Representation of homothetic forward performance processes via ergodic and infinite horizon quadratic BSDE in stochastic factor models*

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Abstract

In an incomplete market, with incompleteness stemming from stochastic factors imperfectly correlated with the underlying stocks, we derive representations of homothetic forward investment performance processes (power, exponential and logarithmic). We develop a connection with ergodic and infinite horizon quadratic BSDE, and with a risk-sensitive control problem. We also develop a connection, for large trading horizons, with a family of traditional homothetic value function processes.

1 Introduction

This paper contributes to the study of homothetic forward performance processes, namely, of power, exponential and logarithmic type, in a stochastic factor market model. Stochastic factors are frequently used to model the predictability of stock returns, stochastic volatility and stochastic interest rates (for an overview of the literature, we refer the reader to the review paper [36]). Forward performance processes were introduced and developed in [25] and [27]

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(see, also, [26], [28] and [29]). They complement the classical expected utility paradigm in which the utility is a deterministic function chosen at a single point in time (terminal horizon). The value function process is constructed backwards in time (as the Dynamic Programming Principle yields) and there is little flexibility to incorporate updating of risk preferences, rolling horizons, learning and other realistic "forward in nature" features, if one requires that time-consistency is being preserved at all times. Forward investment performance criteria alleviate some of these shortcomings and offer a construction of a genuinely dynamic mechanism for evaluating the performance of investment strategies as the market evolves.

In [30] a stochastic PDE (cf. (10) herein) was proposed for the characterization of forward performance processes in a market with Itô-diffusion price processes. It may be viewed as the forward analogue of the finite-dim. classical Hamilton-Jacobi-Bellman (HJB) equation that arises in Markovian models of optimal portfolio choice. Like the HJB equation, the forward SPDE is fully nonlinear and possibly degenerate. In addition, however, it is ill-posed and its volatility coefficient is an input that the investor chooses while, in the classical case, the latter is uniquely obtained from the Itô decomposition of the value function process. These features result in significant technical difficulties and, as a result, the use of the forward SPDE for general market dynamics has been limited. Results for time-monotone processes can be found in [29] and and a connection between optimal portfolios and forward process has been explored in [10]. An axiomatic construction for the case of exponential preferences can be found in [38] and a connnection with risk measures in [37].

When the market coefficients depend explicitly on stochastic factors, there is more structure that can be explored by seeking performance criteria represented as deterministic functions of the factors. As it was first noted in [30], the SPDE reduces to a finite-dim. HJB equation (see (51) therein) that this function is expected to satisfy. Still, however, this HJB equation remains ill-posed and how to solve it remains an open problem. For a single stochastic factor, two cases have been so far analyzed, namely, for power and exponential initial data. The power case was treated in [32] where the homotheticity reduces the forward HJB to a semilinear pde which is, in turn, linearized using a distortion transformation. One then obtains a one-dim. ill-posed heat equation with state dependent coefficients, which is solved using an extention of Widder's theorem. The exponential case was studied in [26] (see, also, [25] and [21] for forward exponential indifference prices).

A detailed discussion on the economic importance of multi-factor modeling of forward performance processes can be found in [31]. Therein, the case of a multi-factor complete market setting is analyzed. The Legendre-Fenchel transformation linearizes the forward SPDE and a multi-dim. ill-posed heat equation with space/time dependent coefficients arises. Its solutions are characterized via an extention of Widder's theorem obtained by the authors. More recently, multi-factors of different (slow and fast) scales were examined in [34] in incomplete markets, and asymptotic expansions were derived for the limiting regimes. Therein, the leading order terms are expressed as time-monotone forward performances with approriate stochastic time-rescaling, resulting from averaging phenomena. The first order terms reflect compiled changes in the investor's preferences based on market changes and his past performance.

Herein, we extend the existing results on forward processes in factor-form by considering an incomplete market with multi-stocks and multi-stochastic factors, and homothetic forward preferences. For such settings, the homotheticity reduces the forward SPDE to an ill-posed multi-dim. semilinear pde, which however *cannot* be linearized. To our knowledge, for such equations no results exist to date. We bypass this problem by constructing factor-form forward processes directly from Markovian solutions of a family of *ergodic quadratic BSDE*. While the form of the latter is suggested by the reduced forward SPDE, we only use results from ergodic equations and not (forward) stochastic optimization. As a byproduct, we use these findings to construct a smooth solution to the ill-posed multi-dim. semilinear pde. To our knowldge, this approach is new. It is quite direct and requires mild assumptions on the dynamics of the factors, essentially the ergodicity condition (4). We, also, provide a connection with a risk-sensitive control problem and the constant appearing in the solution of the ergodic BSDE, thus providing the forward analogue of the results in [12] and [13].

In a different direction, we develop a connection of homothetic forward processes with *infinite horizon quadratic BSDE*. Our contribution is threefold. Firstly, we use their Markovian solutions to construct a new class of homothetic processes in closed form. These processes naturally depend on the parameter, denoted by ρ, appearing in the infinite horizon BSDE. In turn, we show that as $\rho \downarrow 0$ they converge to their Markovian ergodic counterparts. Thirdly, we use these infinite horizon BSDE to establish a connection among the homothetic forward processes we construct and classical analogues, specifically, finite horizon value function processes in the presence of an appropriately chosen terminal endowment. We show that these value functions converge to the homothetic processes as the trading horizon tends to infinity.

In the finite horizon setting, quadratic BSDE was first studied in [20] and has been subsequently analyzed extensively by a number of authors. They constitute one of the most active areas of research in financial mathematics for they offer direct applications to risk measures (see [2]), indifference prices (see [1, 16, 22]), and value functions when the terminal utility is homothetic (see [17]). Several extensions to the latter line of applications include, among others, [23] and [3] where the results in [17] were generalized, respectively, to a continuous martingale setting and to jump-diffusions. We note that in the traditional framework, prices, portfolios, risk measures and value functions are intrinsically constructed "backwards" in time and, thus, BSDE offers the ideal tool for their analysis.

Despite the popularity of quadratic BSDE in the finite horizon setting, neither their ergodic or infinite horizon counterparts have received much attention to date. In an infinite dimensional setting, an ergodic Lipschitz BSDE was introduced by [14] for the solution of an ergodic stochastic control problem; see also [7, 9, 33], and more recently [8] and [18] for various extensions. The infinite horizon quadratic BSDE was first solved in [5] by combining the techniques used in [6] and [20].

We note that both types of ergodic and infinite horizon equations have been so far motivated mainly from theoretical interest. Our results show, however, that both types of equations are natural candidates for the characterization of forward performance processes and their associated optimal portfolios and wealths. It is worth mentioning that both the ergodic and infinite horizon BSDE we consider actually turn out to be Lipschitz, since one can show that the parts corresponding to the relevant processes Z are bounded. In other words, the quadratic growth does not play a crucial role. Indeed, as we show in the Appendix, the existing results from the ergodic Lipschitz BSDE [14] and the infinite horizon Lipschitz BSDE [6] can be readily adapted to solve the forward equations at hand.

The paper is organized as follows. In section 2, we introduce the market model, and review the notion of forward performance process and the forward SPDE. In sections 3, 4 and 5, we construct the corresponding forward performance processes and the associated optimal portfolios and wealth processes using both the ergodic and infinite horizon quadratic BSDE. We also present, in each section, the connection between the forward performance processes and their finite-horizon counterparts. The technical background results on the ergodic and infinite horizon BSDE are presented in the Appendix A.

2 The stochastic factor model and its forward performance process

The market consists of a riskless bond and n stocks. The bond is taken to be the numeraire and the individual (discounted by the bond) stock prices S_t^i, $t \geq 0$, solve, for $i = 1, ..., n$,

$$\frac{dS_t^i}{S_t^i} = b^i(V_t)dt + \sum_{j=1}^{d} \sigma^{ij}(V_t)dW_t^j, \tag{1}$$

with $S_0^i > 0$. The process $W = (W^1, \cdots, W^d)^T$ is a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions . The superscript T denotes the matrix transpose.

The d-dimensional process $V = (V^1, \cdots, V^d)$ models the stochastic factors affecting the dynamics of the stock prices and its components are assumed to solve, for $i = 1, ..., d$,

$$dV_t^i = \eta^i(V_t)dt + \sum_{j=1}^{d} \kappa^{ij} dW_t^j, \tag{2}$$

with $V_0^i \in \mathbb{R}$.

We introduce the following model assumptions.

Assumption 1 *i) The market coefficients $b(v) = (b^i(v))$ and $\sigma(v) = (\sigma^{ij}(v))$, $1 \leq i \leq n, 1 \leq j \leq d$, $v \in \mathbb{R}^d$, are uniformly bounded and the volatility matrix $\sigma(v)$ has full row rank n.*

ii) The market price of risk vector $\theta(v)$, $v \in \mathbb{R}^d$, defined as the solution to the equation $\sigma(v)\theta(v) = b(v)$ and given by $\theta(v) = \sigma(v)^T[\sigma(v)\sigma(v)^T]^{-1}b(v)$, is uniformly bounded and Lipschitz continuous.

Assumption 2 *The drift coefficients of the stochastic factors satisfy the dissipative condition*

$$(\eta(v) - \eta(\bar{v}))^T(v - \bar{v}) \leq -C_\eta|v - \bar{v}|^2, \tag{3}$$

for any $v, \bar{v} \in \mathbb{R}^d$ and a constant C_η large enough. The volatility matrix $\kappa = (\kappa^{ij})$, $1 \leq i, j \leq d$, is a constant matrix with $\kappa\kappa^T$ positive definite and normalized to $|\kappa| = 1$.

The "large enough" property of the above constant C_η will be refined later on when we introduce another auxiliary constant C_v (cf. (40)) related to the drivers of the upcoming BSDE.

A direct application of Gronwall's inequality yields that the stochastic factors V satisfy, for any $v, \bar{v} \in \mathbb{R}^d$, the *exponential ergodicity* condition

$$|V_t^v - V_t^{\bar{v}}|^2 \leq e^{-2C_\eta t}|v - \bar{v}|^2, \tag{4}$$

where the superscript v denotes the dependence on the initial condition. We note that (4) is the only condition needed to be satisfied by the stochastic factors. Any diffusion processes satisfying (4) may serve as a stochastic factor vector.

Next, we consider an investor who starts at time $t = 0$ with initial endowment x and trades among the $(n + 1)$ assets. We denote by $\tilde{\pi} = (\tilde{\pi}^1, \cdots, \tilde{\pi}^n)^T$ the proportions of her total (discounted) wealth in the individual stock accounts. Assuming that the standard self-financing condition holds and using (1), we deduce that her (discounted by the bond) wealth process solves

$$dX_t^\pi = \sum_{i=1}^{n} \tilde{\pi}_t^i X_t^\pi \frac{dS_t^i}{S_t^i} = X_t^\pi \tilde{\pi}_t^T \left(b(V_t)dt + \sigma(V_t)dW_t\right),$$

with $X_0 = x \in \mathbb{D}$, where the set $\mathbb{D} \subseteq \mathbb{R}$ denotes the wealth admissibility domain.

For mere convenience, we will be working throughout with the trading strategies *rescaled by the volatility*, namely,

$$\pi_t^T = \tilde{\pi}_t^T \sigma(V_t). \tag{5}$$

Then, the wealth process solves

$$dX_t^\pi = X_t^\pi \pi_t^T (\theta(V_t)dt + dW_t). \tag{6}$$

For any $t \geq 0$, we denote by $\mathcal{A}_{[0,t]}$ the set of admissible strategies in the trading interval $[0, t]$, given by

$$\mathcal{A}_{[0,t]} = \{(\pi_u)_{u\in[0,t]} : \pi \in L^2_{BMO}[0,t],\ \pi_u \in \Pi \text{ and } X_u^\pi \in \mathbb{D},\ u \in [0,t]\}. \tag{7}$$

The set $\Pi \subseteq \mathbb{R}^d$ is closed and convex, and the space $L^2_{BMO}[0,t]$ defined as

$$L^2_{BMO}[0,t] = \Big\{(\pi_u)_{u\in[0,t]} : \pi \text{ is } \mathbb{F}\text{-progressively measurable and}$$
$$\operatorname*{ess\,sup}_{\tau} E_{\mathbb{P}} \left(\int_\tau^t |\pi_u|^2 du \middle| \mathcal{F}_\tau \right) < \infty \text{ for any } \mathbb{F}\text{-stopping time } \tau \in [0,t] \Big\}.$$

The above integrability condition is also called the *BMO-condition*, since for any $\pi \in L^2_{BMO}[0,t]$,

$$\operatorname*{ess\,sup}_{\tau\in[0,t]} E_{\mathbb{P}} \left(\int_\tau^t \pi_u^T dW_u \middle| \mathcal{F}_\tau \right)^2 = \operatorname*{ess\,sup}_{\tau\in[0,t]} E \left(\int_\tau^t |\pi_u|^2 du \middle| \mathcal{F}_\tau \right) < \infty,$$

and, hence, the stochastic integral $\int_0^s \pi_u^T dW_u$, $0 \leq s \leq t$, is a BMO-martingale.

Then, we define the set of admissible strategies for all $t \geq 0$ as $\mathcal{A} := \cup_{t\geq 0} \mathcal{A}_{[0,t]}$.

Next, we review the notion of the forward investment performance process, introduced and developed in [25]-[30]. Variations and relaxations of the original definition can be also found in [4], [10], [15] and [31].

Definition 1 *A process $U(x,t)$, $(x,t) \in \mathbb{D}\times[0,\infty)$ is a forward investment performance process if*

i) for each $x \in \mathbb{D}$, $U(x,t)$ is $\mathbb{F}$-progressively measurable;

ii) for each $t \geq 0$, the mapping $x \mapsto U(x,t)$ is strictly increasing and strictly concave;

iii) for any $\pi \in \mathcal{A}$ and any $0 \leq t \leq s$,

$$E_{\mathbb{P}} \left(U(X_s^\pi, s) | \mathcal{F}_t \right) \leq U\left(X_t^\pi, t\right), \tag{8}$$

and there exists an optimal portfolio $\pi^* \in \mathcal{A}$ such that, for any $0 \leq t \leq s$,

$$E_{\mathbb{P}} \left(U_s(X_s^{\pi^*}, s) | \mathcal{F}_t \right) = U\left(X_t^{\pi^*}, t\right). \tag{9}$$

As mentioned earlier, it was shown in [30] that $U(x,t)$ is associated with an ill-posed fully nonlinear SPDE, which plays the role of the Hamilton-Jacobi-Bellman equation in the classical finite-dimensional setting. Formally, the forward SPDE is derived by first assuming that $U(x,t)$ admits the Itô decomposition

$$dU(x,t) = b(x,t)dt + a(x,t)^T dW_t,$$

for some $\mathbb{F}$-progressively measurable processes $a(x,t)$ and $b(x,t)$, and that all involved quantities have enough regularity so that the Itô-Ventzell formula can be applied to $U(X_s^\pi, s)$, for all admissible π. The requirements (8) and (9) then yield that, for a chosen volatility process $a(x,t)$, the drift $b(x,t)$ must have a specific form.

In the setting herein, the forward performance SPDE takes the form

$$\begin{aligned} dU(x,t) = \Bigg(& -\frac{1}{2}x^2 U_{xx}(x,t) dist^2 \left(\Pi, -\frac{\theta(V_t)U_x(x,t) + a_x(x,t)}{xU_{xx}(x)} \right) \\ & + \frac{1}{2}\frac{|\theta(V_t)U_x(x,t) + a_x(x,t)|^2}{U_{xx}(x,t)} \Bigg) dt + a(x,t)^T dW_t, \end{aligned} \tag{10}$$

where $dist(\Pi, x)$ represents the distance function from $x \in \mathbb{R}^d$ to Π. Moreover, if a strong solution to (6) exists, say $X_t^{\pi^*}$, when the feedback policy

$$\pi_t^* = Proj_\Pi \left(-\frac{\theta(V_t)U_x(X_t^{\pi^*}, t) + a_x(X_t^{\pi^*}, t)}{X_t^{\pi^*} U_{xx}(X_t^{\pi^*}, t)} \right), \tag{11}$$

is used, then the control process π_t^* is optimal. We note that these arguments are informal and a verification theorem is still lacking.

Herein we bypass these difficulties and construct homothetic forward performance processes and their volatilities using directly the Markovian solutions of associated ergodic and infinite horizon quadratic BSDE. The SPDE is merely used to guess the appropriate form of the involved BSDE.

3 Power forward performance processes

We start with forward performance processes $U(x,t)$ that are homogeneous of degree δ. For simplicity we only consider $\delta \in (0,1)$ since the case $\delta < 0$ follows along similar arguments. For this range of δ, the admissible wealth domain is taken to be $\mathbb{D} = \mathbb{R}_+$.

3.1 Representation via ergodic quadratic BSDE

We first introduce the underlying ergodic quadratic BSDE and provide the main existence and uniqueness result for a Markovian solution. For the reader's convenience, we present the proof in Appendix A.

Proposition 2 *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1.ii and let the set Π be as in (7). Then, the ergodic quadratic BSDE*

$$dY_t = (-F(V_t, Z_t) + \lambda)dt + Z_t^T dW_t, \tag{12}$$

with the driver $F(\cdot,\cdot)$ given by

$$F(V_t, Z_t) = -\frac{1}{2}\delta(1-\delta)dist^2\left(\Pi, \frac{Z_t + \theta(V_t)}{1-\delta}\right) + \frac{1}{2}\frac{\delta}{1-\delta}|Z_t + \theta(V_t)|^2 + \frac{1}{2}|Z_t|^2, \tag{13}$$

admits a unique Markovian solution (Y_t, Z_t, λ), $t \geq 0$.

Specifically, there exist a unique $\lambda \in \mathbb{R}$ and functions $y : \mathbb{R}^d \to \mathbb{R}$ and $z : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t, Z_t) = (y(V_t), z(V_t))$. The function $y(\cdot)$ is unique up to a constant and has at most linear growth, and $z(\cdot)$ is bounded with $|z(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η and C_v are as in (3) and (40), respectively.

The following result yields the power forward performance processes and its volatility in a *factor-form* as well as their associated optimal portfolio and wealth processes.

Theorem 3 *Let $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda), t \geq 0$, be the unique Markovian solution of (12). Then,*

i) the process $U(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, given by

$$U(x,t) = \frac{x^\delta}{\delta} e^{y(V_t) - \lambda t} \tag{14}$$

is a power forward performance process with volatility

$$a(x,t) = \frac{x^\delta}{\delta} e^{y(V_t) - \lambda t} z(V_t). \tag{15}$$

ii) The optimal portfolio weights π_t^* and the associated wealth process X_t^* (cf. (5),(6)) are given by

$$\pi_t^* = Proj_\Pi\left(\frac{z(V_t) + \theta(V_t)}{1-\delta}\right) \text{ and } X_t^* = X_0 \mathcal{E}\left(\int_0^\cdot (\pi_s^*)^T(\theta(V_s)ds + dW_s)\right)_t. \tag{16}$$

Proof. It is immediate that the process $U(x,t)$ is $\mathbb{F}$-progressively measurable, strictly increasing and strictly concave in x, and homogeneous of degree δ. Next, we show that, for any $\pi \in \mathcal{A}$ and any $0 \leq t \leq s$,

$$E_\mathbb{P}\left(\frac{(X_s^\pi)^\delta}{\delta} e^{Y_s - \lambda s} | \mathcal{F}_t\right) \leq \frac{(X_t^\pi)^\delta}{\delta} e^{Y_t - \lambda t},$$

while for π^* given by (16), and any $0 \leq t \leq s$,

$$E_{\mathbb{P}}\left(\frac{(X_s^{\pi^*})^\delta}{\delta}e^{Y_s-\lambda s}|\mathcal{F}_t\right) = \frac{(X_s^{\pi^*})^\delta}{\delta}e^{Y_t-\lambda t}.$$

To this end, Itô's formula yields

$$(X_s^\pi)^\delta = (X_t^\pi)^\delta \exp\left(\int_t^s \delta\left(\pi_u^T\theta(V_u) - \frac{1}{2}|\pi_u|^2\right)du + \int_t^s \delta\pi_u^T dW_u\right).$$

On the other hand, from the ergodic quadratic BSDE (12), we have

$$e^{Y_s-\lambda s} = e^{Y_t-\lambda t}\exp\left(-\int_t^s F(V_u, z(V_u))du + \int_t^s z(V_u)^T dW_u\right).$$

Therefore,

$$(X_s^\pi)^\delta e^{Y_s-\lambda s} = (X_t^\pi)^\delta e^{Y_t-\lambda t}\exp\left(\int_t^s \delta\left(\pi_u^T\theta(V_u) - \frac{1}{2}|\pi_u|^2\right) - F(V_u, z(V_u))du \right.$$
$$\left. + \int_t^s \left(\delta\pi_u^T + z(V_u)^T\right)dW_u\right)$$

and, in turn,

$$E_{\mathbb{P}}\left((X_s^\pi)^\delta e^{Y_s-\lambda s}|\mathcal{F}_t\right)$$
$$= (X_t^\pi)^\delta e^{Y_t-\lambda t}E_{\mathbb{P}}\left(\exp\left(\int_t^s\left(\delta\left(\pi_u^T\theta(V_u) - \frac{1}{2}|\pi_u|^2\right) - F(V_u, z(V_u))\right)du \right.\right.$$
$$\left.\left. + \int_t^s \left(\delta\pi_u^T + z(V_u)^T\right)dW_u\right)\Big|\mathcal{F}_t\right).$$

Next, let $s \geq 0$ and $\pi \in \mathcal{A}$. We define a probability measure, say $\mathbb{Q}^\pi$, by introducing the Radon-Nikodym density process $\mathcal{Z}_u, u \in [0, s]$,

$$\mathcal{Z}_u = \left.\frac{d\mathbb{Q}^\pi}{d\mathbb{P}}\right|_{\mathcal{F}_u} = \mathcal{E}(N)_u, \text{ with } N = \int_0^\cdot \left(\delta\pi_u^T + Z_u^T\right)dW_u. \tag{17}$$

We recall that both π_u and $z(V_u)$, $u \in [0, s]$, satisfy the BMO-condition (up to time s). Therefore, the process N_u, $u \in [0, s]$, is a BMO-martingale and, in turn, $\mathcal{E}(N)$ is in Doob's class $\mathcal{D}$ and, thus, uniformly integrable. Bayes formula then yields

$$E_{\mathbb{P}}\left(\exp\left(\int_t^s (F^\pi(V_u, z(V_u)) - F(V_u, z(V_u)))\,du\right)\frac{\mathcal{Z}_s}{\mathcal{Z}_t}\Big|\mathcal{F}_t\right)$$
$$= E_{\mathbb{Q}^\pi}\left(\exp\left(\int_t^s (F^\pi(V_u, z(V_u)) - F(V_u, z(V_u)))\,du\right)\Big|\mathcal{F}_t\right),$$

where

$$F^{\pi}(V_t, z(V_t)) = -\frac{1}{2}\delta(1-\delta)|\pi_t|^2 + \delta\pi_t^T(z(V_t)+\theta(V_t)) + \frac{1}{2}|z(V_t)|^2$$

$$= -\frac{1}{2}\delta(1-\delta)\left|\pi_t - \frac{z(V_t)+\theta(V_t)}{1-\delta}\right|^2 + \frac{1}{2}\frac{\delta}{1-\delta}|z(V_t)+\theta(V_t)|^2 + \frac{1}{2}|z(V_t)|^2.$$

Using that $F^{\pi}(V_t, z(V_t)) \leq F(V_t, z(V_t))$, we easily deduce that

$$E_{\mathbb{P}}\left((X_s^{\pi})^{\delta} e^{Y_s - \lambda s}|\mathcal{F}_t\right) \leq (X_t^{\pi})^{\delta} e^{Y_t - \lambda t}.$$

Moreover, for $\pi = \pi^*$ as in (16), $F^{\pi^*}(V_t, z(V_t)) = F(V_t, z(V_t))$, and thus

$$E_{\mathbb{P}}\left((X_s^{\pi^*})^{\delta} e^{Y_s - \lambda s}|\mathcal{F}_t\right) = (X_t^{\pi^*})^{\delta} e^{Y_t - \lambda t}.$$

To show (15), we recall (10) and observe that (14) yields

$$dU(x,t) = U(x,t)(-F(V_t, z(V_t)) + \frac{1}{2}|z(V_t)|^2)dt + U(x,t)z(V_t)^T dW_t.$$

The rest of the proof follows easily. ■

3.1.1 Connection with risk-sensitive optimization

We provide an interpretation of the constant λ, appearing in the representation of the forward performance process (14), as the solution of an auxiliary risk-sensitive control problem.

Proposition 4 *Let $T > 0$ and $\pi \in \mathcal{A}$, and define the probability measure $\mathbb{P}^{\pi}$ using the Radon-Nikodym density process $\mathcal{Z}_u$, $u \in [0,T]$,*

$$\mathcal{Z}_u = \left.\frac{d\mathbb{P}^{\pi}}{d\mathbb{P}}\right|_{\mathcal{F}_u} = \mathcal{E}\left(\int_0^{\cdot} \delta\pi_u^T dW_u\right)_u. \tag{18}$$

Let $(y(V_t), z(V_t), \lambda)$, $t \geq 0$, be the unique Markovian solution of the ergodic BSDE (12). Then, λ is the long-term growth rate of the risk-sensitive control problem

$$\lambda = \sup_{\pi\in\mathcal{A}} \limsup_{T\uparrow\infty} \frac{1}{T} \ln E_{\mathbb{P}^{\pi}}\left(e^{\int_0^T L(V_u,\pi_u)du}\right), \tag{19}$$

where $L(V_u, \pi_u) = -\frac{1}{2}\delta(1-\delta)|\pi_u|^2 + \delta\theta(V_u)^T\pi_u$, $0 \leq u \leq T$. The optimal control process π_t^*, $t \geq 0$, is as in (16).

Proof. First we note that the driver $F(\cdot,\cdot)$ in (13) can be written as

$$F(V_t, Z_t) = \sup_{\pi_t\in\Pi}\left(L(V_t,\pi_t) + Z_t^T\delta\pi_t\right) + \frac{1}{2}|Z_t|^2.$$

Therefore, for arbitrary $\tilde{\pi} \in \mathcal{A}$, we rewrite (12) under the probability measure $\mathbb{P}^{\tilde{\pi}}$ as

$$dY_t = \left(- \sup_{\pi_t \in \Pi} \left(L(V_t, \pi_t) + Z_t^T \delta \pi_t \right) + Z_t^T \delta \tilde{\pi}_t + \lambda - \frac{1}{2} |Z_t|^2 \right) dt + Z_t^T dW_t^{\mathbb{P}^{\tilde{\pi}}},$$

where $W_t^{\mathbb{P}^{\tilde{\pi}}} = W_t - \int_0^t \delta \tilde{\pi}_u du$, $t \geq 0$, is a Brownian motion under $\mathbb{P}^{\tilde{\pi}}$. In turn,

$$e^{\lambda T + Y_0} e^{-Y_T} \mathcal{E} \left(\int_0^{\cdot} Z_u^T dW_u^{\mathbb{P}^{\tilde{\pi}}} \right)_T$$

$$= \exp \left(\int_0^T \left(\sup_{\pi_t \in \Pi} \left(L(V_t, \pi_t) + Z_t^T \delta \pi_t \right) - \left(L(V_t, \tilde{\pi}_t) + Z_t^T \delta \tilde{\pi}_t \right) \right) dt \right) e^{\int_0^T L(V_t, \tilde{\pi}_t) dt}.$$

Next, we observe that for any $\tilde{\pi} \in \mathcal{A}$, the first exponential term on the right hand side is bounded below by 1. Taking expectation under $\mathbb{P}^{\tilde{\pi}}$ yields

$$e^{\lambda T + Y_0} E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{-Y_T} \mathcal{E} \left(\int_0^{\cdot} Z_u^T dW_u^{\mathbb{P}^{\tilde{\pi}}} \right)_T \right) \geq E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{\int_0^T L(V_t, \tilde{\pi}_t) dt} \right)$$

and using the definition of $\mathbb{Q}^{\tilde{\pi}}$ in (17),

$$\lambda + \frac{Y_0}{T} + \frac{1}{T} \ln E_{\mathbb{Q}^{\tilde{\pi}}} \left(e^{-Y_T} \right) \geq \frac{1}{T} \ln E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{\int_0^T L(V_t, \tilde{\pi}_t) dt} \right).$$

Next, we recall that there exists a constant, say C, independent of T, such that

$$\frac{1}{C} \leq E_{\mathbb{Q}^{\pi}} \left(e^{-Y_T} \right) \leq C.$$

This follows easily from the linear growth property of the function $y(\cdot)$ and the ergodicity condition (4) (see, for example, [11]). Sending $T \uparrow \infty$ yields that, for any $\tilde{\pi} \in \mathcal{A}$,

$$\lambda \geq \limsup_{T \uparrow \infty} \frac{1}{T} \ln E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{\int_0^T L(V_t, \tilde{\pi}_t) dt} \right),$$

with equality choosing $\tilde{\pi}_t = \pi_t^*$, with π_t^* as in (16). ■

3.1.2 Connection with an ill-posed multi-dimensional semilinear PDE

A byproduct of previous results is the construction of a solution to an ill-posed semilinear pde given in (20) below. The latter is derived from (10) for solutions of the form $U(x,t) = \frac{x^\delta}{\delta} e^{f(V_t,t)}$, for some deterministic function $f : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$.

Similar semilinear equations have been extensively analyzed and used for the representation of indifference prices, risk measures, power and exponential value functions, and others. To our knowledge, their ill-posed version has not been studied except in the one-dimensional case which, however, can be linearized (see [32]).

Proposition 5 *Consider the ill-posed semilinear PDE*

$$f_t + \mathcal{L}f + F(v, \kappa^T \nabla f) = 0, \tag{20}$$

for $(v,t) \in \mathbb{R}^d \times [0,\infty)$*, with* $F(\cdot,\cdot)$ *as in (13), the initial data* $f(v,0) = y(v)$*, where* $y(\cdot)$ *is the function appearing in the Markovian solution of the ergodic quadratic BSDE (12), and* $\mathcal{L}$ *being the infinitesimal generator of the factor process* V*,*

$$\mathcal{L} = \frac{1}{2} \mathit{Trace}\left(\kappa\kappa^T \nabla^2\right) + \eta(v)^T \nabla. \tag{21}$$

Then, equation (20) admits a smooth solution

$$f(v,t) = y(v) - \lambda t$$

with $y(\cdot)$ *and* λ *as in Proposition 2.*

Proof. Assume that the function $y(\cdot) \in C^2(\mathbb{R}^d)$. Applying Itô's formula to $y(V_t)$ yields

$$dy(V_t) = \mathcal{L}y(V_t)dt + \left(\kappa^T \nabla y(V_t)\right)^T dW_t.$$

Using the above equation and (12) yields that $Z_t = z(V_t) = \kappa^T \nabla y(V_t)$, and

$$-\lambda + \mathcal{L}y(V_t) + F(V_t, \kappa^T \nabla y(V_t)) = 0.$$

It remains to show that $y(\cdot) \in C^2(\mathbb{R}^d)$. Indeed, for any $\rho > 0$, consider the semi-linear elliptic PDE

$$\rho y^\rho = \mathcal{L}y^\rho + F\left(v, \kappa^T \nabla y^\rho\right).$$

Classical PDE results yield that the above equation admits a unique bounded solution $y^\rho(\cdot) \in C^2(\mathbb{R}^d)$. Using arguments similar to the ones in Appendix A, we deduce that $|y^\rho(v)| \leq \frac{K}{\rho}$ and $|\nabla y^\rho(v)| \leq \frac{C_v}{C_\eta - C_v}$. Therefore, for any reference point $v^0 \in \mathbb{R}^d$, we have that $\rho y^\rho(v_0)$ is uniformly bounded and that the difference $y^\rho(v) - y^\rho(v_0)$ is equicontinuous. Using a diagonal argument we deduce that there exists a subsequence $\rho_n \downarrow 0$ such that $\rho_n y^{\rho_n}(v_0) \to \lambda$ and $y^{\rho_n}(v) - y^{\rho_n}(v_0) \to y(v)$, uniformly on compact sets of $\mathbb{R}^d$. Since, however, both $\rho_n y^{\rho_n}(v)$ and $\nabla y^{\rho_n}(v)$ are bounded uniformly in ρ_n, $\nabla^2 y^{\rho_n}(v)$ is also bounded on compact sets as it follows from the above equation. This implies a Hölder estimate for $\nabla y^{\rho_n}(v)$ uniformly on compact sets. In turn, standard arguments for elliptic equations give that the limit $y(\cdot) \in C^2(\mathbb{R}^d)$ (see, for example, Theorem 3.3 of [11]). ■

3.1.3 Example: Single stock and single stochastic factor

In (1) and (2), let $n = 1$ and $d = 2$. Then, the stock and the stochastic factor processes follow, respectively,

$$dS_t = b(V_t)S_t dt + \sigma(V_t)S_t dW_t^1,$$

$$dV_t^1 = \eta(V_t)dt + \kappa^1 dW_t^1 + \kappa^2 dW_t^2 \quad \text{and} \quad dV_t^2 = 0,$$

with $\min(\kappa^1, \kappa^2) > 0$, $|\kappa^1|^2 + |\kappa^2|^2 = 1$ and $\sigma(\cdot)$ bounded by a positive constant. This is the one-dimensional model considered in [32]. Let $\Pi = \mathbb{R} \times \{0\}$ so that $\pi_t^2 \equiv 0$. Then, the wealth equation reduces to $dX_t^\pi = X_t^\pi \pi_t^1 \left(\theta(V_t)dt + dW_t^1\right)$ with $\theta(V_t) = b(V_t)/\sigma(V_t)$. In turn, the driver of (12) is

$$F(V_t, Z_t^1, Z_t^2) = \frac{1}{2}\frac{\delta}{1-\delta}|Z_t^1 + \theta(V_t)|^2 + \frac{1}{2}|Z_t^1|^2 + \frac{1}{2}|Z_t^2|^2.$$

By Theorem 3, the optimal portfolio weights are $\left(\pi_t^{*,1}, \pi_t^{*,2}\right) = \left(\frac{Z_t^1 + \theta^1(V_t)}{1-\delta}, 0\right)$.

To find Z_t^1 and Z_t^2, we set $Z_t^i = \kappa^i Z_t$, $i = 1, 2$, for some process Z_t to be determined. Then (12) further reduces to

$$dY_t = \left(-\frac{\hat{\delta}}{2}|Z_t|^2 - \frac{\delta\kappa^1}{1-\delta}\theta(V_t)Z_t - \frac{\delta}{2(1-\delta)}|\theta(V_t)|^2 + \lambda\right) dt$$

$$+ Z_t\left(\kappa^1 dW_t^1 + \kappa^2 dW_t^2\right),$$

with $\hat{\delta} = \frac{1-\delta+\delta|\kappa^1|^2}{1-\delta}$. Next, let $\tilde{Y}_t = e^{\hat{\delta}(Y_t - \lambda t)}$ and $\tilde{Z}_t = \hat{\delta}\tilde{Y}_t Z_t$. Then,

$$d\tilde{Y}_t = -\hat{\delta}\frac{\delta}{2(1-\delta)}|\theta(V_t)|^2 \tilde{Y}_t dt + \tilde{Z}_t d\tilde{W}_t,$$

where $\tilde{W}_t = \kappa^1 W_t^1 + \kappa^2 W_t^2 - \int_0^t \frac{\delta\kappa^1}{1-\delta}\theta(V_u)du$, $t \geq 0$, is a Brownian motion under some probability measure equivalent to $\mathbb{P}$.

Let $\beta_t = \exp\left(\int_0^t \hat{\delta}\frac{\delta}{2(1-\delta)}|\theta(V_u)|^2 du\right)$. Applying Itô's formula to $\tilde{Y}_t\beta_t$ yields that

$$\tilde{Y}_t = \frac{\beta_0}{\beta_t}\tilde{Y}_0 + \int_0^t \frac{\beta_u}{\beta_t}\tilde{Z}_u d\tilde{W}_u.$$

The power forward performance process is given by

$$U(x,t) = \frac{x^\delta}{\delta}(\tilde{Y}_t)^{1/\hat{\delta}} = \frac{x^\delta}{\delta}\left(\frac{\beta_0}{\beta_t}\tilde{Y}_0 + \int_0^t \frac{\beta_u}{\beta_t}\tilde{Z}_u d\tilde{W}_u\right)^{1/\hat{\delta}},$$

which yields an alternative representation to the solution derived in [32] bypassing the solution of the linearized reduced forward SPDE. One can esily deduce that writing $\tilde{Y}_t = \tilde{y}(V_t, t)$ and using the dynamics (2), yields that $\tilde{y}$ must satisfy

$$\tilde{y}_t(v,t) + \frac{1}{2}\tilde{y}_{vv}(v,t) + \left(\eta(v) + \frac{\delta\kappa^1}{1-\delta}\theta(v)\right)\tilde{y}_v(v,t) + \frac{\hat{\delta}\delta}{2(1-\delta)}\theta^2(v)\tilde{y}(v,t) = 0,$$

recovering the result of [32].

3.2 Representation via infinite horizon quadratic BSDE

In this section, we provide an alternative representation of power forward performance processes using the Markovian solutions of a family of infinite horizon quadratic BSDE, parametrized by a positive parameter ρ. The motivation is twofold. Firstly, these non-Markovian solutions provide an approximation of the process $U(x,t)$ as $\rho \downarrow 0$. Secondly, they yield an interesting connection with a family of traditional value function processes in horizon $[0,T]$, as $T \uparrow \infty$.

We start with some background results on infinite horizon quadratic BSDE. Among others, we recall that [6] is one of the first papers in which Girsanov's transformation is used to solve infinite horizon BSDE with Lipschitz driver, while the quadratic driver case was solved in [5]. We refer the reader to [5] for further references.

Proposition 6 *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1.ii and let the set Π be as in (7). Let $\rho > 0$, and consider the infinite horizon quadratic BSDE*

$$dY_t^\rho = (-F(V_t, Z_t^\rho) + \rho Y_t^\rho)\, dt + (Z_t^\rho)^T\, dW_t, \tag{22}$$

where the driver $F(\cdot,\cdot)$ is given in (12). Then, (22) admits a unique Markovian solution (Y_t^ρ, Z_t^ρ), $t \geq 0$. Specifically, for each $\rho > 0$, there exist unique functions $y^\rho : \mathbb{R}^d \to \mathbb{R}$ and $z^\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t^\rho, Z_t^\rho) = (y^\rho(V_t), z^\rho(V_t))$, with $|y^\rho(\cdot)| \leq \frac{K}{\rho}$ and $|z^\rho(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η as in (3), and C_v, K given in (40) and (42), respectively.

The solvability of (22) is an intermediate step to solve (12), and is included in the proof of Proposition 2 in the Appendix.

Theorem 7 *Let $(y^\rho(V_t), z(V_t^\rho))$, $t \geq 0$, be the unique Markovian solution to the infinite horizon quadratic BSDE (22). Then,*

i) the process $U^\rho(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, given by

$$U^\rho(x,t) = \frac{x^\delta}{\delta} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_s) ds} \tag{23}$$

is a power forward performance process with volatility

$$a^\rho(x,t) = \frac{x^\delta}{\delta} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_s) ds} z^\rho(V_t).$$

ii) The optimal portfolio weights $\pi_t^{*,\rho}$ and wealth process $X_t^{*,\rho}$ (cf. (5), (6), $t \geq 0$, are given, respectively, by

$$\pi_t^{*,\rho} = Proj_\Pi \left(\frac{z^\rho(V_t) + \theta(V_t)}{1-\delta} \right), \quad X_t^{*,\rho} = X_0 \mathcal{E}\left(\int_0^\cdot (\pi_s^{*,\rho})^T (\theta(V_s) ds + dW_s) \right)_t.$$

The proof is similar to the one of Theorem 3, and it is thus omitted.

The next result relates the representations $U(x,t)$ in Theorem 3 and $U^{\rho}(x,t)$ in Theorem 7, and their corresponding optimal portfolio strategies. We use the superscript index v to denote the dependence on the initial condition.

Proposition 8 *Let $U^{\rho}(x,t)$ and $U(x,t)$ be the power forward processes related to (22) and (12), respectively. Then, for an arbitrary reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for all $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,*

$$\lim_{\rho_n \downarrow 0} \frac{U^{\rho_n}(x,t)e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1. \tag{24}$$

Moreover, for each $t \geq 0$, the associated optimal portfolio weights π^{*,ρ_n} and π^* satisfy

$$\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t \left|\pi_s^{*,\rho_n} - \pi_s^*\right|^2 ds = 0.$$

Proof. For an arbitrary reference point $v_0 \in \mathbb{R}^d$, from the representations (14) and (23), we obtain that

$$\frac{U^{\rho}(x,t)e^{-y^{\rho}(v_0)}}{U(x,t)} = \exp\left((y^{\rho}(V_t^v) - \int_0^t \rho y^{\rho}(V_u^v)\, du) - (y(V_t^v) - \lambda t) - y^{\rho}(v_0 \right)$$

$$= \exp\left((y^{\rho}(V_t^v) - y^{\rho}(v_0) - y(V_t^v)) - \int_0^t \rho\left(y^{\rho}(V_u^v) - y^{\rho}(v_0)\right) du - (\rho y^{\rho}(v_0) - \lambda)t \right).$$

On the other hand, (56) and (57) yield that there exists a subsequence $\rho_n \downarrow 0$ such that

$$\lim_{\rho_n \downarrow 0} \left(y^{\rho_n}(V_t^v) - y^{\rho_n}(v_0) - y(V_t^v)\right) = 0,$$

$$\lim_{\rho_n \downarrow 0} \rho_n \left(y^{\rho_n}(V_t^v) - y^{\rho_n}(v_0)\right) = 0 \quad \text{and} \quad \lim_{\rho_n \downarrow 0} \left(\rho_n y^{\rho_n}(v_0) - \lambda\right) = 0.$$

We then easily conclude (24).

Finally, the convergence of the optimal portfolio weights follows from the convergence $\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t \left|z^{\rho_n}(V_s^v) - z(V_s^v)\right|^2 ds = 0$, for any $t \geq 0$, as it is shown in the Appendix. ■

3.3 Connection with the classical power expected utility for long horizons

We investigate whether the forward processes $U(x,t)$ and $U^{\rho}(x,t)$ can be interpreted as long-term limits of traditional portfolio choice problems in a finite horizon, say $[0,T]$ as $T \uparrow \infty$. We show that this is indeed the case for a family of expected utility models with appropriately chosen terminal random payoffs.

To this end, let $[0,T]$ be an arbitrary trading horizon and introduce for $\rho > 0$, the value function process

$$u^{\rho}(x,t;T) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{[t,T]}} E_{\mathbb{P}}\left(\frac{(X_T^{\pi}e^{\xi_T})^{\delta}}{\delta}|\mathcal{F}_t, X_t^{\pi}=x\right), \tag{25}$$

for $(x,t) \in \mathbb{R}_+ \times [0,T]$ and the wealth process X_s^{π}, $s \in [t,T]$ solving (6). The payoff ξ_T is defined as

$$\xi_T = -\frac{1}{\delta}\int_0^T \rho Y_t^{\rho,T} dt, \tag{26}$$

with $Y_t^{\rho,T}$ being the solution of the finite-horizon quadratic BSDE

$$Y_t^{\rho,T} = \int_t^T \left(F(V_s, Z_s^{\rho,T}) - \rho Y_s^{\rho,T}\right) ds - \int_t^T \left(Z_s^{\rho,T}\right)^T dW_s, \tag{27}$$

with the driver $F(\cdot,\cdot)$ given in (13). The associated optimal portfolio weights are denoted by $\pi_s^{*,\rho,T}$ for $s \in [t,T]$.

We recall that the traditional (without a terminal payoff) expected utility model for power utility has been solved by using the quadratic BSDE method in [17] for a Brownian motion setting, and in [23] under a general semimartingale framework.

Proposition 9 *i) Let $u^{\rho}(x,t;T)$ and $U^{\rho}(x,t)$ be given in (25) and (23), respectively. Then, for each $\rho > 0$ and $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,*

$$\lim_{T\uparrow\infty} \frac{u^{\rho}(x,t;T)}{U^{\rho}(x,t)} = 1,$$

and the optimal portfolio weights satisfy, for any $t \leq s \leq T$,

$$\lim_{T\uparrow\infty} E_{\mathbb{P}} \int_t^s \left|\pi_u^{*,\rho,T} - \pi_u^{*,\rho}\right|^2 du = 0.$$

ii) Let $U(x,t)$ as in (14). Then, for each arbitrary reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,

$$\lim_{\rho_n\downarrow 0}\lim_{T\uparrow\infty} \frac{u^{\rho_n}(x,t;T)e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1,$$

and the optimal portfolio weights satisfy, for any $t \leq s \leq T$,

$$\lim_{\rho_n\downarrow 0}\lim_{T\uparrow\infty} E_{\mathbb{P}} \int_t^s \left|\pi_u^{*,\rho_n,T} - \pi_u^{*}\right|^2 du = 0.$$

Proof. We only show part i). From Theorem 3.3 in [5] we have that $|Y_t^{\rho,T}| \leq \frac{K}{\rho}$, and therefore, the quantity $\delta\xi_T = -\int_0^T \rho Y_u^{\rho,T} du$ is bounded. On the other hand, the driver $F(\cdot,\cdot)$ satisfies (41) and (42), so along similar arguments used in section 3 of [17], it follows that the value function process is of the form $u^\rho(x,t;T) = \frac{x^\delta}{\delta} e^{Y_t}$, with Y_t being the unique solution of the quadratic BSDE

$$Y_t = \delta\xi_T + \int_t^T F(V_s, Z_s)ds - \int_t^T (Z_s)^T dW_s \tag{28}$$

on $[0,T]$.

Moreover, the optimal portfolio weights are given by $\pi_t^{*,\rho,T} = Proj_\Pi(\frac{Z_t+\theta(V_t)}{1-\delta})$.

Note, however, that the pair of processes $(Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds, Z_t^{\rho,T})$, $t \in [0,T]$, with $(Y_t^{\rho,T}, Z_t^{\rho,T})$ solving (27), also satisfies the above quadratic BSDE (28). Therefore, we must have $Y_t = Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds$, $t \in [0,T]$, and as a consequence, $u^\rho(x,t;T) = \frac{x^\delta}{\delta} \exp\left(Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds\right)$. In turn,

$$\frac{u^\rho(x,t;T)}{U^\rho(x,t)} = \exp\left((Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds) - (Y_t^\rho - \int_0^t \rho Y_s^\rho ds)\right).$$

Using (48) we deduce that $\lim_{T\uparrow\infty} Y_t^{\rho,T} = Y_t^\rho$.

Finally, the convergence of the optimal portfolio weights follows from the convergence of $Z^{\rho,T}$ to Z^ρ in $L_\rho^2[t,\infty)$. ■

3.4 General solutions

If we do not require Markovian solutions for (12) and (22), then there are multiple solutions. For example, for any process $Z \in L^2_{BMO}$, if a pair (Y,λ) satisfies (12), then (Y_t, Z_t, λ), $t \geq 0$, is a (non-Markovian) solution. Herein, we do not examine such solutions but only comment on three examples in the absence of portfolio constraints, $\Pi = \mathbb{R}^d$.

i) Time-monotone case: Let $Z_t \equiv 0$ for $t \geq 0$, and choose (Y,λ) such that

$$Y_t - \lambda t = Y_0 - \int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\theta(V_s)|^2 ds,$$

for any constant Y_0. Then, $(Y_t, 0, \lambda)$ is a solution to the ergodic quadratic BSDE (12) and Theorem 3 yields $U(x,t) = e^{Y_0}\frac{x^\delta}{\delta}e^{-\frac{1}{2}\frac{\delta}{1-\delta}A_t}$ with $A_t = \int_0^t |\theta(V_s)|^2 ds$. This is a *path-dependent* time-monotone forward performance process of power type (see [29] for a general study of such processes). The optimal portfolio weights and wealth process are given, respectively, by $\pi_t^* = \frac{\theta(V_t)}{(1-\delta)}$ and

$$X_t^* = X_0 \exp\left(\int_0^t \frac{1-2\delta}{2(1-\delta)^2}|\theta(V_s)|^2 ds + \frac{1}{1-\delta}\theta(V_s)^T dW_s\right).$$

Variations of this solution with non-zero forward volatility can be constructed as it is shown below. Note, however, that these forward processes essentially correspond to a fictitious market with different risk-premia and, thus, do not constitute genuine new solutions for the original market.

ii) Market view case: Let $Z_t = \phi_t$ with $\phi \in L^2_{BMO}$, and choose (Y, λ) such that (12) is satisfied. Then

$$U(x,t) = \frac{x^\delta}{\delta} e^{Y_0 - \int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\phi_s + \theta(V_s)|^2 ds} \mathcal{E}\left(\int_0^\cdot \phi_s^T dW_s\right)_t,$$

which can be expressed as

$$U(x,t) = e^{Y_0} \frac{x^\delta}{\delta} e^{-\frac{1}{2}\frac{\delta}{1-\delta}A_t^\phi} M_t$$

with $A_t^\phi = \int_0^t |\phi_s + \theta(V_s)|^2 ds$ and $M_t = \mathcal{E}(\int_0^\cdot \phi_s^T dW_s)_t$. This is a power forward process with market view as in [30]. The optimal portfolio weights can be decomposed as the sum of the myopic and non-myopic components, $\pi_t^* = \frac{\theta(V_t)}{(1-\delta)} + \frac{\phi_t}{1-\delta}$, and the optimal wealth process is given by

$$X_t^* = X_0 \exp\left(\int_0^t \frac{(\theta(V_s) + \phi_s)^T((1-2\delta)\theta(V_s) - \phi_s)}{2(1-\delta)^2} ds + \frac{1}{1-\delta}(\theta(V_s) + \phi_s)^T dW_s\right).$$

(3) *Benchmark case*: If the auxiliary process Z_t is parametrized as $Z_t = \delta\phi_t$ with $\phi \in L^2_{BMO}$, then we deduce the alternative representation

$$\begin{aligned} U(x,t) &= \frac{x^\delta}{\delta} e^{Y_0 - \int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\delta\phi_s + \theta(V_s)|^2 ds} \mathcal{E}\left(\int_0^\cdot \delta\phi_s^T dW_s\right)_t \\ &= e^{Y_0} \frac{x^\delta}{\delta} e^{-\int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\phi_s + \theta(V_s)|^2} \left(\mathcal{E}\left(\int_0^\cdot -\phi_s^T(dW_s + \theta(V_s)ds)\right)_t\right)^{-\delta} \\ &= e^{Y_0} \frac{1}{\delta}\left(\frac{x}{M_t}\right)^\delta e^{-\frac{1}{2}\frac{\delta}{1-\delta}A_t}, \end{aligned}$$

with $A_t^\phi = \int_0^t |\phi_s + \theta(V_s)|^2 ds$ and $M_t = \mathcal{E}\left(\int_0^\cdot -\phi_s^T(\theta(V_s)ds + dW_s)\right)_t$. This is the power forward process measured with respect to the benchmark M_t as in [30].

4 Exponential forward performance processes

We examine representations of the exponential forward performance processes. For this family of processes, it is more convenient for the control policy to represent the discounted amount (and not the proportions of the discounted weath) invested in the individual stock accounts. Hence, by introducing $\tilde{\alpha}_t = \tilde{\pi}_t X_t^{\pi}$, and, in turn, rescaling $\tilde{\alpha}_t$ by the volatility process, we deduce that the wealth process solves, for $t \geq 0$,

$$dX_t^{\alpha} = \alpha_t^T(\theta(V_t)dt + dW_t) \tag{29}$$

with $\alpha_t^T = \tilde{\alpha}_t^T \sigma(V_t)$. We take $\alpha \in \mathcal{A}$, and the admissible wealth domain is $\mathbb{D} = \mathbb{R}$.

4.1 Representation via ergodic quadratic BSDE

The results are analogous to the ones derived in the previous section and, thus, we state them without proofs.

Proposition 10 *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1.ii and let the set Π be as in (7). Then, the ergodic quadratic BSDE*

$$dY_t = (-G(V_t, Z_t) + \lambda)dt + Z_t^T dW_t, \tag{30}$$

where the driver $G(\cdot,\cdot)$ is given by

$$G(V_t, Z_t) = \frac{1}{2}\gamma^2 dist^2\left(\Pi, \frac{Z_t + \theta(V_t)}{\gamma}\right) - \frac{1}{2}|Z_t + \theta(V_t)|^2 + \frac{1}{2}|Z_t|^2, \tag{31}$$

admits a unique Markovian solution $(Y_t, Z_t, \lambda), t \geq 0$.

Specifically, there exists a unique $\lambda \in \mathbb{R}$ and functions $y : \mathbb{R}^d \to \mathbb{R}$ and $z : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t, Z_t) = (y(V_t), z(V_t))$. The fucntion $y(\cdot)$ is unique up to a constant and has at most linear growth, and $z(\cdot)$ is bounded with $|z(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η and C_v are as in (3) and (40), respectively.

Theorem 11 *Let $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda)$, $t \geq 0$, be the unique Markovian solution of the ergodic quadratic BSDE (30). Then,*

i) the process $U(x,t)$, $(x,t) \in \mathbb{R} \times [0,\infty)$, given by

$$U(x,t) = -e^{-\gamma x} e^{y(V_t) - \lambda t} \tag{32}$$

is an exponential forward performance process with volatility

$$a(x,t) = -e^{-\gamma x} e^{y(V_t) - \lambda t} z(V_t).$$

ii) The optimal portfolios α_t^* and the optimal wealth process X_t^* are given, respectively, by

$$\alpha_t^* = Proj_{\Pi}\left(\frac{z(V_t) + \theta(V_t)}{\gamma}\right), \; X_t^* = X_0 + \int_0^t (\alpha_t^*)^T(\theta(V_t)dt + dW_t). \tag{33}$$

We recall that exponential forward performance processes have been used for the construction of forward indifference prices (see, among others, [24], [25], [27] and [15]). An axiomatic construction can be found in [38] in a general semimartingale market setting, while their connection with maturity-independent entropy risk measures is developed in [37].

4.2 Representation via infinite horizon quadratic BSDE

In analogy to the results of section 3.2, we derive an alternative representation of the exponential forward performance process using an infinite horizon quadratic BSDE. The proof follows along similar arguments and is, thus, omitted.

Proposition 12 *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1.ii and let the set Π be as in (7). Let $\rho > 0$. Then, the infinite horizon quadratic BSDE*

$$dY_t^\rho = (-G(V_t, Z_t^\rho) + \rho Y_t^\rho)\, dt + (Z_t^\rho)^T\, dW_t, \tag{34}$$

where the driver $G(\cdot,\cdot)$ is given in (30), admits a unique Markovian solution, $t \geq 0$. Specifically, for each $\rho > 0$, there exist unique functions $y^\rho : \mathbb{R}^d \to \mathbb{R}$ and $z^\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t^\rho, Z_t^\rho) = (y^\rho(V_t), z^\rho(V_t))$, with $|y^\rho(\cdot)| \leq \frac{K}{\rho}$ and $|z^\rho(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η as in (3), and C_v, K given in (40) and (42), respectively.

Theorem 13 *Let $(y^\rho(V_t), z^\rho(V_t))$, $t \geq 0$, be the unique Markovian solution to the infinite horizon quadratic BSDE (34). Then,*

i) the process $U^\rho(x,t)$, $(x,t) \in \mathbb{R} \times [0,\infty)$, given by

$$U^\rho(x,t) = -e^{-\gamma x} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_u) du} \tag{35}$$

is an exponential forward performance process with volatility

$$a^\rho(x,t) = -e^{-\gamma x} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_u) du} z^\rho(V_t).$$

ii) The optimal portfolios $\alpha_t^{*,\rho}$ and optimal wealth process $X_t^{*,\rho}$ (cf. (29)), $t \geq 0$, are given, respectively, as in (33) with $z(V_t)$ replaced by $z^\rho(V_t)$.

In line with Proposition 8, we have the following connection between the ergodic and infinite horizon representations for exponential forward performance processes.

Proposition 14 *Let $U^\rho(x,t)$ and $U(x,t)$ be the exponential forward performance processes (32) and (35), respectively. Then, for any reference point*

$v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R} \times [0,\infty)$,

$$\lim_{\rho_n \downarrow 0} \frac{U^{\rho_n}(x,t)e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1.$$

Moreover, for any $t \geq 0$, the associated optimal portfolios satisfy

$$\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t |\alpha_u^{*,\rho_n} - \alpha_u^*|^2 \, du = 0.$$

4.3 Connection with the classical exponential expected utility for long horizons

As in section 3.3, we discuss the relationship between the exponential forward performance process $U(x,t)$ and its traditional finite horizon expected utility analogue with the latter incorporating a terminal random endowment.

To this end, let $\rho > 0$ and $[0,T]$ be an arbitrary trading horizon. Consider a family of maximal expected utility problems

$$u^\rho(x,t;T) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left(-e^{-\gamma(X_T^\alpha + \xi_T)} | \mathcal{F}_t, X_t^\alpha = x \right), \tag{36}$$

for $(x,t) \in \mathbb{R} \times [0,T]$ and the wealth process X_s^α, $s \in [t,T]$, solving (29). The payoff ξ_T is defined as $\xi_T = \frac{1}{\gamma} \int_0^T \rho Y_t^{\rho,T} dt$, where $Y_t^{\rho,T}$ is the solution of the finite horizon quadratic BSDE

$$Y_t^{\rho,T} = \int_t^T \left(G(V_s, Z_s^{\rho,T}) - \rho Y_s^{\rho,T} \right) ds - \int_t^T \left(Z_s^{\rho,T} \right)^T dW_s,$$

with the driver $G(\cdot,\cdot)$ given in (31). The optimal portfolios are denoted by $\alpha_s^{*,\rho,T}$ for $s \in [t,T]$. We have the following convergence result.

Proposition 15 *i) Let $u^\rho(x,t;T)$ and $U^\rho(x,t)$ be given in (36) and (35), respectively. Then, for each $\rho > 0$, and $(x,t) \in \mathbb{R} \times [0,\infty)$,*

$$\lim_{T \uparrow \infty} \frac{u^\rho(x,t;T)}{U^\rho(x,t)} = 1.$$

Moreover, for any $t \leq s \leq T$, the optimal portfolios satisfy

$$\lim_{T \uparrow \infty} E_{\mathbb{P}} \int_t^s \left| \alpha_u^{*,\rho,T} - \alpha_u^{*,\rho} \right|^2 du = 0.$$

ii) Let $U(x,t)$ be as in (32). Then, for any reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R} \times [0,\infty)$,

$$\lim_{\rho_n \downarrow 0} \lim_{T \uparrow \infty} \frac{u^{\rho_n}(x,t;T)e^{-y^{\rho_n}}(v_0)}{U(x,t)} = 1.$$

Moreover, for any $t \leq s \leq T$, the associated optimal portfolios satisfy

$$\lim_{\rho_n \downarrow 0} \lim_{T \uparrow \infty} E_{\mathbb{P}} \int_t^s \left| \alpha_u^{*,\rho_n,T} - \alpha_u^* \right|^2 du = 0.$$

5 Logarithmic forward performance processes

We conclude with the logarithmic forward performance process. The results are similar to the ones in section 3 and, for this, they are stated in an abbreviated manner.

As in Proposition 2, we deduce that the ergodic BSDE

$$dY_t = (-\tilde{F}(V_t) + \lambda)dt + Z_t^T dW_t$$

with $\tilde{F}(V_t) = -\frac{1}{2} dist^2 \{\Pi, \theta(V_t)\} + \frac{1}{2}|\theta(V_t)|^2$ has a unique Markovian solution, say $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda)$ for some functions $y(\cdot)$ and $z(\cdot)$ with similar properties.

We then easily deduce that the process

$$U(x,t) = \ln x + y(V_t) - \lambda t \tag{37}$$

is a logarithmic forward performance process in factor-form with volatility $a(x,t) = z(V_t)$, and that the associated optimal policy is given by $\pi_t^* = Proj_{\Pi}\theta(V_t)$. We also have the interpretation

$$\lambda = \sup_{\pi \in \mathcal{A}} \limsup_{T \uparrow \infty} \frac{1}{T} \ln E_{\mathbb{P}^\pi} \left(e^{\int_0^T \theta(V_u)^T \pi_u du} \right).$$

Moreover, the process $U^\rho(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, given by

$$U^\rho(x,t) = \ln x + y^\rho(V_t) - \int_0^t \rho y^\rho(V_s) ds \tag{38}$$

is a logarithmic forward performance process, where $Y_t^\rho = y^\rho(V_t)$ solves the infinite horizon BSDE

$$dY_t^\rho = \left(-\tilde{F}(V_t) + \rho Y_t^\rho \right) dt + (Z_t^\rho)^T dW_t.$$

The process $U(x,t)$ and $U^\rho(x,t)$ in (37) and (38) are connected in a similar way as their power analogues in Proposition 8. Namely, for an arbitrary reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,

$$\lim_{\rho_n \downarrow 0} (U^{\rho_n}(x,t) - y^{\rho_n}(v_0) - U(x,t)) = 0.$$

Finally, in order to connect $U(x,t)$ and $U^{\rho}(x,t)$ with the classical logarithmic expected utility models, we introduce the logarithmic expected utility problem

$$u^{\rho}(x,t;T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left(\ln X_T^{\pi} + \xi_T | \mathcal{F}_t, X_t^{\pi} = x \right), \tag{39}$$

where $\xi_T = -\int_0^T \rho Y_u^{\rho,T} du$ and $Y^{\rho,T}$ being the unique solution of the BSDE on $[0,T]$,

$$Y_t^{\rho,T} = \int_t^T \left(\tilde{F}(V_u) - \rho Y_u^{\rho,T} \right) du - \int_t^T \left(Z_u^{\rho,T} \right)^T dW_u.$$

Using similar arguments to the ones in Proposition 9, we deduce that for any reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,

$$\lim_{\rho_n \downarrow 0} \lim_{T \uparrow \infty} \left(u^{\rho_n}(x,t;T) - y^{\rho_n}(v_0) - U(x,t) \right) = 0.$$

A Appendix: Solving ergodic and infinite horizon quadratic BSDE

We present background results for Markovian solutions of the ergodic quadratic BSDE (12) and (30). We also obtain existence and uniqueness of bounded Markovian solutions to the infinite horizon quadratic BSDE (22) and (34) as intermediate steps in the proofs of Propositions 2 and 10. We note that some of the results can be readily extended to non-Markovian solutions.

We start with the key observation that, by Assumption 1.ii on the market price of risk process, the definition of the admissible set $\mathcal{A}$, and the Lipschitz continuity of the distance function $dist(\Pi, \cdot)$, the drivers $H = F$, G appearing in (13) and (31) satisfy

$$|H(v,z) - H(\bar{v},z)| \leq C_v(1+|z|)|v-\bar{v}|, \tag{40}$$

$$|H(v,z) - H(v,\bar{z})| \leq C_z(1+|z|+|\bar{z}|)|z-\bar{z}|, \tag{41}$$

and

$$|H(v,0)| \leq K, \tag{42}$$

for any $v, \bar{v}, z, \bar{z} \in \mathbb{R}^d$, where C_v, C_z, $K > 0$ are positive constants.

The main ideas come from Theorem 3.3 in [5], Theorem 3.3 in [6], Theorem 4.4 in [14], and Theorem 2.3 in [20]. We first define the truncation function $q : \mathbb{R}^d \to \mathbb{R}^d$,

$$q(z) = \frac{\min\left(|z|, C_v/(C_\eta - C_v)\right)}{|z|} z \mathbf{1}_{\{z \neq 0\}}, \tag{43}$$

and consider the truncated ergodic BSDE on $[0,\infty)$,

$$dY_t = (-H(V_t, q(Z_t)) + \lambda)\, dt + Z_t^T dW_t, \tag{44}$$

where q is as in (43), and the driver $H(\cdot,\cdot)$ satisfying conditions (40)-(42). We easily deduce that the Lipschitz continuity conditions

$$|H(v,q(z)) - H(\bar{v}, q(z)| \leq \frac{C_\eta C_v}{C_\eta - C_v}|v-\bar{v}|, \tag{45}$$

and

$$|H(v,q(z)) - H(v,q(\bar{z})| \leq C_z \frac{C_\eta + C_v}{C_\eta - C_v}|z-\bar{z}| \tag{46}$$

hold.

If we can then show that the BSDE (44) admits a Markovian solution denoted, say by (Y_t, Z_t, λ) with $|Z_t| \leq \frac{C_v}{C_\eta - C_v}$, $t \geq 0$, then $q(Z_t) = Z_t$, $t \geq 0$. In turn, this process (Y_t, Z_t, λ) would also solve the ergodic quadratic BSDE (12) in Proposition 2 and (30) in Proposition 10, respectively.

We first establish existence of Markovian solutions of (44). For this, we adapt the perturbation technique and the Girsanov's transformation used in Section 4 of [14] in an infinite dimensional setting.

To this end, let $n > 0$, and consider the discounted BSDE with a small discount factor, say $\rho > 0$, on the finite horizon $[0,n]$,

$$Y_t^{\rho,v,n} = \int_t^n (H(V_s^v, q(Z_s^{\rho,v,n})) - \rho Y_s^{\rho,v,n})\, ds - \int_t^n (Z_s^{\rho,v,n})^T\, dW_s, \tag{47}$$

where we use the superscript v to emphasize the initial dependence of the stochastic factor process on its initial data $V_0^v = v$.

From Section 3.1 of [6], we deduce that BSDE (47) admits a unique solution $(Y_t^{\rho,v,n}, Z_t^{\rho,v,n}) \in L^2_{[0,n]}$ with $|Y_t^{\rho,v,n}| \leq \frac{K}{\rho}$, $0 \leq t \leq n$, where

$$L^2\,[0,n] = \{(Y_t)_{t\in[0,n]} : Y \text{ is } \mathbb{F}\text{-progressively mble and } E_{\mathbb{P}}(\int_0^n |Y_t|^2 dt) < \infty\}.$$

On the other hand, parameterizing (47) by the auxiliary horizon n, we obtain (cf. Section 3.1 of [6]) that there exists a process $Y_t^{\rho,v}$, $t \geq 0$, such that

$$\lim_{n\uparrow\infty} Y_t^{\rho,v,n} = Y_t^{\rho,v} \tag{48}$$

for a.e. $(t,\omega) \in [0,\infty)\times\Omega$, and moreover, that for each $\rho > 0$, both $\{Y_t^{\rho,v,n}\}$ and $\{Z_t^{\rho,v,n}\}$ are Cauchy sequences in $L^2_\rho\,[0,\infty]$, where

$$L^2_\rho[0,\infty) = \{(Y_t)_{t\in[0,\infty)} : Y \text{ is } \mathbb{F}\text{-progressively mble and } E_{\mathbb{P}}(\int_0^\infty e^{-2\rho t}|Y_t|^2 dt) < \infty\}.$$

Therefore, there exist limiting processes $(Y_t^{\rho,v}, Z_t^{\rho,v})$, $t \geq 0$, belonging to $L^2_\rho[0,\infty)$, such that

$$\lim_{n\uparrow\infty}(Y_t^{\rho,v,n}, Z_t^{\rho,v,n}) = (Y_t^{\rho,v}, Z_t^{\rho,v}) \tag{49}$$

in $L^2_\rho[0,\infty)$ with $|Y_t^{\rho,v}| \leq \frac{K}{\rho}$. Then, it is easy to show that the process $(Y_t^{\rho,v}, Z_t^{\rho,v})$, $t \geq 0$, is a solution to the BSDE

$$dY_t^{\rho,v} = (-H(V_t^v, q(Z_t^{\rho,v})) + \rho Y_t^{\rho,v})\,dt + (Z_t^{\rho,v})^T\,dW_t. \tag{50}$$

Moreover, we recall that the solution is Markovian in the sense that there exist functions, say $y^\rho(\cdot)$ and $z^\rho(\cdot)$, such that

$$(Y_t^{\rho,v}, Z_t^{\rho,v}) = (y^\rho(V_t^v), z^\rho(V_t^v))\,.$$

Next, using the Girsanov's transformation and adapting the argument in Lemma 4.3 in [14], we claim that the Lipschitz continuity property

$$|y^\rho(V_t^v) - y^\rho(V_t^{\bar{v}})| \leq \frac{C_v}{C_\eta - C_v}|V_t^v - V_t^{\bar{v}}| \tag{51}$$

holds, for any $v, \bar{v} \in \mathbb{R}^d$, and C_v and C_η as in (40) and (3), respectively.

Indeed, define for $t \geq 0$, the differences

$$\Delta Y_t = Y_t^{\rho,v} - Y_t^{\rho,\bar{v}} \quad \text{and} \quad \Delta Z_t = Z_t^{\rho,v} - Z_t^{\rho,\bar{v}}.$$

Then,

$$\begin{aligned} d\,(\Delta Y_t) &= -\left(H(V_t^v, q(Z_t^{\rho,v})) - H(V_t^{\bar{v}}, q(Z_t^{\rho,\bar{v}}))\right)dt + \rho\Delta Y_t dt + (\Delta Z_t)^T\,dW_t \\ &= -\Delta H_t dt + \rho \Delta Y_t dt + (\Delta Z_t)^T\,(dW_t - m_t dt)\,, \end{aligned}$$

where $\Delta H_t = H(V_t^v, q(Z_t^{\rho,v})) - H(V_t^{\bar{v}}, q(Z_t^{\rho,v}))$, and

$$m_t = \frac{H(V_t^{\bar{v}}, q(Z_t^{\rho,v})) - H(V_t^{\bar{v}}, q(Z_t^{\rho,\bar{v}}))}{|\Delta Z_t|^2}\Delta Z_t \mathbf{1}_{\{\Delta Z_t \neq 0\}}.$$

Note that the process m_t is bounded by (46), so we can define the process $\bar{W}_t = W_t - \int_0^t m_u du$, $t \geq 0$, which is a Brownian motion under some measure $\mathbb{Q}^m$ equivalent to $\mathbb{P}$. Hence, for $0 \leq t \leq s < \infty$, taking conditional expectation on $\mathcal{F}_t$ under $\mathbb{Q}^m$ yields

$$\Delta Y_t = \frac{\beta_s}{\beta_t}E_{\mathbb{Q}^m}\left(\Delta Y_s|\mathcal{F}_t\right) + E_{\mathbb{Q}^m}\left(\int_t^s \frac{\beta_u}{\beta_t}\left(\Delta H_u\right)du\middle|\,\mathcal{F}_t\right),$$

where $\beta_t = e^{-\rho t}$. Since the first expectation is bounded by $2K/\rho$, it converges to zero when $s \uparrow \infty$. Moreover, by (45), the second expectation is bounded by

$$E_{\mathbb{Q}^m}\left(\int_t^s \frac{\beta_u}{\beta_t}\left(\Delta H_u\right)du\middle|\,\mathcal{F}_t\right) \leq \frac{C_\eta C_v}{C_\eta - C_v}E_{\mathbb{Q}^m}\left(\int_t^s e^{-\rho(u-t)}|V_u^v - V_u^{\bar{v}}|du\middle|\,\mathcal{F}_t\right)$$

$$\leq \frac{C_\eta C_v}{C_\eta - C_v} \frac{e^{\rho t}\left(e^{-(\rho+C_\eta)t} - e^{-(\rho+C_\eta)s}\right)}{\rho + C_\eta}|v - \bar{v}|,$$

where we used the exponential ergodicity condition (4). Then, (51) follows by letting $s \uparrow \infty$.

Next, assume that $y^\rho(\cdot) \in C^2(\mathbb{R}^d)$. Then, applying Itô's formula to $y^\rho(V_t^v)$ yields

$$dy^\rho(V_t^v) = \mathcal{L}y^\rho(V_t^v)dt + \left(\kappa^T \nabla y^\rho(V_t^v)\right)^T dW_t, \tag{52}$$

where $\mathcal{L}$ is as in (21). In turn, from (50) and (52) we deduce that

$$\kappa^T \nabla y^\rho(V_t^v) = Z_t^{\rho,v}, \tag{53}$$

and (with a slight abuse of notation) for $v \in \mathbb{R}^d$,

$$\rho y^\rho(v) = \mathcal{L}y^\rho(v) + H\left(v, q\left(\kappa^T \nabla y^\rho(v)\right)\right), \tag{54}$$

for $v \in \mathbb{R}^d$. The above equation (54) is a standard semilinear elliptic PDE, and classical PDE results yield that it admits a unique bounded solution $y^\rho(\cdot) \in C^2(\mathbb{R}^d)$, with $|y^\rho(v)| \leq \frac{K}{\rho}$. In addition, recall that (51) yields $|\nabla y^\rho(v)| \leq \frac{C_v}{C_\eta - C_v}$, and thus, by (53) and Assumption 2 on the matrix κ, we obtain that for $t \geq 0$,

$$|Z_t^{\rho,v}| \leq \frac{C_v}{C_\eta - C_v}. \tag{55}$$

Next, we fix a reference point, say $v_0 \in \mathbb{R}^d$. Define the process $\bar{Y}_t^{\rho,v} = Y_t^{\rho,v} - Y_0^{\rho,v_0}$, and consider the perturbed version of BSDE (50), namely,

$$\bar{Y}_t^{\rho,v} = \bar{Y}_s^{\rho,v} + \int_t^s \left(H(V_u^v, q(Z_u^{\rho,v})) - \rho\bar{Y}_u^{\rho,v} - \rho Y_0^{\rho,v_0}\right) du - \int_t^s (Z_u^{\rho,v})^T dW_u,$$

for $0 \leq t \leq s < \infty$. Then $\bar{Y}_t^{\rho,v} = \bar{y}^\rho(V_t^v)$ with $\bar{y}^\rho(\cdot) = y^\rho(\cdot) - y^\rho(v_0)$.

Since $y^\rho(\cdot)$ is Lipschitz continuous, uniformly in ρ, we deduce that $|\bar{y}^\rho(\cdot)| \leq \frac{C_v}{C_\eta - C_v}|v - v_0|$. Moreover, $|\rho y^\rho(v)| \leq K$. Thus, by a diagonal procedure, we can construct a sequence $\rho_n \downarrow 0$ such that, for v in a countable subset of $\mathbb{R}^d$,

$$\lim_{\rho_n \downarrow 0} \rho_n y^{\rho_n}(v_0) = \lambda \quad \text{and} \quad \lim_{\rho_n \downarrow 0} \bar{y}^{\rho_n}(v) = y(v), \tag{56}$$

for some constant λ and some function $y(\cdot)$. Moreover, since the function $\bar{y}^\rho(\cdot)$ is Lipschitz continuous uniformly in ρ, the limit $y(\cdot)$ can be extended to a Lipschitz continuous function defined for all $v \in \mathbb{R}^d$, and

$$\lim_{\rho_n \downarrow 0} \bar{y}^{\rho_n}(v) = y(v), v \in \mathbb{R}^d. \tag{57}$$

Thus, we have $\lim_{\rho_n \downarrow 0} \bar{Y}_t^{\rho_n,v} = y(V_t^v)$ and $\lim_{\rho_n \downarrow 0}\left(\rho_n \bar{Y}_t^{\rho_n}\right) = 0$. Next, define the process $Y_t^v = y(V_t^v)$, $t \geq 0$. Then it is standard to show that there exists

$Z_u^v = z(V_u^v)$, $u \in [t,s]$, in $L^2[t,s]$ such that $\lim_{\rho_n \to 0} Z^{\rho_n,v} = Z^v$ in $L^2[t,s]$, and moreover, that the triplet $(Y_t^v, Z_t^v, \lambda) = (y(V_t^v), z(V_t^v), \lambda)$ is a solution to the truncated ergodic BSDE (44).

Finally, using the latter limit and the fact that $|Z_t^{\rho,v}| \leq C_v/(C_\eta - C_v)$, by (55), we also have that $|Z_t^v| \leq C_v/(C_\eta - C_v)$. Therefore, $q(Z_t^v) = Z_t^v$, $t \geq 0$, and in turn, the triplet (Y^v, Z^v, λ) is also a solution to the quadratic ergodic BSDE (12) and (30) in Propositios 2 and 10, respectively.

From the above arguments, it follows easily, as a by-product, the existence of Markovian solutions to the infinite horizon quadratic BSDE (22) and (34), respectively.

It remains to show the uniqueness of the Markovian solutions to the ergodic BSDE (12) and (30). Indeed, since Z_t, $t \geq 0$, is bounded by $C_v/(C_\eta - C_v)$ in both (12) and (30), the uniqueness can be proved along similar arguments used in Theorem 4.6 in [14] and Theorem 3.11 in [9].

On the other hand, the uniqueness of the Markovian solutions to the infinite horizon quadratic BSDE (22) and (34) easily follows from Section 3.1 in [6] and Theorem 3.3 in [5].

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