An ergodic BSDE approach to forward entropic risk measures: representation and large-maturity behavior*

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Abstract

Using elements from the theory of ergodic backward stochastic differential equations, we study the behavior of forward entropic risk measures. We provide a general represention result and examine their behavior for risk positions of large maturities. We also compare them with their classical counterparts and derive a parity result.

1 Introduction

Risk measures constitute one of the most active areas of research in financial mathematics, for they provide a general axiomatic framework to assess risks. Their universality and wide applicability, together with the wealth of related interesting mathematical questions, have led to considerable theoretical and applied developments (see, among others, [1, 12, 13, 14]). For dynamic convex risk measures, we refer to [8, 22, 23, 38]; see, also, [2, 5, 16, 37] for their connection with nonlinear expectations.

A number of popular risk measures are defined in relation to investment opportunities in a given financial market like, for example, VaR, CVaR, indifference prices, etc. As we discuss in the following subsection, such measures are

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tied to both a horizon and a market model, and these choices are made at initial time with limited, if any, flexibility to be revised.

As a result, issues related to how the risk of upcoming positions of arbitrary maturities can be assessed, model revision can be implemented, time-consistency can be preserved, etc. arise. Some of these questions were addressed by one of the authors and Zitkovic in [42], where an axiomatic construction of the so-called "maturity-independent" risk measures was proposed. For the reader's convenience we provide their definition and some key motivational arguments in the next subsection.

Herein we analyze an important subclass of maturity-independent risk measures, the *forward entropic* ones. They are constructed via the so-called forward exponential performance criteria (see Definition 5) and yield the risk assessment of a position by comparing the optimal investment, under these criteria, with and without it. Like the forward processes, via which they are built, the forward entropic measures are defined for *all* times.

We focus on a stochastic factor model, considering a multi-asset market model and assuming that the dynamics of the assets depend on correlated stochastic factors (see (3) and (4)). Stochastic factors are frequently used to model the dynamics of assets (see, for example, the review paper [41]). In the forward performance setting, the use of stochastic factors is discussed in [34], where the multi-stock/multi-factor complete market case is also solved. The incomplete market case with a single stock/single factor was examined in [35] and, more recently, in [39] for a model with slow and fast stochastic factors.

Our contribution is threefold. Firstly, we provide a general representation result for the forward entropic risk measures. We do so building on a recent work of two of the authors ([25]), who developed a new approach for the construction of homothetic (exponential, power and logarithmic) forward performance processes using elements from the theory of ergodic BSDE. This method bypasses a number of technical difficulties associated with solving an underlying ill-posed stochastic PDE (see (7)) that the forward process is expected to satisfy. This SPDE plays the role of the Hamilton-Jacobi-Bellman equation that the classical value function satisfies, but it is substantially harder to solve due to infinite-dimensionality and ill-posedness.

For the exponential forward family we consider herein, the approach in [25] yields the forward performance criterion in both a factor- and non-factor forms (see (19) and (17), respectively). Furthermore, for the former case, the representation is unique. Having these representation results, we use Definition 5 and solve the involved stochastic optimization problems. We then show that the risk measure satisfies a BSDE whose driver, however, depends on the solution of the aforementioned ergodic BSDE (see Theorem 6). The fact that the risk measure is computed via a BSDE is not surprising since we are dealing with the valuation of a risk position, which is by nature a "backward" problem. However, the new element is the dependence of the driver of this BSDE on the ergodic BSDE.

The second contribution is the asymptotic behavior of the forward entropic risk measure when the maturity of the risk position is very large. For risk positions that are deterministic functions of the stochastic factor vector, we show that their risk measure converges to a constant, which is independent of the initial state of the factors, and, furthermore, provide the convergence rate (see Theorems 10 and 12).

We establish the convergence by approximating the solution of the BSDE that yields the risk measure by the solution of an appropriately chosen ergodic BSDE. The main difficulty comes from the interdependence of the two ergodic BSDE (one used for the construction of the exponential forward process and the other for the specification of the risk measure limit). As we show, this requires different treatment for Markovian and non-Markovian forward exponential criteria.

The third contribution is the derivation of a parity result between the forward and the classical entropic risk measures. We show that the forward measure can be constructed as the difference of two classical entropic measures applied to the risk position and a normalizing factor related to the solution of the ergodic BSDE for the forward criterion (see (50)).

We conclude with an example cast in the single stock/single stochastic factor case. Using the ergodic BSDE approach, we derive a closed form representation of the forward risk measure (see (56)). We also derive a representation of the classical entropic risk measure (see (57)) and, in turn, compute numerically the long-term limits of the two measures for specific risk positions.

The paper is organized as follows. In section 2, we introduce the stochastic factor model and provide background results on exponential forward performance processes. In section 3, we review the definition of the forward entropic risk measures and provide their representation. In section 4, we study their behavior for risk positions of large maturities and, in section 5, we derive the parity result. We conclude in section 6 with the example.

1.1 "Maturity-independent" risk measurement

To facilitate the exposition and motivate the reader, we review the main ideas in [42] and recall the definition of the maturity-independent risk measures.

Maturity independence is a fundamental feature of arbitrage-free pricing and the aim in [42] was to investigate whether there also exist non-linear risk assessment mechanisms preserving this property in a decision theoretic framework.

We start with some general observations on features of risk measures defined in relation to a given market environment. A dynamic risk measure, say $\rho_{t,T}$, is typically defined, for $t \in [0,T]$, where $[0,T]$ is a *fixed* investment horizon with the terminal condition $\rho_{T,T}$ given. Therefore, at initial time $t = 0$, two implicit but crucial assumptions are made.

Firstly, it is indirectly assumed that all risk positions in consideration will be both introduced and mature at times within this pre-chosen horizon. However, in many applications, this is not a very realistic assumption. Indeed, we are frequently faced with the valuation/risk assessment of sequential projects/risk positions without, however, having at initial time $t = 0$ complete knowledge of when they will be introduced, when they will mature, what their sizes will be, etc. (e.g. projects associated with different phases of clinical trials, oil exploration, movie production, and others).

Secondly, at $t = 0$, there is an indirect commitment to the market model for the entire horizon $[0, T]$. As a result, this model choice considerably affects the risk measurement of all risk positions, no matter how short or long their maturities are. Namely, even if a risk position matures at short time t, its risk measure in $[0, t]$ will be affected by the model choice for the entire $[0, T]$. This is a direct consequence of the backward construction of $\rho_{t,T}$, which starts at the end of the horizon T and, thus, naturally requires model specification in $[0, T]$. Therefore, any information of changing market conditions and/or modification of the market model cannot be incorporated at any time beyond time 0, for this will destroy the time-consistency of the dynamic risk measure, as computed starting at time $t = 0$.

To put the above two issues in a more mathematical context, let us assume that, at $t = 0$, we have complete knowledge of all risk positions maturing within $[0, T]$. A new risk position, however, may be introduced at a later time, say $s \in (0, T]$ with maturity $T' > T$. We then see that it is not possible to manage this new risk position in the entire interval $[s, T']$ using the existing risk measure $\rho_{t,T}$, for the latter is *not* defined for times $t \in (T, T']$. Therefore, as soon as the new position is introduced at time s, one needs to *extend* $\rho_{t,T}$, $t \in [s, T]$, to a new measure, say $\tilde{\rho}_{t,T'}$, $t \in [s, T']$, for the new longer horizon $[s, T']$. How can this can be done? If one requires that internal time-consistency is being preserved, the new measure $\tilde{\rho}_{t,T'}$ must satisfy $\rho_{t,T} \equiv \tilde{\rho}_{t,T}$, $t \in [s, T]$. However, it is not clear how to define $\tilde{\rho}_{t,T'}$, for $s \in [T, T']$. To our knowledge, this is a question that has not been satisfactorily addressed to date.

Note that in an arbitrage-free valuation setting, we do not encounter such issues, since the pricing operator - given by the conditional expectation of the position - is defined for *all* times, *independently of the maturity* of each claim in consideration and without the need to know *a priori* what kind of claims/risk positions will be introduced and what maturities they will have.

Additional difficulties might come from the aforementioned model commitment at $t = 0$ for the entire $[0, T]$. As mentioned earlier, when a trading horizon $[0, T]$ is pre-chosen, one implicitly chooses the market model with regards to which the construction of the risk measure will be done. Note that this is also the case even in a robust setting, because one has to pre-commit to a family of models for the entire horizon $[0, T]$. Similarly, even when filtering is incorporated, one also needs to pre-commit to the dynamics of the process with respect to which the filtering is done.

Therefore, even if, at $t = 0$, we have complete knowledge of all risk positions within $[0, T]$, it might be the case that after some time, say $s \in (0, T]$, it is evident - due to new information - that the initial market model is not adequately accurate and, thus, it needs to be improved for the remaining investment horizon $[s, T]$. Then, there are two choices. If time-consistency needs to be preserved, one needs to continue working with the initial model which, however, is now wrong. The second choice is to adapt to a new model but accept that time-consistency is violated. This is by far the most popular choice in practice.

In summary, it seems that dynamic risk measures defined for a fixed horizon do not appear to be flexible enough to accommodate arbitrary upcoming risk positions. Moreover, it appears that model specification is binding for the entire horizon, which precludes dynamic updating as market evolves.

In [42] a class of dynamic risk measures, the so-called *maturity-independent* ones, were axiomatically introduced to remedy the above issues. They are cast in a general multi-dimensional semimartingale model, denoted by S_t, $t \geq 0$, and an associated filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We recall their definition next, assuming without loss of generality that the risk position is introduced at time 0.

We first introduce the space of candidate risk positions,

$$\mathcal{L} = \cup_{t\geq 0}\mathcal{L}^{\infty}\left(\mathcal{F}_t\right). \tag{1}$$

where $\mathcal{L}^{\infty}(\mathcal{F}_t)$ is the space of uniformly bounded $\mathcal{F}_t$-measurable random variables.

Definition 1 *A functional* $\rho : \mathcal{L} \to \mathbb{R}$ *is called a maturity-independent convex risk measure if it has the following properties for all* $\xi, \bar{\xi} \in \mathcal{L}$ *and* $\alpha \in (0,1)$:

i) $\rho(\xi) \leq 0, \forall \xi \geq 0$ *(anti-positivity),*

ii) $\rho(\alpha\xi + (1-\alpha)\bar{\xi}) \leq \alpha\rho(\xi) + (1-\alpha)\rho(\bar{\xi})$ *(convexity),*

iii) $\rho(\xi - m) = \rho(\xi) + m, \forall m \in \mathbb{R}$ *(cash-translativity) and*

iv) for all $t \geq 0$ *and admissible investment strategies* π,

$$\rho(\xi) = \rho\left(\xi + \int_0^t \pi_s \frac{dS_s}{S_s}\right). \tag{2}$$

As it is highlighted in [42], the properties that differentiate the maturity-independent risk measures from their classical finite-horizon counterparts are, from the one hand, the choice of the space $\mathcal{L}$ and, from the other, the validity of the "replication requirement" (2) for *all* possible times.

2 The stochastic factor market model

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, supporting a d-dimensional Brownian motion W. We consider a market of a risk-free bond offering zero interest rate and n risky stocks, with $n \leq d$. The stock price processes S_t^i, $t \geq 0$, solve, for $i = 1, \ldots, n$,

$$\frac{dS_t^i}{S_t^i} = b^i(V_t)dt + \sigma^i(V_t)dW_t. \tag{3}$$

The d-dimensional stochastic process V models the stochastic factors affecting the coefficients of the stock prices, and solves

$$dV_t = \eta(V_t)dt + \kappa dW_t. \tag{4}$$

We introduce the following model assumptions. Throughout, we will be using the superscript A^{tr} to denote the transpose of matrix A.

Assumption 1 *i) The drift and volatility coefficients,* $b^i(v) \in \mathbb{R}$ *and* $\sigma^i(v) \in \mathbb{R}^{1\times d}$, $v \in \mathbb{R}^d$, *are uniformly bounded.*

ii) The volatility matrix, defined as $\sigma(v) = (\sigma^1(v), \ldots, \sigma^n(v))^{tr}$, *has full row rank* n.

iii) The market price of risk, defined as

$$\theta(v) := \sigma(v)^{tr}[\sigma(v)\sigma(v)^{tr}]^{-1}b(v), \tag{5}$$

$v \in \mathbb{R}^d$, *is uniformly bounded and Lipschitz continuous.*

Note that $\theta(v)$ solves $\sigma(v)\theta(v) = b(v)$, $v \in \mathbb{R}^d$, which admits a solution because of (ii) above.

Assumption 2 *i) The drift coefficients* $\eta(v) \in \mathbb{R}^d$ *satisfy a dissipative condition, namely, there exists a large enough constant* $C_\eta > 0$ *such that, for* $v, \bar{v} \in \mathbb{R}^d$,

$$(\eta(v) - \eta(\bar{v}))^{tr}(v - \bar{v}) \leq -C_\eta |v - \bar{v}|^2.$$

ii) The volatility matrix $\kappa \in \mathbb{R}^{d\times d}$ *is positive definite and normalized to* $|\kappa| = 1$.

The "large enough" property will be quantified in the sequel when it is assumed that $C_\eta > C_v > 0$, where the constant C_v appears in the properties of the driver of an upcoming ergodic BSDE (see inequality (25)).

Under Assumption 2, the stochastic factor process V admits a unique invariant measure and it is thus ergodic and, moreover, any two paths will converge to each other exponentially fast.

In this market environment, an investor trades dynamically among the risk-free bond and the risky assets. Let $\tilde{\pi} = (\tilde{\pi}^1, \cdots, \tilde{\pi}^n)^{tr}$ denote the (discounted by the bond) amounts of her wealth in the stocks, which are taken to be self-financing. Then, the (discounted by the bond) wealth process satisfies

$$dX_t^{\tilde{\pi}} = \sum_{i=1}^n \frac{\tilde{\pi}_t^i}{S_t^i} dS_t^i = \tilde{\pi}_t^{tr} \sigma(V_t) \left(\theta(V_t)dt + dW_t\right),$$

with $X_0 = x \in \mathbb{R}$. As in [25], we work with the investment strategies rescaled by the volatility,

$$\pi_t^{tr} := \tilde{\pi}_t^{tr} \sigma(V_t).$$

Then, the wealth process satisfies

$$dX_t^{\pi} = \pi_t^{tr} \left(\theta(V_t)dt + dW_t\right). \tag{6}$$

For any $t \geq 0$, we denote by $\mathcal{A}_{[0,t]}$ the set of the admissible investment strategies in $[0, t]$, defined as

$$\mathcal{A}_{[0,t]} = \{\pi \in \mathcal{L}^2_{BMO}[0, t] : \ \pi_s \in \Pi \text{ for } s \in [0, t]\},$$

where Π is a closed and convex set in $\mathbb{R}^d$, and

$$\mathcal{L}^2_{BMO}[0,t] = \{\pi_s,\ s\in[0,t] : \pi \text{ is } \mathbb{F}\text{-progressively measurable and}$$

$$ess\sup_{\tau} E_{\mathbb{P}}\left[\int_{\tau}^{t} |\pi_s|^2 ds|\mathcal{F}_\tau\right] < \infty \text{ for any } \mathbb{F}\text{-stopping time } \tau\in[0,t]\}.$$

The set of admissible investment strategies for *all* $t\geq 0$ is, in turn, defined as $\mathcal{A} = \cup_{t\geq 0}\mathcal{A}_{[0,t]}$.

2.1 Exponential forward performance processes and representation via ergodic BSDE

For the reader's convenience, we start with some background results. We first recall the definition of the forward performance criterion (see [28, 29, 30, 31]) and, in turn, focus on the exponential class. We then recall its ergodic BSDE representation, established in [25].

Throughout, we use the notation $\mathbb{D} = \mathbb{R}\times[0,\infty)$.

Definition 2 *A process $U(x,t)$, $(x,t)\in\mathbb{D}$, is a forward performance process if:*

i) for each $x\in\mathbb{R}$, $U(x,t)$ is $\mathbb{F}$-progressively measurable,

ii) for each $t\geq 0$, the mapping $x\mapsto U(x,t)$ is strictly increasing and strictly concave,

iii) for all $\pi\in\mathcal{A}$ and $0\leq t\leq s$,

$$U\left(X_t^{\pi},t\right) \geq E_{\mathbb{P}}\left[U(X_s^{\pi},s)|\mathcal{F}_t\right],$$

and there exists an optimal $\pi^*\in\mathcal{A}$ such that,

$$U\left(X_t^{\pi^*},t\right) = E_{\mathbb{P}}\left[U(X_s^{\pi^*},s)|\mathcal{F}_t\right],$$

with X^{π}, X^{π^*} solving (6).

A connection between the forward performance process $U(x,t)$ and a stochastic PDE was proposed in [33]. For the model at hand, this SPDE takes the form

$$dU(x,t) = \left(-\frac{1}{2}U_{xx}(x,t)dist^2\left(\Pi, -\frac{\theta(V_t)U_x(x,t)+a_x\left(x,t\right)}{U_{xx}(x)}\right)\right. \tag{7}$$

$$\left.+\frac{1}{2}\frac{|\theta(V_t)U_x(x,t)+a_x\left(x,t\right)|^2}{U_{xx}(x,t)}\right)dt + a(x,t)^{tr}dW_t,$$

where $dist\left(\Pi,z\right)$ represents the distance function from $z\in\mathbb{R}^d$ to Π (see equation (10) in [25][1]) and the volatility process $a\left(x,t\right)\in\mathcal{F}_t$, $t\geq 0$, is an investor-specific

[1]In [25] the trading strategies are represented as proportions of the wealth.

modeling input. If, furthermore, a strong solution to the wealth SDE (6) exists, say $X_t^{\pi^*}$, when the feedback policy

$$\pi_t^* = Proj_\Pi \left(-\theta(V_t) \frac{U_x(X_t^{\pi^*}, t)}{U_{xx}(X_t^{\pi^*}, t)} - \frac{a_x(X_t^{\pi^*}, t)}{U_{xx}(X_t^{\pi^*}, t)} \right),$$

is used, then this control process π^* is optimal. These arguments are not rigorous and a verification theorem is still lacking. Fundamental difficulties stem from, the one hand, the ill-posed and possibly degenerate nature of equation (7) and, from the other, the specification of the appropriate class of the volatility processes $a(x,t)$, which remains an open question. In a related direction, a connection between the forward performance process and optimal portfolios was developed in [9].

The zero volatility case ($a(x,t) \equiv 0$) was analyzed in detail in [32] (see, also, [3]). The forward process is time-monotone (decreasing), represented as

$$U(x,t) = u(x, A_t), \tag{8}$$

where u is a deterministic function and A a time-increasing process depending on the (realized) risk premium.

When the volatility is not zero, forward performance processes may be classified as follows. One family can be constructed from the time-monotone process using the transformation

$$U(x,t) = u\left(\frac{x}{Y_t}, A_t^{(Y,Z)}\right) Z_t, \tag{9}$$

where Y is the price process of a tradeable asset and Z an exponential martingale. This new class of criteria, though, need to be interpreted correctly because they essentially correspond to modified risk premia, benchmarking and market views in an auxiliary pseudo-market (see [33]), and thus do not constitute genuinely new forward processes in the original market.

An entirely different class are forward processes that are represented as *deterministic functions* of underlying stochastic factors, say

$$U(x,t) = v(x, t, V_t, N_t), \tag{10}$$

for some finite-dimensional function v, and V, N being vectors of processes, with the former affecting only the dynamics of the traded assets, as (4) herein. Such processes cannot be time-monotone and, furthermore, cannot be manufactured by any transformation of type (9). For such performance processes, a random Hamilton-Jacobi-Bellman (HJB) equation can be derived for the function v above (see [31]), and questions about existence, uniqueness and regularity of its solutions as well as about the nature of optimal feedback controls arise. Due to the ill-posed nature of this HJB equation, these questions are much more challenging than similar ones for the classical analogue. Note, however, that even in the classical case, general existence, uniqueness and regularity results for the solutions of the associated HJB equations as well as related verification results are lacking (see, among others, [36]).

Existing works for this class include [35], where the single stock/single factor case was analyzed, and [34] where the multi-stock/multi-factor complete market case was studied. Recently, its incomplete analogue with slow and fast factors was analyzed in [39].

We stress that there is a fundamental difference between the candidate forward performance processes and the classical value function processes in stochastic factor settings. In the classical case, there is a unique factor representation of smooth value function processes, as it follows from their Itô decomposition. In particular, if the terminal utility depends only on wealth, the value function process depends both deterministically and exclusively on (x, t, V_t). In the forward framework, however, this is not the case. Forward processes do not have to be of the functional form (10). Furthermore, even if they are modelled as in (10), dependence on other processes (represented by the vector N_t in (10)), beyond the factors appearing in the asset dynamics, is allowed. For a discussion on the economic interpretation of such forward performance criteria, we refer the reader to [34].

Finally, a third class of forward processes are the ones that are infinite-dimensional and cannot be constructed as extensions of neither (9) nor (10). For this class, there is limited knowledge, if any.

For the stochastic factor model we consider herein, general representation results for homothetic forward criteria (power, logarithmic and exponential) were recently produced by two of the authors in [25]. They extend all previous cases for this class of criteria, and provide the forward processes directly through the solutions of an underlying ergodic BSDE[2]. This approach yields both the Markovian and non-Markovian cases ((9) and (10), respectively), and bypasses the aforementioned difficulties associated with solving the forward SPDE (7). While the form of the involved ergodic BSDE (cf. (14)) is motivated by the homothetic properties and the form of the SPDE (7), the resulting ergodic BSDE is considerably simpler to solve. Moreover, it provides a unique representation result for forward processes with local dependence of the stochastic factors.

For the reader's convenience, we motivate the form of the underlying ergodic BSDE next. To this end, we seek forward exponential criteria that are local functionals of the stochastic factor process V, namely,

$$U(x,t) = -e^{-\gamma x + f(V_t,t)}, \tag{11}$$

for $(x,t) \in \mathbb{D}$, with $\gamma > 0$, and the function $f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ to be determined. Then, using (informally) the SPDE (7), we deduce that f must satisfy the ill-posed semilinear equation

$$f_t + \mathcal{L}f + F\left(v, \kappa^{tr}\nabla f\right) = 0, \tag{12}$$

where $\mathcal{L}$ is the generator of the factor process,

$$\mathcal{L}f = \frac{1}{2}Trace\left(\kappa\kappa^{tr}\nabla^2 f\right) + \eta\left(v\right)^{tr}\nabla f,$$

[2]For an introduction to ergodic BSDE, we refer to [6, 7, 15, 21] and the references therein.

and $F:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is defined as

$$F(v,z):=\frac{1}{2}\gamma^2 dist^2\left\{\Pi,\frac{z+\theta(v)}{\gamma}\right\}-\frac{1}{2}|z+\theta(v)|^2+\frac{1}{2}|z|^2, \tag{13}$$

with $\theta(\cdot)$ as in (5). In Proposition 11 of [25], the following result was proved.

Proposition 3 *Suppose that Assumptions 1 and 2 hold. Then, the ergodic BSDE*

$$dY_t=(-F(V_t,Z_t)+\lambda)dt+Z_t^{tr}dW_t, \tag{14}$$

with the driver $F(\cdot,\cdot)$ given by (13) admits a unique Markovian solution (Y_t,Z_t,λ), $t\geq 0$. *Specifically, there exist a unique $\lambda\in\mathbb{R}$, and functions $y:\mathbb{R}^d\to\mathbb{R}$ and $z:\mathbb{R}^d\to\mathbb{R}^d$ such that*

$$(Y_t,Z_t)=(y(V_t),z(V_t)). \tag{15}$$

The function $y(\cdot)$ is unique up to a constant and has at most linear growth, and $z(\cdot)$ is bounded with $|z(\cdot)|\leq\frac{C_v}{C_\eta-C_v}$, where C_η and C_v are as in Assumption 2. Moreover,

$$|\nabla y(v)|\leq\frac{C_v}{C_\eta-C_v}. \tag{16}$$

We stress that while there exists a unique solution in factor form, equation (14) admits *multiple non-Markovian* solutions. In particular, we may choose $Z_t=z^0(V_t)$, for any smooth function $z^0(\cdot)$ with bounded derivatives of all orders, and any pair (Y,λ) such that (14) holds. Then, $(Y_t,z^0(V_t),\lambda)$ is a non-Markovian solution to (14). Such a non-Markovian solution will be used in Section 4.2 (see (47)).

The next result connects the solutions of the ergodic BSDE (14), Markovian or not, with the exponential forward performance process (11) and its associated optimal policy. For its proof, see Theorem 12 of [25].

Proposition 4 *Suppose that Assumptions 1 and 2 hold, and let (Y,Z,λ) be a solution to (14). Then the process $U(x,t)$, $(x,t)\in\mathbb{D}$, given by*

$$U(x,t)=-e^{-\gamma x+Y_t-\lambda t}, \tag{17}$$

is an exponential forward performance process. Its volatility is

$$a(x,t)=-e^{-\gamma x+Y_t-\lambda t}Z_t,$$

and the associated optimal strategy

$$\pi_t^*=Proj_\Pi\left(\frac{\theta(V_t)}{\gamma}+\frac{Z_t}{\gamma}\right). \tag{18}$$

From the above result and Proposition 3, we deduce that, while there exist multiple exponential forward processes of the general form (17), there exists a *unique* (in the appropriate class) such process in the stochastic factor form

$$U(x,t)=-e^{-\gamma x+y(V_t)-\lambda t}, \tag{19}$$

with the pair $(y(\cdot), \lambda)$ as in Proposition 3.

We conclude mentioning that the most general result for exponential forward processes can be found in [43], where an axiomatic construction is developed for general semimartingale market models.

3 Forward entropic risk measures and ergodic BSDE

Next, we recall the definition of the forward entropic risk measure in relation to the exponential process (17). To simplify the presentation, we assume without loss of generality that the generic risk position is introduced at time $t = 0$, and also remind the reader that $\mathbb{D} = \mathbb{R} \times [0, \infty)$.

Definition 5 *i) Consider the forward exponential performance process* $U(x,t) = -e^{-\gamma x + Y_t - \lambda t}$, $(x,t) \in \mathbb{D}$ *(cf. (17)). Let* $T > 0$ *be arbitrary and consider a risk position* $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_T)$, *introduced at time* $t = 0$.

Its (T-normalized) forward entropic risk measure, denoted by $\rho_t(\xi_T;T)$, $t \in [0,T]$, *is defined as the unique* $\mathcal{F}_t$*-measurable random variable that satisfies the indifference condition*

$$
\begin{aligned}
&ess \sup_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}}\left[U(x + \rho_t(\xi_T;T) + \int_t^T \pi_u^{tr}\left(\theta(V_u)du + dW_u\right) + \xi_T, T) \middle| \mathcal{F}_t \right] \\
&\qquad = ess \sup_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}}\left[U(x + \int_t^T \pi_u^{tr}\left(\theta(V_u)du + dW_u\right), T) \middle| \mathcal{F}_t \right],
\end{aligned}
\tag{20}
$$

for all $(x,t) \in \mathbb{R} \times [0,T]$.

ii) Let the risk position $\xi \in \mathcal{L}$. *Define* $T_\xi = \inf\{T \geq 0 : \xi \in \mathcal{F}_T\}$. *Then, the forward entropic risk measure of* ξ *is defined, for* $t \in [0, T_\xi]$, *as*

$$
\rho_t(\xi) := \rho_t(\xi; T_\xi). \tag{21}
$$

Hence, for $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_T)$, *we have* $\rho_t(\xi_T) = \rho_t(\xi_T;T)$.

We stress that the performance criterion entering in Definition 5 is defined for *all* $t > 0$, and thus one can price claims of arbitrary maturities. This is not, however, the case in the classical framework.

We are now ready to provide one of the main results herein, which is the representation of the forward entropic risk measure using the solutions of associated BSDE and ergodic BSDE. The main idea is to express the forward entropic risk measure process as the solution of a traditional BSDE whose driver, however, *depends* on the solution of the ergodic BSDE (14); see (23) below. This dependence follows from the fact that equation (14) was used to construct the exponential forward performance process (17) that enters into (20) in Definition 5.

Theorem 6 *Consider a risk position $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_T)$, with its maturity $T>0$ be arbitrary. Suppose that Assumptions 1 and 2 hold, and that the process Z in the ergodic BSDE (14) is uniformly bounded. Consider, for $t\in[0,T]$, the BSDE*

$$Y_t^{T,\xi} = -\xi_T + \int_t^T G(V_u, Z_u, Z_u^{T,\xi})du - \int_t^T (Z_u^{T,\xi})^{tr} dW_u, \tag{22}$$

where the driver $G:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ is defined as

$$G(v,z,\hat{z}) := \frac{1}{\gamma}\left(F(v,z+\gamma\hat{z}) - F(v,z)\right), \tag{23}$$

with $F(\cdot,\cdot)$ given by (13). Then the following assertions hold:

i) The BSDE (22) has a unique solution $\left(Y_t^{T,\xi}, Z_t^{T,\xi}\right)$, $t\in[0,T]$, with $Y^{T,\xi}$ being uniformly bounded and $Z^{T,\xi}\in\mathcal{L}^2_{BMO}[0,T]$.

ii) The forward entropic risk measure of ξ_T (cf. (20)) is given, for $t\in[0,T]$, by

$$\rho_t(\xi_T) = Y_t^{T,\xi}. \tag{24}$$

Proof. To facilitate the exposition, we eliminate the ξ-superscript notation in the processes $Y^{T,\xi}, Z^{T,\xi}$. We also recall that in the proof of Proposition 11 in [25], two key inequalities were used, which follow from Assumption 1 and the Lipschitz property of the distance function. Specifically, it can be shown that there exist constants $C_v>0$ and $C_z>0$ such that

$$|F(v_1,z)-F(v_2,z)| \le C_v(1+|z|)|v_1-v_2|, \tag{25}$$

and

$$|F(v,z_1)-F(v,z_2)| \le C_z(1+|z_1|+|z_2|)|z_1-z_2|, \tag{26}$$

for any $v_1,v_2,z_1,z_2\in\mathbb{R}^d$.

(i) First note that, for $t\in[0,T]$, $G(v,Z_t,\hat{z})$ is locally Lipschitz continuous in $\hat{z}$, since a.s.

$$|G(v,Z_t,\hat{z}_1)-G(v,Z_t,\hat{z}_2)| \le C_z(1+2|Z_t|+\gamma|\hat{z}_1|+\gamma|\hat{z}_2|)|\hat{z}_1-\hat{z}_2|,$$

with Z being uniformly bounded. Using, furthermore, that $\xi_T\in\mathcal{L}^\infty(\mathcal{F}_T)$, the assertion follows from Theorems 2.3 and 2.6 of [24] and Theorem 7 of [20].

(ii) Using (17) and that $\rho_t(\xi_T)\in\mathcal{F}_t$, $t\in[0,T]$, we have that

$$ess\sup_{\pi\in\mathcal{A}_{[t,T]}} E_{\mathbb{P}}\left[U(x+\rho_t(\xi_T)+\int_t^T \pi_u^{tr}\left(\theta(V_u)du+dW_u\right)+\xi_T,T)\middle|\mathcal{F}_t\right]$$

$$= e^{-\gamma\rho_t(\xi_T)} ess\sup_{\pi\in\mathcal{A}_{[t,T]}} E_{\mathbb{P}}\left[-e^{-\gamma\left(x+\int_t^T\pi_u^{tr}(\theta(V_u)du+dW_u)\right)+Y_T-\lambda T-\gamma\xi_T}|\mathcal{F}_t\right]. \tag{27}$$

To solve the above problem, we work as follows. Define, for $s\in[t,T]$, the process

$$R_s^\pi := -e^{-\gamma\left(x+\int_t^s\pi_u^{tr}(\theta(V_u)du+dW_u)\right)+Y_s-\lambda s+\gamma Y_s^T}. \tag{28}$$

We will show that it is a supermartingale for any $\pi \in \mathcal{A}_{[t,T]}$, and becomes a martingale for

$$\pi_s^{*,\xi} = \mathcal{P}_\Pi \left(Z_s^T + \frac{Z_s + \theta(V_s)}{\gamma} \right). \tag{29}$$

Indeed, for $0 \le t \le r \le s \le T$, observe that the exponent in (28) satisfies

$$-\gamma \left(x + \int_t^s \pi_u^{tr} \left(\theta \left(V_u \right) du + dW_u \right) \right) + Y_s - \lambda s + \gamma Y_s^T$$

$$= -\gamma \left(x + \int_t^r \pi_u^{tr} \left(\theta \left(V_u \right) du + dW_u \right) \right) + Y_r - \lambda r + \gamma Y_r^T$$

$$-\gamma \left(x + \int_r^s \pi_u^{tr} \left(\theta \left(V_u \right) du + dW_u \right) \right) + \left(Y_s - Y_r \right) - \lambda \left(s - r \right) + \gamma \left(Y_s^T - Y_r^T \right).$$

Using the ergodic BSDE (14) and BSDE (22), respectively, yields

$$\left(Y_s - Y_r \right) - \lambda \left(r - s \right) = -\int_r^s F \left(V_u, Z_u \right) du + \int_r^s Z_u^{tr} dW_u$$

and

$$Y_s^T - Y_r^T = -\frac{1}{\gamma} \int_r^s \left(F \left(V_u, Z_u + \gamma Z_u^T \right) - F \left(V_u, Z_u \right) \right) du + \int_r^s \left(Z_u^T \right)^{tr} dW_u.$$

Combining the above yields

$$E_{\mathbb{P}} \left[-e^{-\gamma \left(x + \int_t^s \pi_u^{tr} (\theta(V_u) du + dW_u) \right) + Y_s - \lambda s + \gamma Y_s^T} \middle| \mathcal{F}_r \right]$$

$$= -e^{-\gamma \left(x + \int_t^r \pi_u^{tr} (\theta(V_u) du + dW_u) \right) + Y_r - \lambda r + \gamma Y_r^T}$$

$$\times E_{\mathbb{P}} \left[e^{\int_r^s \left(-\gamma \pi_u^{tr} \theta(V_u) - F(V_u, Z_u + \gamma Z_u^T) \right) du + \int_r^s \left(-\gamma \pi_u + Z_u + \gamma Z_u^T \right)^{tr} dW_u} \middle| \mathcal{F}_r \right].$$

For $s \in [0,T]$, consider the process $N_s := \int_0^s \left(-\gamma \pi_u + Z_u + \gamma Z_u^T \right)^{tr} dW_u$. Because $\pi, Z^T \in \mathcal{L}^2_{BMO}[0,T]$ and Z is uniformly bounded, we deduce that N is a BMO-martingale.

Next, we define on $\mathcal{F}_T$ a probability measure, denoted by $\mathbb{Q}^\pi$, by $\frac{d\mathbb{Q}^\pi}{d\mathbb{P}} = \mathcal{E}(N)_T$. Then, $\frac{d\mathbb{Q}^\pi}{d\mathbb{P}}\Big|_{\mathcal{F}_s} = \mathcal{E}(N)_s$, which is uniformly integrable due to the BMO-martingale property of the process N. Therefore,

$$e^{\int_r^s \left(-\gamma \pi_u + Z_u + \gamma Z_u^T \right)^{tr} dW_u} = \left(e^{\frac{1}{2} \int_r^s |-\gamma \pi_u + Z_u + \gamma Z_u^T|^2 du} \right) \frac{\mathcal{E}(N)_s}{\mathcal{E}(N)_r},$$

and, thus,

$$E_{\mathbb{P}} \left[-e^{-\gamma \left(x + \int_t^s \pi_u^{tr} (\theta(V_u) du + dW_u) \right) + Y_s - \lambda s + \gamma Y_s^T} \middle| \mathcal{F}_r \right]$$

$$= -e^{-\gamma \left(x + \int_t^r \pi_u^{tr} (\theta(V_u) du + dW_u) \right) + Y_r - \lambda r + \gamma Y_r^T}$$

$$\times E_{\mathbb{P}}\left[e^{\int_r^s\left(-\gamma\pi_u^{tr}\theta(V_u)-F(V_u,Z_u+\gamma Z_u^T)\right)du+\frac{1}{2}\int_r^s|-\gamma\pi_u+Z_u+\gamma Z_u^T|^2du}\frac{\mathcal{E}(N)_s}{\mathcal{E}(N)_r}|\mathcal{F}_r\right]$$

$$= -e^{-\gamma\left(x+\int_t^r\pi_u^{tr}(\theta(V_u)du+dW_u)\right)+Y_r-\lambda r+\gamma Y_r^T}$$

$$\times E_{\mathbb{Q}^\pi}\left[e^{\int_r^s\left(-\gamma\pi_u^{tr}\theta(V_u)-F(V_u,Z_u+\gamma Z_u^T)\right)du+\frac{1}{2}\int_r^s|-\gamma\pi_u+Z_u+\gamma Z_u^T|^2du}|\mathcal{F}_r\right]$$

$$= -e^{-\gamma\left(x+\int_t^r\pi_u^{tr}(\theta(V_u)du+dW_u)\right)+Y_r-\lambda r+\gamma Y_r^T}$$

$$\times E_{\mathbb{Q}^\pi}\left[e^{\int_r^s\left(\left(-\gamma\pi_u^{tr}\theta(V_u)+\frac{1}{2}|-\gamma\pi_u+Z_u+\gamma Z_u^T|^2\right)-F(V_u,Z_u+\gamma Z_u^T)\right)du}|\mathcal{F}_r\right].$$

In turn, if we can show that, for $u\in[r,s]$,

$$-\gamma\pi_u^{tr}\theta(V_u)+\frac{1}{2}|-\gamma\pi_u+Z_u+\gamma Z_u^T|^2\geq F(V_u,Z_u+\gamma Z_u^T),$$

then

$$E_{\mathbb{Q}^\pi}\left[e^{\int_r^s\left(\left(-\gamma\pi_u^{tr}\theta(V_u)+\frac{1}{2}|-\gamma\pi_u+Z_u+\gamma Z_u^T|^2\right)-F(V_u,Z_u+\gamma Z_u^T)\right)du}|\mathcal{F}_r\right]\geq 1,$$

and the supermartingality property would follow. Indeed, after some calculations, we obtain that

$$-\gamma\pi_u^{tr}\theta(V_u)+\frac{1}{2}|-\gamma\pi_u+Z_u+\gamma Z_u^T|^2$$

$$=\frac{\gamma^2}{2}\left|\pi_u-\left(Z_u^T+\frac{Z_u+\theta\left(V_u\right)}{\gamma}\right)\right|^2-\frac{1}{2}\left|Z_u+\gamma Z_u^T+\theta\left(V_u\right)\right|^2+\frac{1}{2}\left|Z_u+\gamma Z_u^T\right|^2.$$

On the other hand, for any $\pi\in\mathcal{A}_{[t,T]}$,

$$\left|\pi_u-\left(Z_u^T+\frac{Z_u+\theta(V_u)}{\gamma}\right)\right|^2\geq\ dist^2\left\{\Pi,Z_u^T+\frac{Z_u+\theta(V_u)}{\gamma}\right\},$$

and using the form of $F(V_u,Z_u+\gamma Z_u^T)$ (cf. (13)) we conclude.

To show that $R^{\pi^{*,\xi}}$ is a martingale for $\pi^{*,\xi}$, defined in (29), observe that

$$\left|\pi_u^{*,\xi}-\left(Z_u^T+\frac{Z_u+\theta(V_u)}{\gamma}\right)\right|^2=\ dist^2\left\{\Pi,Z_u^T+\frac{Z_u+\theta(V_u)}{\gamma}\right\},$$

and the martingale property follows. Note, moreover, that this policy is admissible.

Combining the above, we obtain that $E_{\mathbb{P}}\left[R_T^\pi|\,\mathcal{F}_t\right]\leq R_t^\pi$, and thus, for any $\pi\in\mathcal{A}_{[t,T]}$,

$$E_{\mathbb{P}}\left[-e^{-\gamma\left(x+\int_t^T\pi_s^{tr}(\theta(V_s)ds+dW_s)\right)+Y_T-\lambda T-\gamma\xi_T}\,\middle|\,\mathcal{F}_t\right]\leq -e^{-\gamma x+Y_t-\lambda t+\gamma Y_t^T},$$

where we also used that $Y_T^T=-\xi_T$ (cf. (22)). Similarly,

$$E_{\mathbb{P}}\left[-e^{-\gamma\left(x+\int_t^T(\pi_s^{*,\xi})^{tr}(\theta(V_s)ds+dW_s)\right)+Y_T-\lambda T-\gamma\xi_T}|\mathcal{F}_t\right]=-e^{-\gamma x+Y_t-\lambda t+\gamma Y_t^T}.$$

In other words,

$$ess \sup_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left[-e^{-\gamma\left(x+\int_t^T \pi_s^{tr}(\theta(V_s)ds+dW_s)\right)+Y_T-\lambda T-\gamma\xi_T} \middle| \mathcal{F}_t \right] = -e^{-\gamma x+Y_t-\lambda t+\gamma Y_t^T},$$

which, by (20), implies

$$-e^{-\gamma\rho_t(\xi_T)-\gamma x+Y_t-\lambda t+\gamma Y_t^T} = -e^{-\gamma x+Y_t-\lambda t},$$

and (24) follows. ■

From the above representation, we also obtain the time-consistency property,

$$\rho_t(\xi_T) = Y_t^{T,\xi} = Y_t^{s,-Y_s^{T,\xi}} = Y_t^{s,-\rho_s(\xi_T)} = \rho_t(-\rho_s(\xi_T))$$

for any $0 \leq t \leq s \leq T < \infty$.

The fact that the forward entropic risk measure is obtained via a BSDE (cf. (22)) should *not* suggest that it does not differ from its classical counterpart, which is also given as a solution of a BSDE (see (51) herein).

Firstly, it is natural to expect that the risk measure will be represented by the solution of a BSDE, since the pricing condition (20) is, by nature, set "backwards" in time. On the other hand, the classical entropic risk measure is defined only on $[0,T]$, because the associated traditional exponential value function is only defined on $[0,T]$. In contrast, the forward entropic risk measure is defined for *all* times $t \geq 0$, for the associated forward process $U(x,t)$ (cf. (17)) is defined for $t \geq 0$.

Secondly, the BSDE (22) for the forward risk measure differs from the one for the classical entropic measure, because its driver depends on the process Z_t, which solves the ergodic BSDE that yields the exponential forward criterion.

We conclude mentioning that in addition to the forward entropic measure, one can define the "*hedging strategies*" associated with the risk position ξ_T. As in the classical case, they are defined as the difference of the optimal strategies for the two optimization problems involved in Definition 5. We easily deduce that the related hedging strategy, denoted by $\alpha_{t,T}, t \in [0,T]$, is given by

$$\alpha_{t,T} = \pi_t^{*,\xi} - \pi_t^* \tag{30}$$

$$= Proj_{\Pi}\left(Z_t^{T,\xi} + \frac{Z_t + \theta(V_t)}{\gamma}\right) - Proj_{\Pi}\left(\frac{Z_t + \theta(V_t)}{\gamma}\right).$$

Observe that the first term naturally depends on the maturity of the claim, while the second is independent of it and defined for all times. This is not the case in the classical setting, where both terms depend on the investment horizon.

4 Long-maturity behavior

We study the behavior of the forward entropic risk measure when the maturity of the risk position is long. We focus on European-type positions written only on the stochastic factors, specifically, risk positions represented as

$$\xi_T = -g\left(V_T\right), \tag{31}$$

with $g : \mathbb{R}^d \to \mathbb{R}$ being a uniformly bounded and Lipschitz continuous function with Lipschitz constant C_g.

We recall from Theorem 6, the representation $\rho_t\left(\xi_T\right) = Y_t^{T,\xi}$, where $Y_t^{T,\xi}$ solves the BSDE (22), rewritten below to ease the exposition,

$$Y_t^{T,g} = g(\xi_T) + \int_t^T G(V_s, Z_s, Z_s^{T,g})ds - \int_t^T (Z_s^{T,g})^{tr} dW_s.$$

To examine its long-term behavior, it is thus natural to relate the above BSDE with an ergodic BSDE, given below, and investigate the proximity of their solutions.

To this end, we consider the ergodic BSDE

$$\hat{Y}_t = \hat{Y}_{T'} + \int_t^{T'} \left(G(V_s, Z_s, \hat{Z}_s) - \hat{\lambda}\right) ds - \int_t^{T'} (\hat{Z}_s)^{tr} dW_s \tag{32}$$

for $0 \leq t \leq T' < \infty$, and examine the approximation of $Y_0^{T,g}$ by $\hat{Y}_0 + \hat{\lambda}T$, for large T.

We stress that the driver of the ergodic BSDE (32) depends on the solution Z of the ergodic BSDE (14) of the forward performance process. As we explain next, this creates various technical issues and prompts to analyze the Markovian and non-Markovian forward processes separately.

4.1 Markovian performance criterion

We first consider the case

$$U(x,t) = -e^{-\gamma x + y(V_t) - \lambda t}, \tag{33}$$

(cf. (19)) where $(Y_t, Z_t) = (y(V_t), z(V_t), \lambda)$ as in Proposition 3.

In this case, the driver $G(v, z(v), \hat{z})$ of the ergodic BSDE (32) *depends* on the function $z(v)$. Although, due to the boundedness of the function $z(\cdot)$, the driver G satisfies the locally Lipschitz estimate (26) in $\hat{z}$, it may not satisfy the locally Lipschitz estimate (25) in v, and hence the existence and uniqueness result in [25] might not apply. Moreover, it is not even clear whether the ergodic BSDE (32) is well-posed.

The overcome this issue, we employ an auxiliary BSDE. Writing,

$$Y_t^{T,g} = \left(Y_t^{T,g} + \frac{Y_t - \lambda t}{\gamma}\right) - \frac{Y_t - \lambda t}{\gamma},$$

we observe that the pair $(\hat{Y}_t^{T,g}, \hat{Z}_t^{T,g})$, defined as

$$(\hat{Y}_t^{T,g}, \hat{Z}_t^{T,g}) := \left(Y_t^{T,g} + \frac{Y_t - \lambda t}{\gamma}, Z_t^{T,g} + \frac{Z_t}{\gamma} \right),$$

solves the finite horizon quadratic BSDE

$$\hat{Y}_t^{T,g} = g(V_T) + \frac{Y_T - \lambda T}{\gamma} + \int_t^T \frac{1}{\gamma} F(V_s, \gamma \hat{Z}_s^{T,g}) ds - \int_t^T (\hat{Z}_s^{T,g})^{tr} dW_s, \quad (34)$$

with the driver $F(\cdot, \cdot)$ as in (13).

We study this BSDE next, starting with some auxiliary results. We will be using the superscript v, whenever appropriate, to denote the dependence on the initial condition of the stochastic factor.

Proposition 7 *Suppose that Assumption 2 holds. Then, the following assertions hold, for all* $t \geq 0$.

i) The stochastic factor process satisfies $|V_t^v - V_t^{\bar{v}}|^2 \leq e^{-2C_\eta t} |v - \bar{v}|^2$, $v, \bar{v} \in \mathbb{R}^d$.

ii) Assume that the process V^v *follows*

$$dV_t^v = (\eta(V_t^v) + H(V_t^v))\, dt + \kappa dW_t^H,$$

where $H : \mathbb{R}^d \to \mathbb{R}^d$ *is a measurable bounded function,* $\mathbb{Q}^H$ *is a probability measure equivalent to* $\mathbb{P}$, *and* W^H *is a* $\mathbb{Q}^H$*-Brownian motion. Then, for some constant* $C > 0$, $E_{\mathbb{Q}^H}\left[|V_t^v|^2\right] \leq C\left(1 + |v|^2\right)$. *Furthermore, for any measurable function* $\phi : \mathbb{R}^d \to \mathbb{R}$ *with polynomial growth rate* $\mu > 0$, *and* $v, \bar{v} \in \mathbb{R}^d$,

$$\left| E_{\mathbb{Q}^H}\left[\phi(V_t^v) - \phi(V_t^{\bar{v}})\right] \right| \leq C\left(1 + |v|^{1+\mu} + |\bar{v}|^{1+\mu}\right) e^{-\hat{C}_\eta t}.$$

The constants C *and* $\hat{C}_\eta$ *depend on the function* H *only through* $\sup_{v \in \mathbb{R}^d} |H(v)|$.

The proof of (i) follows from Grönwall's inequality. The first assertion in (ii) is an application of a Lyapunov argument (see Lemma 3.1 of [11]), while the rest follows from the basic coupling estimate in Lemma 3.4 in [21].

Lemma 8 *Suppose that Assumptions 1 and 2 hold, and that the forward performance process* $U(x,t)$ *is given by (33). Then, the following assertions hold:*

i) There exists a unique solution $(\hat{Y}_t^{T,g}, \hat{Z}_t^{T,g}) = (\hat{y}^{T,g}(V_t, t), \hat{z}^{T,g}(V_t, t))$, $t \in [0,T]$, *to the BSDE (34).*

ii) For $(v,t) \in \mathbb{R}^d \times [0, \infty)$, *we have that*

$$|\hat{y}^{T,g}(v,t)| \leq C_T(1 + |v|) \quad (35)$$

for some constant C_T *(which may depend on* T*). Moreover,* $\hat{z}^{T,g}(v,t)$ *is uniformly bounded, namely,*

$$|\hat{z}^{T,g}(v,t)| \leq q \quad \text{with} \quad q = \frac{\gamma C_\eta C_g + C_v}{\gamma(C_\eta - C_v)} + \frac{C_\eta C_v}{\gamma(C_\eta - C_v)^2}, \quad (36)$$

where C_v *is given in (25), and* C_η *and* C_g *in Assumptions 2 and (31), respectively.*

Proof. The linear growth of the function $\hat{y}^{T,g}$ follows from the boundedness of the process $Y_t^{T,g}$ and the linear growth property of the function $y(\cdot)$.

Next, we only show the uniform boundedness of $\hat{z}^{T,g}(v,t)$, since uniqueness would then follow directly. To ease the notation, we drop the g-superscript notation in the processes $\hat{Y}_t^{T,g}, \hat{Z}_t^{T,g}$. To this end, we first note that, for $s \in [t,T]$, $(\hat{Y}_s^T, \hat{Z}_s^T) = (\hat{Y}_s^{T,t,v}, \hat{Z}_s^{T,t,v}) = (\hat{y}^T(V_s^{t,v}, s), \hat{z}^T(V_s^{t,v}, s))$ for some measurable functions $(\hat{y}^T(\cdot,\cdot), \hat{z}^T(\cdot,\cdot))$, with $V_t^{t,v} = v$.

With a slight abuse of notation, we introduce the truncation function $q : \mathbb{R}^d \to \mathbb{R}^d$,

$$q(z) = \frac{\min(|z|, q)}{|z|} z \mathbf{1}_{\{z \neq 0\}}, \tag{37}$$

as well as the truncated version of equation (34),

$$\hat{Y}_t^{T,t,v} = g(V_T^{t,v}) + \frac{Y_T^{t,v} - \lambda T}{\gamma} + \int_t^T \frac{1}{\gamma} F(V_s^{t,v}, \gamma q(\hat{Z}_s^{T,t,v})) ds - \int_t^T (\hat{Z}_s^{T,t,v})^{tr} dW_s.$$

From the form of the driver (13) and (37) we deduce

$$|F(v, \gamma q(z)) - F(\bar{v}, \gamma q(z))| \leq C_v (1 + \gamma q) |v - \bar{v}|, \tag{38}$$

$$|F(v, \gamma q(z)) - F(v, \gamma q(\bar{z}))| \leq C_z (1 + 2\gamma q) \gamma |z - \bar{z}|, \tag{39}$$

for any $v, \bar{v}, z, \bar{z} \in \mathbb{R}^d$. In turn,

$$\begin{aligned}
\hat{Y}_t^{T,t,v} - \hat{Y}_t^{T,t,\bar{v}} &= g(V_T^{t,v}) - g(V_T^{t,\bar{v}}) + \frac{1}{\gamma}(Y_T^{t,v} - Y_T^{t,\bar{v}}) \\
&+ \int_t^T \frac{1}{\gamma} \left[F(V_s^{t,v}, \gamma q(\hat{Z}_s^{T,t,v})) - F(V_s^{t,\bar{v}}, \gamma q(\hat{Z}_s^{T,t,\bar{v}})) \right] ds - \int_t^T \left(\hat{Z}_s^{T,t,v} - \hat{Z}_s^{T,t,\bar{v}} \right)^{tr} dW_s \\
&= g(V_T^{t,v}) - g(V_T^{t,\bar{v}}) + \frac{1}{\gamma}(y(V_T^{t,v}) - y(V_T^{t,\bar{v}})) \\
&+ \int_t^T \frac{1}{\gamma} \left[F(V_s^{t,v}, \gamma q(\hat{Z}_s^{T,t,v})) - F(V_s^{t,\bar{v}}, \gamma q(\hat{Z}_s^{T,t,v})) \right] ds \\
&- \int_t^T \left(\hat{Z}_s^{T,t,v} - \hat{Z}_s^{T,t,\bar{v}} \right)^{tr} (dW_s - M_s ds),
\end{aligned}$$

where the process M_s, $s \in [t,T]$, is defined as

$$M_s := \frac{\left(F(V_s^{t,\bar{v}}, \gamma q(\hat{Z}_s^{T,t,v})) - F(V_s^{t,\bar{v}}, \gamma q(\hat{Z}_s^{T,t,\bar{v}})) \right) \left(\hat{Z}_s^{T,t,v} - \hat{Z}_s^{T,t,\bar{v}} \right)}{\gamma \left| \hat{Z}_s^{T,t,v} - \hat{Z}_s^{T,t,\bar{v}} \right|^2} \mathbf{1}_{\left\{ \hat{Z}_s^{T,t,v} - \hat{Z}_s^{T,t,\bar{v}} \neq 0 \right\}}.$$

Note, however, that M_s is bounded as it follows from inequality (39). Thus, we can define $W_t^M := W_t - \int_0^t M_s ds$, for $0 \leq t \leq T$, which is a Brownian motion under some measure $\mathbb{Q}^M$ equivalent to $\mathbb{P}$, defined on $\mathcal{F}_T$. In turn,

$$|\hat{Y}_t^{T,t,v} - \hat{Y}_t^{T,t,\bar{v}}| = |\hat{y}^T(v,t) - \hat{y}^T(\bar{v},t)|$$

$$\leq \; C_g E_{\mathbb{Q}^M}\left[|V_T^{t,v}-V_T^{t,\bar{v}}||\mathcal{F}_t\right]+\frac{C_v}{\gamma(C_\eta-C_v)}E_{\mathbb{Q}^M}\left[|V_T^{t,v}-V_T^{t,\bar{v}}||\mathcal{F}_t\right]$$
$$+\frac{C_v(1+\gamma q)}{\gamma}E_{\mathbb{Q}^M}\left[\int_t^T|V_s^{t,v}-V_s^{t,\bar{v}}|ds\middle|\mathcal{F}_t\right]$$
$$\leq \; \left(C_g+\frac{C_v}{\gamma(C_\eta-C_v)}+\frac{C_v(1+\gamma q)}{\gamma C_\eta}\right)|v-\bar{v}|,$$

where we used the Lipschitz continuity conditions on $g(v)$, $y(v)$ and $F(v,\gamma q(z))$ with respect to v (cf. (31), (16) and (38) respectively), and the exponential ergodicity condition (i) in Proposition 7. From the relationship $\kappa^{tr}\nabla\hat{y}^T(V_s^{t,v},s)=\hat{Z}_s^{T,t,v}$, we further deduce that $q(\hat{Z}_s^T)=\hat{Z}_s^T$. In other words, the truncation does not play a role, and q is the uniform bound of $\hat{Z}^T$. ■

Next, we establish some auxiliary estimates.

Lemma 9 *Suppose that Assumptions 1 and 2 hold, and that the forward performance process is given by (33). Then, the Markovian solution*

$$(Y_t^{T,g},Z_t^{T,g})=(y^{T,g}(V_t,t),z^{T,g}(V_t,t))$$

to the BSDE (22) has the following properties, for $(v,t)\in\mathbb{R}^d\times[0,T]$ *:*

- *i) For some constant* C, $|y^{T,g}(v,t)|\leq C(1+|v|)$.
- *ii) With the constant* q *given in (36),* $|\nabla y^{T,g}(v,t)|\leq q+\frac{C_v}{\gamma(C_\eta-C_v)}$.
- *iii) With the constant* $\hat{C}_\eta$ *given in Proposition 7,* $|y^{T,g}(v,t)-y^{T,g}(\bar{v},t)|\leq C(1+|v|^2+|\bar{v}|^2)e^{-\hat{C}_\eta(T-t)}$.

Proof. To ease the notation, we eliminate the g-supersripct notation throughout the proof.

Firstly, we recall that $(Y_s^T,Z_s^T)=(Y_s^{T,t,v},Z_s^{T,t,v})=(y^T(V_s^{t,v},s),z^T(V_s^{t,v},s))$, for some measurable functions $(y^T(\cdot,\cdot),z^T(\cdot,\cdot))$, and that

$$Y_t^{T,t,v}=g(V_T^{t,v})+\int_t^T\frac{1}{\gamma}\left(F(V_s^{t,v},\gamma\hat{Z}_s^{T,t,v})-F(V_s^{t,v},Z_s^{t,v})\right)ds$$
$$-\int_t^T\left(\hat{Z}_s^{T,t,v}-\frac{Z_s^{t,v}}{\gamma}\right)^{tr}dW_s. \tag{40}$$

Moreover, the process Z^T is uniformly bounded, since

$$|Z_s^{T,t,v}|=|\hat{Z}_s^{T,t,v}-\frac{Z_s^{t,v}}{\gamma}|\leq q+\frac{C_v}{\gamma(C_\eta-C_v)}.$$

Therefore, the gradient estimate (ii) for $Y_t^{T,t,v}=y^T(v,t)$ follows from the relationship $\kappa^{tr}\nabla y^T(V_s^{t,v},s)=Z_s^{T,t,v}$.

To prove (i), we introduce the quantity

$$H(V_s^{t,v}) := \frac{\left(F(V_s^{t,v}, \gamma \hat{Z}_s^{T,t,v}) - F(V_s^{t,v}, Z_s^{t,v})\right)\left(\hat{Z}_s^{T,t,v} - \frac{Z_s^{t,v}}{\gamma}\right)}{\gamma \left|\hat{Z}_s^{T,t,v} - Z_s^{t,v}/\gamma\right|^2} \mathbf{1}_{\left\{\hat{Z}_s^{T,t,v} - \frac{Z_s^{t,v}}{\gamma} \neq 0\right\}}, \tag{41}$$

and observe that it is uniformly bounded due to (26) and the boundedness of $\hat{Z}^{T,t,v}$ and $Z^{t,v}$. Then, equation (40) can be written as

$$Y_t^{T,t,v} = y^T(v,t)$$

$$= g(V_T^{t,v}) - \int_t^T \left(\hat{Z}_s^{T,t,v} - \frac{Z_s^{t,v}}{\gamma}\right)^{tr} (dW_s - H(V_s^{t,v})ds) = E_{\mathbb{Q}^H}[g(V_T^{t,v})|\mathcal{F}_t],$$

and the assertion follows from the linear growth property of $g(\cdot)$ and the first assertion of part (ii) in Proposition 7.

Finally, for $v, \bar{v} \in \mathbb{R}^d$, by the second assertion of (ii) in Proposition 7, we deduce

$$|y^T(v,t) - y^T(\bar{v},t)| = \left|E_{\mathbb{Q}^H}[g(V_T^{t,v}) - g(V_T^{t,\bar{v}})|\mathcal{F}_t]\right|$$

$$= \left|E_{\mathbb{Q}^H}[g(V_{T-t}^{0,v}) - g(V_{T-t}^{0,\bar{v}})]\right| \leq C\left(1 + |v|^2 + |\bar{v}|^2\right) e^{-\hat{C}_\eta (T-t)},$$

and we conclude. ■

We are now ready for the second main result herein, which yields the asymptotic behavior of the forward entropic risk measure for long maturities. We show that as $T \uparrow \infty$, the risk measure tends to a constant, which is independent of the initial state of the stochastic factor process V. We also provide the rate of this convergence. Recall that, without loss of generality, we have assumed that the risk position is introduced at time $t = 0$.

Theorem 10 *Suppose that Assumptions 1 and 2 hold, and that the forward exponential performance process is given by (33). Consider a risk position ξ_T as in (31), introduced at $t = 0$, with T arbitrary. The following assertions hold:*

i) For any $T > 0$, and some constant C,

$$\frac{|\rho_0(\xi_T)|}{T} \leq \frac{C(1 + |v|)}{T},$$

where v is the initial value of the stochastic factor, $V_0 = v$. In turn,

$$\lim_{T\uparrow\infty} \frac{\rho_0(\xi_T)}{T} = 0.$$

ii) There exists a constant $L = L^g \in \mathbb{R}$, independent of v, such that

$$\lim_{T\uparrow\infty} \rho_0(\xi_T) = \lim_{T\uparrow\infty} y^{T,g}(v,0) = L^g. \tag{42}$$

Moreover, for all $T > 0$,

$$|y^{T,g}(v,0) - L^g| \leq C(1 + |v|^2) e^{-\hat{C}_\eta T}, \tag{43}$$

with the constant $\hat{C}_\eta$ given in Proposition 7.

Proof. Part (i) follows directly from part (i) of Lemma 9 and the representation (24).

To show (ii) we argue as follows. To ease the notation, we eliminate the g-superscript notation throughout. From the first estimate (i) in Lemma 9, we first construct, using a standard diagonal procedure, a sequence $\{T_i\}_{i=1}^{\infty}$ such that $T_i \uparrow \infty$, and $\lim_{T_i \uparrow \infty} y^{T_i}(v,0) = L^g(v)$, for $v \in D$, where D is a dense subset of $\mathbb{R}^d$, and some $L^g(v)$.

Moreover, the second estimate (ii) in Lemma 9 implies that, for $v, \bar{v} \in \mathbb{R}^d$,

$$|y^T(v,0) - y^T(\bar{v},0)| \leq \left(q + \frac{C_v}{\gamma(C_\eta - C_v)}\right)|v - \bar{v}|. \tag{44}$$

Therefore, the limit $L^g(v)$ can be extended to a Lipschitz continuous function, defined for all $v \in \mathbb{R}^d$, and, moreover, we have that

$$\lim_{T_i \uparrow \infty} y^{T_i}(v,0) = L^g(v), \quad v \in \mathbb{R}^d.$$

Indeed, for $v \in \mathbb{R}^d \backslash D$, there exists a sequence $\{v_j\}_{j=1}^{\infty} \subseteq D$ such that $v_j \to v$.

Define $L^g(v) := \lim_{j \uparrow \infty} L^g(v_j)$. Using the estimate (44), we have

$$|y^{T_i}(v,0) - y^{T_i}(v_j,0)| \leq \left(q + \frac{C_v}{\gamma(C_\eta - C_v)}\right)|v - v_j|.$$

Taking $T_i \uparrow \infty$ and since $\lim_{j \uparrow \infty} y^{T_i}(v_j,0) = L^g(v_j)$, we obtain

$$\left|\lim_{T_i \uparrow \infty} y^{T_i}(v,0) - L^g(v_j)\right| \leq \left(q + \frac{C_v}{\gamma(C_\eta - C_v)}\right)|v - v_j|.$$

Sending $j \uparrow \infty$, we deduce that for any $v \in \mathbb{R}^d$, $\lim_{T_i \uparrow \infty} y^{T_i}(v,0) = L^g(v)$.

Next, we show that $L^g(v) \equiv L^g$, a constant function. To this end, by the third estimate (iii) in Lemma 9, we have, for any $v, \bar{v} \in \mathbb{R}^d$, that

$$|y^{T_i}(v,0) - y^{T_i}(\bar{v},0)| \leq C\left(1 + |v|^2 + |\bar{v}|^2\right)e^{-\hat{C}_\eta T_i}.$$

Letting $T_i \uparrow \infty$ yields $\lim_{T_i \uparrow \infty} y^{T_i}(v,0) = \lim_{T_i \uparrow \infty} y^{T_i}(\bar{v},0)$, which means the limit function $L^g(v)$ is independent of v. Thus, it is a constant, denoted as L^g. Moreover, such a constant L^g is independent of the choice of the sequence $\{T_i\}_{i=1}^{\infty}$ (see. pp. 394-395 in [21] for its proof). Therefore, $\lim_{T \uparrow \infty} y^T(v,0) = L^g$.

To prove the convergence rate (43), we argue as follows. For $v \in \mathbb{R}^d$ and $T > 0$, we have, from the proof of Lemma 9, that

$$\begin{aligned}
|y^T(v,0) - L^g| &= \lim_{T' \uparrow \infty} |y^T(v,0) - y^{T'}(v,0)| \\
&= \lim_{T' \uparrow \infty} \left|y^T(v,0) - E_{\mathbb{Q}^H}\left[g(V_{T'}^{0,v})\right]\right| \\
&= \lim_{T' \uparrow \infty} \left|y^T(v,0) - E_{\mathbb{Q}^H}\left[E_{\mathbb{Q}^H}\left[g(V_{T'}^{0,v})|\mathcal{F}_{T'-T}\right]\right]\right|
\end{aligned}$$

$$= \lim_{T'\uparrow\infty} \left| y^T(v,0) - E_{\mathbb{Q}^H}\left[y^{T'}\left(V^{0,v}_{T'-T}, T'-T \right)\right]\right|$$

$$= \lim_{T'\uparrow\infty} \left| y^T(v,0) - E_{\mathbb{Q}^H}\left[y^{T}\left(V^{0,v}_{T'-T}, 0 \right)\right]\right|$$

$$= \lim_{T'\uparrow\infty} \left| E_{\mathbb{Q}^H}\left[y^T(v,0) - y^{T}\left(V^{0,v}_{T'-T}, 0 \right)\right]\right|$$

$$\leq \lim_{T'\uparrow\infty} C E_{\mathbb{Q}^H}\left[1 + |v|^2 + \left| V^{0,v}_{T'-T}\right|^2 \right] e^{-\hat{C}_\eta T} \leq C\left(1+|v|^2\right) e^{-\hat{C}_\eta T},$$

where we used (ii) in Proposition 7 and (iii) in Lemma 9 in the last two inequalities. ■

Next, we establish the asymptotic behavior of the hedging strategy $\alpha_{t,T}$ (cf. (30)) as $T \uparrow \infty$.

Proposition 11 *Under the assumptions of Theorem 10, the hedging strategy satisfies, for any* $s \in [0,T]$,

$$\lim_{T\uparrow\infty} E_{\mathbb{P}}\left[\int_0^s |\alpha_{t,T}|^2 dt \right] = 0. \tag{45}$$

Proof. Again, we drop the superscript g in the proof. We establish the convergence of $Z_t^T = \hat{Z}_t^T - Z_t/\gamma$, namely, that for $s \in [0,T]$, $\lim_{T\uparrow\infty} E_{\mathbb{P}}\left[\int_0^s |Z_t^T|^2 dt\right] = 0$. Then, the convergence of the hedging strategy $\alpha_{t,T}$ would follow from Theorem 6 and the Lipschitz continuity of the projection operator on the convex set Π. To this end, we easily deduce, using (iii) in Lemma 9, that, for $t \in [0,T)$,

$$|y^T(v,t) - L^g| \leq C(1+|v|^2)e^{-\hat{C}_\eta (T-t)}. \tag{46}$$

Applying Itô's formula to $|y^T(V_t,t) - L^g|^2$ and using (40), we in turn have

$$|y^T(v,0) - L^g|^2 + E_{\mathbb{P}}\left[\int_0^s |Z_u^T|^2 du \right]$$

$$= E_{\mathbb{P}}[|y^T(V_s,s) - L^g|^2] + 2E_{\mathbb{P}}\left[\int_0^s |y^T(V_u,u) - L^g| \frac{F(V_u,\gamma\hat{Z}_u^T) - F(V_u,Z_u)}{\gamma} du \right]$$

$$= E_{\mathbb{P}}[|y^T(V_s,s) - L^g|^2] + 2E_{\mathbb{P}}\left[\int_0^s (Z_u^T)^{tr} H_u |y^T(V_u,u) - L^g| du \right]$$

$$\leq E_{\mathbb{P}}[|y^T(V_s,s) - L^g|^2] + \frac{1}{2}E_{\mathbb{P}}\left[\int_0^s |Z_t^T|^2 dt \right] + C E_{\mathbb{P}}\left[\int_0^s |y^T(V_u,u) - L^g|^2 du \right],$$

where the process H, introduced in (41), is uniformly bounded. Hence, (46) further yields that

$$E_{\mathbb{P}}\left[\int_0^s |Z_t^T|^2 dt \right] \leq C e^{-2\hat{C}_\eta T}\left(e^{2\hat{C}_\eta s} E_{\mathbb{P}}[(1+|V_s|^2)^2] + \int_0^s e^{2\hat{C}_\eta u} E_{\mathbb{P}}[(1+|V_u|^2)^2] du \right),$$

and sending $T \uparrow \infty$ we conclude. ■

4.2 Non-Markovian performance criterion

We consider criteria of the form

$$\tilde{U}(x,t) = -e^{-\gamma x + Y_t - \lambda t}, \tag{47}$$

where the triplet (Y_t, Z_t, λ) solves the ergodic BSDE (14) with $Z_t = z^0(V_t)$, for an arbitrary smooth function $z^0(\cdot)$ with bounded derivatives of every order. To differentiate these processes from the Markovian one in (33), we use $\tilde{U}$ to denote them.

In this case, the driver $G(v, z^0(v), \hat{z})$ of equation (32) satisfies the Lipschitz estimates (25) in v and (26) in $\hat{z}$, respectively. In turn, using the Proposition 11 in [25], we deduce that the ergodic BSDE (32) admits a unique Markovian solution $(\hat{Y}, \hat{Z}, \hat{\lambda})$.

On the other hand, it is easy to verify that $(C, 0, 0)$, where C is a generic constant, solves equation (32). Therefore, $(\hat{Y}_t, \hat{Z}_t, \lambda) = (C, 0, 0)$ is the only Markovian solution to (32). We then deduce that the large time behavior of $\rho_0(\xi_T)$ is given by the constant $\hat{Y}_0 + \hat{\lambda}T = C$. Furthermore, from Lemma 4.5 in [21], we obtain analogous estimates to the ones in Lemma 9, namely, that $|y^T(v,t)| \leq C(1+|v|)$, $|\nabla y^T(v,t)| \leq C$ and $|y^T(v,t) - y^T(\bar{v},t)| \leq C(1+|v|^2 + |\bar{v}|^2)e^{-\hat{C}_\eta (T-t)}$, but for different constants.

Combining the above and using arguments similar to the ones used in Theorem 10, we obtain the following.

Theorem 12 *Suppose that Assumptions 1 and 2 hold, and that the exponential performance process is given by $\tilde{U}(x,t)$ in (47). Consider a risk position ξ_T as in (31), introduced at $t=0$, with arbitrary maturity T. Then, the following assertions hold:*

i) Its forward entropic risk measure satisfies $\lim_{T\uparrow\infty} \frac{\rho_0(\xi_T)}{T} = 0$.

ii) There exists a constant $K = K^g \in \mathbb{R}$, independent of v, such that

$$\lim_{T\uparrow\infty} \rho_0(\xi_T) = \lim_{T\uparrow\infty} y^{T,g}(v,0) = K^g.$$

Moreover, for all $T > 0$,

$$|y^{T,g}(v,0) - K^g| \leq C(1+|v|^2)e^{-\hat{C}_\eta T},$$

with the constant $\hat{C}_\eta$ given in Proposition 7.

iii) The associated hedging strategy $\alpha_{t,T}$ satisfies, for $s \in [0,T]$,

$$\lim_{T\uparrow\infty} E_{\mathbb{P}}\left[\int_0^s |\alpha_{t,T}|^2\, dt\right] = 0.$$

5 A parity result between forward and classical entropic risk measures

In this section, we relate the forward entropic risk measure to its classical analogue. In the latter case, the investment horizon is finite, say $[0,T]$, for some fixed T, and the terminal utility is given by

$$U_T(x) = -e^{-\gamma x}, \tag{48}$$

$x \in \mathbb{R}$, $\gamma > 0$. We recall the definition of the entropic risk measure $\rho_{t,T}(\xi_T)$ associated with this utility (see, among others, [4, 10, 17, 18, 19, 20, 26, 27, 40]).

Definition 13 *Let $T > 0$ be fixed, and consider a risk position introduced at $t = 0$, yielding payoff $\xi_T \in \mathcal{F}_T$. Its entropic risk measure is defined as the unique $\rho_{t,T}(\xi_T) \in \mathcal{F}_t$, $t \in [0,T]$, solving the indifference condition*

$$ess \sup_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left[U_T(x + \rho_{t,T}(\xi_T) + \int_t^T \pi_s^{tr} \left(\theta(V_s)ds + dW_s\right) + \xi_T) \middle| \mathcal{F}_t \right] \tag{49}$$

$$= ess \sup_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left[U_T(x + \int_t^T \pi_s^{tr} \left(\theta(V_s)ds + dW_s\right)) \middle| \mathcal{F}_t \right],$$

for all $(x,t) \in \mathbb{R} \times [0,T]$, with $U_T(\cdot)$ as in (48).

Next, we present a decomposition formula that relates the forward entropic risk measure under Markovian performance criteria with its classical counterpart.

Proposition 14 *Assume that $U(x,t)$ is given by (33), and let $\xi_T \in \mathcal{L}^\infty(\mathcal{F}_T)$ be a risk position. Then, for $t \in [0,T]$, the forward $\rho_t(\cdot)$ and classical $\rho_{t,T}(\cdot)$ entropic risk measures satisfy*

$$\rho_t(\xi_T) = \rho_{t,T}(\xi_T - \frac{Y_T - \lambda T}{\gamma}) - \rho_{t,T}(-\frac{Y_T - \lambda T}{\gamma}), \tag{50}$$

where (Y_T, λ) is the unique Markovian solution to the ergodic BSDE (14).

Proof. Let $\bar{\xi}_T := \xi_T - \frac{Y_T - \lambda T}{\gamma}$. Using arguments similar to the ones used in section 3 of [20], we deduce that

$$ess \sup_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left[U_T(x + \rho_{t,T}(\bar{\xi}_T) + \int_t^T \pi_s^{tr} \left(\theta(V_s)ds + dW_s\right) + \bar{\xi}_T) \middle| \mathcal{F}_t \right]$$

$$= -e^{-\gamma x - \gamma \rho_{t,T}(\bar{\xi}_T) + \gamma P_t^{T,\bar{\xi}_T}},$$

where $P^{T,\bar{\xi}_T}$ solves the quadratic BSDE

$$P_t^{T,\bar{\xi}_T} = -\bar{\xi}_T + \int_t^T \frac{1}{\gamma} F\left(V_s, \gamma Q_s^{T,\bar{\xi}_T}\right) ds - \int_t^T \left(Q_s^{T,\bar{\xi}_T}\right)^{tr} dW_s, \tag{51}$$

with $F(\cdot,\cdot)$ as in (13).

The above BSDE (51) admits a unique Markovian solution $(\hat{Y}_t^T, \hat{Z}_t^T) = (\hat{y}^T(V_t,t), \hat{z}^T(V_t,t))$ as stated in Lemma 8. Therefore, we have that $P_t^{T,\bar{\xi}_T} = \hat{Y}_t^T$, and, in turn,

$$\rho_{t,T}(\bar{\xi}_T) = P_t^{T,\bar{\xi}_T} - P_t^{T,0}. \tag{52}$$

By Theorem 6, we then have that $\rho_t(\xi_T) = Y_t^T = \hat{Y}_t^T - \frac{Y_t-\lambda t}{\gamma}$, and in turn, that

$$\rho_t(\xi_T) = P_t^{T,\bar{\xi}_T} - \frac{Y_t - \lambda t}{\gamma}. \tag{53}$$

Taking $\xi_T \equiv 0$, yields $\rho_t(0) = P_t^{T,-\frac{Y_T-\lambda T}{\gamma}} - \frac{Y_t-\lambda t}{\gamma}$, and since $\rho_t(0) = 0$, we obtain that

$$P_t^{T,-\frac{Y_T-\lambda T}{\gamma}} = \frac{Y_t - \lambda t}{\gamma}. \tag{54}$$

Therefore, (53) and (54) yield

$$\rho_t(\xi_T) = P_t^{T,\bar{\xi}_T} - P_t^{T,-\frac{Y_T-\lambda T}{\gamma}}$$

$$= P_t^{T,\bar{\xi}_T} - P_t^{T,0} - (P_t^{T,-\frac{Y_T-\lambda T}{\gamma}} - P_t^{T,0})$$

$$= \rho_{t,T}(\bar{\xi}_T) - \rho_{t,T}(-\frac{Y_T - \lambda T}{\gamma}),$$

where we used (52) in the last equality. ■

6 An example

We provide an example in which we derive explicit formulae for both the forward and classical entropic risk measures. We also provide numerical results for their long-maturity limits.

To this end, we consider a market with a single stock whose dynamics depends on a single stochastic factor driven by a 2-dimensional Brownian motion, namely,

$$dS_t = b(V_t)\,S_t dt + \sigma(V_t)\,S_t dW_t^1,$$

$$dV_t = \eta(V_t)\,dt + \kappa_1 dW_t^1 + \kappa_2 dW_t^2,$$

for positive constants κ_1, κ_2.

We assume that $|\kappa_1|^2 + |\kappa_2|^2 = 1$, the functions $b(\cdot)$ and $\sigma(\cdot)$ are uniformly bounded, and that $\eta(\cdot)$ satisfies the dissipative condition in Assumption 2. We also choose $\Pi = \mathbb{R} \times \{0\}$, so that $\pi_{2,t} \equiv 0$. Then, the wealth equation (6) becomes $dX_t^{\pi_1} = \pi_{1,t}\left(\theta(V_t)dt + dW_t^1\right)$, where $\theta(V_t) = \frac{b(V_t)}{\sigma(V_t)}$.

In turn, for $z = (z_1, z_2)^{tr}$, the drivers in (13) and (23) become

$$F\left(v, (z_1, z_2)^{tr}\right) = -\frac{1}{2}|z_1 + \theta(v)|^2 + \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2|^2$$

$$= -\theta(v) z_1 - \frac{1}{2}|\theta(v)|^2 + \frac{1}{2}|z_2|^2,$$

and, for $\hat{z} = (\hat{z}_1, \hat{z}_2)^{tr}$,

$$G\left(v, (z_1, z_2)^{tr}, (\hat{z}_1, \hat{z}_2)^{tr}\right) = \frac{1}{\gamma}\left(F(v, (z_1 + \gamma\hat{z}_1, z_2 + \gamma\hat{z}_2)^{tr}) - F(v, (z_1, z_2)^{tr})\right)$$

$$= -\theta(v)\hat{z}_1 + z_2\hat{z}_2 + \frac{\gamma}{2}|\hat{z}_2|^2.$$

Let $Z_{1,t}^T := \kappa_1 Z_t^T$ and $Z_{2,t}^T =: \kappa_2 Z_t^T$ for some process Z_t^T to be determined. Then, equation (22) becomes

$$dY_t^T = -\left(\left(-\kappa_1\theta(V_t) + \kappa_2 Z_{2,t}\right) Z_t^T + \frac{\gamma|\kappa_2|^2}{2}|Z_t^T|^2\right) dt + Z_t^T\left(\kappa_1 dW_t^1 + \kappa_2 dW_t^2\right),$$

with $Y_T^T = g(V_T)$. Let $\tilde{Y}_t^T := e^{\gamma|\kappa_2|^2 Y_t^T}$ and $\tilde{Z}_t^T := \gamma|\kappa_2|^2\tilde{Y}_t^T Z_t^T$. In turn,

$$d\tilde{Y}_t^T = \tilde{Z}_t^T\left(\left(\kappa_1\theta(V_t) - \kappa_2 Z_{2,t}\right) dt + \kappa_1 dW_t^1 + \kappa_2 dW_t^2\right),$$

with $\tilde{Y}_T^T = e^{\gamma|\kappa_2|^2 g(V_T)}$. Since $\theta(\cdot)$ and Z_2 are uniformly bounded, the process

$$B_t := \int_0^t \left(\kappa_1\theta(V_s) - \kappa_2 Z_{2,s}\right) ds + \kappa_1 dW_s^1 + \kappa_2 dW_s^2,$$

$t \geq 0$, is a Brownian motion under some measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, with

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := \mathcal{E}\left(-\int_0^{\cdot} \left(\kappa_1\theta(V_s) - \kappa_2 Z_{2,s}\right)\left(\kappa_1 dW_s^1 + \kappa_2 dW_s^2\right)\right)_t. \tag{55}$$

Hence, $d\tilde{Y}_t^T = \tilde{Z}_t^T d\tilde{W}_t^T$, and, thus, $\tilde{Y}_t^T = E_{\mathbb{Q}}\left[e^{\gamma|\kappa_2|^2 g(V_T)}|\mathcal{F}_t\right]$.

In turn, the forward entropic risk measure has the closed-form representation

$$\rho_t(\xi_T) = Y_t^T = \frac{1}{\gamma|\kappa_2|^2}\ln E_{\mathbb{Q}}\left[e^{\gamma|\kappa_2|^2 g(V_T)}|\mathcal{F}_t\right]. \tag{56}$$

Similarly, for the classical entropic risk measure, we have the representation $\rho_{t,T}(\xi_T) = P_t^{T,\xi_T} - P_t^{T,0}$, with P_t^{T,ξ_T} being the unique solution to (51) with $\bar{\xi}_T$ replaced by ξ_T. Direct calculations then show that

$$\rho_{t,T}(\xi_T) = P_t^{T,\xi_T} - P_t^{T,0}$$

$$= g(V_T) + \int_t^T G\left(V_s, \gamma Q_s^{T,0}, Q_s^{T,\xi_T} - Q_s^{T,0}\right) ds - \int_t^T \left(Q_s^{T,\xi_T} - Q_s^{T,0}\right)^{tr} dW_s.$$

Further calculations yield the closed-form representation

$$\rho_{t,T}(\xi_T) = P_t^{T,\xi_T} - P_t^{T,0} = \frac{1}{\gamma|\kappa_2|^2}\ln E_{\mathbb{Q}^T}\left[e^{\gamma|\kappa_2|^2 g(V_T)}|\mathcal{F}_t\right], \tag{57}$$

where the measure $\mathbb{Q}^T$, defined on $\mathcal{F}_T$, is equivalent to $\mathbb{P}$ and satisfying

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\int_0^{\cdot}\left(\kappa_1\theta\left(V_s\right) - \kappa_2\gamma Q_{2,s}^{T,0}\right)\left(\kappa_1 dW_s^1 + \kappa_2 dW_s^2\right)\right)_t. \tag{58}$$

Note that in this single stock/single factor example, the only difference between the forward and classical entropic risk measures is their respective measures $\mathbb{Q}$ and $\mathbb{Q}^T$ (cf. (55) and (58)).

In the forward case, the pricing measure $\mathbb{Q}$ is determined by the component Z_2, appearing in the ergodic BSDE representation of the forward performance process (17). It is naturally *independent* of the maturity T. In the classical case, however, the pricing measure $\mathbb{Q}^T$ is determined by the component $Q_2^{T,0}$, coming from the exponential utility maximization (49) with zero risk position (cf. (51)), which depends critically on the maturity T. For $t \in [0,T]$, the two measures are related as

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \left.\frac{d\mathbb{Q}^T}{d\mathbb{P}}\right|_{\mathcal{F}_t}\left(\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}\right)^{-1}$$

$$= e^{\int_0^t \kappa_1\theta(V_s)\kappa_2(\gamma Q_{2,s}^{T,0} - Z_{2,s})ds}\mathcal{E}\left(\int_0^{\cdot}\kappa_2\left(\gamma Q_{2,s}^{T,0} - Z_{2,s}\right)\left(\kappa_1 dW_s^1 + \kappa_2 dW_s^2\right)\right)_t.$$

Finally, we conclude with numerical results for $\rho_0(\xi_T)$ and $\rho_{0,T}(\xi_T)$, taking T as large as possible, with $\eta(v) = -\alpha v$, $\theta(v) = (K_2 - |v|)_+$ and $g(v) = (K_1 - |v|)_+$ for two positive constants K_1, K_2. The graphs in Figures 1 and 2 confirm the large maturity behavior of both the forward and classical entropic risk measures. However, it is not clear yet about the relationship between such two limiting constants.

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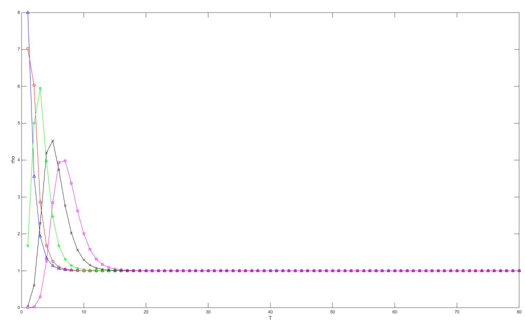
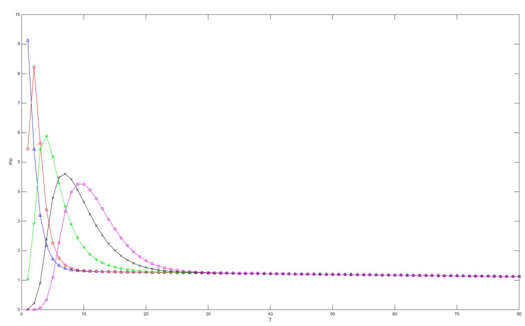
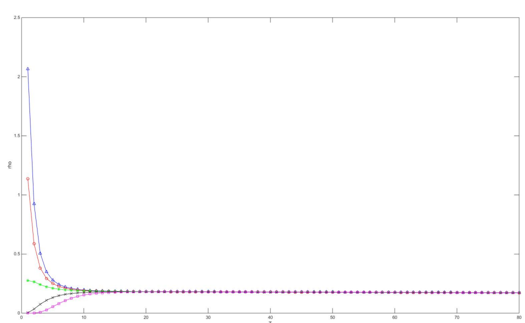


(a) $\kappa_1 = 0.9$, $\kappa_2 = 0.1$

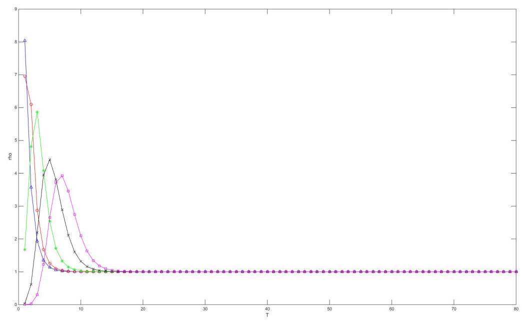


(b) $\kappa_1 = 0.5$, $\kappa_2 = 0.5$

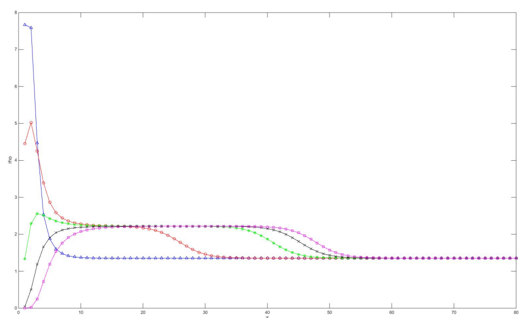


(c) $\kappa_1 = 0.0$, $\kappa_2 = 1.0$

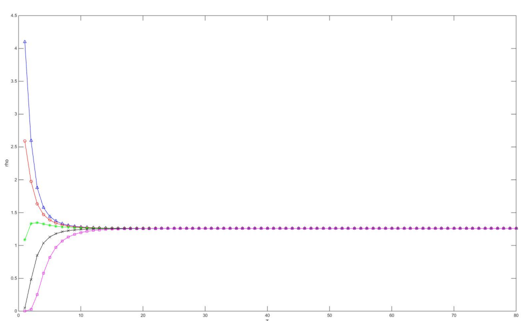
Figure 1: Forward entropic risk measure against the maturity T with $\gamma = 1$, $\alpha = 0.1$, $K_1 = K_2 = 10$; blue upward-pointing triangle for $v_0 = 5$, red circle for $v_0 = 7.5$, green asterisk for $v_0 = 10$, black cross for $v_0 = 12.5$ and magenta square $v_0 = 15$.



(a) $\kappa_1 = 0.9$, $\kappa_2 = 0.1$



(b) $\kappa_1 = 0.5$, $\kappa_2 = 0.5$



(c) $\kappa_1 = 0.0$, $\kappa_2 = 1.0$

Figure 2: Classical entropic risk measure against the maturity T with $\gamma = 1$, $\alpha = 0.1$, $K_1 = K_2 = 10$; blue upward-pointing triangle for $v_0 = 5$, red circle for $v_0 = 7.5$, green asterisk for $v_0 = 10$, black cross for $v_0 = 12.5$ and magenta square $v_0 = 15$.