Noname manuscript No.
(will be inserted by the editor)

Dynamically consistent investment under model uncertainty: the robust forward criteria

Sigrid Källblad · Jan Obłój · Thaleia Zariphopoulou

Received: date / Accepted: date

Abstract We combine forward investment performance processes and ambiguity averse portfolio selection. We introduce *robust forward criteria* which address the ambiguity in specification of the model, the risk preferences and the investment horizon. They encode the evolution of dynamically consistent ambiguity averse preferences.

We first focus on establishing dual characterizations of the robust forward criteria. This is advantageous as the dual problem amounts to a search for an infimum whereas the primal problem features a saddle point. We then study in detail the so–called time monotone criteria. We solve explicitly the example of an investor who starts with a logarithmic utility and applies a quadratic penalty function. Such an investor builds a dynamic estimate of the

The authors are grateful to two anonymous reviewers for their insightful and helpful comments and to the Oxford-Man Institute of Quantitative Finance for its support. The first author carried out most of this research during her D.Phil. degree at University of Oxford and was supported by Santander Graduate Scholarship. The second author gratefully acknowledges support from St John's College in Oxford as well as the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. 33542. The third author gratefully acknowledges support from NSF DMS-RTG-0636586 Grant.

S. Källblad
CMAP, Ecole Polytechnique, Paris, France.
E-mail: sigrid.kallblad@cmap.polytechnique.fr

J. Obłój
Mathematical Institute, the Oxford-Man Institute of Quantitative Finance and St John's College, University of Oxford, Oxford, UK.
E-mail: Jan.Obloj@maths.ox.ac.uk

T. Zariphopoulou
Depts. of Mathematics and IROM, The University of Texas at Austin, Austin 78712, USA and the Oxford-Man Institute of Quantitative Finance, University of Oxford, UK.
E-mail: zariphop@math.utexas.edu

market price of risk $\hat{\lambda}$ and updates her stochastic utility in accordance with the so-perceived elapsed market opportunities. We show that this leads to a time consistent optimal investment policy given by a fractional Kelly strategy associated with $\hat{\lambda}$. The leverage is proportional to the investor's confidence in her estimate $\hat{\lambda}$.

Keywords Robust forward criteria · Optimal investment · Model uncertainty · Ambiguity aversion · Dynamic consistency · Time consistency · Duality theory

Mathematics Subject Classification (2010) 91B16 · 91G10 · 91B06 · 91G80 · 49N15 · 62C20

JEL Classification G11 · D81

1 Introduction

This paper is a contribution to optimal investment as a problem of normative decisions under uncertainty. This topic is central to financial economics and mathematical finance, and the relevant body of research is large and diverse. Within it, the expected utility maximisation (EUM), with its axiomatic foundation going back to von Neumann and Morgenstern [66] and Savage [59], is probably the most widely used and extensively studied framework. In a continuous time setting, it was first applied to the optimal portfolio selection by Merton [48] who proposed a stochastic optimization problem of the form

$$\max_{\pi} \mathrm{E}^{\mathbb{P}} \left[U(X_T^{\pi}) \right], \tag{1.1}$$

where $\mathbb{P}$ is the historical probability measure, T the trading horizon and $U(\cdot)$ the investor's utility at T.

Despite the popularity of the above model, there has been a considerable amount of criticism of the model fundamentals $(\mathbb{P}, T, U)$, for these inputs might be ambiguous, inflexible, not very amenable to applications and difficult to specify. First, there are numerous issues regarding elucidation and choice of the utility function U. Some authors argue that the concept of utility per se is elusive and one should look for different, more pragmatic criteria to use in order to quantify the risk preferences of the investor. We refer the reader to an old note of F. Black [9] where the criterion is the choice of the optimal portfolio, see also He and Huang [29] and Cox, Hobson and Obłój [14], and to Sharpe [64] and Monin [49] where the criterion is a targeted wealth distribution. Another line of research accepts the utility as an appropriate device to rank outcomes but challenges the classical EUM, for empirical evidence shows that investors feel differently with respect to gains and losses. Among others, see Hershey and Schoemaker [33] and Kahneman and Tversky [35] which then led to the development of the area of behavioural finance (see e.g. Barberis [5] and Jin and Zhou [34]). A third line generalises the concept of utility and moves away from a terminal-horizon deterministic utility, as $U(\cdot)$ above, by allowing state- and path-dependence which can alleviate several drawbacks of the classical setting. One of the best known paradigms are the *recursive utilities*, see e.g. Duffie [19], El Kaouri et al. [23], Skiadas [65]. State-dependent utilities have also been considered in static frameworks, see e.g. Drèze [18], Karni [40].

Second, the investment horizon T might not be fixed and/or a priori known. Such situations arise, for example, in investment problems with rolling horizons or problems in which the horizon needs to be modified due to inflow of new funds, new market opportunities, or new investment options and obligations. In this context it is natural to study under which model conditions and preference structure one could extend the standard investment problem beyond a pre-specified horizon in a time consistent manner, see e.g. Källblad [36]. It is also interesting to study utilities that are not biased by the horizon choice, as the *horizon-unbiased utilities* introduced by Henderson and Hobson [30]; see also Choulli et al. [13].

Last but not least, an investor frequently faces a significant ambiguity as to which market model to use, specifically, how to determine the probability measure $\mathbb{P}$. This is often referred to as the *Knightian uncertainty*, in reference to the original contribution of Knight [43]. Weakening of the independence axiom to account for the *ambiguity aversion*, motivated by the Ellsberg [24] paradox, led to the generalised *robust* EUM paradigm in Gilboa and Schmeidler [28]. It built on earlier contributions, including Anscombe and Aumann [2] and Schmeidler [63], and has since been followed and extended by a large number of works; we refer the reader to Maccheroni et al. [46], Schied [61] and to Föllmer, Schied and Weber [26], and the references therein, for an overview.

Our work herein was motivated by the above considerations of the triplet of model inputs $(\mathbb{P}, T, U)$. We propose a framework that alleviates some of the above shortcomings in a unified manner, combining elements from the classical robustness theory and the recently developed forward investment performance approach. We now briefly introduce the latter before describing our main contributions.

In the absence of model uncertainty, forward performance processes were introduced by Musiela and Zariphopoulou [52, 53]. It is an adapted stochastic criterion parametrized by wealth and time, denoted by $U(x,t)$, $t \geq 0$, and constructed "forward-in-time". Specifically, given today's profile $U(x,t)$, the forward process for an *arbitrary* upcoming investment period $T > t$, $U(x,T)$, is specified so that

$$\begin{aligned} U(x,t) &\geq \mathrm{E}^{\mathbb{P}}\left[U(X_T^{\pi}, T) \middle| \mathcal{F}_t, X_t = x \right] \quad \text{for any admissible } \pi, \\ \text{and } U(x,t) &= \mathrm{E}^{\mathbb{P}}[U(X_T^{\pi^*}, T) | \mathcal{F}_t, X_t = x] \quad \text{for the optimal } \pi^*. \end{aligned}$$

This allows considerable flexibility in incorporating changing market opportunities and investors' attitudes in a dynamically–consistent manner. In contrast, in the classical formulation, the value function is constructed in a similar manner but in the opposite time direction: the utility criterion is first chosen at the end of the horizon and then the Dynamic Programming Principle generates the solution from T to previous times. The computation of the value function involves the underlying model for market dynamics for the entire investment horizon and there is no *a priori* mechanism to extend the investment problem beyond T in a dynamically–consistent manner. This induces significant limitations, as discussed below in our motivating example in Section 2.1.

In this paper we build an analogous decision framework for an agent who faces model ambiguity. As in the classical robust EUM, we consider an investor in a stochastic market environment for which she does not know the "true" model. Instead, she describes the market reality through relative weighting of stochastic models with some models being more likely than others, some being excluded all together, etc. These views are expressed by a penalty function and are updated dynamically in time. The investor's personal evaluation of wealth is expressed through her preferences. When considering a given investment horizon, say T, the investor aims to maximise the robust expected utility (max-min) functional, similarly to Maccheroni et al. [46] and Schied [61]. However, we generalise their criterion by considering stochastic preferences. These preferences evolve forward in time, taking into account the model ambiguity, and are defined for all investment horizons. Accordingly, we call them *robust forward criteria.* They are encoded by pairs of utility fields and penalty functions which are dynamically consistent.

Our theoretical focus is on defining and further characterising the new investment criteria. We consider their duals and establish an appropriate duality result. Similarly to Schied [61], as well as Quenez [58] and Schied and Wu [62], the duality proof proceeds by using an appropriate min-max theorem and then applying a model-specific duality result to the inner maximizaiton. However, unlike [61] which relied on results of Kramkov and Schachermayer [44], we view the inner maximization problems under the fixed reference measure $\mathbb{P}$ but featuring stochastic utility functions and apply the duality in Žitković [67]. Our proofs involve a number of technical and conceptual novelties. We prove relevant conjugacy relations and the existence of a dual optimizer for a class of utility functions which are allowed to be *stochastic* and finite on the *entire* real line. This means that the dynamic consistency conditions are imposed *jointly* on the penalty function and the utility random field. Unlike for convex risk measures or the classical EUM, the dynamic aspects of robust portfolio optimization seem to have been studied only for specific examples, see e.g. Laeven and Stadje [45], Müller [50]. We provide general results which, in particular, highlight the necessity of a conditional stability property of the penalty functions, see (*iv*) in Definition 2.3, in the past only considered for dynamic risk measures. Further, we also obtain the equivalence between dynamic consistency in the primal and dual domain and characterize the latter via a suitable submartingale property. While these are natural properties which are well understood in other contexts, e.g. classical EUM, they appear to be novel in the context of robust portfolio optimization. We use the dual formulation to study the question of time consistency of the optimal strategies. We show that in general, both in our framework as well as in the classical robust EUM, the optimal strategies may fail to be time consistent. This is caused by possibly arbitrary dynamics of the penalty functions. We show that time consistency of the optimal strategies is guaranteed under suitable assumptions of dynamic consistency of the penalty functions.

Apart from the theoretical contribution, we also construct and solve explicitly a practically relevant example which showcases the advantages of our approach. We consider an investor who starts with a logarithmic utility and applies a quadratic penalty function. Specifically, the investor builds a dynamic estimate of the market price of risk, say $\hat{\lambda}$, and updates her stochastic utility in accordance with the so-perceived elapsed market opportunities. We show that this leads to a time consistent optimal investment policy given by a fractional Kelly strategy associated with $\hat{\lambda}$. The leverage is a function of the investor's confidence in the estimate $\hat{\lambda}$. This solution is both intuitive and relevant since it corresponds to strategies often followed by large investors in practice. In the classical robust EUM approach, for a fixed time horizon $[0, T]$, such behaviour is consistent with the simplest setting of a complete market and constant penalty weighting and is essentially the only explicit example available in the classical approach, see Hernández-Hernández and Schied [32]. In a more complex setting — e.g. of incomplete market and/or general adapted penalty weight — this structure is lost, the solution is described with PDE or BSDE methods and the optimal investment strategies may depend on the setting and on the investment horizon T. This complexity is due to an entangled nature of solving the problem backwards and having a deterministic boundary constraint at T. Our approach, in contrast, does not suffer from such drawbacks and offers a solution which holds in great generality. We discuss this in detail in Section 2.1.

The rest of the paper is organised as follows. In Section 2, the market model is specified, the robust forward criteria are introduced and the motivating example is studied. In Section 3, equivalent dual characterizations of robust forward criteria are established. Then, in Section 4, we study the link between dynamic consistency of penalty functions and time consistency of optimal investment strategies. In particular, we discuss a simple example of criteria leading to time inconsistent optimal investment strategies. Section 5 is devoted to a, mostly formal, discussion of various classes of criteria. Our aim is to illustrate the flexibility of the notion and the fact that interesting preferences might be identified under additional evolutionary requirements. In particular, time-monotone criteria are linked to a specific PDE. We also argue that for each robust forward criterion, there exists a specific (standard) forward criterion in the reference market producing the same optimal behaviour. The proofs are deferred to Section 6.

2 Robust forward criteria: motivation and definition

In order to motivate and illustrate the upcoming definition, we first consider an example in Section 2.1: a robust forward criterion which combines logarithmic preferences with a quadratic penalty structure for model ambiguity. The example is of particular interest as it gives theoretical justification for fractional Kelly strategies often used in practice. In Section 2.2 we then provide the general setup and definition.

2.1 A motivating example: non-volatile criteria yielding fractional Kelly strategies

Consider a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \widehat{\mathbb{P}})$ with the filtration spanned by a two-dimensional $\widehat{\mathbb{P}}$-Brownian motion $\hat{W}_t = (\hat{W}_t^1, \hat{W}_t^2)$, $t \geq 0$, and a market with a zero-interest bond and a stock whose price $(S_t)_{t\geq 0}$ solves

$$dS_t = S_t \sigma_t \left(\hat{\lambda}_t dt + d\hat{W}_t^1 \right), t \geq 0, \tag{2.1}$$

for some $\mathbb{F}$-progressively measurable processes $\hat{\lambda}_t$ and $\sigma_t > 0$, $t \geq 0$. An investor acting in this incomplete market chooses the number of shares, denoted $(\pi_t)_{t\geq 0}$, to buy of the risky asset. Her wealth process then follows the dynamics

$$dX_t^\pi = \pi_t \sigma_t S_t \left(\hat{\lambda}_t dt + d\hat{W}_t^1 \right), \quad X_0 = x.$$

The set of admissible strategies is defined as

$$\mathcal{A} := \Big\{ \pi : (\pi_t) \text{ adapted, } (X_t^\pi) \text{ well-defined and } X_t^\pi > 0 \text{ a.s. for all } t > 0 \Big\},$$

and we also write $\mathcal{A}^x$ when we want to stress the dependence on the initial wealth $X_0 = x$. We denote by $\mathcal{A}_t^x$ the analogous set of strategies on $[t, \infty)$ starting from $X_t^\pi = x$.

Before we introduce model uncertainty, let us discuss this simple setup to highlight the differences between the classical problem (1.1) and the forward performance criteria. An investor solving (1.1) with a time horizon T and utility function $U(x) = \ln(x)$ is myopic and simply follows the growth optimal, or the Kelly [41], strategy which invests $\hat{\lambda}_t/\sigma_t$ fraction of wealth in the risky asset, $\pi_t^* = \frac{\hat{\lambda}_t}{\sigma_t S_t} X_t^{\pi^*}$, see Bansal and Lehmann [4] and Kardaras et al. [39] and the references therein for details. While π^* does not rely on T, nor on the particular dynamics of $\hat{\lambda}$ in the future, the value function of the investor with wealth x at time t very much does and is given by

$$\ln(x) + \frac{1}{2}\hat{\mathrm{E}}\left[\int_t^T \hat{\lambda}_u^2 du \Big| \mathcal{F}_t \right].$$

In contrast, the analogous time-monotone forward performance process, which generates the same optimal investment strategy, is given by

$$U(x,t) = \ln x - \frac{1}{2}\int_0^t \hat{\lambda}_s^2 ds,$$

which puts value in the context of the elapsed market opportunities instead. This allows considerable flexibility in reassessing upcoming market evolution, in a dynamically–consistent manner. Crucially, as we show below, this setup behaves much more naturally when model uncertainty is introduced.

Suppose now that the investor acknowledges model ambiguity. She builds, and updates dynamically, her best estimate of reality $\widehat{\mathbb{P}}$ (or equivalently $\hat{\lambda}$) but she is aware that it might be inaccurate. So the investor considers various other models and quantifies their relative likelihood via a penalty function γ. Specifically, when making decisions over the interval $[t,T]$, we only consider measures $\mathbb{Q} \sim \widehat{\mathbb{P}}$ on $\mathcal{F}_T$. We denote by $\mathcal{P}$ the set of all $\mathbb{F}$-progressively measurable processes $(\nu_t)_{t\geq 0}$ such that $\int_0^T (\nu_t)^2 dt < \infty$ a.s. for all $T>0$, and let the martingale $(D_t^\eta)_{t\in[0,T]}$ be given by

$$D_t^\eta := \mathcal{E}\left(\int \eta_s^1 d\hat{W}_s^1 + \int \eta_s^2 d\hat{W}_s^2\right)_t. \tag{2.2}$$

Any measure $\mathbb{Q} \sim \widehat{\mathbb{P}}$ on $\mathcal{F}_T$ then admits a process $\eta_t = (\eta_t^1, \eta_t^2) \in \mathcal{P}\times\mathcal{P}$, $t \leq T$, such that $\frac{d\mathbb{Q}}{d\widehat{\mathbb{P}}}|_{\mathcal{F}_T} = D_T^\eta$. We write $\mathbb{Q} = \mathbb{Q}^\eta$ and, for the present example, assign it the penalty

$$\gamma_{t,T}(\mathbb{Q}^\eta) := \begin{cases} \mathrm{E}^{\mathbb{Q}^\eta}\left[\int_t^T \frac{\delta_u}{2}\left|\eta_u\right|^2 du \middle| \mathcal{F}_t\right] & \text{if } \mathrm{E}^{\mathbb{Q}^\eta}\left[\int_t^T \hat{\lambda}_s^2 ds\right] < \infty \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

for some adapted, non–negative process (δ_t) which controls the strength of the penalisation (cf. also (5.3) below), i.e. (δ_t) quantifies[1] the investor's trust in the estimate $\widehat{\mathbb{P}}$. Note that it is natural to expect $\gamma_{t,T}(\cdot)(\omega)$ to have a global minimum at $\widehat{\mathbb{P}}_{|\mathcal{F}_T}$. We let $\mathcal{Q}_{t,T}$ denote the set of $\mathbb{Q}^\eta$ with a.s. finite penalty at time t. Finally, we assume that there exists $\kappa > 1/2$ such that $\widehat{\mathrm{E}}\big[\exp\big(\kappa \int_0^T \hat{\lambda}_s^2 ds\big)\big] < \infty$ for all $T>0$. This is a convenient integrability assumption which can be interpreted as $\widehat{\mathbb{P}}$ being *reasonable*. We then have the following result, the proof of which is reported in Section 6.

[1] For $\delta_t \equiv \delta$ constant, the penalty function defined in (2.3) corresponds to the entropic penalty function $\gamma(\mathbb{Q}) = \delta H(\mathbb{Q}|\widehat{\mathbb{P}})$, the properties of which imply that the optimization problem in (2.6) can be reformulated as a pure maximization problem with a modified utility function (if considering utility from intertemporal consumption such penalty functions still yield non-trivial problems; see among others [10,65]). For δ_t a general process, the situation is however different.

Proposition 2.1 *Given the investor's choice of* $(\hat{\lambda}_t)$ *and* (δ_t) *as above, let*

$$\bar{\eta}_t := \left[-\hat{\lambda}_t/(1+\delta_t), 0\right] \quad \text{and} \quad \bar{\pi}_t := \frac{\delta_t}{1+\delta_t}\frac{\hat{\lambda}_t}{\sigma_t}\frac{X_t^{\bar{\pi}}}{S_t}, \tag{2.4}$$

and

$$U(x,t) := \ln x - \frac{1}{2}\int_0^t \frac{\delta_s}{1+\delta_s}\hat{\lambda}_s^2 ds, \quad t \geq 0, x \in \mathbb{R}_+. \tag{2.5}$$

Recall that the penalty γ *is given by* (2.3) *and assume that* $\gamma_{0,T}(\bar{\eta}) < \infty$ *for* $T > 0$. *Then, for all* $0 \leq t \leq T < \infty$,

$$U(x,t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_t^x} \operatorname*{ess\,inf}_{\mathbb{Q}^\eta \in \mathcal{Q}_{t,T}} \mathrm{E}^{\mathbb{Q}^\eta}\left[U(X_T^\pi, T) + \gamma_{t,T}(\mathbb{Q}^\eta) \middle| \mathcal{F}_t \right], \tag{2.6}$$

and the optimum is attained for the saddle point $(\bar{\eta}, \bar{\pi})$ *given in* (2.4).

The investment strategy given in (2.4) corresponds to strategies used in practice by some of the large fund managers. Specifically, it is a fractional Kelly strategy, where the investor invests in the growth optimal (Kelly) portfolio corresponding to her best estimate of the market price of risk $\hat{\lambda}$. However she is not fully invested but instead chooses a leverage[2] proportional to her trust in the estimate $\hat{\lambda}$. If $\delta_t \nearrow \infty$ (infinite trust in the estimation), then $\bar{\pi}_t S_t / X_t^{\bar{\pi}} \nearrow \hat{\lambda}_t/\sigma_t$ which is the Kelly strategy associated with the most likely model $\widehat{\mathbb{P}}$. On the other hand, if $\delta_t \searrow 0$ (no trust in the estimation), then $\bar{\pi}_t \searrow 0$ and the optimal behaviour is to invest nothing. We stress that $\hat{\lambda}$ and δ are the investor's arbitrary inputs. In particular, there is no assumption that $\hat{\lambda}$ is a good estimate of some "true" market price of risk λ. For the dynamic consistency (2.6), it is only crucial that the investor's utility function (2.5) evolves in function of the *investor's perception of market*.

The above solution is intuitive, practically relevant and robust. It is very insightful to compare it with the classical robust EUM framework. The latter would fix an investment horizon T and take $U(x,T) = \ln x$ with (2.6) defining the value field for $t \leq T$. For some simple setups, e.g. a complete market with $\delta_s \equiv \delta$, this would lead to the same optimal investment strategy $\bar{\pi}$ in (2.4), cf. Hernández-Hernández and Schied [32]. However in more general setups, the optimal strategy would not be explicit, would depend on T and on the set of measures $\mathcal{Q}_{t,T}$ in a complex way, see e.g. [32,45,50]. The robust EUM entangles model ambiguity with horizon specification in a rather complex way leading to loss of the intuitive structure of the solution. There are further important advantages of our approach. The classical robust EUM would result in a value function which is defined on $[0,T]$ and has a non-trivial volatility while (2.5) is defined for all time horizons simultaneously and is monotone in

[2] In practice, the leverage has often a risk interpretation, e.g. it is adjusted to achieve a targeted level of volatility for the fund. It is adjusted rarely in comparison to the dynamic updating of the estimate $\hat{\lambda}$. Similarly, in our framework, the trust in one's estimation methods is likely to be adjusted on a much slower scale than the changes to the estimate itself.

time; see Section 5 for further discussion of such structural properties within the forward context.

We believe that the above example showcases the advantages of our approach over the classical robust EUM. Our idea behind the *robust forward criteria*, which we introduce formally below, is to take the condition of dynamic consistency (2.6) as the defining property, and study the corresponding class of investment criteria. Specifically, we say that a pair of mappings, namely a utility (random) field $U : \Omega\times[0,\infty)\times\mathbb{R} \to \mathbb{R}$ and a penalty function $\gamma : \{\mathbb{Q} \sim \mathbb{P}\} \to \mathbb{R}$, is a robust forward criterion if they satisfy this property for all $0 \leq t \leq T < \infty$; see Definition 2.4. With this terminology, the pair (U, γ) defined in Proposition 2.1 is a robust forward criterion for which the fractional Kelly strategy is optimal. This class of preferences provides dynamically consistent investment criteria which are well-defined for *all* investment horizons.

2.2 Definition of robust forward performance criteria

We now specify our general market model and define the robust forward criteria.

2.2.1 The market model and notation

The market consists of $d+1$ securities whose prices $(S^0_t; S_t) = (S^0_t, S^1_t, ..., S^d_t)$, $t \geq 0$, are modeled as a $(d+1)$-dimensional càdlàg semi-martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ satisfies the usual conditions. We let $S^0 \equiv 1$ and assume S to be locally bounded. A portfolio process $\pi = (\pi_t)_{t\in[0,\infty)}$ is an $\mathbb{F}$-predictable process which is S-integrable on $[0, T]$, for each $T > 0$, and denotes the number of shares held in the risky asset. The associated wealth process X^π is given by

$$X^\pi_t = \int_0^t \pi_u dS_u, \qquad t \geq 0.$$

The set of admissible portfolio processes available to the investor is denoted by $\mathcal{A}$ and is typically a subset of all portfolio processes.

For each $T > 0$, $\mathcal{M}^e_T$ denotes the set of equivalent local martingale measures. That is, the set of measures $\mathbb{Q}$ on $\mathcal{F}_T$ such that $\mathbb{Q} \sim \mathbb{P}_{|\mathcal{F}_T}$ and each component of S is a $\mathbb{Q}$-local martingale. Similarly, $\mathcal{M}^a_T$ denotes the set of absolutely continuous local martingale measures. The corresponding sets of density processes are denoted, respectively, by $\mathcal{Z}^e_T$ and $\mathcal{Z}^a_T$. Put differently,

$$\mathcal{Z}^e_T = \left\{ Z = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_{|\mathcal{F}_T}} : \mathbb{Q} \in \mathcal{M}^e_T \right\},$$

and similarly for $\mathcal{Z}^a_T$. We impose the following assumption.

Assumption 1 *The set $\mathcal{M}^e_T$ is non-empty for each $T > 0$.*

This assumption is referred to as the absence of arbitrage (NFLVR) *on finite horizons*; see Section 2 in [67] for further discussion. Note that while

$$\mathcal{M}^e_{T_1} = \{\mathbb{Q}|_{\mathcal{F}_{T_1}} : \mathbb{Q} \in \mathcal{M}^e_{T_2}\}, \quad \text{for all } 0 \leq T_1 \leq T_2,$$

there need *not* exist a set $\mathcal{M}^e$ of probability measures equivalent to $\mathbb{P}$ such that $\mathcal{M}^e_T = \{\mathbb{Q}|_{\mathcal{F}_T} : \mathbb{Q} \in \mathcal{M}^e\}$, for all $T > 0$. As argued in [67], the condition of NFLVR on finite horizons implies that, for each $\mathbb{Q} \in \mathcal{M}^e_T$, the density process $Z^{\mathbb{Q}}_t = \mathrm{E}\big[\frac{d\mathbb{Q}}{d\mathbb{P}_{|\mathcal{F}_T}}|\mathcal{F}_t\big]$, $t \in [0, T]$, might be extended to a strictly positive martingale $(Z_t)_{t\in[0,\infty)}$ such that $Z_0 = 1$ and ZS is a local martingale. The set of all such processes Z will be denoted by $\mathcal{Z}^e$. In particular, NFLVR on finite horizons holds, if and only if, $\mathcal{Z}^e$ is non-empty. If the condition of strict positivity is replaced by the one of non-negativity, the obtained family is denoted by $\mathcal{Z}^a$. For any $\mathbb{Q} \ll \mathbb{P}$, we use the notation $Z^{\mathbb{Q}}_{t,T} := Z^{\mathbb{Q}}_T/Z^{\mathbb{Q}}_t$, with the convention that $Z^{\mathbb{Q}}_{t,T} \equiv 1$ on $\{Z^{\mathbb{Q}}_t = 0\}$.

2.2.2 Utility random fields and penalty funtions

The robust forward criteria, which we introduce below, combine two elements: a utility random field $U(\omega, x, t)$, $t \geq 0$, and a family of penalty functions $\gamma_{t,T}(\mathbb{Q})$, for $0 \leq t \leq T$ and $T \geq 0$. The component $U(\omega, \cdot, t)$ models investor's preferences at time t and may depend on the past $(\omega_s)_{s\leq t}$. The investor faces ambiguity about the "true model" for the dynamics of financial assets and forms a view about the relative plausibility of different probability measures. This is reflected in $\gamma_{t,T}(\mathbb{Q})(\omega)$ which gives the weighting of measure $\mathbb{Q}$ on $\mathcal{F}_T$. In Section 2.1 we considered the case of U defined on $\mathbb{R}_+$. From now on, we focus on the case of U defined on $\mathbb{R}$. This simplifies some aspects of the duality theory, as explained in Section 3 below. Alterations of our abstract definitions to the case of U on $\mathbb{R}_+$ are immediate.

Definition 2.2 *A random field is a mapping $U : \Omega\times\mathbb{R}\times[0,\infty) \to \mathbb{R}$, which is measurable with respect to the product of the optional σ-algebra on $\Omega \times [0,\infty)$ and $\mathcal{B}(\mathbb{R})$. A utility random field is a random field which satisfies the following conditions:*

i) For all $t \in [0,\infty)$, the mapping $x \to U(\omega, x, t)$ is $\mathbb{P}$-a.s. a strictly concave and strictly increasing $C^1(\mathbb{R})$-function which satisfies the Inada conditions

$$\lim_{x\to-\infty} \frac{\partial}{\partial x}U(\omega, x, t) = \infty \quad \textit{and} \quad \lim_{x\to\infty} \frac{\partial}{\partial x}U(\omega, x, t) = 0.$$

ii) $\mathbb{P}$-a.s., for all $x \in \mathbb{R}$ the mapping $t \to U(\omega, x, t)$ is càdlàg on $[0,\infty)$.

iii) For each $x \in \mathbb{R}$ and $T \in [0,\infty)$, $U(\cdot, x, T) \in L^1(\mathcal{F}_T)$.

In what follows, we suppress ω from the notation and simply write $U(x,t)$.

Definition 2.3 *For $0 \le t \le T < \infty$, a mapping $\gamma_{t,T} : \Omega \times \{\mathbb{Q} \sim \mathbb{P}_{|\mathcal{F}_T}\} \to \mathbb{R}_+ \cup \{\infty\}$, is called a penalty function if*

i) $\gamma_{t,T}$ *is $\mathcal{F}_t$-measurable,*

ii) $\mathbb{Q} \to \gamma_{t,T}(\mathbb{Q})$ *is convex a.s,*

iii) for $\kappa \in L^\infty_+(\mathcal{F}_t)$, $\mathbb{Q} \to \mathrm{E}[\kappa\gamma_{t,T}(\mathbb{Q})]$ is weakly lower semicontinuous,

iv) for $\mathbb{Q}_1, \mathbb{Q}_2 \sim \mathbb{P}|_{\mathcal{F}_T}$ and $A \in \mathcal{F}_t$ with $\mathbb{P}(A) > 0$, if $\mathrm{E}^{\mathbb{Q}_1}[\mathbb{1}_B|\mathcal{F}_t] = \mathrm{E}^{\mathbb{Q}_2}[\mathbb{1}_B|\mathcal{F}_t]$ $\mathbb{P}$-a.s. on A for any $B \in \mathcal{F}_T$, then $\gamma_{t,T}(\mathbb{Q}_1) = \gamma_{t,T}(\mathbb{Q}_2)$, $\mathbb{P}$-a.s. on A.

Moreover, for a given utility random field $U(x,t)$ and a set of admissible strategies $\mathcal{A}$, we say that $(\gamma_{t,T})$, $0 \le t \le T < \infty$, is an admissible family of penalty functions if for all $T > 0$ and $\pi \in \mathcal{A}$, $\mathrm{E}^{\mathbb{Q}}[U(X^\pi_T, T)]$ is well defined in $\mathbb{R} \cup \{\infty\}$ for all $\mathbb{Q} \in \mathcal{Q}_{t,T}$, $t \le T$, where $\mathcal{Q}_{t,T}$ is the set of measures on $\mathcal{F}_T$ given by

$$\mathcal{Q}_{t,T} := \left\{\mathbb{Q} \sim \mathbb{P}_{|\mathcal{F}_T} \text{ and } \gamma_{t,T}(\mathbb{Q}) < \infty \text{ a.s.}\right\}. \tag{2.7}$$

Condition (iv) above simply says that if at time t an investor considering $[t,T]$ can not tell apart $\mathbb{Q}^1$ from $\mathbb{Q}^2$ then she assigns them the same penalty. To the best of our knowledge, such condition has not been invoked previously in the context of robust portfolio optimization but is required here since, unlike previous works, we consider a dynamic problem and prove *conditional* conjugacy relations. Analogous condition has appeared before in the context of dynamic risk measures, see Definition 3.11 of *local property* of a penalty function in Cheridito et al. [12] or *pasting property* in Lemma 3.3 in Klöppel and Schweizer [42]. Its importance here becomes apparent in the proof of Lemma 6.4.

In the above definition, $\mathcal{Q}_{t,T}$ is the set of feasible measures considered at time t when investing over $[t,T]$. It may depend on t and T but is non-random. Both larger and smaller sets could be used, e.g. the (random) set of measures $\mathbb{Q}$ with $\gamma_{t,T}(\mathbb{Q})(\omega) < \infty$ or the set of measures $\mathbb{Q}$ with $\mathrm{E}\big[\gamma_{t,T}(\mathbb{Q})\big] < \infty$. However, for many natural penalty functions, these different choices lead to the same value function. Finally, note that we do not impose any regularity or consistency assumptions on $\gamma_{t,T}(\mathbb{Q})$ in the time variables. These are not necessary for the abstract results in Section 3 and will be introduced later when they appear naturally, see Assumption 3.

2.2.3 Robust forward performance criteria

We are now ready to introduce the robust forward criteria. As highlighted above, these are pairs (U,γ) which exhibit a dynamic consistency akin to the dynamic programming principle.

Definition 2.4 *Let U be a utility random field, $\mathcal{A}$ a set of admissible strategies and γ an admissible family of penalty functions. We say that (U,γ) is a* robust forward criterion *if for all $0 \le t \le T < \infty$ and all $\xi \in L^\infty(\mathcal{F}_t)$*

$$U(\xi,t) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \left\{ \mathrm{E}^{\mathbb{Q}}\left[U\left(\xi + \int_t^T \pi_s dS_s, T\right) \middle| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\} \quad a.s. \tag{2.8}$$

We note that the above definition is well posed. Indeed, given the assumptions on U and γ, the conditional expectations in (2.9) are well-defined (extended-valued) random variables. Since all $\mathbb{Q} \in \mathcal{Q}_{t,T}$ are equivalent to $\mathbb{P}$, for each $\pi \in \mathcal{A}$, the essential infimum is also well-defined (extended-valued) with respect to the reference measure $\mathbb{P}$. The set of admissible strategies $\mathcal{A}$ which we consider is specified below. In general, in particular if U were defined on $\mathbb{R}_+$, one might need to take $\mathcal{A}$ which depends on (ξ,t). Note that our definition of robust forward criteria does not require the existence of optimal investment strategies. In that aspect, we follow the approach in [67] rather than the original definition in [52,53]. This is particularly helpful for the duality theory developed in Section 3.

The optimisation in (2.8) fits within the robust EUM paradigm, as discussed in the Introduction. The crucial difference is that we require (2.8) to hold for all pairs $t \le T$. We refer to (2.8) as the *dynamic consistency* property of (U,γ). In absence of model ambiguity, (2.8) provides a direct extension of the notion of self-generating utility fields studied in [67] and, consequently, of the notion of forward performance criteria, see the Introduction and Section 5.

To relate (2.8) to the more classical dynamic programming principle it is useful to introduce the family of classical value functions $\{u(\cdot;t,T) : 0 \le t \le T < \infty\}$, with $u(\cdot;t,T) : L^\infty(\mathcal{F}_t) \to L^0(\mathcal{F}_t;\mathbb{R}\cup\{\infty\})$ given by

$$u(\xi;t,T) := \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \left\{ \mathrm{E}^{\mathbb{Q}}\left[U\left(\xi + \int_t^T \pi_s dS_s, T\right) \middle| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\}. \tag{2.9}$$

Then, (U,γ) is a robust forward criterion if and only if for all $0 \le t \le T < \infty$ and all $\xi \in L^\infty(\mathcal{F}_t)$

$$U(\xi,t) = u(\xi,t,T) \quad \text{a.s.}$$

This then implies a familiar DPP (or martingale optimality principle):

$$\begin{aligned} u(\xi,t,T) &= U(\xi,t) = u(\xi,t,r) \\ &= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,r}} \left\{ \mathrm{E}^{\mathbb{Q}}\left[U\left(\xi + \int_t^r \pi_s dS_s, r\right) \middle| \mathcal{F}_t \right] + \gamma_{t,r}(\mathbb{Q}) \right\} \\ &= \operatorname*{ess\,sup}_{\pi\in\mathcal{A}} \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,r}} \left\{ \mathrm{E}^{\mathbb{Q}}\left[u\left(\xi + \int_t^r \pi_s dS_s, r, T\right) \middle| \mathcal{F}_t \right] + \gamma_{t,r}(\mathbb{Q}) \right\}, \end{aligned} \tag{2.10}$$

for $0 \le t \le r \le T$ and $\xi \in L^\infty(\mathcal{F}_t)$.

The setting of (2.9) corresponds to a very general robust EUM but we note that it also has its limitations. For example, the penalty associated to a given measure, $\gamma_{t,T}(\mathbb{Q})$, is fixed and independent of wealth. This has important implications for the time consistency of optimal investment strategies. Indeed, as we show in Proposition 4.3, when $(\gamma_{t,T})$ are dynamically consistent, and if we have saddle points $(\pi^{t,T}, \mathbb{Q}^{t,T})$ solving (2.9), then $\mathbb{Q}^{t,r} = \mathbb{Q}^{t,T}|_{\mathcal{F}_r}$, $t \le r \le T$, and also the optimal investment strategies are time consistent. However, in all generality, we could have (dynamically consistent) robust forward criteria which lead to time inconsistent optimal strategies. An example is given in Section 4. Independence of $\gamma_{t,T}(\mathbb{Q})$ from investor's wealth is also contrary to the empirical evidence, as discussed in behavioural finance, see e.g. Kahneman and Tversky [35], which points to the importance of investor's reference point for judging scenarios. In consequence, we believe it might be interesting to study generalisations of the problem in (2.9). Within the framework of robust EUM, these are possible using quasi-concave utility functionals introduced in Cerreia-Vioglio et al. [11]. Their use for (classical) optimal investment problem has been recently investigated by Källblad [37].

3 Dual characterization of robust forward criteria

Dual methods have proved useful for the study of optimal investment problems. Although the main focus here is on the evolution of the preferences themselves rather than on the optimal strategy, this still applies. Specifically, the dual problem amounts to the search for an infimum whereas the primal problem features a saddle point. In consequence, the robust forward criteria are easier to characterize in the dual rather than the primal domain. The aim of this section is to establish equivalence of dynamic consistency in the primal and dual domains.

We focus on utility random fields which are finite on the entire real line. The reasons are twofold. First, we complement the work of Schied [61] where only utilities defined on the positive half-line were studied. Second, this simplifies certain technical aspects, see e.g. [27], and allows us to focus on the novelty of our setting. We note that allowing for negative wealth usually complicates the choice of an appropriate set of admissible strategies yielding the existence of an optimizer, cf. [57, 60]. This is not a concern for us since we do not require the existence of a primal optimiser and hence, without loss of generality, we can restrict to the set of bounded wealth processes[3]. Accordingly, we set in

[3] Indeed, the utility field defined on the entire real line does not possess any singularities (cf. Assumption 2 below). The value field defined with respect to a more general (but feasible) set of admissible strategies would therefore coincide with the one defined with respect to bounded strategies. Definition 2.4 would still apply, since the notion of robust forward criteria is a consistency requirement placed on the preferences themselves, without a reference to an optimal strategy. In consequence, for utility fields defined on the entire real line, robust forward criteria may be studied and characterized without exactly specifying the

Definitions 2.2 and 2.4 $\mathcal{A} = A_{bd}$, the set of all portfolios producing bounded wealth processes. Specifically, $\mathcal{A}_{bd} = \bar{\mathcal{A}} \cap (-\bar{\mathcal{A}})$, where $\bar{\mathcal{A}}$ is the set of all admissible portfolio processes for which, for any $T > 0$, there exists a constant $c > 0$ such that $X_t^\pi \geq -c$, $0 \leq t \leq T$, a.s.

For a given utility random field U, the associated dual random field $V : \Omega \times [0, \infty) \times (0, \infty) \to \mathbb{R}$, is given by

$$V(y,t) = \sup_{x \in \mathbb{R}} \big(U(x,t) - xy \big) \qquad \text{for } t \geq 0,\ y \geq 0. \tag{3.1}$$

The notion of dynamic consistency in the dual domain is then naturally defined as follows.

Definition 3.1 *A pair (V, γ), consisting of a dual random field and a family of penalty functions, is* dynamically consistent *(or self-generating) if for all $0 \leq t \leq T < \infty$, and all $\eta \in L_+^0(\mathcal{F}_t)$,*

$$V(\eta, t) = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_{t,T}} \operatorname*{ess\,inf}_{Z \in \mathcal{Z}_T^a} \left\{ \mathrm{E}^{\mathbb{Q}} \left[V\left(\eta Z_{t,T} / Z_{t,T}^{\mathbb{Q}}, T \right) \Big| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\}. \tag{3.2}$$

For later use we also introduce the dual value field. Specifically, for $0 \leq t \leq T < \infty$, let $v(\cdot; t, T) : L_+^0(\mathcal{F}_t) \to L^0(\mathcal{F}_t; \mathbb{R} \cup \{\infty\})$ be given by

$$v(\eta; t, T) := \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{Q}_{t,T}} \operatorname*{ess\,inf}_{Z \in \mathcal{Z}_T^a} \left\{ \mathrm{E}^{\mathbb{Q}} \left[V\left(\eta Z_{t,T} / Z_{t,T}^{\mathbb{Q}}, T \right) \Big| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}) \right\}. \tag{3.3}$$

It follows that a pair (V, γ), consisting of a dual random field and a family of penalty functions, is dynamically consistent if for all $0 \leq t \leq T < \infty$, and all $\eta \in L_+^0(\mathcal{F}_t)$,

$$V(\eta, t) = v(\eta; t, T), \quad \text{a.s.}$$

3.1 Equivalence between primal and dual dynamic consistency

We first introduce the following technical assumption:

Assumption 2 *For each $T > 0$ and $0 \leq t \leq T$, the set $\mathcal{Q}_{t,T}$ is convex and weakly compact and the set $\{ZU^-(x,T) : Z \in \mathcal{Q}_{t,T}\}$ is uniformly integrable, for all $x \in \mathbb{R}$. Furthermore, if $\kappa \in L_+^\infty(\mathcal{F}_t)$ and $\mathbb{Q} \in \mathcal{Q}_{t,T}$ are such that $\kappa Z_{t,T}^{\mathbb{Q}} U(x,T) \in L^1$, for all $x \in \mathbb{R}$, then*

$$\tilde{U}(x,T) := \mathbb{1}_{\{\kappa = 0\}} U(x,T) + \mathbb{1}_{\{\kappa > 0\}} Z_{t,T}^{\mathbb{Q}} U(x,T), \quad x \in \mathbb{R}, \tag{3.4}$$

satisfies the non-singularity condition in Definition 3.3 in [67].

domain of optimization; see also Remark 3.8 in [67]. We note also that, since the preferences are stochastic, the exact specification of a feasible set of admissible, but not necessarily bounded, strategies would be highly involved.

Assumption 2 implies that $U(x,t)$ itself satisfies the non-singularity condition. For further discussion of this concept, we refer to Remark 3.4 in [67]. Given that the set $\mathcal{Q}_{t,T}$ is weakly compact, a sufficient condition for Assumption 2 to hold, is that $U(x,t)$ is (x,ω)-uniformly bounded from below by a deterministic utility function. Then, it also trivially holds that any family of penalty functions is admissible. Due to convexity, weak compactness of $\mathcal{Q}_{t,T}$ is equivalent to closedness in L^0 (cf. Lemma 3.2 in [62]).

Next, we present the first main result, which yields the conjugacy relations between the functions $u(x;t,T)$ and $v(y;t,T)$. We stress that even for $t=0$, Theorem 3.2 differs from Theorem 2.4 in [61] in that $U(\cdot,T)$ is defined on the entire real line and allowed to be stochastic, and moreover we do not impose any finiteness assumptions. The proof is reported in Section 6.1.

Theorem 3.2 *Let $U(x,t)$, $t\geq 0$, be a utility random field, $\gamma_{t,T}$ an admissible family of penalty functions and $V(y,t)$ the associated dual random field. Assume that Assumption 2 holds.*

Then, for all $\xi\in L^\infty(\mathcal{F}_t)$, $\eta\in L^0_+(\mathcal{F}_t)$ and $0\leq t\leq T<\infty$,

$$u(\xi;t,T)=\operatorname*{ess\,inf}_{\eta\in L^0_+(\mathcal{F}_t)}\big(v(\eta;t,T)+\xi\eta\big)\quad a.s. \tag{3.5}$$

and

$$v(\eta;t,T)=\operatorname*{ess\,sup}_{\xi\in L^\infty(\mathcal{F}_t)}\big(u(\xi;t,T)-\xi\eta\big)\quad a.s. \tag{3.6}$$

In consequence, the combination of a utility random field $U(x,t)$ and a family of penalty functions $\gamma_{t,T}$ is dynamically consistent, if and only if, the combination of the dual random field $V(y,t)$ and $\gamma_{t,T}$ is dynamically consistent.

Similarly to the non-robust case, see [67], the dual problem admits a solution even though the primal problem may not as we restricted to using bounded wealth strategies.

Proposition 3.3 *Let (U,γ) be a utility random field and a family of penalty functions for which Assumption 2 holds. Let (V,γ) be the corresponding dual pair given by (3.1). Then, for each $\eta\in L^0_+$ and $t\leq T<\infty$, there exist $\mathbb{Q}\in\mathcal{Q}_{t,T}$ and $Z\in\mathcal{Z}^a_T$ attaining the infimum in (3.2).*

We provide the proof in Section 6.1, but remark that the fact that the second component of the optimizer lies in $\mathcal{M}^a_T$ (as opposed to a larger set of finitely additive measures) is a consequence of the utility function being finite on the entire real line (see [67] and also [6,60]).

We assumed above that the set of measures $\mathcal{Q}_{t,T}$, defined in (2.7), is weakly compact. Replacing the set $\mathcal{Q}_{t,T}$ in the definition of $u(\cdot;t,T)$ by the set $\mathcal{Q}^a_{t,T}$ of absolutely continuous measures for which the penalty is finite a.s., Theorem 3.2 still holds under the assumption that $\mathcal{Q}^a_{t,T}$ is weakly compact. This holds, for example, for all penalty functions associated with coherent risk measures continuous from below; see also the closing remarks in Section 4. In order to genuinely extend to absolutely continuous measures and allow for $\mathcal{Q}^a_{t,T}$ in the definition of $v(\cdot;t,T)$, one would need to extend the definition of $Z^{\mathbb{Q}}V(\eta/Z^{\mathbb{Q}})$ to the null-sets of $\mathbb{Q}$ in a suitable way (preserving lower semicontinuity), analogously to the case of utility functions defined on $\mathbb{R}_+$ in [61]. Such an extension would enable proving Proposition 3.3 using the weak compactness of the level sets rather than of $\mathcal{Q}^a_{t,T}$, an assumption which holds for all penalty functions associated with convex risk measures continuous from below. However, at present, we do not know how to carry out such an extensions and, more importantly, how to then prove the conjugacy relations (3.5) and (3.6) relying on compactness of the level sets rather than of $\mathcal{Q}^a_{t,T}$.

4 Dynamic consistency of penalty functions and time consistency of the optimal investment strategies

The definition of robust forward criteria requires the *combined* criterion consisting of $U(x,t)$ and $\gamma(\cdot)$ to be dynamically consistent (cf. Definition 2.4). In this section we further investigate this assumption and relate it to the dynamic consistency of the penalty functions and the time consistency optimal investment strategies. The corresponding proofs are reported in Section 6.2.

Assumption 3 *For any $T>0$ and $\mathbb{Q}\sim\mathbb{P}|_{\mathcal{F}_T}$, the family of penalty functions $(\gamma_{t,T})$ is càdlàg in $t\le T$, $\gamma_{t,t}\equiv 0$ and*

$$\gamma_{s,T}(\mathbb{Q})=\gamma_{s,t}(\mathbb{Q}_{|\mathcal{F}_t})+\mathrm{E}^{\mathbb{Q}}\left[\gamma_{t,T}(\mathbb{Q})|\,\mathcal{F}_s\right],\quad s\le t\le T. \tag{4.1}$$

Moreover,

$$\mathcal{Q}_{s,T}=\tilde{\mathcal{Q}}_{s,T}, \tag{4.2}$$

where

$$\tilde{\mathcal{Q}}_{s,T}:=\left\{\mathbb{Q}\sim\mathbb{P}_{|\mathcal{F}_T}:Z^{\mathbb{Q}}_T=Z^{\mathbb{Q}_0}_tZ^{\mathbb{Q}_1}_{t,T},\ \ \mathbb{Q}_0\in\mathcal{Q}_{s,t},\ \ \mathbb{Q}_1\in\mathcal{Q}_{t,T},s\le t\le T\right\}.$$

For any penalty function satisfying (4.1), $\mathcal{Q}_{t,T}\subseteq\tilde{\mathcal{Q}}_{t,T}$. However, in general, stability under pasting (4.2) may fail. It may be recovered if different definitions of $\mathcal{Q}_{t,T}$ are used, e.g. with measures satisfying $\mathrm{E}[\gamma_{t,T}(\mathbb{Q})]<\infty$, see the remarks below on penalty functions associated with risk measures.

The additional structure resulting from Assumption 3 allows us to consider the question of whether, for a fixed $T>0$, the value field $u(x,t;T)$ associated with a general utility field satisfies *itself* the Dynamic Programming Principle for $t\le T$ as in (2.10). We show that, under suitable assumptions on the penalty function, this is the case. For particular choices of preferences, this property was used to address the ambiguity averse problem by stochastic control methods in [31,32,50]. The proof proceeds by first establishing appropriate consistency in the dual domain and then applying Theorem 3.2.

Proposition 4.1 *Let $U(x,t)$ be a utility random field and $\gamma_{t,T}$ an admissible family of penalty functions, and let $u(\cdot;t,T)$ be the associated value field. Suppose Assumptions 2 and 3 hold. Then, for $0 \le s \le t \le T$,*

$$u(x;s,T) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{bd}} \operatorname*{ess\,inf}_{\mathcal{Q}_{s,t}} \left\{ \mathrm{E}^{\mathbb{Q}}\left[u\Big(x + \int_s^t \pi_u dS_u; t, T\Big)\Big| \mathcal{F}_s \right] + \gamma_{s,t}(\mathbb{Q}) \right\}. \quad (4.3)$$

For the case of standard (non–robust) utility maximization and deterministic utility functions, it is well known that the value process satisfies the DPP; also referred to as the martingale optimality principle, see [20]. Proposition 4.1 shows that a similar consistency property holds for certain ambiguity averse criteria. However, the value field associated with a general penalty function may fail to be dynamically consistent, see [61] for counterexamples. Hence, while standard forward performance criteria might be viewed as a generalisation, to all times $t \ge 0$, of value functions associated with stochastic utility functions, in the robust setting our Definition 2.4 enforces additional structure by imposing the dynamic consistency requirement (2.8) on the pair (U,γ). In general, this is weaker than the assumption of dynamic consistency of γ. Indeed, as illustrated by the next example, there are dynamically consistent pairs (U,γ) where the penalty function γ itself is not dynamically consistent. Such robust forward criteria may lead to time *inconsistent* optimal investment strategies.

Example 4.2 *We work in the setting of Section 2.1. We set $\hat{\lambda} \equiv 0$ and fix a family of bounded random variables $(\lambda^{t,T})$ with $0 \le t \le T$, with each $\lambda^{t,T}$ being $\mathcal{F}_t$–measurable and $(\lambda^{t,T})^2 \le K$, for some $K > 0$. In turn, let*

$$\gamma_{t,T}(\mathbb{Q}^\eta) := \begin{cases} \frac{1}{2}(T-t)(K-(\lambda^{t,T})^2) & \text{if } (\eta^1_u, \eta^2_u) = (\lambda^{t,T}, 0),\ t \le u \le T, \\ +\infty & \text{otherwise.} \end{cases} \quad (4.4)$$

Let $U(x,t) := \ln x - \frac{t}{2}K$ and $\eta^{t,T}_u := 0$ for $u < t$ and $\eta^{t,T}_u := (\lambda^{t,T}, 0)$ for $t \le u \le T$. By definition $\mathcal{Q}_{t,T} = \{\mathbb{Q}^{\eta^{t,T}}\}$ and, therefore, using classical results on log utility maximisation, we have that

$$\begin{aligned} u(\xi,t,T) &= \ln \xi + \frac{1}{2}\mathrm{E}^{\mathbb{Q}^{\eta^{t,T}}}\left[\int_t^T |\eta^{t,T}_u|^2 du \Big| \mathcal{F}_t\right] - \frac{T}{2}K + \gamma_{t,T}(\mathbb{Q}^{\eta^{t,T}}) \\ &= \ln \xi + \frac{1}{2}(T-t)(\lambda^{t,T})^2 - \frac{T}{2}K + \frac{1}{2}(T-t)(K-(\lambda^{t,T})^2) \\ &= \ln \xi - \frac{t}{2}K = U(\xi,t), \quad t \le T. \end{aligned}$$

We easily conclude that (U,γ) is a robust forward criterion and that dynamic consistency holds. Meanwhile, the resulting optimal strategy, at time t when investing for the horizon $[t,T]$, is $\bar{\pi}^{t,T}_u = \frac{\lambda^{t,T}}{\sigma_t} X^{\bar{\pi}^{t,T}}_u$, $t \le u \le T$. Even when considering classical robust portfolio optimisation on $[0,T]$ these may be time inconsistent since we may have $\frac{\lambda^{t,T}}{\sigma_u} \ne \frac{\lambda^{u,T}}{\sigma_u}$ for $t \le u \le T$. In our context of *forward criteria, when T is not fixed, the "optimal strategy" might be further* horizon-inconsistent *in the sense that we may have $\bar{\pi}_t^{t,T} \neq \bar{\pi}_t^{t,T_1}$ for $t \leq T < T_1$. Hence, the "optimal strategy" is not really a well defined concept since it may depend not only on when we make the decision but also on which horizon we want to consider. This due to fundamental inconsistencies in the beliefs about feasible market models and violation of* (4.1)*.*

Observe that in this example property (4.1) is violated in a rather simplistic way. Indeed, at any time t, looking to invest on $[t, T]$, the investor believes that only one model is feasible. This is a degenerate case since the choice of this model changes arbitrary with t and T and there is no consistency requirement. Consider, for example, the extreme situation when all $\lambda^{t,T}$ are constant and T is fixed. Then, at time zero, the investor picks possibly different models which she will choose to believe in when making investment decisions at t for horizon $[t, T]$. It is not surprising that this might lead to time inconsistent investment strategies. However the flexibility of fixing the penalty $\gamma_{t,T}$ means that the dynamic consistency of the value functions, (2.10) on $[0, T]$, or (2.8) in general, may be preserved.

In Example 4.2, lack of time consistency of optimal strategies is inherited from lack of dynamic consistency of the penalty functions, i.e. from violation of (4.1). In contrast, when the penalty functions are consistent, we recover the time consistency of the optimisers.

Proposition 4.3 *Let (U, γ) be a robust forward criterion such that Assumptions 2 and 3 hold. Moreover, assume that for each $0 \leq t < T < \infty$ and $\xi \in L^\infty(\mathcal{F}_t)$, there is a saddle point $(\pi^{t,T}(\xi), \mathbb{Q}^{t,T}(\xi))$ for which $u(\xi, t; T)$ is attained (cf.* (2.9)*). Then, the saddle point may be taken to be time consistent in that $\mathbb{Q}^{t,T}(\xi) = \mathbb{Q}^{t,\bar{T}}(\xi)_{|\mathcal{F}_T}$, for all $t \leq T \leq \bar{T}$, and for $0 \leq t \leq u \leq T \leq \bar{T}$,*

$$\pi_u^{t,T}(\xi) = \pi_u^{t,\bar{T}}(\xi) \quad \text{and} \quad \pi_u^{t,T}(\xi) = \pi_u^{u,T}\left(\xi + \int_t^u \pi_s^{t,T} dS_s\right).$$

Furthermore, for $x > 0$, there exists a process $\bar{\pi}_t$, $t \geq 0$, and a positive martingale Y_t, $t \geq 0$, such that, for all $0 \leq t < T < \infty$, $u(x + \int_0^t \bar{\pi}_s dS_s; t, T)$ is attained for $\pi^{t,T} = \bar{\pi}$ and $\mathbb{Q}^{t,T} = \bar{\mathbb{Q}}^T$, with $\frac{\mathrm{d}\bar{\mathbb{Q}}^T}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_T}} = Y_T$.

The above result, combined with Example 4.2, shows that the dynamic consistency of penalty functions (4.1) is a necessary and sufficient condition for time consistency of optimal investment strategies. This applies both to the robust forward criteria studied here as well as to the classical robust expected utility maximisation on a fixed horizon. It leads to interesting open questions. First, the economic and empirical justification for (4.1) remains unclear. In fact, it is a non-trivial requirement and, for example, penalty functions associated to convex risk measures do not, in general, satisfy (4.1); see also Remark 3.5 in Schied [61]. Second, what are the generalisations of the optimisation problem in (2.9) which would preserve time consistency of optimal strategies while (4.1) is violated?

Next, we show that the dynamic consistency property of penalty functions leads to a characterization of robust forward criteria in terms of a certain "weighted submartingale" property of the dual field. This will be used to derive an equation allowing us to investigate particular classes and examples of robust forward criteria.

Proposition 4.4 *Let $U(x,t)$ be a utility random field and $\gamma_{t,T}$ an admissible family of penalty functions such that Assumption 2 holds. In addition, assume either that Assumption 3 holds, or that* (4.1) *holds and, for all $T>0$, $U(x,T)\in L^1(\mathcal{F}_T,\mathbb{Q})$ for all $\mathbb{Q}\in\tilde{\mathcal{Q}}_{0,T}$. Let $V(y,t)$ be the dual field given in* (3.1)*. Then, the following two statements are equivalent:*

i) (U,γ) is a robust forward criterion;

ii) For each $y>0$ and all $t\le T<\infty$, the inequality

$$V(yZ_t/Z_t^{\mathbb{Q}},t)\le \mathrm{E}^{\mathbb{Q}}\left[V(yZ_T/Z_T^{\mathbb{Q}},T)\Big|\,\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q}),\qquad (4.5)$$

holds for all $Z\in\mathcal{Z}_T^a$ and $\mathbb{Q}\in\mathcal{Q}_{t,T}$. Furthermore, there exists $Z\in\mathcal{Z}^a$ and a positive martingale Y_t, $t\ge 0$, such that, for all $t\le T<\infty$, $\mathbb{Q}_T\in\mathcal{Q}_{0,T}$, with $\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_T}}=Y_T$, and (4.5) *holds as equality for Z_T and $\mathbb{Q}_T$.*

We conclude this section with brief remarks on penalty functions $\gamma_{t,T}$ associated with convex risk measures (see [28,46] and [26]). Such penalty functions satisfy properties i) - iv) of Definition 2.3 but not necessarily the weak compactness condition in Assumption 2. However the latter may be recovered under suitable further conditions. To wit, recall that the level sets of convex risk measures (continuous from below) are weakly compact (see Lemma 4.1 in [61]). In particular, our compactness assumption is satisfied by any coherent risk measure which only assigns finite penalty to equivalent measures, see [31] for an example. Regarding Assumption 3, time consistency of convex risk measures is characterized by property (4.1) and any time consistent coherent risk measure[4] admits the pasting property (4.2) (cf. Corollary 1.26 in [1]). However, in general, (4.1) does not imply (4.2). Nevertheless, any convex risk measure admits a robust representation where $\mathcal{Q}_{t,T}$ is replaced by the set $\{\mathbb{Q}\sim\mathbb{P}|_{\mathcal{F}_T}:\mathrm{E}[\gamma_{t,T}(\mathbb{Q})]<\infty\}$, which in turn satisfies (4.2). This property is crucial to proving the equivalence between time consistency of the risk measure and property (4.1) (see, e.g. Theorem 17 in [1] or [8]). Assuming (4.2) is therefore consistent with the use of time consistent penalty functions associated with risk measures.

[4] For our case when all measures in $\mathcal{Q}_{t,T}$ are equivalent to the reference measure, even more explicit results hold for the coherent risk measures; see [16,25,42].

5 The structure of robust forward criteria and representative cases

In this section we study the structure of robust forward criteria and subsequently discuss specific cases. Throughout, we assume the Brownian setup of Section 2.1 and the discussion is mostly formal. We start with the structure of forward criteria and focus on the non-uniqueness of robust forward criteria for given initial preferences. Examples of classes where the uniqueness may be recovered are then provided in Sections 5.2 and 5.3. Particular attention is paid to robust forward criteria with no volatility (cf. (5.6) below), a class to which the main example studied in Section 2.1 belongs. Such criteria are characterised by a specific evolutionary property and linked to a certain PDE (equation (5.7) below). In Section 5.4, we show that for each robust forward criterion there exists a (standard) forward criterion, in the fixed reference market producing the same optimal behaviour.

5.1 The structure and non-uniqueness of robust forward criteria

In the standard, model-specific setting, the forward performance criteria of Musiela and Zariphopoulou [52, 53] are not uniquely specified from the initial condition. This is due to the flexibility of the investor to choose the volatility of her criterion. Indeed, recall that a (standard) forward performance criterion (admitting an Itô decomposition) satisfies the SPDE

$$dU(x,t) = \frac{1}{2}\frac{\left|\lambda_t U_x(x,t) + \sigma_t\sigma_t^+ a_x(x,t)\right|^2}{U_{xx}(x,t)}dt + a(x,t)\cdot d\hat{W}_t, \quad t \geq 0, \qquad (5.1)$$

equipped with an initial condition, say $U(x,0) = u_0(x)$. Similarly, the value process in the classical EUM problem satisfies (under appropriate regularity conditions) the SPDE (5.1) on the interval $[0,T)$. However the equation is then equipped with a terminal condition $U(x,T) = U(x)$ and constitutes a backward SPDE; see e.g. [47]. For a given terminal condition $U(x)$, when recovering the value process from this backward SPDE, the (unique) solution consists of the pair $(U(x,t), a(x,t))$ which are both simultaneously obtained. Due to the volatility component $a(x,t)$, there might however exist multiple stochastic terminal conditions, for all of which $U(\cdot,0)$ coincide. Put differently, for a given initial condition $u_0(x)$, the forward SPDE (5.1) might have multiple solutions which are catalogued by their volatility $a(x,t)$. In the forward approach it is then down to the investor herself to specify this volatility. Similarly in the robust setting, even with a fixed penalty function, in order to specify robust forward criteria uniquely, we expect to impose further constraints. These could be either on the form of the primal/dual field or on the choice of volatility structure. We discuss both below.

From the financial perspective, compared with classical utility maximization, the forward formulation considers different inputs to the investment problem; this for the standard as well as the robust case. In the classical setup, the investor's preferences are fully characterized via the spatial behaviour of the utility function at a future date and the rest is derived. In the forward setting, the initial condition $u_0(x)$ along with a requirement of dynamic consistency are a-priori fixed. The former might serve as a proxy for the investor's prior investments and the latter formalise the aim to invest in a manner which is consistent therewith. In order to pin down a unique investment preference, the investor needs then specify additional evolutionary properties of the utility field.

5.2 Imposing constraints on the dual field

We start with a logarithmic example. Namely, we assume that $V(y,t)$ admits the representation

$$V(y,t) = -\ln y + \int_0^t b_s ds + \int_0^t a_s \cdot d\hat{W}_s, \tag{5.2}$$

for some processes b_t and a_t which do not depend on y. Further, we assign to the measure $\mathbb{Q}^\eta$ (cf. (2.2)) the penalty[5]

$$\gamma_{t,T}(\mathbb{Q}) := \mathrm{E}^{\mathbb{Q}}\left[\int_t^T g_u(\eta_u)du \middle| \mathcal{F}_t\right], \tag{5.3}$$

for some $g : [0,\infty) \times \omega \times \mathbb{R}^2 \to [0,\infty]$, such that $g_t(\cdot)$ is proper, convex and lower semicontinuous, and moreover satisfies the coercivity condition, $g_t(\eta) \geq -a+b|\eta|^2$ for some constants a and b (cf. (8.6) in [26]). For example, the choice of $g_t(\eta) = |\eta|^2$ for $|\eta| \leq \overline{g}$, and $g_t(\eta) = \infty$ otherwise, ensures that $\gamma_{t,T}$ satisfies both Assumptions 2 and 3.[6] We let $\mathcal{Q} = \cap_{T>0}\mathcal{Q}_{0,T}$.

We assume that $(\hat{\lambda}_t)$ is in $\mathcal{P}$ and let $Z_t^\nu = \mathcal{E}\big(-\int \hat{\lambda}_s d\hat{W}_s^1 - \int \nu_s d\hat{W}_s^2\big)_t$, for $\nu \in \mathcal{P}$. Note that $\mathcal{Z}^e = \{Z^\nu : \nu \in \mathcal{P}$ and Z_t^ν is a $\widehat{\mathbb{P}}$-martingale on $[0,\infty)\}$. In particular, the assumption of NFLVR on finite horizons, implies that $Z^\nu \in \mathcal{Z}^e$ for $\nu_t \equiv 0$. According to Lemma 6.6 below, in order for the pair (V,γ) to satisfy (3.2), it suffices[7] that for all $Z^\nu \in \mathcal{Z}^e$ and $\mathbb{Q}^\eta \in \mathcal{Q}$, the process

$$M_t^{\eta\nu} := V\Big(yZ_t^\nu/D_t^\eta, t\Big) + \int_0^t g_s(\eta_s)ds \tag{5.4}$$

[5] Recall that according to [15], it holds within a Brownian filtration that a dynamic penalty function is time consistent (cf. (4.1)) if and only if it is representable as in (5.3).

[6] This follows e.g. from Lemma 3.1 in [31] and the fact that $\mathcal{Q}_{t,T}$ is weakly compact if and only if it is closed in L^0; see also discussion below Assumption 2 herein.

[7] The stronger assumptions on $\gamma_{t,T}$ in Lemma 6.6 are used only to argue the necessity.

is a $\mathbb{Q}^\eta$-submartingale, and that there exist ν^* and η^* for which it is a martingale. We recall that $\mathbb{Q}^\eta$ is given by $\frac{d\mathbb{Q}^\eta}{d\mathbb{P}}|\mathcal{F}_t = D_t^\eta$, with D_t^η specified in (2.2). A straightforward application of Itô-Ventzell's formula and formal minimization over ν_t, yields that in order for $M_t^{\eta\nu}$ to satisfy this condition, the processes a_t and b_t must satisfy the relation:

$$b_t = -\inf_\eta \left\{ g_t(\eta) + \frac{\left(\eta^1 + \hat{\lambda}_t\right)^2}{2} + a_t \cdot \eta \right\}, \quad a.s., \ t \geq 0. \tag{5.5}$$

We then see that for a given initial condition and a fixed penalty $g_t(\cdot)$, specifying the volatility process a_t typically leads to a unique robust forward criterion, for the drift is then given via (5.5). Another approach to pin down a unique U might be to consider fields which are Markovian. For example, within a (Markovian) stochastic factor model, we could require that U is represented as a deterministic function of the underlying factors. This function must then solve a specific equation, closely related to the HJB equation associated with the classical value function within the same factor model. However, in the forward setting, the equation has to be solved forwards in time and is therefore ill-posed. We refer to [54] for a study of such criteria in a model-specific setup.

5.3 Imposing constraints on the volatility structure: non–volatile criteria

We now consider constraints on the volatility process. Specifically, we consider the class of criteria for which the volatility of the dual field is identically zero,

$$dV(y,t) = V_t(y,t)dt, \quad t \geq 0. \tag{5.6}$$

We will refer to this class as *non–volatile*, or *time–monotone*, criteria. For standard forward criteria, this additional assumption specifies an interesting class of preferences; we refer the reader to [7, 52] for further details. Similarly as in the example in Section 5.2, a straightforward application of Itô-Ventzell's formula and formal minimization over ν_t, yields that in order for $M_t^{\eta\nu}$ (cf. (5.4)) to be a submartingale for each choice of ν and η, and a martingale at optimum, the random convex function $V(y,t)$ must solve the equation

$$V_t(y,t) + \inf_\eta \left\{ g_t(\eta) + \frac{y^2 V_{yy}(y,t)}{2} \left(\eta_t^1 + \hat{\lambda}_t\right)^2 \right\} = 0, \quad a.s., \ t \geq 0. \tag{5.7}$$

This is a random PDE, as opposed to the SPDE we obtained before. Note that (5.7) implies that non-volatile criteria are, in fact, monotone in time which justifies the terminology. We studied an instance of this equation in Section 2.1 where the criterion (2.5) is logarithmic as well as non-volatile, and its appropriate form could, formally, be obtained by substituting the dual Ansatz $V(y,t) = -\ln y + \int_0^t b_s ds$ into either of equations (5.5) or (5.7).

Equation (5.7) might be viewed as a (dual) Hamilton-Jacobi-Bellman equation. In particular, a verification theorem stating that every well-behaved (convex) solution to (5.7) constitutes a robust forward criterion might be proven. However, to prove existence or explicitly solve this equation is hard. In order to illustrate this, consider the case of no model-uncertainty, which corresponds to $g_t(\eta) = \infty$, $\eta \neq 0$. Then, equation (5.7) reduces to the random equation

$$V_t(y,t) + \frac{\hat{\lambda}_t^2}{2} y^2 V_{yy}(y,t) = 0 \quad a.s., \quad t \geq 0. \tag{5.8}$$

This equation characterizes standard non-volatile criteria in a model with market price of risk $(\hat{\lambda}_t)$. Equation (5.8), see [7, 52], is closely related to the (ill-posed) backward heat equation whose solutions only exist for a specific class of initial conditions, as characterised by Widder's theorem. We easily see that Equation (5.7) inherits difficulties related to the equation being ill–posed, but in addition it is fully non-linear. Moreover, we also need to ensure that its solution is adapted.

5.4 Equivalent standard (non-robust) forward criteria

We conclude with some remarks on the existence of equivalent forward criteria within a non–robust setting. First, returning to the example in Section 2.1, observe that the optimal strategy $\bar{\pi}$ in (2.4) can also be interpreted as the Kelly strategy associated with an auxiliary market with market price of risk $\bar{\lambda}_t := \hat{\lambda}_t + \bar{\eta}_t^1 = \frac{\delta_t}{1+\delta_t}\hat{\lambda}_t$, $t \geq 0$, which can be interpret as the market price of risk $\hat{\lambda}$ the investor considers most likely, adjusted by the investor's trust in that estimation. This is an instance of a general phenomenon related to the existence of a saddle point $(\bar{\pi}, \bar{\eta})$. Indeed, if a saddle point exists for all $t \leq T$, a robust forward criterion (U, γ) with penalty function given by (5.3) produces the same investment strategies as the standard forward criterion

$$\tilde{U}(x,t) \;:=\; U(x,t) + \int_0^t g_s(\bar{\eta}_s)ds, \tag{5.9}$$

specified in a fictitious market with market price of risk $\bar{\lambda}_t = \hat{\lambda}_t + \bar{\eta}_t^1$, $t \geq 0$. In turn, an application of Bayes' rule implies that the optimal strategy associated with the criterion (5.9) is also optimal for a forward criterion specified in the *reference* market; namely,

$$D_t^{\bar{\eta}}\tilde{U}(x,t) \;=\; D_t^{\bar{\eta}}\Big(U(x,t) + \int_0^t g_s(\bar{\eta}_s)ds\Big). \tag{5.10}$$

Note that if $U(x,t)$ is a non-volatile criterion, then $D_t^{\bar{\eta}}\tilde{U}(x,t)$ is in general volatile (cf. Theorem 4 in [51] for examples).

For the class of robust forward criteria for which the above formalism can be made rigorous, the following holds: if the robust forward criterion admits an optimal strategy, then that strategy is optimal also for a specific standard (non-robust) forward criterion viewed in the reference market. Naturally, the latter criterion is defined in terms of the optimal $\bar{\eta}_t$, which is part of the solution to the robust problem and not a priori known. Nevertheless, on a more abstract level, this implies that *viewed as a class of preference criteria, forward criteria can be argued to be 'closed' under the introduction of a certain type of model uncertainty.* For a similar conclusion in terms of the use of different numeraires, see Theorem 2.5 in [22] or Section 5.1 in [21]. An analogue result was first shown for stochastic differential utilities in [65]. In both cases, the results are possible since the notions are general enough to allow for stochastic preferences.

The advantage of properly formulating a robust forward criteria is to disentangle the impact of the preferences originating, respectively, from risk and model-ambiguity, see Section 2.1. In consequence, the inverse question to the above observations appears of great interest: under what conditions a given (volatile non-robust) forward criterion can be written as a non-volatile robust forward criterion with respect to some non-trivial penalty function? Finally, we also remark that the analysis herein and, thus, the above discussion, is restricted to measures equivalent to $\mathbb{P}$. Considering absolutely continuous measures introduces further complexity (cf. [61] for the static case) but should not alter the main conclusions; see also remarks in Section 3.1. In contrast, considering a larger set of possibly mutually singular measures would require new insights, see [17, 56].

6 Proofs

6.1 Proofs of Theorem 3.2 and Proposition 3.3

We start by introducing relevant notation from Zitković [67] since we then apply the duality therein in our proofs, see (6.8) below. Then, in Section 6.1.1, we prove conjugacy relations and existence of a dual optimizer for a specific auxiliary $\mathcal{F}_0$-measurable problem. In Section 6.1.2, Theorem 3.2 and Proposition 3.3 are proven via a reduction to the auxiliary problem.

Let $0 \leq t \leq T < \infty$ and κ a random variable in $L^\infty_+(\mathcal{F}_t)$. We will typically consider $\kappa = \mathbb{1}_A$, $A \in \mathcal{F}_t$, and use it to localise arguments to a set. We also use the notation $Z_{t,T} \in \mathcal{Z}^a_T$ and $Z \in \mathcal{Q}_{t,T}$ to denote, respectively, an element in $\{Z_{t,T} : Z \in \mathcal{Z}^a_T\}$ and $\{Z^{\mathbb{Q}} : \mathbb{Q} \in \mathcal{Q}_{t,T}\}$. The L^p-spaces, $p \in [0, \infty]$, are defined with respect to $(\Omega, \mathcal{F}_T, \mathbb{P}_{|\mathcal{F}_T})$. Let $\mathcal{K}_{t,T} := \left\{\int_t^T \pi_s dS_s : \pi \in \mathcal{A}_{bd}\right\}$ and $\mathcal{C}_{t,T} := \left(\mathcal{K}_{t,T} - L^0_+\right) \cap L^\infty$. The optimization over $\mathcal{K}_{t,T}$ in (2.9) might then be replaced by optimization over $\mathcal{C}_{t,T}$. For $\mathbb{Q} \in \mathcal{Q}_{t,T}$, we introduce the function

$$u_\kappa^{\mathbb{Q}}(\xi) = \sup_{g\in\mathcal{C}_{t,T}} \mathrm{E}\left[\kappa Z_{t,T}^{\mathbb{Q}} U(\xi+g,T)\right], \qquad \xi \in L^\infty(\mathcal{F}_t).$$

Next, let $\mathcal{D}_{t,T} := \left\{\zeta^* \in (L^\infty)^* : \langle \zeta^*, \zeta\rangle \leq 0 \text{ for all } \zeta \in \mathcal{C}_{t,T}\right\}$ and, for $\eta \in L^1_+(\mathcal{F}_t)$, let $\mathcal{D}^\eta_{t,T} := \left\{\zeta^* \in \mathcal{D}_{t,T} : \langle \zeta^*, \xi\rangle = \langle \eta, \xi\rangle, \text{ for all } \xi \in L^\infty(\mathcal{F}_t)\right\}$. Recall that according to Lemma A.4 in [67],

$$\zeta^* \in \mathcal{D}_{t,T} \cap L^1_+ \quad \text{if and only if} \quad \zeta^* = \eta Z_{t,T}, \tag{6.1}$$

for some $\eta \in L^1_+(\mathcal{F}_t)$ and $Z_{t,T} \in \mathcal{Z}^a_T$. Note that the proof of this result uses that the market satisfies NFLVR on finite horizons. We also define the function $\mathbb{V}^{\mathbb{Q}}_\kappa : \mathcal{D}_{t,T} \to (-\infty, \infty]$ by

$$\mathbb{V}^{\mathbb{Q}}_\kappa(\zeta^*) := \begin{cases} \mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T} V\left(\zeta^*/\left(\kappa Z^{\mathbb{Q}}_{t,T}\right), T\right)\right], & \zeta^* \in L^1_+ \text{ and } \{\zeta^* > 0\} \subseteq \{\kappa > 0\}, \\ \infty, & \text{otherwise;} \end{cases} \tag{6.2}$$

and the function $v^{\mathbb{Q}}_\kappa : L^1(\mathcal{F}_t) \to (-\infty, \infty]$ by

$$v^{\mathbb{Q}}_\kappa(\eta) := \begin{cases} \inf_{\zeta^* \in \mathcal{D}^\eta_{t,T}} \mathbb{V}^{\mathbb{Q}}_\kappa(\zeta^*), & \eta \in L^1_+(\mathcal{F}_t), \\ \infty, & \eta \in L^1(\mathcal{F}_t) \setminus L^1_+(\mathcal{F}_t). \end{cases} \tag{6.3}$$

Finally, we introduce auxiliary value functions $u_\kappa : L^\infty(\mathcal{F}_t) \to (-\infty, \infty]$ and $v_\kappa : L^1(\mathcal{F}_t) \to (-\infty, \infty]$ given, respectively, by

$$u_\kappa(\xi) = \sup_{g\in\mathcal{C}_{t,T}} \inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \mathrm{E}\Big[\kappa\Big(Z^{\mathbb{Q}}_{t,T} U(\xi+g,T) + \gamma_{t,T}(\mathbb{Q})\Big)\Big],$$

and

$$v_\kappa(\eta) = \inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \Big(v^{\mathbb{Q}}_\kappa(\eta) + \mathrm{E}\left[\kappa \gamma_{t,T}(\mathbb{Q})\right]\Big).$$

6.1.1 Results for the auxiliary value functions u_κ and v_κ

We establish in this section results for the $\mathcal{F}_0$-measurable value functions u_κ and v_κ introduced above. First, we consider the existence of a dual optimizer.

Proposition 6.1 *Let $\eta \in L^1_+(\mathcal{F}_t)$. Then, there exists $(\bar\zeta^*, \bar{\mathbb{Q}}) \in \mathcal{D}^\eta_{t,T} \times \mathcal{Q}_{t,T}$ such that*

$$v_\kappa(\eta) = \mathbb{V}^{\bar{\mathbb{Q}}}_\kappa(\bar\zeta^*) + \mathrm{E}\big[\kappa \gamma_{t,T}(\bar{\mathbb{Q}})\big].$$

Moreover, the function $v_\kappa(\eta)$ is convex and lower semicontinuous with respect to the weak topology.

Proof Since $\eta \in L^1_+(\mathcal{F}_t)$, we deduce that

$$v_\kappa^{\mathbb{Q}}(\eta) = \inf_{\zeta^* \in \mathcal{D}^\eta_{t,T}} \mathbb{V}_\kappa^{\mathbb{Q}}(\zeta^*), \qquad \mathbb{Q} \in \mathcal{Q}_{t,T}. \tag{6.4}$$

Hence, let $(\zeta_n^*, \mathbb{Q}_n) \in \mathcal{D}^\eta_{t,T} \times \mathcal{Q}_{t,T}$, a sequence such that

$$\mathbb{V}_\kappa^{\mathbb{Q}_n}(\zeta_n^*) + \mathrm{E}\big[\kappa\gamma_{t,T}(\mathbb{Q}_n)\big] \to v_\kappa(\eta). \tag{6.5}$$

Since $\mathcal{Q}_{t,T}$ is weakly compact, there is a subsequence $(\zeta_n^*, \mathbb{Q}_n)$ such that $\mathbb{Q}_n$ converges a.s. to some $\bar{\mathbb{Q}} \in \mathcal{Q}_{t,T}$. From the Banach-Alaouglu theorem, we have that $\mathcal{D}_{t,T}$ is weak*-compact. Hence, there exists a (further) subsequence such that ζ_n^* converges in the weak*-topology to some $\bar{\zeta}^* \in \mathcal{D}_{t,T}$. Since, for any $\xi \in L^\infty(\mathcal{F}_t)$, $\langle \bar{\zeta}^*, \xi\rangle = \lim_{n\to\infty}\langle \zeta_n^*, \xi\rangle = \langle \eta, \xi\rangle$, we have that $\bar{\zeta}^* \in \mathcal{D}^\eta_{t,T}$.

Next, according to Definition 2.3, the mapping $\mathbb{Q} \mapsto \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right]$ is l.s.c. with respect to a.s. convergence. In order to argue that $v_\kappa(\eta)$ is attained for $(\bar{\zeta}^*, \bar{\mathbb{Q}})$, it therefore only remains to argue joint lower semicontinuity of the mapping $(\zeta^*, \mathbb{Q}) \mapsto \mathbb{V}_\kappa^{\mathbb{Q}}(\zeta^*)$. Due to Assumption 2, Proposition A.3 in [67] may be applied to the (auxiliary) utility field $\tilde{U}(x,t) = Z^{\mathbb{Q}}_{t,T} U(x,T)$ to deduce that

$$\mathbb{V}_\kappa^{\mathbb{Q}}(\zeta^*) = \sup_{\zeta \in L^\infty} \left(\mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T} U(\zeta, T)\right] - \langle \zeta^*, \zeta\rangle \right). \tag{6.6}$$

For each $\zeta \in L^\infty$, the set $\left\{Z^{\mathbb{Q}}_{t,T} U^-(\zeta,T) : \mathbb{Q} \in \mathcal{Q}_{t,T}\right\}$ is uniformly integrable due to Assumption 2. Application of Fatou's Lemma to the corresponding positive part then yields lower semicontinuity, with respect to a.s. convergence, of the first term in (6.6) as a function of $\mathbb{Q} \in \mathcal{Q}_{t,T}$. The second term is continuous in ζ^* with respect to weak*-convergence. Since the pointwise supremum preserves lower semicontinuity, this allows us to conclude the joint lower semicontinuity of the mapping $(\zeta^*, \mathbb{Q}) \mapsto \mathbb{V}_\kappa^{\mathbb{Q}}(\zeta^*)$ with respect to the product topology on $\mathcal{D}^\eta_{t,T} \times \mathcal{Q}_{t,T}$.

The convexity of $v_\kappa(\eta)$ follows immediately from the joint convexity of the mapping $(\zeta^*, \mathbb{Q}) \to \mathbb{V}_\kappa^{\mathbb{Q}}(\zeta^*) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right]$ (cf. (6.6)). In order to argue the lower semicontinuity of $v_\kappa(\eta)$, let $\eta_\alpha \in L^1_+$ be a sequence such that $\eta_\alpha \to \eta$ weakly and let $(\zeta_\alpha^*, \mathbb{Q}_\alpha) \in \mathcal{D}^{\eta_\alpha}_{t,T} \times \mathcal{Q}_{t,T}$ such that $v_\kappa(\eta_\alpha) = \mathbb{V}_\kappa^{\mathbb{Q}_\alpha}(\zeta_\alpha^*) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q}_\alpha)\right]$. By use of the same arguments as above, one obtains a subsequence $(\zeta_\alpha^*, \mathbb{Q}_\alpha)$ converging in the product topology to some $(\zeta^*, \mathbb{Q}) \in \mathcal{D}_{t,T} \times \mathcal{Q}_{t,T}$. Since $\langle \zeta^*, \xi\rangle = \lim_{n\to\infty}\langle \eta_\alpha, \xi\rangle = \langle \eta, \xi\rangle$, $\xi \in L^\infty(\mathcal{F}_t)$, it follows that $\zeta^* \in \mathcal{D}^\eta_{t,T}$. The joint lower semicontinuity of the mapping $(\zeta^*, \mathbb{Q}) \to \mathbb{V}_\kappa^{\mathbb{Q}}(\zeta^*) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right]$, then yields the lower semicontinuity of $v_\kappa(\eta)$. □

In order to establish the conjugacy relations for u_κ and v_κ, we first recall a result from [67]. To this end, let $\mathbb{Q} \in \mathcal{Q}_{t,T}$, $\kappa \in L^\infty_+(\mathcal{F}_t)$ and $U(x,t)$ such that $\tilde{U}(x,T)$, defined in (3.4), satisfies the non-singularity condition in Definition 3.3 in [67], and $\tilde{U}(x,T) \in L^1$, $x \in \mathbb{R}$. Note that on $\{\kappa > 0\}$,

$\tilde V(y,T) = Z^{\mathbb{Q}}_{t,T} V\big(y/Z^{\mathbb{Q}}_{t,T}, T\big)$. Application of Propositions A1 and A3 in [67] to the (auxiliary) stochastic utility function[8] $\tilde U(x,T)$, $x \in \mathbb{R}$, then yields

$$u^{\mathbb{Q}}_{\kappa}(\xi) = \inf_{\zeta^* \in \mathcal{D}_{t,T}} \left(\mathbb{V}^{\mathbb{Q}}_{\kappa}(\zeta^*) + \langle \zeta^*, \xi\rangle\right), \qquad \xi \in L^\infty(\mathcal{F}_t). \tag{6.7}$$

According to (6.1), for each $\zeta^* \in \mathcal{D}_{t,T} \cap L^1_+$, there exists $\eta \in L^1_+(\mathcal{F}_t)$ such that $\zeta^* \in \mathcal{D}^{\eta}_{t,T}$. Combined with the definitions of $\mathbb{V}^{\mathbb{Q}}_{\kappa}$ and $v^{\mathbb{Q}}_{\kappa}$, (6.7) therefore implies that

$$u^{\mathbb{Q}}_{\kappa}(\xi) = \inf_{\eta \in L^1(\mathcal{F}_t)} \left(v^{\mathbb{Q}}_{\kappa}(\eta) + \langle \xi, \eta\rangle\right), \qquad \xi \in L^\infty(\mathcal{F}_t). \tag{6.8}$$

We also establish an auxiliary Lemma. To this end, for $\xi, \kappa \in L^\infty(\mathcal{F}_t)$, $\mathbb{Q} \in \mathcal{Q}_{t,T}$ and $g \in \mathcal{C}_{t,T}$, let

$$J_{\kappa,\xi}(\mathbb{Q}, g) := \kappa \mathrm{E}\left[Z^{\mathbb{Q}}_{t,T} U(\xi + g, T) \middle| \mathcal{F}_t\right] + \kappa \gamma_{t,T}(\mathbb{Q}). \tag{6.9}$$

Lemma 6.2 *Suppose that Assumption 2 holds and let $\xi \in L^\infty(\mathcal{F}_t)$, $\kappa \in L^\infty(\mathcal{F}_t)$ and $g \in \mathcal{C}_{t,T}$. Then, the mapping $\mathbb{Q} \mapsto \mathrm{E}[J_{\kappa,\xi}(\mathbb{Q}, g)]$ is weakly lower-semicontinuous on the convex and weakly compact set $\mathcal{Q}_{t,T}$.*

Proof Recall that there exists $c > 0$ such that $\xi + g \geq -c$, a.s. Moreover, $\{Z_n U^-(-c,T)\}$, $Z_n \in \mathcal{Q}_{t,T}$, is uniformly integrable due to Assumption 2. Fatou's Lemma then yields that the function $Z \to \mathrm{E}[\kappa Z U(\xi + g, T)]$ is lower-semicontinuous with respect to a.s.-convergence on $\mathcal{Q}_{t,T}$. Since the set $\mathcal{Q}_{t,T}$ is convex and weakly compact according to Assumption 2, it is uniformly integrable. Hence, the mapping $Z \to \mathrm{E}[\kappa Z U(\xi + g, T)]$ is also lower-semicontinuous with respect to convergence in L^1 and, thus, weakly lower-semicontinuous since the function is convex. According to Definition 2.3, also $Z \to \mathrm{E}[\kappa \gamma_{t,T}(Z)]$ is convex and weakly lower-semicontinuous on $\mathcal{Q}_{t,T}$, which completes the proof. □

We now establish the conjugacy relations between u_κ and v_κ. This result is the cornerstone in the proof of the conditional versions in Theorem 3.2 below. As in previous works, see [58, 61, 62], we use a minimax theorem in order to reformulate the robust problem as the infimum over a class of non-robust criteria. We then apply duality to each of the inner maximization problems. Unlike Schied [61], who used EUM duality results of Kramkov and Schachermayer [44], we apply relation (6.8) to suitably defined (stochastic) utility fields considered under the fixed reference measure. This is of technical as well as conceptual importance and makes key use of Assumption 2.

[8] Note that although $Z^{\mathbb{Q}}_{t,T} U(x,T) \in L^1$, it is not a priori clear whether $Z^{\mathbb{Q}}_{t,s} U(s,x) \in L^1(\mathcal{F}_s)$, for $t < s < T$. Hence, it is not clear whether the associated random field is actually a utility field in the sense of Definition 2.2 (the field could easily be adjusted in order for the utility and path regularity conditions to hold). However, Proposition A1 in [67] only makes use of the slice $U(x,T)$ and can therefore be applied under the given assumptions.

Proposition 6.3 *Suppose (U,γ) satisfy Assumption 2. Then, for all $\xi \in L^\infty(\mathcal{F}_t)$ and $\eta \in L^1_+(\mathcal{F}_t)$, it holds that*

$$u_\kappa(\xi) = \inf_{\eta \in L^1(\mathcal{F}_t)} \big(v_\kappa(\eta) + \langle \xi, \eta\rangle\big) \quad \text{and} \quad v_\kappa(\eta) = \sup_{\xi \in L^\infty(\mathcal{F}_t)} \big(u_\kappa(\xi) - \langle \xi, \eta\rangle\big).$$

Proof Since, for each $Z \in \mathcal{Q}_{t,T}$, $g \to \mathrm{E}[\kappa Z U(\xi+g)]$ is concave on the convex set $\mathcal{C}_{t,T}$, according to Lemma 6.2 we might apply the lopsided minimax theorem (cf. Chapter 6 in [3]) to obtain

$$\begin{aligned} u_\kappa(\xi) &= \sup_{g\in\mathcal{C}_{t,T}} \inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \mathrm{E}\left[\kappa\left(Z^{\mathbb{Q}}_{t,T} U\left(\xi+g,T\right) + \gamma_{t,T}(\mathbb{Q})\right)\right] \\ &= \inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \left(\sup_{g\in\mathcal{C}_{t,T}} \mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T} U\left(\xi+g,T\right)\right] + \mathrm{E}\big[\kappa\gamma_{t,T}(\mathbb{Q})\big]\right) \\ &= \inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \left(u^{\mathbb{Q}}_\kappa(\xi) + \mathrm{E}\big[\kappa\gamma_{t,T}(\mathbb{Q})\big]\right), \end{aligned} \tag{6.10}$$

where the last equality follows directly from the definition of $u^{\mathbb{Q}}_\kappa$. Note that due to concavity, if $U(x^0,T) \in L^1$ for some $x^0 \in \mathbb{R}$, then $U(x,T) \in L^1$ for all $x \in \mathbb{R}$. Since $\mathbb{P} \in \mathcal{Q}_{t,T}$ and $U(x,T) \in L^1$ due to assumption, we can then, w.l.o.g., replace the set $\mathcal{Q}_{t,T}$ in (6.10) by

$$\mathcal{Q}^\kappa_{t,T} := \big\{\mathbb{Q} \in \mathcal{Q}_{t,T} : \kappa Z^{\mathbb{Q}}_{t,T} U(x,T) \in L^1, x \in \mathbb{R}\big\}. \tag{6.11}$$

Next, according to Assumption 2, for all $\mathbb{Q} \in \mathcal{Q}^\kappa_{t,T}$, it holds that $\tilde U(x,T)$ defined in (3.4) satisfies the non-singularity condition in Definition 3.3 in [67], and that $\tilde U(x,T) \in L^1$, $x \in \mathbb{R}$. Then, the conjugacy relation (6.8) yields

$$\begin{aligned} u_\kappa(\xi) &= \inf_{\mathbb{Q}\in\mathcal{Q}^\kappa_{t,T}} \left(\inf_{\eta\in L^1(\mathcal{F}_t)} \left(v^{\mathbb{Q}}_\kappa(\eta) + \langle\xi,\eta\rangle\right) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right]\right) \\ &= \inf_{\eta\in L^1(\mathcal{F}_t)} \left(\inf_{\mathbb{Q}\in\mathcal{Q}^\kappa_{t,T}} \left(v^{\mathbb{Q}}_\kappa(\eta) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right]\right) + \langle\xi,\eta\rangle\right) \\ &= \inf_{\eta\in L^1(\mathcal{F}_t)} \big(v_\kappa(\eta) + \langle\xi,\eta\rangle\big), \end{aligned}$$

where it remains to argue the last step. To this end, note that for each $\zeta^* \in \mathcal{D}^\eta_{t,T}$, $\eta \in L^1(\mathcal{F}_t)$, it holds that

$$\begin{aligned} &\mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T} V\left(\zeta^*/\kappa Z^{\mathbb{Q}}_{t,T}, T\right)\right] + \mathrm{E}[\xi\eta] \\ &\quad\geq \mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T}\left(U\left(\xi+g,T\right) - \zeta^*(\xi+g)/\kappa Z^{\mathbb{Q}}_{t,T}\right)\right] + \mathrm{E}[\xi\eta] \\ &\quad+\mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T} U\left(\xi+g,T\right)\right] - \mathrm{E}\left[\zeta^*(\xi+g)\right] + \mathrm{E}[\xi\eta] \;\geq\; \mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T} U\left(\xi+g,T\right)\right]. \end{aligned}$$

Hence, $\mathcal{Q}^\kappa_{t,T}$ can be replaced by $\mathcal{Q}_{t,T}$ without loss of generality. This completes the proof of the first conjugacy relation. To argue that v_κ is the convex conjugate of u_κ it suffices to argue that v_κ is convex and weakly lower semi-continuous, which follows from Proposition 6.1. □

6.1.2 Proofs of Theorem 3.2 and Proposition 3.3

We are now ready to prove the main results of Section 3.1. Our setting is dynamic, which in this generality appears novel even in the context of the classical robust EUM, compare e.g. Schied [61]. In consequence, we need to reduce the conditional formulations to the $\mathcal{F}_0$-measurable case. This is done with the help of the following auxiliary lemma which uses crucially condition (*iv*) from Definition 2.3 of penalty functions. Recall from (6.9) the definition of $J_{\kappa,\xi}(\mathbb{Q}, g)$, $\mathbb{Q} \in \mathcal{Q}$, $g \in \mathcal{C}_{t,T}$.

Lemma 6.4 *For fixed $\xi \in L^\infty(\mathcal{F}_t)$, $\kappa \in L^\infty(\mathcal{F}_t)$ and $g \in \mathcal{C}_{t,T}$, it holds that*

$$\mathrm{E}\left[\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}} J_{\kappa,\xi}(\mathbb{Q}, g)\right] = \inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}} \mathrm{E}\Big[J_{\kappa,\xi}(\mathbb{Q}, g)\Big].$$

Proof The inequality '$\leq$' is trivial. To show the reverse inequality, let $J(\mathbb{Q}) := J_{\kappa,\xi}(\mathbb{Q}, g), \mathbb{Q} \in \mathcal{Q}_{t,T}$. It suffices to argue that the set $\big\{J(\mathbb{Q}) : \mathbb{Q} \in \mathcal{Q}_{t,T}\big\}$ is downwards directed. Indeed, according to Neveu [55], there then exists a sequence $\mathbb{Q}_n \in \mathcal{Q}_{t,T}$ such that $J(\mathbb{Q}_n)$ is decreasing and $\operatorname{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}} J(\mathbb{Q}) = \lim_{n\to\infty} J(\mathbb{Q}_n)$. The result then follows by use of the monotone convergence theorem. To this end, let $\mathbb{Q}_1$, $\mathbb{Q}_2 \in \mathcal{Q}_{t,T}$ and define a set $A := \{J(\mathbb{Q}_1) \leq J(\mathbb{Q}_2)\} \in \mathcal{F}_t$. Further, let the measure $\bar{\mathbb{Q}}$ be given by $\frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} := 1\!\!1_A Z^{\mathbb{Q}_1}_{t,T} + 1\!\!1_{A^c} Z^{\mathbb{Q}_2}_{t,T}$. Using Definition 2.3 iv), we have that $\gamma_{t,T}(\bar{\mathbb{Q}}) = 1\!\!1_A \gamma_{t,T}(\mathbb{Q}_1) + 1\!\!1_{A^c}\gamma_{t,T}(\mathbb{Q}_2)$. Hence, $\bar{\mathbb{Q}} \in \mathcal{Q}_{t,T}$ and $J(\bar{\mathbb{Q}}) = \min\{J(\mathbb{Q}_1), J(\mathbb{Q}_2)\}$, a.s. In consequence, $\big\{J(\mathbb{Q}) : \mathbb{Q} \in \mathcal{Q}_{t,T}\big\}$ is closed under minimization and, thus, downwards directed. □

First, we establish the existence of a dual optimizer.

Proof of Proposition 3.3. Recall that $\eta \in L^1_+(\mathcal{F}_t)$ is fixed. Further, according to Proposition 6.1, for any $\kappa \in L^\infty(\mathcal{F}_t)$, $v_\kappa(\eta)$ is attained for some pair $(\bar{\zeta}^*, \bar{\mathbb{Q}}) \in \mathcal{D}^\eta_{t,T} \times \mathcal{Q}_{t,T}$. Observe that if $v_\kappa(\eta) < \infty$, then $\mathbb{V}^{\bar{\mathbb{Q}}}_\kappa(\bar{\zeta}^*) < \infty$, and it follows from (6.2) that $\bar{\zeta}^* \in L^1_+$. Therefore $\bar{\zeta}^* \in \mathcal{D}^\eta_{t,T} \cap L^1_+$ and according to (6.1), there exists $\bar{Z} \in \mathcal{Z}^a_T$ such that $\bar{\zeta}^* = \eta \bar{Z}_{t,T}$. We now argue that for $\kappa := (\max\{1, v(\eta; t, T\})^{-1} \in L^\infty(\mathcal{F}_t)$, the thus defined pair $(\bar{Z}_{t,T}, \bar{\mathbb{Q}})$ attains the essential infimum in (3.2). First, note that $\kappa \in [0, 1]$ and, w.l.o.g., we may assume that $\{\kappa > 0\} \neq \emptyset$. Further, by definition, $v_\kappa(\eta) < \infty$. Hence, the above pair $(\bar{Z}_{t,T}, \bar{\mathbb{Q}})$ is well-defined. Next, suppose contrary to the claim that there exist $\varepsilon > 0$, $\mathbb{Q}' \in \mathcal{Q}_{t,T}$, $Z'_{t,T} \in \mathcal{Z}^a_T$ and a set $B \in \mathcal{F}_t$ with $\mathbb{P}(B) > 0$, such that on B,

$$\mathrm{E}^{\mathbb{Q}'}\left[V\left(\eta \frac{Z'_{t,T}}{Z^{\mathbb{Q}'}_{t,T}}, T\right) \Big| \mathcal{F}_t\right] + \gamma_{t,T}(\mathbb{Q}') + \varepsilon < \mathrm{E}^{\bar{\mathbb{Q}}}\left[V\left(\eta \frac{\bar{Z}_{t,T}}{Z^{\bar{\mathbb{Q}}}_{t,T}}, T\right) \Big| \mathcal{F}_t\right] + \gamma_{t,T}(\bar{\mathbb{Q}}). \tag{6.12}$$

Note that $B \subseteq \{\kappa > 0\}$. Moreover, w.l.o.g. (scaling if necessary), we may assume that $B \subseteq \{\kappa = 1\}$. Define the random variable $\tilde{\zeta}^* \in L^1_+$ by $\tilde{\zeta}^* :=$

$\eta(Z'_{t,T}\mathbb{1}_B+\bar{Z}_{t,T}\mathbb{1}_{B^c})$. It follows that $\tilde{\zeta}^*\in\mathcal{D}^\eta_{t,T}$ and, thus, $\tilde{\zeta}^*=\eta\tilde{Z}_{t,T}$ for some $\tilde{Z}_{t,T}\in\mathcal{Z}^a_T$. Further, we define $\tilde{\mathbb{Q}}\in\mathcal{Q}_{t,T}$ via $\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}:=\mathbb{1}_B Z^{\mathbb{Q}'}_{t,T}+\mathbb{1}_{B^c}Z^{\bar{\mathbb{Q}}}_{t,T}$. Taking expectations on both sides of (6.12) and applying property iv) of Definition 2.3, then yields

$$\mathbb{V}^{\tilde{\mathbb{Q}}}_\kappa(\tilde{\zeta}^*)+\mathrm{E}\big[\kappa\gamma_{t,T}(\tilde{\mathbb{Q}})\big]-\varepsilon\mathbb{P}(B)\le\mathbb{V}^{\bar{\mathbb{Q}}}_\kappa(\bar{\zeta}^*)+\mathrm{E}\big[\kappa\gamma_{t,T}(\bar{\mathbb{Q}})\big],$$

which contradicts the choice of $(\bar{\zeta}^*,\bar{\mathbb{Q}})$ as the minimizer. □

We now turn to Theorem 3.2. We argue by contradiction; assuming that the conditional conjugacy relations does not hold, taking expectations and applying Proposition 6.3 and Lemma 6.4 yields a contradiction which allows us to conclude.

Proof of Theorem 3.2 First, we consider assertion (3.5). In order to verify that the (weak) inequality '≤' holds, note that we trivially have the inequality

$$u(\xi;t,T)\le\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}}\left(\operatorname*{ess\,sup}_{g\in\mathcal{C}_{t,T}}\mathrm{E}^{\mathbb{Q}}\left[U(\xi+g,T)|\,\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q})\right). \tag{6.13}$$

Since $\mathrm{E}^{\mathbb{Q}}[g]\le 0$, for all $\mathbb{Q}\in\mathcal{M}^a_T$, $g\in\mathcal{C}_{t,T}$, and $U(x,T)\le V(y,T)+xy$, for all $x\in\mathbb{R}$, $y\ge 0$, it follows immediately from (6.13) that, for all $\eta\in L^1_+(\mathcal{F}_t)$,

$$\begin{aligned}u(\xi;t,T)&\le\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}}\left(\operatorname*{ess\,inf}_{Z\in\mathcal{Z}^a_T}\mathrm{E}^{\mathbb{Q}}\left[V\left(\eta Z_{t,T}/Z^{\mathbb{Q}}_{t,T},T\right)\Big|\,\mathcal{F}_t\right]+\xi\eta+\gamma_{t,T}(\mathbb{Q})\right)\\&=v(\eta;t,T)+\xi\eta.\end{aligned}$$

To prove the reverse inequality, we argue by contradiction and assume that there exist $\xi\in L^\infty(\mathcal{F}_t)$, $\varepsilon>0$ and $A\in\mathcal{F}_t$ such that

$$\begin{aligned}&\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}}\Big(\mathrm{E}^{\mathbb{Q}}\left[U(\xi+g,T)|\,\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q})\Big)+\varepsilon\mathbb{1}_A\\&\qquad\le\mathrm{E}^{\mathbb{Q}}\left[V\left(\eta Z_{t,T}/Z^{\mathbb{Q}}_{t,T},T\right)\Big|\,\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q})+\xi\eta,\end{aligned}$$

for all $g\in\mathcal{K}_{t,T}$, $Z_{t,T}\in\mathcal{Z}^a_T$, $\mathbb{Q}\in\mathcal{Q}_{t,T}$ and $\eta\in L^1_+(\mathcal{F}_t)$. Observe that $u(\xi;t,T)<\infty$ a.s. on A and, w.l.o.g., we may assume that there is $M<\infty$ such that $u(\xi;t,T)\le M$ a.s. on A. Multiplying the latter inequality by $\kappa=\mathbb{1}_A$, taking expectations on both sides and applying Lemma 6.4, we then obtain

$$\begin{aligned}&\inf_{\mathbb{Q}\in\mathcal{Q}_{t,T}}\mathrm{E}\Big[\kappa\Big(Z^{\mathbb{Q}}_{t,T}U(\xi+g,T)+\gamma_{t,T}(\mathbb{Q})\Big)\Big]+\varepsilon P(A)\\&\qquad\le\mathrm{E}\left[\kappa Z^{\mathbb{Q}}_{t,T}V\left(\frac{\eta}{\kappa}\frac{Z_{t,T}}{Z^{\mathbb{Q}}_{t,T}},T\right)\right]+\mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right]+\mathrm{E}\left[\kappa\xi\eta\right],\end{aligned}$$

where the expression in the first expectation on the right hand side is defined to be zero on A^c. According to (6.1), we have that for every $\zeta^*\in\mathcal{D}^\eta_{t,T}\cap L^1_+$

with $\eta \in L^1_+(\mathcal{F}_t)$, there exists $Z_{t,T} \in \mathcal{Z}^a_T$ such that $\zeta^* = \eta Z_{t,T}$. Using this and taking the supremum over $g \in \mathcal{K}_{t,T}$, we deduce that

$$u_\kappa(\xi) + \varepsilon P(A) \leq \mathbb{V}^{\mathbb{Q}}_\kappa(\zeta^*) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right] + \langle \xi, \eta \rangle, \tag{6.14}$$

for all $\eta \in L^1_+(\mathcal{F}_t)$ such that $\eta = \eta 1\!\!1_A$, $\mathbb{Q} \in \mathcal{Q}_{t,T}$ and $\zeta^* \in \mathcal{D}^\eta_{t,T} \cap L^1_+$. Therefore, for any $\eta \in L^1_+(\mathcal{F}_t)$ and $\mathbb{Q} \in \mathcal{Q}_{t,T}$, the above inequality holds for all $\zeta^* \in \mathcal{D}^\eta_{t,T}$. Indeed, if $\zeta^* \notin L^1_+$ or $\eta \neq \eta 1\!\!1_A$, then it holds that $\mathbb{V}^{\mathbb{Q}}_\kappa(\zeta^*) = \infty$ (cf. (6.2)). Hence,

$$u_\kappa(\xi) + \varepsilon P(A) \leq v^{\mathbb{Q}}_\kappa(\eta) + \mathrm{E}\left[\kappa\gamma_{t,T}(\mathbb{Q})\right] + \langle \xi, \eta \rangle,$$

for all $\eta \in L^1_+(\mathcal{F}_t)$ and $\mathbb{Q} \in \mathcal{Q}_{t,T}$ and. In turn, since $u_\kappa(\xi) \leq M < \infty$ due to the above choice of κ, we obtain

$$u_\kappa(\xi) < u_\kappa(\xi) + \varepsilon P(A) \leq \inf_{\eta \in L^1(\mathcal{F}_t)} \left(v_\kappa(\eta) + \langle \xi, \eta \rangle\right),$$

and according to Proposition 6.3, this yields the required contradiction.

Next, we turn to relation (3.6). Note that assertion (3.5) implies that for all $\eta \in L^1(\mathcal{F}_t)$ and $\xi \in L^\infty(\mathcal{F}_t)$, $v(\eta; t, T) \geq u(\xi; t, T) - \xi\eta$. Hence, the inequality "$\geq$" follows directly. The reverse inequality follows by similar arguments as above; specifically, by arguing by contradiction and then applying Lemma 6.4 and Proposition 6.3. □

6.2 Proofs of Propositions 4.1, 4.3 and 4.4

In order to prove the results in Section 4, we first establish two lemmas.

Lemma 6.5 *Let $T > 0$ and let V be a dual random field and γ an admissible family of penalty functions such that either Assumption 3 holds, or* (4.1) *holds and $v^-(\zeta; t, T) \in L^1(\mathcal{F}_t; \mathbb{Q})$ for all $\zeta \in L^0(\mathcal{F}_t)$ and $\mathbb{Q} \in \tilde{\mathcal{Q}}_{0,T}$, $t \leq T$. Then, the pair (v, γ), where $v(\cdot; t, T)$ is the dual value field, satisfies* (3.2) *on* $[0, T]$.

Proof Fix $0 \leq s < t < T < \infty$. For $\mathbb{Q} \in \mathcal{Q}_{0,T}$, we use the convention $\gamma_{0,t}(\mathbb{Q}) = \gamma_{0,t}(\mathbb{Q}_{|\mathcal{F}_t})$. Let $Z \in \mathcal{Z}^a_t$ and $\mathbb{Q} \in \mathcal{Q}_{s,t}$. Using Proposition 3.3, we denote by $Z^* \in \mathcal{Z}^a_T$ and $\mathbb{Q}^* \in \mathcal{Q}_{t,T}$ the optimal elements for which $v\big(\eta Z_{s,t}/Z^{\mathbb{Q}}_{s,t}; t, T\big)$ is attained. Then, it holds that

$$\begin{aligned}
&\mathrm{E}\left[Z^{\mathbb{Q}}_{s,t} v\left(\eta \frac{Z_{s,t}}{Z^{\mathbb{Q}}_{s,t}}; t, T\right) \big| \mathcal{F}_s\right] + \gamma_{s,t}(\mathbb{Q}) \qquad (6.15)\\
&\quad = \mathrm{E}\left[Z^{\mathbb{Q}}_{s,t}\left(\mathrm{E}\left[Z^{\mathbb{Q}^*}_{t,T} V\left(\eta \frac{Z_{s,t}}{Z^{\mathbb{Q}}_{s,t}} \frac{Z^*_{t,T}}{Z^{\mathbb{Q}^*}_{t,T}}, T\right) \big| \mathcal{F}_t\right] + \gamma_{t,T}\left(Z^{\mathbb{Q}^*}_{t,T}\right)\right) \big| \mathcal{F}_s\right] + \gamma_{s,t}(\mathbb{Q})\\
&\quad = \mathrm{E}\left[Z^{\mathbb{Q}}_{s,t} Z^{\mathbb{Q}^*}_{t,T} V\left(\eta \frac{Z_{s,t} Z^*_{t,T}}{Z^{\mathbb{Q}}_{s,t} Z^{\mathbb{Q}^*}_{t,T}}, T\right) \big| \mathcal{F}_s\right] + \gamma_{s,T}\left(Z^{\mathbb{Q}}_{s,t} Z^{\mathbb{Q}^*}_{t,T}\right) \quad \geq \quad v(\eta; s, T),
\end{aligned}$$

where it was used that $Z_t Z^*_{t,T} \in \mathcal{Z}^a_T$ and that $\bar{\mathbb{Q}} \in \mathcal{Q}_{s,T}$, with $\frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{P}_{|\mathcal{F}_T}} = Z^{\mathbb{Q}}_t Z^{\mathbb{Q}^*}_{t,T}$. Indeed, (4.2) yields immediately that $\bar{\mathbb{Q}} \in \mathcal{Q}_{s,T}$. For the case when (4.1) holds and $v^-(\zeta; s, T) \in L^1(\mathcal{F}_T; \bar{\mathbb{Q}})$, $\zeta \in L^0(\mathcal{F}_T)$, the fact that $v(\eta; s, t)$ is finite implies that $\mathrm{E}^{\mathbb{Q}}\big[\gamma_{t,T}(\mathbb{Q}^*)\,|\mathcal{F}_s\big] < \infty$ and, thus, $\bar{\mathbb{Q}} \in \mathcal{Q}_{s,T}$.

Next, let $Z \in \mathcal{Z}^a_T$ and $\mathbb{Q} \in \mathcal{Q}_{0,T}$ be the optimal objects for which the infimum in $v(\eta; s, T)$ is attained. Note that due to (4.1), the fact that $\mathbb{Q} \in \mathcal{Q}_{0,T}$ implies $\mathbb{Q} \in \mathcal{Q}_{t,T}$ and $\mathbb{Q}_{|\mathcal{F}_t} \in \mathcal{Q}_{s,t}$. It follows that

$$\begin{aligned} v(\eta; s, T) &= \mathrm{E}\left[Z^{\mathbb{Q}}_{s,T} V\left(\eta \frac{Z_{s,T}}{Z^{\mathbb{Q}}_{s,T}}, T\right) |\mathcal{F}_s \right] + \gamma_{s,T}\left(Z^{\mathbb{Q}}_{s,T}\right) \\ &= \mathrm{E}\left[Z^{\mathbb{Q}}_{s,t} \left(\mathrm{E}\left[Z^{\mathbb{Q}}_{t,T} V\left(\eta \frac{Z_{s,t}}{Z^{\mathbb{Q}}_{s,t}} \frac{Z_{t,T}}{Z^{\mathbb{Q}}_{t,T}}, T\right) |\mathcal{F}_t \right] + \gamma_{t,T}\left(Z^{\mathbb{Q}}_{t,T}\right) \right) |\mathcal{F}_s \right] + \gamma_{s,t}(\mathbb{Q}) \\ &\geq \mathrm{E}\left[Z^{\mathbb{Q}}_{s,t} v\left(\eta \frac{Z_{s,t}}{Z^{\mathbb{Q}}_{s,t}}; t, T\right) |\mathcal{F}_s \right] + \gamma_{s,t}(\mathbb{Q}) \quad \geq \quad v(\eta; s, T), \end{aligned} \tag{6.16}$$

where the last inequality is due to (6.15). Hence, equality must hold and we easily conclude. □

Lemma 6.6 *Let $U(x,t)$ be a utility random field and let $V(y,t)$ be the corresponding dual field defined in* (3.1)*. Further, let $\gamma_{t,T}$ be a family of penalty functions satisfying* (4.1)*. Then, the following two statements are equivalent:*

i) the pair (V, γ) satisfies (3.2) *for all $t \leq T < \infty$,*

ii) for each $y > 0$ and all $t \leq T < \infty$,

$$V(yZ_t/Z^{\mathbb{Q}}_t, t) \leq \mathrm{E}^{\mathbb{Q}}\left[V(yZ_T/Z^{\mathbb{Q}}_T, T) \Big| \mathcal{F}_t \right] + \gamma_{t,T}(\mathbb{Q}), \tag{6.17}$$

for all $\mathbb{Q} \in \mathcal{Q}_{t,T}$ and $Z \in \mathcal{Z}^a_T$. Moreover, for each $\bar{T} > 0$, there exists $\bar{\mathbb{Q}} \in \mathcal{Q}_{0,\bar{T}}$ and $\bar{Z} \in \mathcal{Z}^a_{\bar{T}}$, such that (6.17) *holds with equality for all $t \leq T \leq \bar{T}$.*

Furthermore, if either a) $\mathcal{Q}_{0,T} = \tilde{\mathcal{Q}}_{0,T}$, $T > 0$, or b) for any $T > 0$ and $\zeta \in L^0(\mathcal{F}_T)$, $V^-(\zeta, T) \in L^1(\mathcal{F}_T; \mathbb{Q})$ for all $\mathbb{Q} \in \tilde{\mathcal{Q}}_{0,T}$. Then, i) and ii) are equivalent to the following condition:

iii) For each $y > 0$ and all $t \leq T < \infty$, (6.17) *holds for all $\mathbb{Q} \in \mathcal{Q}_{t,T}$ and $Z \in \mathcal{Z}^a_T$. Moreover, there exist $Z \in \mathcal{Z}^a$ and a sequence $\big(\mathbb{Q}_{T^i}\big)$, $i \in \mathbb{N}$, with $\mathbb{Q}_{T^i} = \mathbb{Q}_{T^{i+1}}|_{\mathcal{F}_{T^i}}$ and $\mathbb{Q}_T := \mathbb{Q}_{T^i}|_{\mathcal{F}_T} \in \mathcal{Q}_{0,T}$, $T^i \geq T$, such that for all $0 < t < T < \infty$,* (6.17) *holds with equality for $\mathbb{Q}_T$ and Z_T.*

Proof First, we show that i) implies ii). To this end, assume that (V, γ) satisfies (3.2) for all $t \leq T < \infty$. Let $y > 0$, $\tilde{Z} \in \mathcal{Z}^a_T$ and $\tilde{\mathbb{Q}} \in \mathcal{Q}_{t,T}$. Further, let

$\eta := y\tilde{Z}_t/Z_t^{\tilde{\mathbb{Q}}}$. Then, it follows that

$$V(y\tilde{Z}_t/Z_t^{\tilde{\mathbb{Q}}},t) = \operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}_{t,T}}\operatorname*{ess\,inf}_{Z\in\mathcal{Z}_T^a}\left\{\mathrm{E}^{\mathbb{Q}}\left[V\left(y\frac{\tilde{Z}_t Z_{t,T}}{Z_t^{\tilde{\mathbb{Q}}}Z_{t,T}^{\mathbb{Q}}},T\right)\middle|\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q})\right\}$$
$$\leq \mathrm{E}^{\tilde{\mathbb{Q}}}\big[V\left(y\tilde{Z}_T/Z_T^{\tilde{\mathbb{Q}}},T\right)\big|\mathcal{F}_t\big]+\gamma_{t,T}(\tilde{\mathbb{Q}}),$$

which yields (6.17). Next, let $\bar{Z}\in\mathcal{Z}_{\bar{T}}^a$ and $\bar{\mathbb{Q}}\in\mathcal{Q}_{0,\bar{T}}$ be the optimal objects for which $v(y,0;\bar{T})$ is attained; their existence is ensured by Proposition 3.3. Let $Z^T := \mathrm{E}[\bar{Z}|\mathcal{F}_T]$ and $\mathbb{Q}_T := \bar{\mathbb{Q}}|\mathcal{F}_T$. Note that $\bar{\mathbb{Q}}\in\mathcal{Q}_{0,\bar{T}}$, implies that $\bar{\mathbb{Q}}_T\in\mathcal{Q}_{0,T}$ and $\bar{\mathbb{Q}}\in\mathcal{Q}_{T,\bar{T}}$. Hence, use of (3.2) combined with the same arguments as used in (6.16) (which uses (4.1)) gives

$$\begin{aligned}v(y;0,\bar{T}) &= \mathrm{E}^{\bar{\mathbb{Q}}}\left[V\left(y\bar{Z}_{\bar{T}}/Z_{\bar{T}}^{\bar{\mathbb{Q}}},\bar{T}\right)\right]+\gamma_{0,\bar{T}}(\bar{\mathbb{Q}})\\ &\geq \mathrm{E}^{\bar{\mathbb{Q}}}\left[v\left(y\bar{Z}_T/Z_T^{\bar{\mathbb{Q}}};T,\bar{T}\right)\right]+\gamma_{0,T}(\bar{\mathbb{Q}}) \;\geq\; v(y;0,T).\end{aligned}\tag{6.18}$$

Applying once again (3.2) yields that (6.18) must hold with equality. Thus, $v(y;0,T)$ is attained for Z^T and $\mathbb{Q}_T$, $T\leq\bar{T}$. We now claim that for $t\leq T\leq\bar{T}$, (6.17) holds as equality for $\bar{Z}$ and $\bar{\mathbb{Q}}$. Indeed, assume contrary to the claim that there exists $\varepsilon>0$ and $A\in\mathcal{F}_t$, $\mathbb{P}(A)>0$, such that

$$V(y\bar{Z}_t/Z_t^{\bar{\mathbb{Q}}},t)+\varepsilon\mathbb{1}_A\leq \mathrm{E}^{\bar{\mathbb{Q}}}\left[V(y\bar{Z}_T/Z_T^{\bar{\mathbb{Q}}},T)\middle|\mathcal{F}_t\right]+\gamma_{t,T}(\bar{\mathbb{Q}}).$$

Taking expectation under $\bar{\mathbb{Q}}$ and using (4.1) we, then, obtain

$$\mathrm{E}^{\bar{\mathbb{Q}}}\left[V(y\bar{Z}_t/Z_t^{\bar{\mathbb{Q}}},t)\right]+\gamma_{0,t}(\bar{\mathbb{Q}})+\varepsilon\bar{\mathbb{Q}}(A)\;\leq\;\mathrm{E}^{\bar{\mathbb{Q}}}\left[V(y\bar{Z}_T/Z_T^{\bar{\mathbb{Q}}},T)\right]+\gamma_{0,T}(\bar{\mathbb{Q}}),\tag{6.19}$$

which, since $v(y;0,t)$ is attained for $\bar{Z}_t=Z^t$ and $Z_t^{\bar{\mathbb{Q}}}=Z_t^{\mathbb{Q}_t}$, yields the contradiction $v(y;0,t)<v(y;0,T)$.

In order to prove that ii) implies i), it suffices to show that, for any $0<t<T<\infty$ and $\eta\in L_+^0(\mathcal{F}_t)$, it holds that

$$V(\eta,t)\leq \mathrm{E}^{\mathbb{Q}}\big[V\big(\eta Z_{t,T}/Z_{t,T}^{\mathbb{Q}},T\big)\big|\mathcal{F}_t\big]+\gamma_{t,T}(\mathbb{Q}),\tag{6.20}$$

for all $\mathbb{Q}\in\mathcal{Q}_{t,T}$ and $Z\in\mathcal{Z}_T^a$ and that there exists some $\hat{Q}\in\mathcal{Q}_{t,T}$ and $\hat{Z}\in\mathcal{Z}_T^a$ for which equality holds. Note that (6.17) implies that for a simple, positive and $\mathcal{F}_t$-measurable random variable $\tilde{\eta}=\sum_{k=1}^n y_k\mathbb{1}_{A_k}$, we have that

$$V(\tilde{\eta}Z_t/Z_t^{\mathbb{Q}},t)\leq \mathrm{E}^{\mathbb{Q}}\left[V(\tilde{\eta}Z_T/Z_T^{\mathbb{Q}},T)\middle|\mathcal{F}_t\right]+\gamma_{t,T}(\mathbb{Q}),\tag{6.21}$$

for all $\mathbb{Q}\in\mathcal{Q}_{t,T}$ and $Z\in\mathcal{Z}_T^a$. Using a similar argument to the one used in the proof of Theorem 3.14 in [67], this implies that (6.21) holds for arbitrary $\tilde{\eta}\in L_+^0(\mathcal{F}_t)$. For any $\mathbb{Q}\in\mathcal{Q}_{t,T}$ and $Z\in\mathcal{Z}_T^a$, (6.20) is then obtained by letting $\tilde{\eta}=\eta Z_t^{\mathbb{Q}}/Z_t$. Equality in (6.20) follows by a similar argument where all the inequalities become equalities by the choice of $\mathbb{Q}_T \in \mathcal{Q}_{t,T}$ and $Z^T \in \mathcal{Z}^a_T$ for which (6.17) holds with equality.

Assertion iii) trivially implies ii). Hence, it only remains to show that i) yields iii). To this end, let $T_1 < T_2$. Further, let $Z^1 \in \mathcal{Z}^a_{T_1}$ and $\mathbb{Q}_1 \in \mathcal{Q}_{T_1}$ be the optimal arguments for which $v(y;0,T_1)$ is attained; their existence is ensured by Proposition 3.3. In turn, let $\mathbb{Q}^* \in \mathcal{Q}_{T_2}$ and $Z^* \in \mathcal{Z}_{T_2}$ be the optimal arguments for which $v\big(yZ^1_{T_1}/Z^{\mathbb{Q}_1}_{T_1};T_1,T_2\big)$ is attained, and define $\mathbb{Q}_2$ and Z^2 as follows:

$$\frac{\mathrm{d}\mathbb{Q}_2}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_{T_2}}} = Z^{\mathbb{Q}_1}_{T_1} Z^{\mathbb{Q}^*}_{T_1,T_2} \qquad \text{and} \qquad Z^2 = Z^1_{T_1} Z^*_{T_1,T_2}.$$

By use of the same argument as in (6.15) (which makes use of (4.1) and (4.2)) combined with the fact that (V,γ) satisfy (3.2) for $t \le T < \infty$, it follows that $Z^2 \in \mathcal{Z}^a_{T_2}$, $\mathbb{Q}_2 \in \mathcal{Q}_{0,T_2}$ and

$$\begin{aligned} v(y;0,T_1) &= \mathrm{E}\left[Z^{\mathbb{Q}^1}_{T_1} Z^{\mathbb{Q}^*}_{T_1,T_2} V\left(y\frac{Z^1_{T_1} Z^*_{T_1,T_2}}{Z^{\mathbb{Q}^1}_{T_1} Z^{\mathbb{Q}^*}_{T_1,T_2}}, T_2\right)\right] + \gamma_{0,T_2}\left(Z^{\mathbb{Q}_1}_{T_1} Z^{\mathbb{Q}^*}_{T_1,T_2}\right) \\ &= \mathrm{E}^{\mathbb{Q}^2}\left[V\left(yZ^2_{T_2}/Z^{\mathbb{Q}^2}_{T_2}, T_2\right)\right] + \gamma_{0,T_2}\left(\mathbb{Q}^2\right) \;\; \ge \;\; v(y;0,T_2). \end{aligned} \quad (6.22)$$

Therefore, equality must hold and, thus, $v(y;0,T_2)$ is attained for Z^2 and $\mathbb{Q}_2$. As argued above (cf. (6.16)), it follows for any $T < T_2$, that $v(y;0,T)$ is attained for $Z = Z^2_T$ and $\mathbb{Q} = \mathbb{Q}_2|_{\mathcal{F}_T}$. Subsequent repetition of the above pasting procedure then yields $Z \in \mathcal{Z}^a$ and a sequence $\big(\mathbb{Q}_{T^i}\big)$, $i \in \mathbb{N}$, with $\mathbb{Q}_{T^i} = \mathbb{Q}_{T^{i+1}}|_{\mathcal{F}_{T^i}}$ and $\mathbb{Q}_T := \mathbb{Q}_{T^i}|_{\mathcal{F}_T} \in \mathcal{Q}_{0,T}$, $T^i \ge T$, such that for all $T > 0$, $v(y;0,T)$ is attained for Z_T and $\mathbb{Q}_T$. In turn, by once again using arguments similar to the ones used to show that i) implies ii), we obtain that for any $t < T < \infty$, (6.17) holds as equality for Z_T and $\mathbb{Q}_T$. Hence, iii) holds and we conclude. □

We now argue that the results in Section 4 follow from the above lemmas. First, Theorem 3.2 and Lemma 6.5 readily yield Proposition 4.1. Further, according to Proposition 3.9 in [67], the fact that $U(x,T) \in L^1(\mathcal{F}_T,\mathbb{Q})$ for all $\mathbb{Q} \in \tilde{\mathcal{Q}}_{0,T}$, $T > 0$, implies that assumption b) of Lemma 6.6 holds. Hence, combined with Theorem 3.2, Lemma 6.6 yields Proposition 4.4.

Next, we establish Proposition 4.3. To this end, w.l.o.g., let $t = 0$ and $x \in \mathbb{R}$. Recall that $u(\cdot;0,T)$ and $v(\cdot;0,T)$ satisfy the conjugacy relations (see Theorem 3.2) and let $y^* > 0$ the value for which the infimum in (3.5) is attained; y^* is independent of T since $u(x;0,T) = U(x,0)$, $T \ge 0$. By use of the same arguments as in the proof of Lemma 6.6 (cf. i) implies iii)), it follows that there exist $Z \in \mathcal{Z}^a$ and a positive martingale Y_t, $t \ge 0$, such that for $T \ge 0$, $\mathbb{Q}_T \in \mathcal{Q}_{0,T}$ with $\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}|_{\mathcal{F}_T}} := Y_T$, and $v(y^*;0,T)$ is attained for Z_T and $\mathbb{Q}_T$. Due to the conjugacy relations and the existence of a saddle point, it follows (see e.g. the proof of Theorem 2.6 in [61]) that

$$u(x;0,T) \;=\; \sup_{\pi\in\mathcal{A}} \mathrm{E}^{\mathbb{Q}_T}\left[U\left(x + \int_0^T \pi_s dS_s, T\right)\right] + \gamma_{0,T}(\mathbb{Q}_T), \quad T > 0, \quad (6.23)$$

and, further, the supremum in (6.23) is attained for

$$\bar{X}_T = -V'\left(y^*\frac{Z_T}{Y_T}, T\right), \quad T > 0. \tag{6.24}$$

It follows from (6.24) that $\bar{X}_T = x - \int_0^T dF_t$, with $F_t := V'\big(y^*\frac{Z_t}{Y_t}, t\big)$. In consequence, $\bar{\pi}_0^{0,T} = \bar{\pi}_0^{0,\bar{T}}$, $0 \le T \le \bar{T}$. To argue that $\pi_u^{t,T}(\xi) = \pi_u^{u,T}\left(\xi + \int_t^u \pi_s^{t,T} dS_s\right)$, $t \le u \le T$, assume contrary to the claim that there exists $\varepsilon > 0$ and $A \in \mathcal{F}_u$ such that

$$\mathrm{E}^{\mathbb{Q}_T}\Big[U\Big(x+\int_0^T \bar{\pi}_s^{0,T} dS_s, T\Big)|\mathcal{F}_u\Big]+\gamma_{u,T}(\mathbb{Q}_T)+\varepsilon \mathbb{1}_A \;\le\; u\Big(x+\int_0^u \bar{\pi}_s^{0,T} dS_s, u; T\Big). \tag{6.25}$$

Taking expectations under $\mathbb{Q}_u$, using that (U,γ) satisfy (2.8) and that (4.1) holds, then yields (cf. (6.19))

$$\mathrm{E}^{\mathbb{Q}_T}\Big[U\Big(x + \int_0^T \bar{\pi}_s^{0,T} dS_s, T\Big)\Big] + \gamma_{0,T}(\mathbb{Q}_T) < \mathrm{E}^{\mathbb{Q}_u}\Big[U\Big(x + \int_0^u \bar{\pi}_s^{0,T} dS_s, u\Big)\Big] + \gamma_{0,u}(\mathbb{Q}_u), \tag{6.26}$$

which gives the contradiction $u(x,0;T) < u(x,0;u)$. Similarly, assuming the reverse strict inequality in (6.25) also gives a contradiction, and we easily conclude.

6.3 Proof of Proposition 2.1

Proof Fix $0 \le t \le T < \infty$. To alleviate the notation, let $L_t = \int_0^t \hat{\lambda}_u d\hat{W}_u$ and $M_t = \int_0^t \frac{\hat{\lambda}_u}{1+\delta_u} d\hat{W}_u$. Recall that $\hat{\mathrm{E}}\left[\mathrm{e}^{\kappa\langle L\rangle_T}\right] < \infty$. Take $p, \tilde{p} > 1$ such that $p^2\tilde{p}^2 \le 2\kappa$ and, with $1/p + 1/q = 1$ and $1/\tilde{p} + 1/\tilde{q} = 1$, $\tilde{q}\left(\frac{p^2\tilde{p}}{2} - \frac{p}{2}\right) = \frac{p\tilde{p}(p\tilde{p}-1)}{2(\tilde{p}-1)} \le \kappa$. We then have that

$$\begin{aligned}
\mathrm{E}^{\mathbb{Q}^{\bar{\eta}}}\Big[\int_0^T \hat{\lambda}_s^2 ds\Big] &= \hat{\mathrm{E}}\left[D_T^{\bar{\eta}} \int_0^T \hat{\lambda}_s^2 ds\right] \\
&= \hat{\mathrm{E}}\left[D_T^{\bar{\eta}}\langle L\rangle_T\right] \le \Big(\hat{\mathrm{E}}[(D_T^{\bar{\eta}})^p]\Big)^{1/p}\Big(\hat{\mathrm{E}}[\langle L\rangle_T^q]\Big)^{1/q} \\
&\le \left(\hat{\mathrm{E}}\left[\mathrm{e}^{-p\tilde{p}M_T - \frac{p^2\tilde{p}^2}{2}\langle M\rangle_T}\right]\right)^{\frac{1}{p\tilde{p}}}\left(\hat{\mathrm{E}}\left[\mathrm{e}^{\kappa\langle M\rangle_T}\right]\right)^{\frac{1}{p\tilde{q}}}\Big(\hat{\mathrm{E}}[\langle L\rangle_T^q]\Big)^{\frac{1}{q}} < \infty,
\end{aligned}$$

where we used the assumed integrability of $\langle L\rangle_T$, together with $\langle M\rangle_T \le \langle L\rangle_T$, for the last two terms. The same coupled with Novikov's condition gives that the first term is equal to one. It follows that $\gamma_{t,T}(\mathbb{Q}^{\bar{\eta}}) < \infty$. Next, let

$$N_u^{\pi,\eta} := U(X_u^\pi, u) + \int_t^u \frac{\delta_s}{2}|\eta_s|^2 ds, \quad u \ge t.$$

Then, it suffices to show that $\mathrm{E}^{\bar\eta}\left[N_T^{\pi,\bar\eta}|\mathcal{F}_t\right] \le N_t^{\pi,\bar\eta}$, for all $\pi\in\mathcal{A}_t^x$, and that $\mathrm{E}^{\eta}\left[N_T^{\bar\pi,\eta}|\mathcal{F}_t\right] \ge N_t^{\bar\pi,\eta}$, for all $\mathbb{Q}^\eta\in\mathcal{Q}_{t,T}$. For simplicity, and w.l.o.g., we establish the claim for $t=0$. For $\pi\in\mathcal{A}^x$, the wealth process then satisfies

$$dX_t^\pi = \pi_t\sigma_t S_t\left[\left(\hat\lambda_t+\eta_t^1\right)dt + dW_t^\eta\right],\quad t\le T,\quad X_0^\pi = x,$$

where W_t^η is a Brownian motion under $\mathbb{Q}^\eta$. Due to the form of $U(x,t)$ and $\bar\pi$, a straightforward application of Itô's Lemma yields

$$\begin{aligned}
dN_t^{\bar\pi,\eta} &= \frac{\delta_t}{1+\delta_t}\hat\lambda_t\left[\left(\hat\lambda_t+\eta_t^1\right)dt+dW_t^\eta\right] - \frac{1}{2}\left(\frac{\delta_t}{1+\delta_t}\hat\lambda_t\right)^2 dt\\
&\quad -\frac{1}{2}\frac{\delta_t}{1+\delta_t}\hat\lambda_t^2 dt + \frac{\delta_t}{2}\left[\left(\eta_t^1\right)^2+\left(\eta_t^2\right)^2\right]dt\\
&= \frac{\delta_t}{1+\delta_t}\hat\lambda_t\eta_t^1 dt + \frac{1}{2}\frac{\delta_t}{(1+\delta_t)^2}\hat\lambda_t^2 dt + \frac{\delta_t\hat\lambda_t}{1+\delta_t}dW_t^\eta + \frac{\delta_t}{2}\left[\left(\eta_t^1\right)^2+\left(\eta_t^2\right)^2\right]dt\\
&= \frac{\delta_t}{2}\left[\left(\frac{\hat\lambda_t+(1+\delta_t)\,\eta_t^1}{1+\delta_t}\right)^2+\left(\eta_t^2\right)^2\right]dt + \frac{\delta_t}{1+\delta_t}\hat\lambda_t dW_t^\eta.
\end{aligned}$$

Note that the quantity $\delta_t/(1+\delta_t)\in(0,1)$ and, thus, by the definition of $\gamma_{t,T}$ in (2.3), we deduce that the process $\int_0^t\frac{\delta_s}{1+\delta_s}\hat\lambda_s dW_s^\eta$ is a martingale under $\mathbb{Q}^\eta$. It follows that $N_t^{\bar\pi,\eta}$ is a submartingale for all $\mathbb{Q}^\eta\in\mathcal{Q}_{0,T}$ and a martingale for $\bar\eta$ as specified in (2.4). On the other hand, it holds that

$$\begin{aligned}
N_T^{\pi,\bar\eta} &= U(X_T^\pi,T) + \int_0^T\frac{\delta_s}{2}(\bar\eta_s)^2 ds \;=\; \ln X_T^\pi - \int_0^T\frac{1}{2}\frac{\delta_s}{1+\delta_s}\hat\lambda_s^2 - \frac{1}{2}\frac{\delta_s}{(1+\delta_s)^2}\hat\lambda_s^2 ds\\
&= \ln X_T^\pi - \frac{1}{2}\int_0^T\left[\frac{\delta_s}{1+\delta_s}\hat\lambda_s\right]^2 ds \;=\; \ln X_T^\pi - \frac{1}{2}\int_0^T\left(\hat\lambda_s+\bar\eta_s^1\right)^2 ds.
\end{aligned}$$

Since $\mathrm{E}^{\mathbb{Q}^{\bar\eta}}\left[\ln X_T^\pi\right]\le\mathrm{E}^{\mathbb{Q}^{\bar\eta}}\left[\ln X_T^{\bar\pi}\right]$ for any strategy $\pi\in\mathcal{A}^x$, we conclude that

$$\mathrm{E}^{\mathbb{Q}^{\bar\eta}}\left[N_T^{\pi,\bar\eta}\right]\le\mathrm{E}^{\mathbb{Q}^{\bar\eta}}\left[\ln X_T^{\bar\pi}\right]-\mathrm{E}^{\mathbb{Q}^{\bar\eta}}\left[\frac{1}{2}\int_0^T\left(\hat\lambda_s+\bar\eta_s^1\right)^2 ds\right] = \ln x = N_0,$$

where the equality follows by a direct computation (see, also, p. 721 in [38]).

□

References

1. B. Acciaio and I. Penner. Dynamic risk measures. *Advanced Mathematical Methods for Finance,* ed. Di Nunno and Øksendal, Springer, pages 1–34, 2011.
2. F. J. Anscombe and R. J. Aumann. A definition of subjective probability. *The Annals of Mathematical Statistics*, 34(1):199–205, 1963.
3. J. Aubin and I. Ekeland. *Applied Nonlinear Analysis.* New York, John Wiley & Sons, 1984.

4. R. Bansal and B. N. Lehmann. Growth-optimal portfolio restrictions on asset pricing models. *Macroeconomic Dynamics*, 1(02):333–354, 1997.
5. N. Barberis and R. Thaler. A survey of behavioral finance. *Handbook of the Economics of Finance*, 1:1053–1128, 2003.
6. F. Bellini and M. Frittelli. On the existence of minimax martingale measures. *Mathematical Finance*, 12(1):1–21, 2002.
7. F. Berrier, L. Rogers, and M. Tehranchi. A characterization of forward utility functions. Available at: http://www.statslab.cam.ac.uk/∼mike/papers/forward-utility-consumption.pdf, 2009.
8. J. Bion-Nadal. Time consistent dynamic risk processes. *Stoch. Process. Appl.*, 119(2):633–654, 2009.
9. F. Black. Investment and consumption through time. Financial Note No. 6B. Arthur D. Little, Inc., 1968.
10. G. Bordigoni, A. Matoussi, and M. Schweizer. A stochastic control approach to a robust utility maximization problem. *Stochastic analysis and applications*, 2:125–151, 2007.
11. S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Uncertainty averse preferences. *Journal of Economic Theory*, 146:1275–1330, 2011.
12. P. Cheridito, F. Delbaen, and M. Kupper. Dynamic monetary risk measures for bounded discrete-time processes. *Electron. J. Probab.*, 11:no. 3, 57–106, 2006.
13. T. Choulli, C. Stricker, and J. Li. Minimal Hellinger martingale measures of order q. *Finance and Stochastics*, 11(3):399–427, 2007.
14. A. Cox, D. Hobson, and J. Obłój. Utility theory front to back – inferring utility from agents' choices. *Int. J. Theor. Appl. Finan.*, 17(3):1450018, 2014.
15. F. Delbaen, S. Peng, and E. Gianin. Representation of the penalty term of dynamic concave utilities. *Finance Stochastics*, 14:449–472, 2010.
16. F. Delbaen. The structure of m–stable sets and in particular of the set of risk neutral measures. In *In Memoriam Paul-André Meyer*, pages 215–258. Springer, 2006.
17. L. Denis and M. Kervarec. Optimal investment under model uncertainty in nondominated models. *SIAM Journal on Control and Optimization*, 51(3):1803–1822, 2013.
18. J. H. Drèze. Les fondements logiques de l'utilité cardinale et de la probabilité subjective. *La Decision, Colloques Internationaux de CNRS*, pages 73–87, 1961.
19. D. Duffie and L. G. Epstein. Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, pages 353–394, 1992.
20. N. El Karoui. *Ecole d'Ete de Probabilites de Saint-Flour IX-1979.* Springer Verlag, 1981.
21. N. El Karoui and M. Mrad. An exact connection between two solvable SDEs and a nonlinear utility stochastic PDE. *SIAM J. Financial Mathematics*, 4(1):697–736, 2013.
22. N. El Karoui and M. Mrad. Stochastic utilities with a given optimal portfolio: Approach by stochastic flows. ArXiv:1004.5192v2, 2013.
23. N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.
24. D. Ellsberg. Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, pages 643–669, 1961.
25. H. Föllmer and I. Penner. Convex risk measures and the dynamics of their penalty functions. *Statistics & Decisions*, 24(1):61–96, 2006.
26. H. Föllmer, A. Schied, and S. Weber. Robust preferences and robust portfolio choice. *Handbook of Numerical Analysis*, 15:29–87, 2009.
27. H. Föllmer and A. Gundel. Robust projections in the class of martingale measures. *Illinois J. Math.*, 50(1-4):439–472 (electronic), 2006.
28. I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.
29. H. He and C. F. Huang. Consumption-portfolio policies: An inverse optimal problem. *Journal of Economic Theory*, 62(2):257 – 293, 1994.
30. V. Henderson and D. Hobson. Horizon-unbiased utility functions. *Stochastic Process. Appl.*, 117(11):1621–1641, 2007.
31. D. Hernández-Hernández and A. Schied. Robust utility maximization in a stochastic factor model. *Statistics & Decisions*, 24(1):109–125, 2006.

32. D. Hernández-Hernández and A. Schied. A control approach to robust utility maximization with logarithmic utility and time-consistent penalties. *Stoch. Process. Appl.*, 117(8):980–1000, 2007.
33. J. C. Hershey and P. J. H. Schoemaker. Probability versus certainty equivalence methods in utility measurement: Are they equivalent? *Management Science*, 31(10):1213–1231, 1985.
34. H. Jin and X. Y. Zhou. Behavioral portfolio selection in continuous time. *Math. Finance*, 18(3):385–426, 2008.
35. D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 1979.
36. S. Källblad. *Topics in portfolio choice: qualitative properties, time consistency and investment under model uncertainty.* PhD thesis, University of Oxford, 2013.
37. S. Källblad. Risk-and ambiguity-averse portfolio optimization with quasiconcave utility functionals. *arXiv preprint arXiv:1311.7419*, 2013.
38. I. Karatzas, J. Lehoczky, S. Shreve, and G. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optim.*, 29(3):702–730, 1991.
39. C. Kardaras, J. Obłój, and E. P. The numéraire property and long-term growth optimality for drawdown-constrained investments. *Math. Finance*, 2014. electronic. DOI: 10.1111/mafi.12081.
40. E. Karni. *Decision making under uncertainty: the case of state-dependent preferences.* Harvard University Press Cambridge, 1985.
41. J. L. Kelly, Jr. A new interpretation of information rate. *Bell System Technical Journal*, 35:917–926, 1956.
42. S. Klöppel and M. Schweizer. Dynamic indifference valuation via convex risk measures. *Math. Finance*, 17(4):599–627, 2007.
43. F. Knight. *Risk, Uncertainty and Profit.* Boston: Houghton Mifflin, 1921.
44. D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investments in incomplete markets. *Annals of Applied Probability*, 9:904–950, 1999.
45. R. J. A. Laeven and M. Stadje. Robust portfolio choice and indifference valuation. *Math. Oper. Res.*, 39(4):1109–1141, 2014.
46. F. Maccheroni, M. Marinacci, and A. Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74:1447–1498, 2006.
47. M. Mania and R. Tevzadze. Backward stochastic partial differential equations related to utility maximization and hedging. *Journal of Mathematical Sciences*, 153(3):291–380, 2008.
48. R. Merton. Lifetime portfolio selection under uncertainty: the continuous time case. *The Review of Economics and Statistics*, 51:247–257, 1969.
49. P. Monin. On a dynamic adaptation of the distribution builder approach to investment decisions. *Quantitative Finance*, 14(5):749–760, 2014.
50. M. Müller. *Market completion and Robust utility maximization.* Dissertation, Humboldt Universität zu Berlin, 2005.
51. M. Musiela and T. Zariphopoulou. Portfolio choice under dynamic investment performance criteria. *Quantitative Finance*, 9(2):161–170, 2009.
52. M. Musiela and T. Zariphopoulou. Portfolio choice under space-time monotone performance criteria. *SIAM J. Financial Mathematics*, 1(1):326–365, 2010.
53. M. Musiela and T. Zariphopoulou. Stochastic partial differential equations and portfolio choice. In *Contemporary Quantitative Finance*, pages 195–216. Springer, 2010.
54. S. Nadtochiy and M. Tehranchi. Optimal investment for all time horizons and Martin boundary of space-time diffusions. ArXiv:1308.2254, 2013.
55. J. Neveu. *Discrete Parameter Martingales.* North-Holland Publishing Company, 1975.
56. M. Nutz. Utility maximization under model uncertainty in discrete time. *Mathematical Finance*, 2014.
57. K. Owari. On admissible strategies in robust utility maximization. *Mathematics and Financial Economics*, 6(2):77–92, 2012.
58. M. Quenez. Optimal portfolio in a multiple-priors model. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, pages 291–321. Springer, 2004.

59. L. J. Savage. *The Foundations of Statistics*. John Wiley & Sons Inc., New York, 1954.
60. W. Schachermayer. Optimal investment in incomplete markets when wealth may become negative. *Annals of Applied Probability*, 11(3):694–734, 2001.
61. A. Schied. Optimal investments for risk- and ambiguity-averse preferences: a duality approach. *Finance and Stochastics*, 11:107–129, 2007.
62. A. Schied and C. Wu. Duality theory for optimal investments under model uncertainty. *Statistics & Decisions*, 23(3):199–217, 2005.
63. D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica: Journal of the Econometric Society*, pages 571–587, 1989.
64. W. Sharpe. Individual risk and return preferences: A preliminary survey. Stanford University, Available at: http://www.stanford.edu/∼wfsharpe/art/rrsurvey/vienna2001.htm, 2001.
65. C. Skiadas. Robust control and recursive utility. *Finance and Stochastics*, 7(4):475–489, 2003.
66. J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, N. J., 1947. 2d ed.
67. G. Zitković. A dual characterization of self-generation and exponential forward performances. *Annals of Applied Probability*, 19:2176–2210, 2009.