5}$$

We then view $\{(S_t, Y_t) : t = 0, 1, ...\}$ as a two-dimensional stochastic process defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The filtration $\mathcal{F}_t$ is generated by S_i and Y_i, or, equivalently, by the random variables ξ_i and η_i, for $i = 0, 1, ..., t$. We also consider the filtration $\mathcal{F}_t^S$ generated only by S_i, $i = 0, 1, ..., t$. The real (historical) probability measure on Ω and $\mathcal{F}$ is denoted by $\mathbb{P}$.

We introduce the sets

$$A_t = \{\omega : \xi_t(\omega) = \xi_t^u\} \qquad \text{and} \qquad B_t = \{\omega : \eta_t(\omega) = \eta_t^u\}, \tag{6}$$

and assign the single-period conditional probabilities $\mathbb{P}(A_t B_t | \mathcal{F}_{t-1})$,
$\mathbb{P}(A_t B_t^c | \mathcal{F}_{t-1})$, $\mathbb{P}(A_t^c B_t | \mathcal{F}_{t-1})$, $\mathbb{P}(A_t^c B_t^c | \mathcal{F}_{t-1})$, for $t = 1, 2, ...$

Throughout, we will be using the notation AB to denote the intersection $A \cap B$ of sets A and B. We will be also using the notations "$Z \in \mathcal{F}_t$" or "Z is $\mathcal{F}_t$-mble" interchangeably to state that a generic random variable Z is $\mathcal{F}_t$-measurable.

An investor starts at $t = 0$ with endowment $X_0 = x$, $x \in \mathbb{R}$, and trades between the stock and the bond, following self-financing strategies. The number of shares of stock held in his portfolio over the time interval $[t-1, t)$, $t = 1, 2, ...$, is denoted by α_t. The set of *admissible* policies is denoted by $\mathcal{A}$ and consists of all sequences $\alpha = \{\alpha_1, \alpha_2, .., \alpha_t, ...\}$, where each term α_t is a real-valued $\mathcal{F}_{t-1}$-mble random variable.

The investor's wealth is, then, given, for $t = 1, 2, ...$, by

$$X_t^\alpha = x + \sum_{i=1}^{t} \alpha_i \triangle S_i, \tag{7}$$

where the price increment $\triangle S_i = S_i - S_{i-1}$.

The performance of the various investment strategies is measured via a stochastic criterion, the so-called *forward performance* process, which measures the output of admissible portfolios and gives a selection criterion as follows: a strategy is deemed optimal if it generates a wealth process whose average performance is maintained over time. Specifically, the average performance of this strategy, at any future date, conditionally on today's information preserves the performance of this strategy up until today. Any strategy that fails to maintain the average performance over time is, then, sub-optimal. We formalize this more rigorously below.

Definition 1 *An $\mathcal{F}_t$-adapted process $U_t(x)$ is a forward performance process if, for $t = 0, 1, ...$,*

i) the mapping $x \to U_t(x)$, $x \in \mathbb{R}$, is strictly increasing and strictly concave, *ii) for each $\alpha \in \mathcal{A}$,*

$$U_t\left(X_t^{\alpha}\right) \geq E_{\mathbb{P}}\left(U_{t+1}\left(X_{t+1}^{\alpha}\right)\middle|\,\mathcal{F}_t\right), \tag{8}$$

iii) there exists $\alpha^* \in \mathcal{A}$ for which

$$U_t\left(X_t^{\alpha^*}\right) = E_{\mathbb{P}}\left(U_{t+1}\left(X_{t+1}^{\alpha^*}\right)\middle|\,\mathcal{F}_t\right). \tag{9}$$

The concept of forward performance process was introduced by two of the authors in [12] for the binomial model at hand in a single-period setting. It was subsequently extended to Itô-diffusion markets, and we refer the reader, among others, to [14], [15], [17], and [19], and references therein.

Characterizing the entire family of forward performance processes remains an open question and is being currently investigated by the authors and others. In the case of Itô-diffusion markets, a stochastic PDE was derived in [16] for the forward performance process. The novel element therein is the forward performance volatility process, which is an investor-specific input. As a result, forward performance processes are not in general unique.

Special classes of volatilities were proposed in [15], which can be interpreted as zero-volatility cases for alternative market settings under a different numeraire and/or market views. More recent works on the forward SPDE include [3], [8], [19], [20] and [22]. For a complete study of the zero-volatility case see [17].

Herein, we study discrete-time forward processes and focus on analyzing the associated indifference prices. Because in the classical expected utility framework such prices have been constructed primarily for exponential risk preferences, we are interested in a similar class of criteria as well.

2.1 An exponential forward performance process

We look for a forward performance process of the form

$$U_t\left(x\right) = -e^{-\gamma x + H_{0,t}},\ x \in \mathbb{R} \ \text{ and } \gamma > 0,$$

for an appropriately chosen process $H_{0,t}$, satisfying $H_{0,0} = 0$ and $H_{0,t} \in \mathcal{F}_t$, $t = 1, 2, ...$

As mentioned earlier, forward performance processes are not unique, for they depend critically on the choice of their volatility process. Herein, we focus on a forward process of the above form which, as we show below, turns out to also be decreasing in time, for each x. We choose to start with this class of discrete-time forward criteria because they provide the simplest direct extension of the zero-volatility case in Itô-diffusion markets, which also turn out to be time-monotone processes (see [17]).

For general semimartingale markets, exponential forward processes were analyzed in [27], and subsequently used for the construction of maturity-independent entropic risk measures in [26].

We proceed with some auxiliary results. For $t = 1, 2, ...$, we denote by $\mathcal{Q}_t$ the set of equivalent martingale measures defined on $\mathcal{F}_t$. We also denote (with a slight abuse of notation) by $\mathbb{Q}$ its generic element and recall the conditional risk neutral probabilities

$$q_t = \mathbb{Q}\left(A_t \middle| \mathcal{F}_{t-1} \right) = \frac{1-\xi_t^d}{\xi_t^u - \xi_t^d}, \tag{10}$$

with A_t and ξ_t^d, ξ_t^u as in (6) and (4).

Definition 2 *The process h_t, $t = 1, 2, ...$, is defined by*

$$h_t = q_t \ln \frac{q_t}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1} \right)} + (1-q_t) \ln \frac{1-q_t}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1} \right)}, \tag{11}$$

with q_t and A_t as in (10) and (6), respectively.

Note that actually $h_t \in \mathcal{F}_{t-1}$ and, moreover,

$$e^{-h_t} = \left(\frac{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1} \right)}{q_t} \right)^{q_t} \left(\frac{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1} \right)}{1-q_t} \right)^{1-q_t}. \tag{12}$$

We present one of the main results next.

Theorem 3 *Let h as in (11) and $\gamma > 0$. Then, for $t = 1, 2, ...$ and $x \in \mathbb{R}$, the process*

$$U_t(x) = -e^{-\gamma x + \Sigma_{i=1}^t h_i}, \tag{13}$$

with $U_0(x) = -e^{-\gamma x}$, is a forward performance.

The policy given, for $i = 1, 2, ..., t$, by

$$\alpha_i^* = \frac{1}{\gamma S_{i-1} \left(\xi_i^u - \xi_i^d \right)} \ln \frac{\left(\xi_i^u - 1 \right) \mathbb{P}\left(A_i \middle| \mathcal{F}_{i-1} \right)}{\left(1 - \xi_i^d \right) \mathbb{P}\left(A_i^c \middle| \mathcal{F}_{i-1} \right)} \tag{14}$$

is optimal and generates the optimal wealth process

$$X_t^* = x + \frac{1}{\gamma} \sum_{i=1}^t \frac{\xi_i - 1}{\left(\xi_i^u - \xi_i^d \right)} \ln \frac{\left(\xi_i^u - 1 \right) \mathbb{P}\left(A_i \middle| \mathcal{F}_{i-1} \right)}{\left(1 - \xi_i^d \right) \mathbb{P}\left(A_i^c \middle| \mathcal{F}_{i-1} \right)}. \tag{15}$$

We first present the following auxiliary result.

Lemma 4 *For $i = 1, 2, ...$, and h_i as in (11), we have*

$$\sup_{\alpha_i \in \mathcal{F}_{i-1}} E_{\mathbb{P}} \left(-e^{-\gamma \alpha_i \Delta S_i} \middle| \mathcal{F}_{i-1} \right) = -e^{-h_i}, \tag{16}$$

with the maximum occurring at α_i^* given in (14).

Proof. We have, with A_i as in (6),

$$E_{\mathbb{P}}\left(-e^{-\gamma\alpha_i\Delta S_i}\middle|\mathcal{F}_{i-1}\right)$$
$$=E_{\mathbb{P}}\left(-e^{-\gamma\alpha_i S_{i-1}(\xi_i^u-1)}\mathbf{1}_{A_i}\middle|\mathcal{F}_{i-1}\right)+E_{\mathbb{P}}\left(-e^{-\gamma\alpha_i S_{i-1}\left(\xi_i^d-1\right)}\mathbf{1}_{A_i^c}\middle|\mathcal{F}_{i-1}\right)$$
$$=-\left(e^{-\gamma\alpha_i S_{i-1}(\xi_i^u-1)}\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)+e^{-\gamma\alpha_i S_{i-1}\left(\xi_i^d-1\right)}\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)\right),$$

where we used the measurability properties of the involved quantities. Direct differentiation then yields that the optimum occurs at (14). Then, the first term above becomes

$$-e^{-\gamma\alpha_i^* S_{i-1}(\xi_i^u-1)}\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)=-\left(e^{\gamma\alpha_i^* S_{i-1}\left(\xi_i^u-\xi_i^d\right)}\right)^{-\frac{\xi_i^u-1}{\xi_i^u-\xi_i^d}}\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)$$
$$=-\left(\frac{(\xi_i^u-1)\,\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)}{\left(1-\xi_i^d\right)\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)}\right)^{-\frac{\xi_i^u-1}{\xi_i^u-\xi_i^d}}\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)$$
$$=-\left(\frac{1-q_i}{q_i}\right)^{-(1-q_i)}\left(\frac{\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)}{\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)}\right)^{-(1-q_i)}\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)$$
$$=-\left(\frac{q_i}{1-q_i}\right)^{1-q_i}\left(\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)\right)^{q_t}\left(\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)\right)^{1-q_i},$$

where we used (10). Similarly,

$$-e^{-\gamma\alpha_i^* S_{i-1}\left(\xi_i^d-1\right)}\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)$$
$$=-\left(\frac{1-q_i}{q_i}\right)^{q_i}\left(\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)\right)^{q_i}\left(\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)\right)^{1-q_i}.$$

Therefore,

$$E_{\mathbb{P}}\left(-e^{-\gamma\alpha_i^*\Delta S_i}\middle|\mathcal{F}_{i-1}\right)$$
$$=-\left(\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)\right)^{q_i}\left(\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)\right)^{1-q_i}\left(\left(\frac{q_i}{1-q_i}\right)^{1-q_i}+\left(\frac{1-q_i}{q_i}\right)^{q_i}\right)$$
$$=-\left(\frac{\mathbb{P}\left(A_i\middle|\mathcal{F}_{i-1}\right)}{q_i}\right)^{q_i}\left(\frac{\mathbb{P}\left(A_i^c\middle|\mathcal{F}_{i-1}\right)}{q_i}\right)^{1-q_i}(q_i+(1-q_i))=-e^{-h_i},$$

where we used (12). ■

We continue with the proof of Theorem 3.

Proof. Requirement (i) in Definition 1 follows directly. Next, we establish (8). Using (7) and (13), we need to show that, for $t\geq 0$ and $\alpha_i\in\mathcal{F}_{i-1}$, $i=1,...,t+1$, and $x\in\mathbb{R}$,

$$-e^{-\gamma\left(x+\Sigma_{i=1}^t\alpha_i\Delta S_i\right)+\Sigma_{i=1}^t h_i}\geq E_{\mathbb{P}}\left(-e^{-\gamma\left(x+\Sigma_{i=1}^{t+1}\alpha_i\Delta S_i\right)+\Sigma_{i=1}^{t+1}h_i}\,|\mathcal{F}_t\right).$$

The above inequality reduces to

$$E_{\mathbb{P}}\left(-e^{-\gamma\alpha_{t+1}\Delta S_{t+1}}\left|\mathcal{F}_t\right.\right)\leq -e^{-h_{t+1}},$$

and we easily conclude using Lemma 4.

To show (9) we work as follows. Let X_t^*, $t=0,1,...,$ given by (15). We need to establish

$$-e^{-\gamma X_t^*+\Sigma_{i=1}^{t}h_i}=E_{\mathbb{P}}\left(-e^{-\gamma X_{t+1}^*+\Sigma_{i=1}^{t+1}h_i}\left|\mathcal{F}_t\right.\right).$$

Using that $X_{t+1}^*=X_t^*+\alpha_{t+1}^*\bigtriangleup S_{t+1}$ and the measurability of the involved quantities, the above equality simplifies to $E_{\mathbb{P}}\left(e^{-\gamma\alpha_{t+1}^*\Delta S_{t+1}+h_{t+1}}\left|\mathcal{F}_t\right.\right)=1,$ and we conclude using Lemma 4 once more. ■

We note how $U_t(x)$ is constructed from one period to the next: at each time t,

$$\begin{aligned}U_t(x)&=U_{t-1}(x)\,e^{h_t}\\&=U_{t-1}(x)\left(\frac{q_t}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}\right)^{q_t}\left(\frac{1-q_t}{\mathbb{P}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}\right)^{1-q_t},\end{aligned}\qquad(17)$$

(cf. (13) and (11)). In other words, to construct $U_t(x)$, we need $U_{t-1}(x)$ and the single-period conditional risk neutral and historical probabilities q_t and $\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)$, measuring the movement of the traded asset for the next upcoming period only, conditionally on today's information. In other words, the forward process is constructed by progressive, forward in time, "single-period" model updates. Moreover, the forward performance process incorporates the market information from initial time 0 up to current time t, "path-by-path", as the term $H_{0,t}=e^{\Sigma_{i=1}^{t}h_i}$ indicates. Thus, $U_t(x)$ evolves in perfect alignment with the market, forward in time.

This is not the case in the classical expected utility framework. For a trading horizon, say $[0,T]$, the classical value process is of the form $V_{t,T}(x)=-e^{-\gamma x+\mathcal{H}_{t,T}}$, with $\mathcal{H}_{t,T}$ being the aggregate minimal entropy conditionally on $\mathcal{F}_t$, till the end of the investment horizon $[t,T]$ (see, for example, [21]). Therefore, for any time $t\in[0,T]$, its construction uses the model specification for the entire remaining investment time $[t,T]$, and incorporates the market information in a much coarser manner, through the term $\mathcal{H}_{t,T}$ associated with the average aggregate relative entropy from t to T, conditionally on $\mathcal{F}_t$.

3 Forward exponential indifference valuation

In this section, we recall the notion of the writer's forward exponential indifference price and provide an iterative algorithm for its construction. Such prices were first introduced in [12] (see, also, [11]) for European claims in a single period model. They were subsequently studied in diffusion models with stochastic volatility in [13], and for American-type claims in [7].

Herein, we consider a generic claim, written on *both* the traded stock and the non-traded factor, say at time t_0, taken for simplicity to be $t_0=0$. The claim matures at $t > 0$ yielding payoff C_t, represented as an $\mathcal{F}_t$-mble random variable.

For convenience, we eliminate the "exponential" terminology and also occasionally rewrite some quantities for the reader's convenience.

Definition 5 *Consider a claim, written at time $t_0 = 0$ and yielding at $t > 0$ payoff $C_t \in \mathcal{F}_t$. Let U_s, $s = 0, 1, ..., t$, be the forward performance process given by*

$$U_s(x) = -e^{-\gamma x + \Sigma_{i=1}^{s} h_i}$$

and h as in Definition 1 (cf. (13) and (11)).

For $s = 0, 1, ..., t-1$, the writer's forward indifference price is defined as the amount $\nu_s(C_t) \in \mathcal{F}_s$ such that, for all wealth levels $x \in \mathbb{R}$,

$$U_s(x) = \sup_{\alpha_{s+1},...,\alpha_t} E_{\mathbb{P}}\left(U_t\left(x + \nu_s(C_t) + \Sigma_{i=s+1}^{t} \alpha_i \Delta S_i - C_t\right) \middle| \mathcal{F}_s\right), \tag{18}$$

with $\alpha_i \in \mathcal{F}_{i-1}$, $i = s+1, ..., t$, and $\nu_t(C_t) = C_t$.

Similarly to the classical setting, the above indifference pricing condition reflects the indifference of the writer between two scenaria: start at s with wealth x and trade optimally till t without taking the claim in consideration, or start at s with wealth x and also accept the compensation $\nu_s(C_t)$, then trade optimally (with initial wealth $x + \nu_s(C_t)$) till t and also fulfill the liability C_t, at time t.

For the reader's convenience, we start with the construction of the indifference price $\nu_{t-1}(C_t)$, just one period before maturity. Its form will motivate the upcoming choices of the pricing functionals as well as the specification of the emerging pricing measure, for all previous times.

Lemma 6 *At time $t-1$, the indifference price $\nu_{t-1}(C_t)$ is given by*

$$\nu_{t-1}(C_t) = q_t \frac{1}{\gamma} \ln\left(\frac{E_{\mathbb{P}}\left(e^{\gamma C_t} \mathbf{1}_{A_t} \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t \middle| \mathcal{F}_{t-1}\right)} \right) + (1 - q_t) \frac{1}{\gamma} \ln \frac{E_{\mathbb{P}}\left(e^{\gamma C_t} \mathbf{1}_{A_t^c} \middle| \mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c \middle| \mathcal{F}_{t-1}\right)}, \tag{19}$$

with q_t and A_t as in (10) and (6).

Proof. We need to show that for $x \in \mathbb{R}$,

$$U_{t-1}(x) = \sup_{\alpha_t \in \mathcal{F}_{t-1}} E_{\mathbb{P}}\left(-e^{-\gamma(x + \nu_{t-1}(C_t) + \alpha_t \Delta S_t - C_t) + \Sigma_{i=1}^{t} h_i} \middle| \mathcal{F}_{t-1}\right),$$

with $\nu_{t-1}(C_t)$ as in (19). Using (13) and the measurability of the involved quantities, the above reduces to showing

$$\sup_{\alpha_t \in \mathcal{F}_{t-1}} E_{\mathbb{P}}\left(-e^{-\gamma(\nu_{t-1}(C_t) + \alpha_t \Delta S_t - C_t) + h_t} \middle| \mathcal{F}_{t-1}\right) = 1. \tag{20}$$

We have,

$$E_{\mathbb{P}}\left(-e^{-\gamma(\alpha_t \Delta S_t - C_t)} \middle| \mathcal{F}_{t-1}\right)$$

$$= E_{\mathbb{P}}\left(-e^{-\gamma\alpha_t S_{t-1}(\xi_t^u-1)}e^{\gamma C_t}\mathbf{1}_{A_t}\middle|\,\mathcal{F}_{t-1}\right) + E_{\mathbb{P}}\left(-e^{-\gamma\alpha_t S_{t-1}\left(\xi_t^d-1\right)}e^{\gamma Z}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right)$$

$$= -\left(e^{-\gamma\alpha_t S_{t-1}(\xi_t^u-1)}E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t}\middle|\,\mathcal{F}_{t-1}\right) + e^{-\gamma\alpha_t S_{t-1}\left(\xi_t^d-1\right)}E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right)\right)$$

$$= -\left(e^{-\gamma\alpha_t S_{t-1}(\xi_t^u-1)}Z_{t-1}^1 + e^{-\gamma\alpha_t S_{t-1}\left(\xi_t^d-1\right)}Z_{t-1}^2\right),$$

with the random variables Z_{t-1}^1, Z_{t-1}^2 defined as

$$Z_{t-1}^1 = E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t}\middle|\,\mathcal{F}_{t-1}\right) \quad \text{and} \quad Z_{t-1}^2 = E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right). \tag{21}$$

The optimum above occurs at

$$\alpha_t^{*,C_t} = \frac{1}{\gamma S_{t-1}\left(\xi_t^u - \xi_t^d\right)}\ln\frac{\left(\xi_t^u-1\right)Z_{t-1}^1}{\left(1-\xi_t^d\right)Z_{t-1}^2} = \frac{1}{\gamma S_{t-1}\left(\xi_t^u - \xi_t^d\right)}\ln\frac{\left(1-q_t\right)Z_{t-1}^1}{q_t Z_{t-1}^2}.$$

In turn,

$$E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_t^{*,C_t}\Delta S_t - C_t\right)}\middle|\,\mathcal{F}_{t-1}\right)$$

$$= -\left(\left(\frac{1-q_t}{q_t}\frac{Z_{t-1}^1}{Z_{t-1}^2}\right)^{-(1-q_t)}Z_{t-1}^1 + \left(\frac{1-q_t}{q_t}\frac{Z_{t-1}^1}{Z_{t-1}^2}\right)^{q_t}Z_{t-1}^2\right)$$

$$= -\left(\frac{Z_{t-1}^1}{q_t}\right)^{q_t}\left(\frac{Z_{t-1}^2}{1-q_t}\right)^{1-q_t}.$$

Therefore,

$$E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_t^{*,C_t}\Delta S_t - C_t\right)}\middle|\,\mathcal{F}_{t-1}\right) = -\exp\left(\ln\left(\left(\frac{Z_{t-1}^1}{q_t}\right)^{q_t}\left(\frac{Z_{t-1}^2}{1-q_t}\right)^{1-q_t}\right)\right)$$

$$= -\exp\left(q_t\ln\frac{Z_{t-1}^1}{q_t} + (1-q_t)\ln\frac{Z_{t-1}^2}{1-q_t}\right).$$

Next, observe that

$$q_t\ln\frac{Z_{t-1}^1}{q_t} + (1-q_t)\ln\frac{Z_{t-1}^2}{1-q_t}$$

$$= q_t\ln\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t}\middle|\,\mathcal{F}_{t-1}\right)}{q_t} + (1-q_t)\ln\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right)}{1-q_t}$$

$$= q_t\ln\left(\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t}\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}\right) + (1-q_t)\ln\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}$$

$$-\left(q_t\ln\frac{\mathbb{P}\left(A_t|\,\mathcal{F}_{t-1}\right)}{q_t} + (1-q_t)\ln\frac{\mathbb{P}\left(A_t^c|\,\mathcal{F}_{t-1}\right)}{1-q_t}\right)$$

$$= \gamma\nu_{t-1}\left(C_t\right) - h_t.$$

Therefore,

$$\sup_{\alpha_t \in \mathcal{F}_{t-1}} E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_t^{*,C_t}\Delta S_t - C_t\right)}\Big|\,\mathcal{F}_{t-1}\right) = -e^{\gamma\nu_{t-1}(C_t)-h_t},$$

and (20) follows. ■

Next, we make the following key observations. First, let us define the random variable

$$\mathcal{C}^{(t-1,t)}\left(C_t\right) := \frac{1}{\gamma}\ln\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\big|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t\right|\mathcal{F}_{t-1})}\mathbf{1}_{A_t} + \frac{1}{\gamma}\ln\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\big|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c\right|\mathcal{F}_{t-1})}\mathbf{1}_{A_t^c}, \tag{22}$$

and observe that $\mathcal{C}^{(t-1,t)}\left(C_t\right) \in \mathcal{F}_t^S$. In particular, it can be expressed as

$$\mathcal{C}^{(t-1,t)}\left(C_t\right) = \frac{1}{\gamma}\ln E_{\mathbb{P}}\left(e^{\gamma C_t}\big|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right). \tag{23}$$

In turn, observe that (19) yields that the indifference price is the conditional expectation of $\mathcal{C}^{(t-1,t)}\left(C_t\right)$ under any martingale measure, namely, for all $\mathbb{Q} \in \mathcal{Q}_t$,

$$\nu_{t-1}\left(C_t\right) = E_{\mathbb{Q}}\left(\mathcal{C}^{(t-1,t)}\left(C_t\right)\Big|\,\mathcal{F}_{t-1}\right). \tag{24}$$

What the above tells us is that the indifference price $\nu_{t-1}\left(C_t\right)$ is constructed via a two-step pricing procedure. In the first step, the claim's payoff C_t is "*distorted*", conditionally on $\mathcal{F}_{t-1}\vee\mathcal{F}_t^S$, and the intermediate payoff $\mathcal{C}^{(t-1,t)}\left(C_t\right)$ is created. The payoff is nonlinear and $\mathcal{F}_t^S$-mble. In the second step, the indifference price is produced as the *arbitrage-free* price of this intermediate payoff $\mathcal{C}^{(t-1,t)}\left(C_t\right)$.

Note that if $C_t \in \mathcal{F}_t^S$, then, $\mathcal{C}^{(t-1,t)}\left(C_t\right) = C_t$ and, naturally, $\nu_{t-1}\left(C_t\right) = E_{\mathbb{Q}}\left(C_t\right|\mathcal{F}_{t-1})$. In general, the price can be represented as the non-linear expression

$$\nu_{t-1}\left(C_t\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\ln E_{\mathbb{P}}\left(e^{\gamma C_t}\big|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)\Big|\,\mathcal{F}_{t-1}\right),$$

which involves an *inner non-linear* expression under the *historical* measure, and an *outer conditional expectation* under any *martingale* measure.

Next, we pose the question whether we can actually express the price $\nu_{t-1}\left(C_t\right)$ as

$$\nu_{t-1}\left(C_t\right) = \mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}\left(C_t\right),$$

for an appropriate *indifference pricing non-linear functional* and a *specific* martingale measure (not necessarily unique) $\mathbb{Q}^* \in \mathcal{Q}_t$. This will provide an intuitively pleasing non-linear analogue of forward indifference prices to their arbitrage-free counterparts.

To this end, the values of the payoff $\mathcal{C}^{(t-1,t)}\left(C_t\right)$ suggest that we should seek a martingale measure $\mathbb{Q}^* \in \mathcal{Q}_t$ such that

$$\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t}\big|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t\right|\mathcal{F}_{t-1})} = \frac{E_{\mathbb{Q}^*}\left(e^{\gamma C_t}\mathbf{1}_{A_t}\big|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^*\left(A_t\right|\mathcal{F}_{t-1})}$$

and

$$\frac{E_{\mathbb{P}}\left(e^{\gamma C_t}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c\middle|\,\mathcal{F}_{t-1}\right)}=\frac{E_{\mathbb{Q}^*}\left(e^{\gamma C_t}\mathbf{1}_{A_t^c}\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{Q}^*\left(A_t^c\middle|\,\mathcal{F}_{t-1}\right)}$$

We, then, see that it suffices for the candidate measure $\mathbb{Q}^*$ to satisfy

$$\frac{\mathbb{Q}^*\left(A_tB_t\middle|\,\mathcal{F}_{t-1}\right)}{q_t}=\frac{\mathbb{P}\left(A_tB_t\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t\middle|\,\mathcal{F}_{t-1}\right)},\quad \frac{\mathbb{Q}^*\left(A_tB_t^c\middle|\,\mathcal{F}_{t-1}\right)}{q_t}=\frac{\mathbb{P}\left(A_tB_t^c\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t\middle|\,\mathcal{F}_{t-1}\right)} \tag{25}$$

$$\frac{\mathbb{Q}^*\left(A_t^cB_t\middle|\,\mathcal{F}_{t-1}\right)}{1-q_t}=\frac{\mathbb{P}\left(A_t^cB_t\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c\middle|\,\mathcal{F}_{t-1}\right)},\quad \frac{\mathbb{Q}^*\left(A_t^cB_t^c\middle|\,\mathcal{F}_{t-1}\right)}{1-q_t}=\frac{\mathbb{P}\left(A_t^cB_t^c\middle|\,\mathcal{F}_{t-1}\right)}{\mathbb{P}\left(A_t^c\middle|\,\mathcal{F}_{t-1}\right)}. \tag{26}$$

In turn, we observe that, under such $\mathbb{Q}^*$, the intermediate payoff $\mathcal{C}^{(t-1,t)}\left(C_t\right)$ retains its form, in that under both $\mathbb{P}$ and $\mathbb{Q}^*$,

$$\mathcal{C}^{(t-1,t)}\left(C_t\right)=\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma C_t}\middle|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)=\frac{1}{\gamma}\ln E_{\mathbb{P}}\left(e^{\gamma C_t}\middle|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right).$$

We then see that if we define the non-linear pricing functional

$$\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}\left(Z\right):=E_{\mathbb{Q}^*}\left(\mathcal{C}^{(t-1,t)}\left(Z\right)\middle|\,\mathcal{F}_t\right)=E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma Z}\middle|\,\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)\middle|\,\mathcal{F}_{t-1}\right), \tag{27}$$

for a generic $Z\in\mathcal{F}_t$, we can actually express the indifference price at $t-1$ in the desired concise form

$$\nu_{t-1}\left(C_t\right)=\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}\left(C_t\right). \tag{28}$$

Observe that the measure $\mathbb{Q}^*$ is used in both expectations in (27) since the outer one is applied to an $\mathcal{F}_t^S-$mble random variable.

Notice that despite the fact that both forward random functionals U_t and U_{t-1}, entering in the derivation of $\nu_{t-1}\left(C_t\right)$, are path-dependent through the terms $\Sigma_{i=1}^t h_i$ and $\Sigma_{i=1}^{t-1} h_i$ appearing in their exponents (cf. (13)), the indifference price $\nu_{t-1}\left(C_t\right)$ takes a substantially simplified "single-period" form.

Furthermore, the involved conditional probabilities of the emerging pricing measure $\mathbb{Q}^*$ have also "single-period" dependence, since they are determined *exclusively* by q_t and $\mathbb{P}\left(A_t\middle|\,\mathcal{F}_{t-1}\right)$ (cf. (25),(26)).

A natural question arises, given the nonlinearity of $\mathcal{C}^{(t-1,t)}\left(C_t\right)$, whether it can be interpreted as a certainty equivalent of some form. In section 4, we show that this is indeed the case. Specifically, we establish that

$$\mathcal{C}^{(t-1,t)}\left(C_t\right)=-U_t^{(-1)}\left(E_{\mathbb{P}}\left(U_t\left(-C_t\right)\middle|\mathcal{F}_{t-1}\vee\mathcal{F}_t^S\right)\right).$$

This also yields a natural interpretation of $\nu_{t-1}\left(C_t\right)$ as the arbitrage-free price of a payoff with certainty equivalent characteristics.

In the next section, we show how to extend the above constructions and interpetations to all previous periods $t-2,t-3,...,$ define analogous to (23), (24) pricing functionals and specify a pricing measure from the martingale ones satisfying similar to (25),(26) properties.

4 The (writer's) forward indifference pricing algorithm

Motivated by the form of the indifference price $\nu_{t-1}(C_t)$ in (28), we seek an analogous price representation,

$$\nu_s(C_t) = \mathcal{E}_{\mathbb{Q}^*}^{(s,t)}(C_t),$$

for an appropriate chosen multi-period valuation functional $\mathcal{E}_{\mathbb{Q}^*}^{(s,t)}$ and a pricing measure $\mathbb{Q}^*$, for $s = 0, 1, ..., t$. As for the case $s = t-1$, the main challenge is how to incorporate the path dependence of the forward functionals U_s and U_t (appearing in Definition 5) coming from the terms $\Sigma_{i=1}^s h_i$ and $\Sigma_{i=1}^t h_i$ (cf. (13)).

We propose such a multi-period pricing functional of an iterative form, $\mathcal{E}_{\mathbb{Q}^*}^{(s,t)}(\cdot) = \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}\left(\mathcal{E}_{\mathbb{Q}^*}^{(s+1,s+2)} ... \mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}(\cdot)\right)$, with the single-period pricing functionals resembling (27). We also show that the appropriate pricing measure $\mathbb{Q}^*$ is a martingale one that has similar to (25),(26) local properties. Furthermore, we prove that it actually minimizes the reverse relative entropy in $[0,t]$ over all martingale measures defined on $\mathcal{F}_t$.

4.1 The forward indifference pricing measure, and the single- and multi-step valuation functionals

We start with some introductory results and notation. For $t = 1, 2, ...,$ recall that $\mathcal{Q}_t$ is the set of equivalent martingale measures and $\mathbb{Q}$ its generic element.

For $s = 1, 2, ..., t$, we have

$$\mathbb{Q}\left(\xi_s \in \left\{\xi_s^d, \xi_s^u\right\}, \eta_s \in \left\{\eta_s^d, \eta_s^u\right\} | \mathcal{F}_{s-1}\right) = \mathbb{Q}\left(A_s B_s | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s B_s}$$
$$+\mathbb{Q}\left(A_s B_s^c | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s B_s^c} + \mathbb{Q}\left(A_s^c B_s | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s^c B_s} + \mathbb{Q}\left(A_s^c B_s^c | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s^c B_s^c},$$

and, similarly, for the historical measure,

$$\mathbb{P}\left(\xi_s \in \left\{\xi_s^d, \xi_s^u\right\}, \eta_s \in \left\{\eta_s^d, \eta_s^u\right\} | \mathcal{F}_{s-1}\right) = \mathbb{P}\left(A_s B_s | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s B_s}$$
$$+\mathbb{P}\left(A_s B_s^c | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s B_s^c} + \mathbb{P}\left(A_s^c B_s | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s^c B_s} + \mathbb{P}\left(A_s^c B_s^c | \mathcal{F}_{s-1}\right) \mathbf{1}_{A_s^c B_s^c},$$

with ξ_s, η_s, A_s, B_s as in (4),(5) and (6).

We will be using the *condensed* notation for the conditional distributions

$$\mathbb{Q}\left(\xi_s, \eta_s | \mathcal{F}_{s-1}\right) \triangleq \mathbb{Q}\left(\xi_s \in \left\{\xi_s^d, \xi_s^u\right\}, \eta_s \in \left\{\eta_s^d, \eta_s^u\right\} | \mathcal{F}_{s-1}\right) \tag{29}$$

and

$$\mathbb{P}\left(\xi_s, \eta_s | \mathcal{F}_{s-1}\right) \triangleq \mathbb{P}\left(\xi_s \in \left\{\xi_s^d, \xi_s^u\right\}, \eta_s \in \left\{\eta_s^d, \eta_s^u\right\} | \mathcal{F}_{s-1}\right). \tag{30}$$

Next, we seek a martingale measure that *minimizes,* for $s = 1, 2, ..., t$, the conditional expectation

$$-E_{\mathbb{P}}\left(\ln \frac{\mathbb{Q}\left(\xi_s, \eta_s | \mathcal{F}_{s-1}\right)}{\mathbb{P}\left(\xi_s, \eta_s | \mathcal{F}_{s-1}\right)} \middle| \mathcal{F}_{s-1}\right). \tag{31}$$

For this, we need to find the minimizers of

$$-\left(\mathbb{P}\left(A_s B_s\left|\mathcal{F}_{s-1}\right.\right)\ln\frac{\mathbb{Q}\left(A_s,B_s\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(A_s,B_s\left|\mathcal{F}_{s-1}\right.\right)}+\mathbb{P}\left(A_s B_s^c\left|\mathcal{F}_{s-1}\right.\right)\ln\frac{\mathbb{Q}\left(A_s B_s^c\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(A_s B_s^c\left|\mathcal{F}_{s-1}\right.\right)}\right.$$

$$\left.+\mathbb{P}\left(A_s^c B_s\left|\mathcal{F}_{s-1}\right.\right)\ln\frac{\mathbb{Q}\left(A_s^c B_s\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(A_s^c B_s\left|\mathcal{F}_{s-1}\right.\right)}+\mathbb{P}\left(A_s^c B_s^c\left|\mathcal{F}_{s-1}\right.\right)\ln\frac{\mathbb{Q}\left(A_s^c B_s^c\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(A_s^c B_s^c\left|\mathcal{F}_{s-1}\right.\right)}\right).$$

Direct calculations yield the above quantity is minimized if one chooses

$$\frac{\mathbb{Q}^*\left(\left.A_s B_s\right|\mathcal{F}_{s-1}\right)}{q_s}=\frac{\mathbb{P}\left(\left.A_s B_s\right|\mathcal{F}_{s-1}\right)}{\mathbb{P}\left(\left.A_s\right|\mathcal{F}_{s-1}\right)},\quad \frac{\mathbb{Q}^*\left(\left.A_s B_s^c\right|\mathcal{F}_{s-1}\right)}{q_s}=\frac{\mathbb{P}\left(\left.A_s B_s^c\right|\mathcal{F}_{s-1}\right)}{\mathbb{P}\left(\left.A_s\right|\mathcal{F}_{s-1}\right)}, \tag{32}$$

$$\frac{\mathbb{Q}^*\left(\left.A_s^c B_s\right|\mathcal{F}_{s-1}\right)}{1-q_s}=\frac{\mathbb{P}\left(\left.A_s^c,B_s\right|\mathcal{F}_{s-1}\right)}{\mathbb{P}\left(\left.A_s^c\right|\mathcal{F}_{s-1}\right)},\quad \frac{\mathbb{Q}^*\left(\left.A_s^c B_s^c\right|\mathcal{F}_{s-1}\right)}{1-q_s}=\frac{\mathbb{P}\left(\left.A_s^c,B_s^c\right|\mathcal{F}_{s-1}\right)}{\mathbb{P}\left(\left.A_s^c\right|\mathcal{F}_{s-1}\right)}. \tag{33}$$

Indeed, observe that the function $f(z)=-\left(\left(\ln\frac{z}{\alpha}\right)\alpha+\left(\ln\frac{c-z}{\beta}\right)\beta\right)$, with $\alpha,\beta,c\in(0,1)$ achieves for $z\in[0,c]$ a minimum at the point $z^*=\frac{\alpha}{\alpha+\beta}c$. Applying this for the triplets $(\alpha,\beta,c)=\left(\mathbb{P}\left(\left.A_s B_s\right|\mathcal{F}_{s-1}\right),\mathbb{P}\left(\left.A_s B_s^c\right|\mathcal{F}_{s-1}\right),q_s\right)$ and $(\alpha,\beta,c)=\left(\mathbb{P}\left(\left.A_s^c B_s\right|\mathcal{F}_{s-1}\right),\mathbb{P}\left(\left.A_s^c B_s^c\right|\mathcal{F}_{s-1}\right),1-q_s\right)$, respectively, we conclude.

We then consider a martingale measure $\mathbb{Q}^*$, defined on $\mathcal{F}_t$, satisfying, for $t=1,2,...$, the conditional properties (32) and (33), and we claim that it is well defined. Indeed, for $t=1,2,...$, we have

$$\mathbb{Q}^*\left(\xi_1\in\left\{\xi_1^d,\xi_1^u\right\},...,\xi_t\in\left\{\xi_t^d,\xi_t^u\right\},\eta_1\in\left\{\eta_1^d,\eta_1^u\right\},...,\eta_t\in\left\{\eta_t^d,\eta_t^u\right\}\right) \tag{34}$$

$$=\prod_{s=1}^{t}\mathbb{Q}^*\left(\xi_s\in\left\{\xi_s^d,\xi_s^u\right\},\eta_s\left\{\eta_s^d,\eta_s^u\right\}\middle|\mathcal{F}_{s-1}\right)=\prod_{s=1}^{t}\mathbb{Q}^*\left(\left.\xi_s,\eta_s\right|\mathcal{F}_{s-1}\right),$$

with each term being well defined from (32) and (33).

Next, we derive the following characterization result in terms of the reverse relative entropy measure, in which we use the self-evident condensed expressions $\mathbb{Q}(\xi_1,...\xi_t,\eta_1,...,\eta_t)$ and $\mathbb{P}(\xi_1,...\xi_t,\eta_1,...,\eta_t)$ to denote the joint distributions of $(\xi_1,...\xi_t,\eta_1,...,\eta_t)$ under $\mathbb{Q}$ and $\mathbb{P}$, respectively.

Proposition 7 *For $t=1,2,...$, let $\mathcal{Q}_t$ be the set of equivalent martingale measures and $\mathbb{Q}^*\in\mathcal{Q}_t$ defined as in (34). Then $\mathbb{Q}^*$ minimizes the reverse relative entropy $\mathcal{H}_t$, defined as*

$$\mathcal{H}_t=-E_{\mathbb{P}}\left(\ln\frac{\mathbb{Q}(\xi_1,...,\xi_t,\eta_1,...,\eta_t)}{\mathbb{P}(\xi_1,...,\xi_t,\eta_1,...,\eta_t)}\right), \tag{35}$$

for $\mathbb{Q}\in\mathcal{Q}_t$.

Proof. First, observe that (35) can be written as

$$\mathcal{H}_t=-\sum_{s=1}^{t}E_{\mathbb{P}}\left(\ln\frac{\mathbb{Q}\left(\xi_s,\eta_s\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(\xi_s,\eta_s\left|\mathcal{F}_{s-1}\right.\right)}\right).$$

and, in turn,

$$\mathcal{H}_t = -\sum_{s=1}^{t} E_{\mathbb{P}}\left(E_{\mathbb{P}}\left(\ln \frac{\mathbb{Q}\left(\xi_s,\eta_s\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(\xi_s,\eta_s\left|\mathcal{F}_{s-1}\right.\right)}\right) \middle| \mathcal{F}_{s-1}\right),$$

and we easily conclude. ■

The next result shows a key property of the measure $\mathbb{Q}^*$. It also provides equalities (37), (38) which will play a main role in the construction of the forward indifference prices. Its proof follows easily.

Proposition 8 *i) The martingale measure $\mathbb{Q}^*$ defined as in (34) satisfies, for $t = 1, 2, ...,$ and $s = 1, 2, ..., t,$*

$$\mathbb{Q}^*\left(Y_s\left|\mathcal{F}_{s-1} \vee \mathcal{F}_s^S\right.\right) = \mathbb{P}\left(Y_s\left|\mathcal{F}_{s-1} \vee \mathcal{F}_s^S\right.\right), \tag{36}$$

with the stochastic factor Y_s given in (5).

ii) Moreover, if Z is an $\mathcal{F}_s$-mble random variable and A_s as in (6), we have

$$\frac{E_{\mathbb{P}}\left(Z\mathbf{1}_{A_s}\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(A_s\left|\mathcal{F}_{s-1}\right.\right)} = \frac{E_{\mathbb{Q}^*}\left(Z\mathbf{1}_{A_s}\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{Q}^*\left(A_s\left|\mathcal{F}_{s-1}\right.\right)} \tag{37}$$

and

$$\frac{E_{\mathbb{P}}\left(Z\mathbf{1}_{A_s^c}\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{P}\left(A_s^c\left|\mathcal{F}_{s-1}\right.\right)} = \frac{E_{\mathbb{Q}^*}\left(Z\mathbf{1}_{A_t^c}\left|\mathcal{F}_{s-1}\right.\right)}{\mathbb{Q}^*\left(A_s^c\left|\mathcal{F}_{s-1}\right.\right)}. \tag{38}$$

We introduce the following single- and multi-period forward pricing functionals.

Definition 9 *For $t > 0$, let $\mathbb{Q}^*$be the martingale measure as in (36) and, for $s = 0, 1, ..., t-1$, let Z be an $\mathcal{F}_{s+1}$-mble random variable in $(\Omega, \mathcal{F}, \mathbb{P})$. We define*

i) the single-step forward price functional

$$\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z) = E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)|\mathcal{F}_s\right) \tag{39}$$

and,

ii) the multi-step forward price functional, $0 \leq s < s' \leq t$,

$$\mathcal{E}_{\mathbb{Q}^*}^{(s,s')}(Z) = \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(\mathcal{E}_{\mathbb{Q}^*}^{(s+1,s+2)}(...\mathcal{E}_{\mathbb{Q}^*}^{(s'-1,s')}(Z))). \tag{40}$$

Remark 10 *We caution the reader that, in general, for $s' > s+1$ and $Z \in \mathcal{F}_{s'}$,*

$$\mathcal{E}_{\mathbb{Q}^*}^{(s,s')}(Z) \neq E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s'}^S\right.\right)|\mathcal{F}_s\right). \tag{41}$$

The reader familiar with existing indifference pricing algorithms (see, among others, [1], [10], [11], [23], [24]), might find the form of $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ and of $\mathcal{E}_{\mathbb{Q}^*}^{(s,s')}$ identical to the ones appearing in these references. This is *not*, however, the case. The results herein are, not only, derived for entirely different risk preference criteria but, also, for more general incomplete market environments, since the nested market model (bond and stock) is incomplete. Moreover, the involved measure is not the minimal entropy one but, rather, the minimal reverse entropy measure.

Another difference, as we show in Proposition 16, is that $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ provides an intuitively pleasing *direct analogue* of the arbitrage-free price of a conditional certainty equivalent, while in the classical exponential utility such analogy fails.

The following auxiliary result will be used repeatedly in the construction of the forward pricing algorithm.

Lemma 11 *Let $t>0$, $s=0,1,...,t-1$, and $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ be as in (39). Then, if Z is an $\mathcal{F}_{s+1}$-mble random variable,*

$$\sup_{\alpha_{s+1}\in\mathcal{F}_s} E_{\mathbb{P}}\left(-e^{-\gamma(\alpha_{s+1}\Delta S_{s+1}-Z)+h_{s+1}}\middle|\,\mathcal{F}_s\right)=-e^{\gamma\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z)},$$

with h_s as in (11).

Proof. The proof follows by analogous arguments as the ones used to show Lemma 6. For this, we only highlight the main steps. We have

$$E_{\mathbb{P}}\left(-e^{-\gamma(\alpha_{s+1}\Delta S_{s+1}-Z)}\middle|\,\mathcal{F}_s\right)=-\left(e^{-\gamma\alpha_{s+1}S_s\left(\xi_{s+1}^u-1\right)}Z_s^1+e^{-\gamma\alpha_{s+1}S_s\left(\xi_{s+1}^d-1\right)}Z_s^2\right)$$

with $Z_s^1=E_{\mathbb{P}}\left(e^{\gamma Z}\mathbf{1}_{A_{s+1}}\middle|\,\mathcal{F}_s\right)$ and $Z_s^2=E_{\mathbb{P}}\left(e^{\gamma Z}\mathbf{1}_{A_{s+1}^c}\middle|\,\mathcal{F}_s\right)$. The optimum occurs at the point

$$\alpha_{s+1}^{*,Z}=\frac{1}{\gamma S_s\left(\xi_{s+1}^u-\xi_{s+1}^d\right)}\ln\frac{\left(1-q_{s+1}\right)Z_s^1}{q_{s+1}Z_s^2}$$

at which we have

$$E_{\mathbb{P}}\left(-e^{-\gamma\left(\alpha_{s+1}^*\Delta S_{s+1}-Z\right)}\middle|\,\mathcal{F}_s\right)=-\left(\frac{Z_s^1}{q_{s+1}}\right)^{q_{s+1}}\left(\frac{Z_s^2}{1-q_{s+1}}\right)^{1-q_{s+1}}.$$

$$=-\exp\left(q_{s+1}\ln\frac{Z_s^1}{q_{s+1}}+\left(1-q_{s+1}\right)\ln\frac{Z_s^2}{1-q_{s+1}}\right).$$

Working as in the proof of Lemma 6, we express the above quantity with respect to the $\mathbb{Q}^*$ measure,

$$q_{s+1}\ln\frac{Z_s^1}{q_{s+1}}+\left(1-q_{s+1}\right)\ln\frac{Z_s^2}{1-q_{s+1}}$$

$$=q_{s+1}\ln\frac{E_{\mathbb{Q}^*}\left(e^{\gamma Z}\mathbf{1}_{A_{s+1}}\middle|\,\mathcal{F}_s\right)}{\mathbb{Q}^*\left(A_{s+1}\middle|\,\mathcal{F}_s\right)}+\left(1-q_{s+1}\right)\ln\frac{E_{\mathbb{Q}^*}\left(e^{\gamma Z}\mathbf{1}_{A_{s+1}^c}\middle|\,\mathcal{F}_s\right)}{\mathbb{Q}^*\left(A_{s+1}^c\middle|\,\mathcal{F}_s\right)}-h_{s+1} \quad (42)$$

where, we used the definition of h_{s+1}, the measurability of Z and the second part of Proposition 4, and the definition of $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$. We easily conclude. ■

We are now ready to present the forward indifference pricing algorithm.

Theorem 12 *Consider a claim, introduced at time $t_0 = 0$, yielding at time $t > 0$, payoff $C_t \in \mathcal{F}_t$, and $\nu_s(C_t)$ be defined as in (18). Let, also, $\mathbb{Q}^*$ be as in (36), and $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^*}^{(s,s')}$ as in (39) and (40), respectively. The following statements hold:*

i) The forward indifference price, $\nu_s(C_t)$, is given, for $s = 0, 1, ..., t-1$, by the iterative algorithm

$$\nu_t(C_t) = C_t,$$

$$\nu_s(C_t) = \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(\nu_{s+1}(C_t)) \tag{43}$$

$$= E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma\nu_{s+1}(C_t)}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)|\mathcal{F}_s\right).$$

ii) The forward indifference price process $\nu_s(C_t) \in \mathcal{F}_s$ and satisfies, for $s = 0, 1, ..., t-1$,

$$\nu_s(C_t) = \mathcal{E}_{\mathbb{Q}^*}^{(s,t)}(C_t). \tag{44}$$

iii) The forward indifference price algorithm is consistent across time in that, for $0 \le s \le s' < t$, the semigroup property

$$\nu_s(C_t) = \mathcal{E}_{\mathbb{Q}^*}^{(s,s')}(\mathcal{E}_{\mathbb{Q}^*}^{(s',t)}(C_t)) \tag{45}$$

$$= \mathcal{E}_{\mathbb{Q}^*}^{(s,s')}(\nu_{s'}(C_t)) = \nu_s(\mathcal{E}_{\mathbb{Q}^*}^{(s',t)}(C_t))$$

holds.

Proof. Assertions (43) and (44) were proved in Lemma 6 for $s = t-1$.

To show (43) for $s = t-2$, we first observe that representation (13) yields with repeated use of Lemma 11 and (13), and $\alpha_{t-1} \in \mathcal{F}_{t-2}, \alpha_t \in \mathcal{F}_{t-1}$,

$$\sup_{\alpha_{t-1},\alpha_t} E_{\mathbb{P}}\left(U_t(X_t - C_t)|\mathcal{F}_{t-2}\right) = \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(\sup_{\alpha_t} E_{\mathbb{P}}\left(U_t(X_t - C_t)|\mathcal{F}_{t-1}\right)|\mathcal{F}_{t-2}\right)$$

$$= \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma(x+\alpha_{t-1}\Delta S_{t-1})+\Sigma_{i=1}^{t-1}h_i}\sup_{\alpha_t} E_{\mathbb{P}}\left(-e^{-\gamma(\alpha_t\Delta S_t - C_t)+h_t}|\mathcal{F}_{t-1}\right)|\mathcal{F}_{t-2}\right)$$

$$= \sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(e^{-\gamma(x+\alpha_{t-1}\triangle S_{t-1})+\Sigma_{i=1}^{t-1}h_i}\left(-e^{\gamma\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}(C_t)}\right)|\mathcal{F}_{t-2}\right)$$

$$= e^{\Sigma_{i=1}^{t-2}h_i}\sup_{\alpha_{t-1}} E_{\mathbb{P}}\left(-e^{-\gamma(x+\alpha_{t-1}\Delta S_{t-1}-\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}(C_t))+h_{t-1}}|\mathcal{F}_{t-2}\right)$$

$$= -e^{-\gamma\left(x-\mathcal{E}_{\mathbb{Q}^*}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}(C_t)\right)\right)+\Sigma_{i=1}^{t-2}h_i} = U_{t-2}\left(x - \mathcal{E}_{\mathbb{Q}^*}^{(t-2,t-1)}\left(\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}(C_t)\right)\right)$$

$$= U_{t-2}\left(x - \mathcal{E}_{\mathbb{Q}^*}^{(t-2,t)}(C_t)\right).$$

The rest of the assertions follow along similar albeit tedious arguments. ■

We conclude with the case of multiple claims. Before we present the general result, let us consider the simple case of two claims, written at $t = 0$ and maturing at $t-1$ and t, yielding payoffs $C_{t-1} \in \mathcal{F}_{t-1}$ and $C_t \in \mathcal{F}_t$, respectively.

Then, since $C_{t-1} \in \mathcal{F}_t$, we have

$$\nu_{t-1}(C_{t-1} + C_t) = \mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}(C_{t-1} + C_t)$$

$$= E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma(C_{t-1}+C_t)}\left|\mathcal{F}_{t-1} \vee \mathcal{F}_t^S\right.\right)|\mathcal{F}_{t-1}\right) = C_{t-1} + \nu_{t-1}(C_t).$$

Trivially, one may view $C_{t-1} + \nu_{t-1}(C_t)$ as a new claim maturing at time $t-1$, and price it iteratively for $s = t-2, t-3, .., 0$. The assumption that all claims are written at time $t = 0$ can be easily removed. Note, however, what in both cases (i.e. common or varying inscription times), the market model needs to be specified at time 0 till the longest a priori known maturity.

Corollary 13 *Let $t = 1, 2, ...$ and $s = 0, 1, ..., t-1$. Consider claims $C_s, ..., C_j, ...C_t$ with $C_j \in \mathcal{F}_j$, $j = s, ..., t$, written at $t = 0$. The forward indifference price, $\nu_s(\Sigma_{j=s}^t C_j)$, is given, for $s = 0, 1, ..., t-1$, by the iterative algorithm*

$$\nu_t\left(C_t\right) = C_t,$$

$$\nu_s(\Sigma_{j=s}^t C_j) = C_s + \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(C_{s+1} + \nu_{s+1}(\Sigma_{j=s+2}^t C_j))$$

$$= C_s + E_{\mathbb{Q}^*}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma\left(C_{s+1}+\nu_{s+1}(\Sigma_{j=s+2}^t C_j)\right)}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)|\mathcal{F}_s\right).$$

ii) The forward indifference price process $\nu_s(\Sigma_{j=s}^t C_j) \in \mathcal{F}_s$ and satisfies, for $s = 0, 1, ..., t$,

$$\nu_s(\Sigma_{j=s}^t C_j)$$

$$= C_s + \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}\left(C_{s+1} + \mathcal{E}_{\mathbb{Q}^*}^{(s+1,s+2)}\left(C_{s+2} + ...\mathcal{E}_{\mathbb{Q}^*}^{(t-1,t)}\left(C_t\right)\right)\right).$$

An interesting case arises when there is *no a priori* knowledge at initial time 0 about all incoming claims and their maturities.

For example, consider the case that a single claim, C_t, is written at time 0 that matures at time t, but it is not known whether additional claims will arrive. Then, at time $s \in (0, t]$, a new claim, say $\tilde{C}_{t'}$, arrives with expiration t'. If $t' < t$, then its valuation is easily accommodated by the above Corollary.

If, however, $t' > t$, then one first needs to specify at time s the market model for the period $(t, t']$, and, in turn, employ the forward exponential criterion for times $t+1, t+2, ..., t'$, and price by indifference. This can be readily done, however, since the forward process can be defined for all times, sequentially forward in time.

Note that in the traditional expected utility framework, such flexibility does *not* exist. Indeed, once the investment horizon $[0, t]$ is prespecified at time 0, only claims maturing at times up to t can be priced. Any claim arriving later and with maturity beyond t cannot be priced, because the expected utility problem cannot be extended beyond t unless time-consistency is violated.

5 Properties of forward exponential indifference prices

The forward indifference pries is constructed via the *optimal* behavior of the investor with and without the claim in consideration. As such, it incorporates and reflects the individual risk preferences. Due to the exponential choice, it is independent of the investor's wealth.

- **Time consistency**

  The forward pricing operator $\mathcal{E}_{\mathbb{Q}^*}^{(s,t)}$ is time consistent, in that the price at any intermediate time, say s, can be thought as the price of a claim equal to the corresponding indifference price at a future time s', namely,

$$\nu_s(C_t) = \nu_s(\nu_{s'}(C_t)), \quad 0 \leq s \leq s' \leq t.$$

  This property is reflected in (45).

- **Scaling and monotonicity properties**

  The following properties of $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z)$, $Z, Z' \in \mathcal{F}_{s+1}$ hold:

i) The mapping $\gamma \to \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z;\gamma)$ is increasing and continuous, and

$$\lim_{\gamma \to 0^+} \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z;\gamma) = E_{\mathbb{Q}^*}(Z) \quad \text{and} \quad \lim_{\gamma \to \infty} \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z;\gamma) = E_{\mathbb{Q}^*} \|Z\|_{L^\infty_{\mathbb{Q}^*\{\cdot|S_s\}}}.$$

Moreover,

$$\lim_{\gamma \to 0} \frac{\partial}{\partial \gamma} \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z;\gamma) = \frac{1}{2} E_{\mathbb{Q}^*}\left(Var_{\mathbb{Q}^*}(Z\,|S_s)\right),$$

and, thus,

$$\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z;\gamma) = E_{\mathbb{Q}^*}(Z) + \frac{1}{2}\gamma E_{\mathbb{Q}^*}\left(Var_{\mathbb{Q}^*}(Z\,|S_s)\right) + o(\gamma).$$

The assertions follow by routine arguments, and their proof is ommitted.

iii) For $\alpha \in (0,1)$, Hölder's inequality gives

$$\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(\alpha Z + (1-\alpha)Z') \leq \alpha \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z) + (1-\alpha)\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z').$$

iv) For $\alpha > 1$, Jensen's inequality yields

$$\alpha \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z) \leq \mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(\alpha Z),$$

and the reverse inequality for $\alpha \in (0,1)$.

v) Let $Z = \tilde{Z} + \bar{Z}$, such that $\tilde{Z} \in \mathcal{F}_{s+1}$ and $\bar{Z} \in \mathcal{F}_{s+1}^S$ Then,

$$\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}(Z) = \mathcal{E}_{\mathbb{Q}^*}^{(s,t)}\left(\tilde{Z}\right) + E_{\mathbb{Q}^*}\left(\bar{Z}\middle| \mathcal{F}_s\right).$$

- **A two-step iterative construction**

The forward indifference price is constructed via an iterative pricing scheme which starts at the claim's maturity and is applied backwards in time in (43). The scheme has local and dynamic properties.

Dynamically, at each time interval, say $(s, s+1)$, the price $\nu_s(C_t)$ is computed via the single-step forward price functional $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$, applied to the end of the period payoff. The latter turns out to be the indifference price $\nu_{s+1}(C_t)$, as discussed earlier. The functional $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ is independent of the specific payoff.

Locally, the pricing role of $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ is similar to its single-period counterpart, developed in [10], in that it is non-linear and produces the price in two sub-steps. In the first sub-step, the end of the period payoff $\nu_{s+1}(C_t)$ is distorted and produces an intermediate payoff, say $\mathcal{C}^{(s,s+1)}(C_t)$, given by

$$\mathcal{C}^{(s,s+1)}(C_t) = \frac{1}{\gamma} \ln E_{\mathbb{Q}^*} \left(e^{\gamma \nu_{s+1}(C_t)} \left| \mathcal{F}_s \vee \mathcal{F}_{s+1}^S \right. \right). \tag{46}$$

This payoff is replicable and is, in turn, priced by expectation, yielding

$$\nu_s(C_t) = E_{\mathbb{Q}^*}(\mathcal{C}^{(s,s+1)}(C_t) \,|\mathcal{F}_s). \tag{47}$$

In the first step, the conditioning is with regards to $\mathcal{F}_s \vee \mathcal{F}_{s+1}^S$ while, in the second, it is only with respect to $\mathcal{F}_s$.

- **Analogies with the static certainty equivalent**

The classical certainty equivalent is a static pricing rule, yielding the price of a generic claim, say Z, as

$$CE\,(Z) = -u^{(-1)}\left(E_{\mathbb{P}}\left(u\left(-Z\right)\right)\right), \tag{48}$$

for a concave and increasing utility function u (see, for example, [6]). Notice that, in contrast to the indifference prices, the above price is derived in the *absence* of any trading activity. Notice, also, that the measure appearing above is the historical probability measure and not any martingale one.

Given that the forward price is constructed taking into account the investor's risk preferences, shall one expect that they would provide *multi-period analogues* of the static certainty equivalent rule? This is not obvious and, as a matter of fact, such analogy fails in the classical setting.

In seeking a multi-period analogue of (48), it is natural to assume that the role of u and $u^{(-1)}$ will be played by the process $U_t\left(x\right)$ and its spatial inverse $U_t^{(-1)}\left(x\right)$, with the latter given, for $t = 1, 2, ...$, by

$$U_t^{(-1)}\left(x\right) = -\frac{1}{\gamma} \ln\left(-x\right) + \frac{1}{\gamma} \sum_{i=1}^{t} h_i, \tag{49}$$

and $U_0^{(-1)}\left(x\right) = -\frac{1}{\gamma} \ln\left(-x\right)$, for $x \in \mathbb{R}^-$ and h as in (11).

We now consider an analogue of the certainty equivalent, defined, for $Z \in \mathcal{F}_{s+1}$, as

$$CE^{(s,s+1)}(Z) := -U_{s+1}^{(-1)}\left(E_{\mathbb{P}}\left(U_{s+1}(-Z)\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)\right). \tag{50}$$

Lemma 14 *Let $t>0$ and $s=0,1,...,t$, and $\mathbb{Q}^*$ be the forward indifference pricing measure. Then, for any $Z \in \mathcal{F}_{s+1}$, the following assertions hold:*

i) The dynamic certainty equivalent $CE^{(s,s+1)}(Z)$ satisfies

$$CE^{(s,s+1)}(Z) = \frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right). \tag{51}$$

ii) Moreover, $CE^{(s,s+1)}(Z)$ is invariant under $\mathbb{P}$ and $\mathbb{Q}^*$, namely,

$$-U_{s+1}^{(-1)}\left(E_{\mathbb{P}}\left(U_{s+1}(-Z)\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)\right) = -U_{s+1}^{(-1)}\left(E_{\mathbb{Q}^*}\left(U_{s+1}(-Z)\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)\right).$$

Proof. To establish (51), we first observe that, under the measure $\mathbb{Q}^*$,

$$E_{\mathbb{Q}^*}\left(U_{s+1}(-Z)\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right) = E_{\mathbb{Q}^*}\left(-e^{\gamma Z + \Sigma_{i=1}^{s+1} h_i}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)$$

$$= -e^{\Sigma_{i=1}^{s+1} h_i} E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right),$$

where we used that $\Sigma_{i=1}^{s+1} h_i$ is $\mathcal{F}_s$-mble. In turn, the forms of (13) and (49) yield

$$-U_{s+1}^{(-1)}\left(E_{\mathbb{Q}^*}\left(U_{s+1}(-Z)\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)\right)$$

$$= \frac{1}{\gamma}\ln\left(e^{\Sigma_{i=1}^{s+1} h_i} E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right)\right) - \frac{1}{\gamma}\sum_{i=1}^{s+1} h_i$$

$$= \frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right).$$

Using property (36), however, we have that

$$\frac{1}{\gamma}\ln E_{\mathbb{Q}^*}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right) = \frac{1}{\gamma}\ln E_{\mathbb{P}}\left(e^{\gamma Z}\left|\mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right.\right),$$

and the rest of the proof follows easily. ■

The above results yield the following representation of the forward indifference price.

Proposition 15 *Consider a claim C_t at time 0 and yielding payoff C_t at time $t>0$. For $s=0,1,..,t$, its forward indifference price $\nu_s(C_t)$ is given as the arbitrage-free price of the conditional certainty equivalent (cf. (50)) of the indifference price at the end of the period, namely,*

$$\nu_s(C_t) = E_{\mathbb{Q}^*}\left(CE^{(s,s+1)}\left(\nu_{s+1}(C_t)\right)\left|\mathcal{F}_s\right.\right).$$

- **The pricing measure**

As we have already established, the pricing measure $\mathbb{Q}^*$ is the one that minimizes the reverse relative entropy (cf. Proposition 7). It has the intuitively pleasing property (36), in that, for each period $[s-1,s)$, the conditional on $\mathcal{F}_{s-1} \vee \mathcal{F}_s^S$ distribution of the stochastic factor Y_s is the *same* under both $\mathbb{P}$ and $\mathbb{Q}^*$.

- **Dependence on the maturity of the claim**

The forward pricing functionals $\mathcal{E}_{\mathbb{Q}^*}^{(s,s+1)}$ and $\mathcal{E}_{\mathbb{Q}^*}^{(s,t)}$ are independent of the claim's maturity. Indeed, neither their form or the involved measure depend on the time t that the claim matures. This does not mean that the price is independent of the claim's maturity, an obvious wrong conclusion. Rather, it says that the forward pricing operator *per se* does not depend on the specific maturity.

This setting is very much aligned with the one in complete markets where the pricing operator, given by the conditional expectation of the (discounted) payoff, is independent of the claim's maturity.

- **Comparison with the traditional exponential utility valuation**

We conclude commenting on some distinct features of the forward and classical exponential indifference prices. To make the notation more familiar with the traditional setting, we assume that the claim matures at time T and that we consider the classical expected utility problem in $[0,T]$ with utility $U_T(x) = -e^{-\gamma x}$, $x \in \mathbb{R}$, $\gamma > 0$.

We recall that the case of a binomial model with exponential preferences in which a claim is written exclusively on a non-traded asset but in a complete nested (stock and bond) market model was studied in [23], [24], [10] and [11]. These results were subsequently generalized by the authors in [18] for a setting like the one herein. Similar results for power utilities were analyzed in [9].

Let us denote by $\mu_{s,T}(C_T)$, $s = 1,2,...,T-1$, the traditional exponential indifference price of the claim C_T and by $V_{s,T}(x)$ the associated value function processes. There are several differences between the prices $\mu_{s,T}(C_T)$ and $\nu_t(C_T)$. As it was shown in [18], the classical price is also computed iteratively,

$$\mu_{s,T}(C_T) = E_{\mathbb{Q}_T^{me}}\left(\frac{1}{\gamma}\ln E_{\mathbb{Q}_T^{me}}\left(e^{\gamma\mu_{s+1,T}(C_T)}\middle| \mathcal{F}_s \vee \mathcal{F}_{s+1}^S\right)\middle| \mathcal{F}_s\right),$$

where $\mu_{s+1,T}(C_T)$ is the indifference price of the claim at the end of the period $(s, s+1]$.

The measure $\mathbb{Q}_T^{me}$ is the minimal relative entropy one and its density depends crucially on the horizon choice T, while this not the case with $\mathbb{Q}^*$. As a result, the form of $\mu_{s,T}(\cdot)$ also depends on the horizon choice, while the form of $\nu_t(\cdot)$ does not.

Another difference, is that the classical price has no natural interpretation as the arbitrage-free price of a dynamic conditional certainty equivalent. Indeed, it can be shown[1] that, if Z is $\mathcal{F}_{s+1}$-mble, then

$$\frac{1}{\gamma}\ln E_{\mathbb{Q}_T^{me}}\left(e^{\gamma Z}\middle|\,\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)\neq -V_{s+1,T}^{(-1)}\left(E_{\mathbb{P}}\left(V_{s+1,T}\left(-Z\right)\middle|\mathcal{F}_s\vee\mathcal{F}_{s+1}^S\right)\right).$$

Finally, as discussed at the end of Section 4**,** the forward indifference valuation mechanism is applicable for claims arriving at arbitrary future times, known a priori or not. This is because the forward criterion can be defined sequentially as time progresses and market evolves. This is not the case, however, in the classical case.

A detailed comparative study between the traditional and forward exponential indifference prices, and their respective measures is being carried out by two of the authors in [25]

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