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PREDICTABLE FORWARD PERFORMANCE PROCESSES: THE BINOMIAL CASE*

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Abstract. We introduce a new class of forward performance processes that are endogenous and predictable with regard to an underlying market information set and, furthermore, are updated at discrete times. We analyze in detail a binomial model whose parameters are random and updated dynamically as the market evolves. We show that the key step in the construction of the associated predictable forward performance process is to solve a single-period *inverse* investment problem, namely, to determine, period-by-period and conditionally on the current market information, the end-time utility function from a given initial-time value function. We reduce this inverse problem to solving a functional equation and establish conditions for the existence and uniqueness of its solutions in the class of inverse marginal functions.

Key words. portfolio selection, forward performance processes, binomial model, inverse investment problem, functional equation, predictability

AMS subject classifications. 91G10, 91G80, 60H30

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1. Introduction.

The classical portfolio selection paradigm is based on three fundamental ingredients: a given investment horizon, $[0, T]$; a performance function (such as a utility or a risk-return trade-off), $U_T(\cdot)$, applied at the *end* of the horizon; and a market model which yields the random investment opportunities available over $[0, T)$. This triplet is exogenously and entirely specified at initial time $t = 0$.

Once these ingredients are chosen, one then solves for the optimal strategy $\pi^*(\cdot)$ and derives the value function $U_0(\cdot)$ at $t = 0$ as the expectation of the terminal utility of optimal wealth. The value function thus stipulates the best possible performance value achievable from each and every amount of initial wealth, and hence it can be in turn considered as a performance criterion at $t = 0$ that is *consistent* with the terminal performance criterion $U_T(\cdot)$. Here, $U_T(\cdot)$ is exogenous, and $\pi^*(\cdot)$ and $U_0(\cdot)$ are endogenous. The model therefore entails a *backward* approach in time, from $U_T(\cdot)$ to $U_0(\cdot)$. This is also in accordance with the celebrated dynamic programming principle (DPP), otherwise known as Bellman's principle of optimality.

Despite its classical mathematical foundations and theoretical appeal, this approach nonetheless has several shortcomings. First, it relies heavily on the model selection for the *entire* investment horizon, which is not practical, especially if the horizon is long. The second difficulty is the precommitment, at the initial time, to a terminal utility. Indeed, it is clearly difficult to assess and specify the performance function when the investment horizon is sufficiently long. Moreover, a performance

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criterion naturally depends on time and state (either state of nature or state of the agent's circumstances). It is more plausible that one knows the utility or the resulting preferred allocations for now or the immediate future, and then preserves them under certain consistency criteria (see, for example, the old note of Black [1]). Third, it is very seldom the case that an optimal investment problem "terminates" at a single horizon T or that T is a priori known when the investment activity is first set.

The above considerations have led to the development of the so-called *forward performance measurement*, initially proposed by [4] and later extended by the same authors in a series of papers (see [5, 7, 6, 8]) and by others (see, for example, [2, 9]) in continuous-time market settings. The main idea of the forward approach is that instead of fixing, as in the classical setting, an investment horizon, a market model, and a terminal utility, one starts with an initial performance measurement and updates it *forward in time* as the market and other underlying stochastic factors evolve. The evolution of the forward process is dictated by a forward-in-time version of the DPP, and thus it ensures time-consistency across all different times.

Most of the existing results on forward performance measurement have so far focused exclusively on continuous-time, Itô-diffusion settings, in which both trading and performance valuation are carried out continuously in time. It was shown in [7] that the forward process is associated with an ill-posed infinite-dimensional stochastic partial differential equation (SPDE) in the same way that the classical value function satisfies (in Markovian models) the finite-dimensional Hamilton–Jacobi–Bellman (HJB) equation. This performance SPDE has been subsequently studied in [2, 10, 9] and, more recently, in [12] for asset price factors evolving at different time scales. Despite the technical challenges that this forward SPDE presents (ill-posedness, high or infinite dimensionality, degeneracies, and volatility specification), the continuous-time cases are tractable because stochastic calculus can be employed and infinitesimal arguments can be in turn developed.

However, the continuous-time setting has a major drawback in that it is hard to see how exactly the performance criterion evolves from one instant to the next. This evolution is lost at the infinitesimal level and hidden behind the (generally intractable) stochastic PDE.

The aim of this paper is to introduce and study forward investment performance processes that are *discrete* in time, while trading can be either discrete or continuous in time. We will develop an iterative mechanism through which an investor updates/predicts her performance criterion at the next investment period, based on both her current performance and her assessment of the upcoming market dynamics in the next period. This predictability will be present in an explicit and transparent manner.

In addition to the conceptual motivation described above, there are also practical considerations in studying the discrete-time predictable forward performance. Indeed, in investment practice, trading occurs at discrete times and not continuously. More importantly, typically performance criteria are directly or indirectly determined by individuals, such as higher-level managers or clients, and not by the portfolio manager. These "performance evaluators" use information sets that are different, both in terms of content and updating frequency, from the ones used by the portfolio manager. Moreover, even if trading can occur at extremely high frequencies (hence almost close to continuous trading), performance assessment/update takes place at a much slower pace; e.g., a senior manager will not keep track of the performance of a portfolio or update the performance criterion as frequently as the subordinate portfolio manager in charge of that portfolio.

In this paper, we will consider a (possibly indefinite) series of time points, $0 = t_0$, $t_1, \ldots, t_n, \ldots$, at which the performance measurement is evaluated and updated. The (short) period between any given two neighboring points will be called an *evaluation period*. We define our forward performance processes in a completely analogous way to their continuous-time counterparts. However, we choose to work with processes that are *predictable* with regard to the information at the most recent evaluation time. We elaborate on this requirement later on.

To highlight the key ideas of predictable forward performance processes, we start our analysis with a simple, yet still rich enough, setting. The market consists of two securities, a riskless asset and a stock whose price evolves according to a binomial model at times $0 = t_0, t_1, \ldots, t_n, \ldots$, at which the forward performance evaluation also occurs. The market model is more general than the standard binomial tree, in that the asset returns and their probabilities are estimated/determined only one period ahead. Such a setting allows for *"real-time"* dynamic updating of the underlying parameters, as the market evolves from one period to the next.

The definition of a discrete-time predictable forward performance process (see Definition 2.1) dictates that in each evaluation period $[t_n, t_{n+1})$, the initial performance function $U_n(\cdot)$ is nothing other than the value function of an expected utility maximization problem in this period, with $U_{n+1}(\cdot)$ being the terminal utility function. Therefore, in generating a predictable forward performance process, we need to solve, in each period, an investment problem where the value function is given and the terminal utility function is to be found. This problem, which we term a *single-period inverse investment* problem, then needs to be solved sequentially "period-by-period," conditionally on the dynamically updated information at the beginning of this period. It turns out that the key to solving this problem is a linear functional equation, which relates the inverse marginal processes at the beginning and the end of each evaluation period. We analyze this equation in detail and establish conditions for existence and uniqueness of the solutions in the class of inverse marginal functions.

Once such a single-period inverse investment is solved, then, starting from $[0, t_1)$ and proceeding iteratively *forward in time*, a predictable performance process is constructed together with the optimal allocations and their wealth processes.[1]

The paper is structured as follows. In section 2, we introduce the notion of predictable forward performance processes in a general market setting. We then formulate a binomial model with random, dynamically updated parameters in section 3. In section 4, we apply the definition of predictable forward performance processes to the binomial model and show that their construction reduces to solving an inverse investment problem. In section 5, this inverse problem is shown to be equivalent to solving a functional equation. We derive sufficient existence and uniqueness conditions as well as the explicit solution to the functional equation in section 6. Finally, we present the general construction algorithm in section 7 and conclude in section 8. Proofs of the main results are relegated to appendices.

2. Predictable forward performance processes: A general definition.

In this section, we introduce the concept of discrete-time predictable forward performance processes in a general market model. Starting with the next section, we will restrict the market setting to a binomial model with random, dynamically updated

[1] In this paper we assume that both the updating and trading take place at the same time. As discussed above, this does not have to be the case. However, we choose to study this parsimonious model in order to highlight the significance of updating the performance measurement in discrete times, without getting into too much technicality.

parameters and provide a detailed discussion on the existence and construction of such performance processes.

The investment paradigm is cast in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ augmented with a filtration $(\mathcal{F}_t)$, $t \geq 0$. We denote by $\mathcal{X}(t,x)$ the set of all admissible wealth processes X_s, $s \geq t$, starting with $X_t = x$ and such that X_s is $\mathcal{F}_s$-measurable. The term "admissible" is, for now, generic and will be specified once a specific market model is introduced in what follows.

We call a function $U : \mathbb{R}^+ \to \mathbb{R}^+$ a *utility (or performance) function* if $U \in C^2(\mathbb{R}^+)$, $U' > 0$, $U'' < 0$, and U satisfies the Inada conditions $\lim_{x\to 0^+} U'(x) = \infty$ and $\lim_{x\to\infty} U'(x) = 0$.

For any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, the set of *$\mathcal{G}$-measurable utility (or performance) functions* is defined as

$$\mathcal{U}(\mathcal{G}) = \big\{ U : \mathbb{R}^+ \times \Omega \to \mathbb{R} \,\big|\, U(x,\cdot) \text{ is } \mathcal{G}\text{ -measurable for each } x \in \mathbb{R}^+, \\ \text{and } U(\cdot,\omega) \text{ is a utility function a.s.} \big\} .$$

In other words, the elements of $\mathcal{U}(\mathcal{G})$ are entirely known (predicted) based on $\mathcal{G}$, as they are predictable with regard to the information contained in $\mathcal{G}$. Alternatively, we may think of $U \in \mathcal{U}(\mathcal{G})$ as a deterministic utility function, given the information in $\mathcal{G}$.

Next, we define the discrete predictable forward performance processes. To ease the notation, we skip the ω-argument throughout.

DEFINITION 2.1. *Let discrete time points $0 = t_0 < t_1 < \cdots < t_n < \cdots$ be given. A family of random functions $\{U_0, U_1, U_2, \dots\}$ is a predictable forward performance process with respect to $(\mathcal{F}_t)$ if, for $X_n = X_{t_n}$ and $\mathcal{F}_n = \mathcal{F}_{t_n}$, $n = 0, 1, 2, \dots$, the following conditions hold:*

(i) *U_0 is a deterministic utility function and $U_n \in \mathcal{U}(\mathcal{F}_{n-1})$.*

(ii) *For any initial wealth $x > 0$ and any admissible wealth process $X = \{X_n\}_{n=0}^\infty \in \mathcal{X}(0,x)$,*

$$U_{n-1}(X_{n-1}) \geq E_{\mathbb{P}}\left[U_n(X_n) \middle| \mathcal{F}_{n-1}\right].$$

(iii) *For any initial wealth $x > 0$, there exists an admissible wealth process $X^* = \{X_n^*\}_{n=0}^\infty \in \mathcal{X}(0,x)$ such that*

$$U_{n-1}\left(X_{n-1}^*\right) = E_{\mathbb{P}}\left[U_n\left(X_n^*\right) \middle| \mathcal{F}_{n-1}\right].$$

This definition is analogous to its continuous-time counterpart (see [5]), except for condition (i). This condition is superfluous in a continuous-time model but fundamental in a discrete-time one. It explicitly requires that the performance function at the *next* upcoming assessment time is *entirely determined* from the information up to the *present* time (hence the name "predictable forward").

On the other hand, as in the continuous-time case, properties (ii)–(iii) draw from Bellman's principle of optimality, which stipulates that the processes $U_n(X_n)$ and $U_n(X_n^*)$, $n = 0, 1, \dots$, are, respectively, a supermartingale and a martingale with respect to the filtration $(\mathcal{F}_n)$. Since the Bellman principle underlines time-consistency, properties (ii)–(iii) directly ensure that the investment problem is time-consistent under the predictable forward performance criterion.

Hence, the above performance measurement is essentially *endogenized* by the time-consistency requirements (ii)–(iii).[2]

Definition 2.1 already suggests a general scheme for constructing predictable forward performance functions in discrete times. Indeed, starting from an initial datum U_0, given at time $t_0 = 0$, the entire family $U_1, \ldots, U_n, \ldots$ can be obtained by determining U_n from U_{n-1} iteratively, $n = 1, 2, \ldots$, in the way described below.

Properties (ii)–(iii) dictate that, for each trading period $[t_{n-1}, t_n]$, we have

$$U_{n-1}\left(X_{n-1}^*\right) = \operatorname*{ess\,sup}_{X_n \in \mathcal{X}(t_{n-1}, X_{n-1}^*)} E_{\mathbb{P}}\left[U_n(X_n)\middle|\,\mathcal{F}_{n-1}\right]. \tag{2.1}$$

At instant t_{n-1}, since $\mathcal{F}_{n-1}$ is realized, the random functions U_{n-1} and U_n are both deterministic and so is X_{n-1}^*. This in turn suggests that we should consider the following "*single-period*" investment problem (conditional on $\mathcal{F}_{n-1}$):

$$U_{n-1}(x) = \operatorname*{ess\,sup}_{X_n \in \mathcal{X}_{n-1,n}(x)} E_{\mathbb{P}}\left[U_n(X_n)|\mathcal{F}_{n-1}\right] \tag{2.2}$$

for $x > 0$, where, with a slight abuse of notation, we use $\mathcal{X}_{n-1,n}(x)$ to denote the set of admissible wealths at t_n starting at t_{n-1} with wealth x.

Therefore, if we are able to determine, for each $n = 1, 2, \ldots$, a performance function $U_n \in \mathcal{U}\left(\mathcal{F}_{n-1}\right)$, such that the pair (U_{n-1}, U_n) satisfies (2.2), then we will have an iterative scheme to construct the *entire* predictable forward performance process, starting from U_0.

One readily recognizes that (2.2) would be the classical expected utility problem if the objective were to derive U_{n-1} from U_n, with U_n being a deterministic utility function. Therefore, what we consider now is an *inverse* investment problem in that we are given its initial value function, and we seek a terminal utility that is consistent with the latter, with both of these functions being deterministic (conditionally on $\mathcal{F}_{n-1}$).

We make the following very important observation. Definition 2.1 of the predictable forward criterion might at first indicate that we need to choose the full model at $t_0 = 0$, in that we need to completely specify both the levels of the stock return process and the related probabilities for *all* future times $t_1, t_2, \ldots$. As mentioned earlier, this is a very stringent requirement in the traditional framework. However, this is *not* the case in the forward setting.

Indeed, as the analysis will show in the binomial model we analyze herein, in order to construct the predictable forward criterion and the associated optimal portfolios and wealths, we *only* need to know at the beginning of each period, say $[t_{n-1}, t_n)$, the transition probabilities, p_n, and the values u_n, d_n of the return R_n. In other words, we only need to specify at t_{n-1} the *single-step* model input (p_n, u_n, d_n). This triplet is thus $\mathcal{F}_{n-1}$- measurable, and, as such, it captures accurately and in "real-time" the evolution of the market in $(0, t_{n-1}]$. There is no need to specify at t_{n-1} any model input beyond (p_n, u_n, d_n).

We also remark that, herein, we are not concerned with the specific mechanism that yields the "real-time" updated model input (e.g., (p_n, u_n, d_n) for the binomial setting) at $\mathcal{F}_{t-1}$. It may be the outcome of a dynamic sequential learning procedure,

[2]Note that the predictability of risk preferences is implicitly present in the *classical* expected utility in finite horizon settings, say $[0, T]$, in which trading is continuous, and a deterministic utility for a single horizon T is prechosen at initial time $t_0 = 0$, and it is thus $\mathcal{F}_0$-measurable. A fundamental difference, however, is that the terminal utility function in the classical theory is exogenous instead of endogenous.

or it may be provided exogenously from a specialist. The crucial point is that there is no requirement that it be a priori modeled for the entire optimization period.

To the best of our knowledge, such inverse discrete-time problems have not been considered in the literature. In this paper we start with the binomial case, in which the parameters—both the transition probabilities and price levels—are not known a priori but are updated period-by-period as the market moves. As we will see, while the binomial case is one of the simplest discrete-time market models, its analysis is sufficiently rich, and its results reveal the key economic insights regarding the predictable forward performance criteria.

3. A binomial market model with random, dynamically updated parameters. We consider a market with two traded assets, a riskless bond and a stock. The bond is taken to be the numeraire and assumed, without loss of generality, to offer zero interest rate.[3] The stock price at times $t_0, t_1, \dots$ evolves according to a binomial model that we now specify.

Let R_n be the total return of the stock over period $[t_{n-1}, t_n)$. Here, R_n is a random variable with two values $u_n > d_n$. We assume that R_n, u_n, and d_n, $n = 1, 2, \dots$, are all random variables in a measurable space $(\Omega, \mathcal{F})$ augmented with a filtration $(\mathcal{F}_n)$, $n = 1, 2, \dots$, with $\mathcal{F}_n$ representing the information available at t_n. Moreover, we assume that R_n is $\mathcal{F}_n$-measurable and that its values, u_n and d_n, are $\mathcal{F}_{n-1}$-measurable. In other words, the high and low return levels for each investment period are known at the beginning of this period, while the realized return is known at the end.

The historical measure $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ and the following standard no-arbitrage conditions are satisfied. As mentioned earlier, the specific values of the transition probabilities, say $p_1, p_2, \dots$, are not a priori specified at $t_0 = 0$. Rather, it is assumed that they are provided at the beginning of the corresponding trading period; namely, in the trading period $[t_{n-1}, t_n)$, p_n is provided at t_{n-1} and as such it is $\mathcal{F}_{n-1}$-measurable. The only standing assumption (see (ii) below) is that these probabilities satisfy the natural no-arbitrage conditions.

Assumption 3.1. For all $n = 1, 2, \dots$,
(i) $0 < d_n < 1 < u_n$, $\mathbb{P}$-a.s.;
(ii) the transition probabilities p_n satisfy $0 < p_n < 1$.

The investor trades between the stock and the bond using self-financing strategies. She starts at $t_0 = 0$ with total wealth $x > 0$ and rebalances her portfolio at times t_n, $n = 1, 2, \dots$. At the beginning of each period, say $[t_n, t_{n+1})$, she chooses the amount π_{n+1} to be invested in the stock (and the rest in the bond) for this period. In turn, her wealth process, denoted by X_n^π, $n = 1, 2, \dots$, evolves according to the wealth equation

$$X_{n+1}^\pi = X_n^\pi + \pi_{n+1}(R_{n+1} - 1),$$

with $X_0 = x$.

The investor is allowed to short the stock, but her wealth can never become negative; thus, π_{n+1} must satisfy

$$-\frac{X_n^\pi}{u_{n+1} - 1} \le \pi_{n+1} \le \frac{X_n^\pi}{1 - d_{n+1}}; \quad n = 1, 2, \dots . \tag{3.1}$$

We call an investment strategy $\pi = \{\pi_n\}_{n=1}^\infty$ *admissible* if it is self-financing, π_n

[3]If the bond price follows a predictable stochastic process, the analysis herein is valid as long as one works appropriately in discounted units.

is $\mathcal{F}_{n-1}$-measurable, and (3.1) is satisfied $\mathbb{P}$-a.s. A wealth process $X = \{X_n^\pi\}_{n=0}^\infty$ is then admissible if the strategy π that generates it is admissible.

We recall that $\mathcal{X}(n,x)$ is the set of admissible wealth processes $\{X_m\}_{m=n}^\infty$, starting with $X_n = x$.

We also introduce the auxiliary "single-step" set of admissible portfolios π_{n+1}, chosen at t_n for the trading period $[t_n, t_{n+1})$ and assuming wealth x at t_n, by

$$\mathcal{A}_{n,n+1}(x) = \Big\{\pi_{n+1} : \pi_{n+1} \text{ is } \mathcal{F}_n\text{-measurable}, \\ -\frac{x}{u_{n+1}-1} \le \pi_{n+1} \le \frac{x}{1-d_{n+1}},\ x>0\Big\},$$

as well as the corresponding set of admissible wealth processes by

$$\mathcal{X}_{n,n+1}(x) = \left\{x + \pi_{n+1}R_{n+1} : \pi_{n+1} \in \mathcal{A}_{n,n+1}(x),\ x>0\right\}.$$

Remark 3.2. Our problem formulation and results can be readily generalized for multiasset, complete markets. In the interest of readability, however, we opt to keep the current single-asset model.

4. Problem statement and reduction to the single-period inverse investment problem. In this section, we consider predictable forward performance processes in the binomial model and show that their construction reduces to solving a series of single-period inverse investment problems.

The investor starts with an initial utility U_0 and updates her performance criteria at times $t_1, t_2, \dots,$ with the associated performance functions $U_1, U_2, \dots$ satisfying Definition 2.1.

We now present the procedure that yields the construction of a predictable forward performance process that starts from U_0 and determines U_n from U_{n-1}, iteratively for $n = 1, 2, \dots$.

At $t_0 = 0$, (2.1) becomes

(4.1)
$$U_0(x) = \operatorname*{ess\,sup}_{X_1 \in \mathcal{X}(0,x)} E_{\mathbb{P}}\left[U_1(X_1)\Big|\mathcal{F}_0\right] = \sup_{\pi_1 \in \mathcal{A}_{0,1}(x)} E_{\mathbb{P}}\left[U_1\Big(x + \pi_1(R_1 - 1)\Big)\right]; \quad x>0.$$

Since the market parameters (u_1, d_1, p_1) and the initial datum U_0 are known at t_0, finding a deterministic ($\mathcal{F}_0$-measurable) U_1 reduces to the single-period inverse investment problem discussed in section 2. Let us for the moment assume that we are able to solve this inverse problem to obtain U_1.

At $t = t_1$, the investor observes the realization of the stock return R_1 and estimates the parameters (u_2, d_2, p_2) for the second trading period $[t_1, t_2)$. Setting $n = 2$ in (2.1) then yields

$$U_1\left(X_1^*(x)\right) = \operatorname*{ess\,sup}_{X_2 \in \mathcal{X}(1, X_1^*(x))} E_{\mathbb{P}}\left[U_2(X_2)\middle|\,\mathcal{F}_1\right], \tag{4.2}$$

where $X_1^*(x)$ is the optimal wealth generated at t_1, starting at x at $t_0 = 0$, from the previous period.

It follows from the classical expected utility theory (see also Theorem 5.2 below) that $X_1^*(x) = I_1(\rho_1 U_0'(x))$, $x>0$, where $I_1 = (U_1')^{-1}$, and ρ_1 is the pricing kernel over the period $[0, t_1)$, given by

$$\rho_1 = \frac{1-d_1}{p_1(u_1-d_1)}\mathbf{1}_{\{R_1=u_1\}} + \frac{u_1-1}{(1-p_1)(u_1-d_1)}\mathbf{1}_{\{R_1=d_1\}}.$$

The mapping $x \to X_1^*(x)$ is strictly increasing for each $x > 0$ and of full range, since I_1 and U_0' are both strictly decreasing functions, $\rho_1 > 0$, and the Inada conditions yield $X_1^*(0) = 0$ and $X_1^*(\infty) = \infty$.

Since $X_1^*(x)$ is $\mathcal{F}_1$-measurable and the parameters (u_2, d_2, p_2), together with U_1, are all known at $t = t_1$, we deduce that (4.2) reduces, with a slight abuse of notation, to finding $U_2(\cdot) \in \mathcal{U}(\mathcal{F}_1)$ such that

$$U_1(x) = \operatorname*{ess\,sup}_{\pi_2 \in \mathcal{A}_{1,2}(x)} E_{\mathbb{P}}\left[U_2\left(x + \pi_2(R_2 - 1)\right) \middle| \mathcal{F}_1\right]; \quad x > 0,$$

with U_1 given. In other words, one needs to solve yet another single-period inverse investment problem that is mathematically identical to (4.1).

At $t = t_n$, in exactly the same manner as above, we have to solve

$$U_n(x) = \operatorname*{ess\,sup}_{\pi_{n+1} \in \mathcal{A}_{n,n+1}(x)} E_{\mathbb{P}}\left[U_{n+1}\left(x + \pi_{n+1}(R_{n+1} - 1)\right) \middle| \mathcal{F}_n\right]; \quad x > 0,$$

thereby deriving U_{n+1} from U_n, with $U_{n+1} \in \mathcal{U}(\mathcal{F}_{n+1})$ and with the parameters (u_n, d_n, p_n) known at t_n.

Thus, all the terms of a predictable forward performance process can be obtained, starting from any arbitrary initial wealth $x > 0$ and proceeding iteratively, solving a "period-by-period" inverse optimization problem. Moreover, as we show in the next section, we also concurrently derive the optimal portfolio and wealth processes.

To summarize, the crucial step in the entire predictable forward construction is to solve this *single-period inverse investment problem*. We do this in the next section.

5. The single-period inverse investment problem.

We focus on the analysis of the inverse investment problem (4.1). To ease the presentation, we introduce a simplified notation. We set $t_0 = 0, t_1 = 1$, and $R_1 = R$, taking values u and d, $u > 1$ and $0 < d < 1$, with probability $0 < p < 1$ and $1 - p$, respectively. We recall the risk neutral probabilities

$$q = \frac{1-d}{u-d} \quad \text{and} \quad 1 - q = \frac{u-1}{u-d}$$

and the pricing kernel

$$\rho_1 = \rho^u \mathbf{1}_{\{R=u\}} + \rho^d \mathbf{1}_{\{R=d\}} := \frac{q}{p}\mathbf{1}_{\{R=u\}} + \frac{1-q}{1-p}\mathbf{1}_{\{R=d\}}. \tag{5.1}$$

The investor starts with wealth $X_0 = x > 0$ and invests the amount π in the stock. Her wealth at $t = 1$ is then given by the random variable $X = x + \pi(R - 1)$. The no-bankruptcy constraint (3.1) becomes $\underline{\pi}(x) \le \pi \le \overline{\pi}(x)$, with

$$\underline{\pi}(x) = -\frac{x}{u-1} < 0 \quad \text{and} \quad \overline{\pi}(x) = \frac{x}{1-d} > 0.$$

We denote the set of admissible portfolios as

$$\mathcal{A}(x) = \{\pi \in \mathbb{R}, \text{ and } \underline{\pi}(x) \le \pi \le \overline{\pi}(x),\ x > 0\}.$$

Given an initial utility function U_0, we then seek a deterministic performance function U_1, such that

$$U_0(x) = \sup_{\pi \in \mathcal{A}(x)} E_{\mathbb{P}}\left[U_1\left(x + \pi(R - 1)\right)\right]; \quad x > 0. \tag{5.2}$$

Let $\mathcal{U}$ be the set of deterministic utility functions. We introduce the set of *inverse marginal functions* $\mathcal{I}$,

$$\mathcal{I} := \left\{ I \in C^1(\mathbb{R}^+) : I' < 0, \lim_{y\to\infty} I(y) = 0, \lim_{y\to 0^+} I(y) = \infty \right\}. \tag{5.3}$$

Note that if functions U and I satisfy $I = (U')^{-1}$, then U is a utility function if and only if I is an inverse marginal function.

Assuming for now that a utility function U_1 satisfying (5.2) exists, we consider the inverse marginal functions

$$I_0 = (U_0')^{-1} \quad \text{and} \quad I_1 = (U_1')^{-1}.$$

Our main goal in this section is to show that the inverse investment problem (5.2) reduces to a functional equation in terms of I_0 and I_1; see (5.4) below.

The following theorem is one of the main results of this paper, establishing a direct relationship between the inverse marginals at the beginning and the end of the trading period $[0, 1]$, when the corresponding utilities are related by (5.2).

THEOREM 5.1. *Let $U_0, U_1 \in \mathcal{U}$ satisfy the optimization problem* (5.2)*. Then, their inverse marginals I_0 and I_1 must satisfy the linear functional equation*

$$I_1(ay) + bI_1(y) = (1 + b)\, I_0(c\,y); \quad y > 0, \tag{5.4}$$

where

$$a = \frac{1-p}{p}\frac{q}{1-q}, \quad b = \frac{1-q}{q}, \quad \text{and} \quad c = \frac{1-p}{1-q}. \tag{5.5}$$

Proof. From standard arguments, we deduce that for all $x > 0$, there exists an optimizer $\pi^*(x)$ for (5.2) satisfying the first-order condition

$$p(u-1)U_1'(x + \pi^*(x)(u-1)) + (1-p)(d-1)U_1'(x + \pi^*(x)(d-1)) = 0. \tag{5.6}$$

Indeed, let $f(\pi) := \mathbb{E}\left[U_1\left(x + \pi(R-1)\right)\right]$. By concavity of $U_1(\cdot)$, one has

$$f''(\pi) = \mathbb{E}\left[(R-1)^2 U_1''\left(x + \pi(R-1)\right)\right] \le 0; \quad \underline{\pi}(x) < \pi < \overline{\pi}(x).$$

Furthermore,

$$f'(\underline{\pi}(x)) = p(u-1)U_1'(0) + (1-p)(d-1)U_1'\Big(x + \underline{\pi}(x)(d-1)\Big) = +\infty$$

and

$$f'(\overline{\pi}(x)) = p(u-1)U_1'\Big(x + \overline{\pi}(x)(u-1)\Big) + (1-p)(d-1)U_1'(0) = -\infty,$$

where we used the Inada condition $U_1'(0) = +\infty$ and the fact that $x + \overline{\pi}(x)(d-1) = x + \underline{\pi}(x)(u-1) = 0$ by the definitions of $\overline{\pi}(x)$ and $\underline{\pi}(x)$. Therefore, for any $x > 0$, there exists a unique $\pi^*(x) \in (\underline{\pi}(x), \overline{\pi}(x))$ such that $f'(\pi^*(x)) = 0$, and (5.6) follows.

On the other hand, we have from (5.2) that

$$U_0(x) = p\,U_1(x + \pi^*(x)(u-1)) + (1-p)\,U_1(x + \pi^*(x)(d-1)).$$

Differentiating the above equation yields

$$U_0'(x) = p\,U_1'(x+\pi^*(x)(u-1)) + (1-p)\,U_1'(x+\pi^*(x)(d-1)) + (\pi^*)'(x)\Big(p(u-1)U_1'\big(x+\pi^*(x)(u-1)\big) + (1-p)(d-1)U_1'\big(x+\pi^*(x)(d-1)\big)\Big),$$

and, using (5.6), one obtains

$$U_0'(x) = p\,U_1'(x+\pi^*(x)(u-1)) + (1-p)\,U_1'(x+\pi^*(x)(d-1)). \tag{5.7}$$

Solving the linear system (5.6)–(5.7) gives

$$U_1'(x+\pi^*(x)(u-1)) = \frac{(1-d)}{p(u-d)}U_0'(x)$$

and

$$U_1'(x+\pi^*(x)(d-1)) = \frac{(u-1)}{(1-p)(u-d)}U_0'(x).$$

Therefore, the optimal allocation function $\pi^*(x)$ satisfies

$$\begin{cases} x+\pi^*(x)(u-1) &= I_1\left(\frac{1-d}{p(u-d)}U_0'(x)\right), \\ x+\pi^*(x)(d-1) &= I_1\left(\frac{u-1}{(1-p)(u-d)}U_0'(x)\right), \end{cases} \tag{5.8}$$

from which we obtain the solution

$$\pi^*(x) = \frac{1}{u-d}\left(I_1\left(\frac{1-d}{p(u-d)}U_0'(x)\right) - I_1\left(\frac{u-1}{(1-p)(u-d)}U_0'(x)\right)\right);\quad x>0.$$

Substituting the above in either of the equations in (5.8) yields

$$\frac{1-d}{u-d}\,I_1\left(\frac{1-d}{p(u-d)}U_0'(x)\right) + \frac{u-1}{u-d}\,I_1\left(\frac{u-1}{(1-p)(u-d)}U_0'(x)\right) = x.$$

Changing variables $x = I_0\left(\frac{(1-p)(u-d)}{u-1}y\right)$, $y>0$, the above becomes

$$I_1\left(\frac{(1-p)(1-d)}{p(u-1)}\,y\right) + \frac{u-1}{1-d}\,I_1(y) = \frac{u-d}{1-d}\,I_0\left(\frac{(1-p)(u-d)}{u-1}y\right);\quad y>0.$$

Noting (5.5), we conclude. ∎

The next theorem shows how to recover U_1 from I_1 and derives the optimal portfolio $\pi^*(x)$ and its wealth $X^*(x)$.

THEOREM 5.2. *Let U_0 be a utility function, let I_0 be its inverse marginal, and let I_1 be an inverse marginal solving the functional equation* (5.4). *Let also ρ_1 be the pricing kernel given by* (5.1). *Then, the following statements hold:*

(i) *The function U_1 defined by*

$$U_1(x) := U_0(1) + E_{\mathbb{P}}\left[\int_{I_1(\rho_1 U_0'(1))}^{x} I_1^{-1}(\xi)d\xi\right];\quad x>0, \tag{5.9}$$

is a well-defined utility function.

(ii) *We have*

$$U_0(x) = \sup_{\pi \in \mathcal{A}(x)} E_{\mathbb{P}}\left[U_1\left(x + \pi(R-1)\right)\right]; \quad x > 0.$$

(iii) *The optimal wealth* $X_1^*(x)$ *and the associated optimal investment allocation* $\pi^*(x)$ *are given, respectively, by*

$$X_1^*(x) = I_1(\rho_1 U_0'(x)) = X^{*,u}(x)\mathbf{1}_{\{R=u\}} + X^{*,d}(x)\mathbf{1}_{\{R=d\}}$$

and

$$\pi^*(x) = \frac{X^{*,u}(x) - X^{*,d}(x)}{u-d},$$

with

$$X^{*,u} = I_1\left(\frac{q}{p}U_0'(x)\right) \quad \text{and} \quad X^{*,d} = I_1\left(\frac{1-q}{1-p}U_0'(x)\right).$$

Proof. See Appendix A. ∎

Remark 5.3. As shown in the proof of Theorem 5.2, we can replace (5.9) with

$$U_1(x) := U_0(c) + E_{\mathbb{P}}\left[\int_{I_1(\rho_1 U_0'(c))}^{x} I_1^{-1}(\xi)d\xi\right]; \quad x > 0,$$

for any arbitrary constant $c > 0$. The choice of c changes neither the value of $U_1(x)$ nor the optimal policies.

As the results of Theorem 5.2 indicate, the inverse investment problem (5.2) essentially reduces to solving the functional equation (5.4). We study this equation next.

6. A functional equation for inverse marginals.

In this section, we analyze the linear functional equation (5.4), with I_0 given and I_1 to be found, for positive constants a, b, c, given by (5.5). We provide conditions for the existence and uniqueness of its solutions and, in particular, solutions in the class of inverse marginal functions.

First, we note that the solution to (5.4) is known in the literature for $b < 0$ (e.g., [11]). Unfortunately, in our case $b = \frac{1-q}{q} > 0$, for which we are not aware of any results, to the best of our knowledge.

When $a = 1$, the unique solution is trivially $I_1(y) = I_0(y)$. This is economically intuitive. If $p = q$, then essentially there is no risk premium to exploit. As a result, when $r = 0$ as assumed herein, the pricing kernel becomes a constant, $\rho = 1$, and the optimal wealth reduces to $X^*(x) = x$. In turn, the value function (at $t = 0$) coincides with the terminal utility. So the forward performance remains constant, $U_0(x) = U_1(x)$, and thus their inverse marginals I_0 and I_1 coincide.[4] Indeed, there is no reason to modify the performance function in a market with no investment opportunities.

Henceforth we assume that $a \neq 1$. We start with an example showing that a general solution of (5.4) may not be unique, even if we restrict the solutions to inverse marginal functions.

[4]This is also in accordance with the time-monotone forward processes in the continuous-time setting. For example, in [6] it is shown that this forward performance is given by $U(x,t) = u(x, \int_0^t |\lambda_s|^2 ds)$, with $u(x,t)$ being a deterministic function, and the process λ is the market price of risk. If $\lambda \equiv 0$, then $U(x,t) = u(x,0) = U(x,0)$ for all $t \geq 0$.

Example 6.1. Let $I_0(y) = y^{\log_a b}$, $y > 0$, for constants $a, b > 0$ such that $\log_a b < 0$. It is easy to check that the function $I_1(y) = \delta y^{\log_a b}$, $y > 0$, with $\delta = \frac{(1+b)}{2b\, c^{-\log_a b}} > 0$, is a solution to (5.4).

However, this particular solution is not the only solution. Indeed, consider any differentiable antiperiodic function, say $\Theta(z) = -\Theta(z + \ln a)$, for which there exists a constant $M > 0$ such that

$$\sup_{z\in\mathbb{R}} \left(|\Theta(z)|, |\Theta'(z)|\right) < M < -\delta \frac{\log_a b}{1 - \log_a b} = -\frac{(1+b)\log_a b}{2b\, c^{-\log_a b}(1 - \log_a b)}.$$

For instance, $\Theta(x) = M \sin(\frac{x}{\ln a}\pi)$ is such a function. One can then directly check that the function

$$\tilde{I}_1(y) = y^{\log_a b}\left(\delta + \Theta(\ln y)\right); \quad y > 0,$$

is a solution.

As a matter of fact, both solutions I_1 and $\tilde{I}_1$ are inverse marginals. This is obvious for I_1. As for $\tilde{I}_1$, we have $\lim_{y\to\infty} \tilde{I}_1(y) = 0$ since $\log_a b < 0$. Moreover, it follows from the inequality $\tilde{I}_1(y) \geq y^{\log_a b}(\delta - M)$, $y > 0$, that $\lim_{y\to 0^+} \tilde{I}_1(y) = \infty$. Furthermore,

$$\begin{aligned}\tilde{I}_1'(y) &= y^{\log_a b - 1} \log_a b \left(\delta + \Theta(\ln y) + \frac{\Theta'(\ln y)}{\log_a b}\right)\\ &\leq y^{\log_a b - 1} \log_a b \left(\delta - \frac{M \log_a b - M}{\log_a b}\right) < 0; \quad y > 0.\end{aligned}$$

Thus, in general, there is no uniqueness even among inverse marginal functions.

The above example suggests that we need additional conditions to ensure uniqueness. To identify these conditions, we first note that (5.4) is a functional equation of the more general form

$$F\left(f(y)\right) = g(y)F(y) + h(y), \tag{6.1}$$

with f, g, and h given functions, and $y \in \mathcal{Y} \subseteq \mathbb{R}$ and F to be found. The equations of this type have been studied in the literature; see [3] and the references therein for a general exposition.

In general, such equations have many solutions. A trivial example is $F(y+1) = F(y)$, $y \in \mathbb{R}$, for which any periodic function with period 1 is a solution. Such nonuniqueness often renders the underlying equation inapplicable for concrete problems, where a single well-defined solution is usually needed. For the general equation (6.1), conditions for the uniqueness of solutions usually limit the set of solutions by imposing an additional assumption on $F(y_0)$, where y_0 is a fixed point for f: $f(y_0) = y_0$. In the example of the equation $F(y+1) = F(y)$, $y \in \mathbb{R}$, if we require a solution to be such that $\lim_{y\to\infty} F(y) = a \in \mathbb{R}$, then $F = a$ becomes the only possible solution. Note here that ∞ is actually a fixed point of the function $f(y) = y + 1$.

For (5.4), we have that $f(y) = ay$, $g(y) = -b$, and $h(y) = (1+b)\, G(cy)$. Therefore, uniqueness conditions should impose additional assumptions on F at $y_1 = 0$ and $y_2 = \infty$, which are the fixed points of $f(y) = ay$.

We start with the following auxiliary result in which we provide *general* uniqueness conditions for (5.4). Afterwards, we will strengthen the results for the family of inverse marginals.

LEMMA 6.2. *Let I_0 be given. Then, there exists at most one solution to* (5.4)*, say I, satisfying* $\lim_{y\to 0^+} y^{-\log_a b} I(y) = 0$. *Similarly, there exists at most one solution satisfying* $\lim_{y\to\infty} y^{-\log_a b} I(y) = 0$.

Proof. See Appendix B. ∎

We note that the function $\tilde{I}_1$ in Example 6.1 satisfies neither condition in Lemma 6.2, and thus uniqueness fails.

Next, we state the main result for this section, which provides sufficient conditions for existence and uniqueness of solutions to (5.4) that are inverse marginal functions.

THEOREM 6.3. *Let I_0 in (5.4) be an inverse marginal, i.e., $I_0 \in \mathcal{I}$ with $\mathcal{I}$ defined in (5.3). Define the functions*

$$\Phi_0(y) = I_0(a\,c\,y) - bI_0(c\,y) \quad \text{and} \quad \Psi_0(y) = y^{-\log_a b} I_0(c\,y); \;\; y > 0. \tag{6.2}$$

The following assertions hold:

(i) *If Φ_0 is strictly increasing and either $a > 1$ and $\lim_{y\to\infty} \Psi_0(y) = 0$ or $a < 1$ and $\lim_{y\to 0^+} \Psi_0(y) = 0$, then a solution of (5.4) is given by*

$$I_1(y) = \frac{1+b}{b} \sum_{m=0}^{\infty} (-1)^m\, b^{-m} I_0(\,a^m\, c\, y); \;\; y > 0. \tag{6.3}$$

(ii) *If Φ_0 is strictly decreasing and either $a > 1$ and $\lim_{y\to 0^+} \Psi_0(y) = 0$ or $a < 1$ and $\lim_{y\to\infty} \Psi_0(y) = 0$, then a solution of (5.4) is given by*

$$I_1(y) = (1+b) \sum_{m=0}^{\infty} (-1)^m b^m I_0(a^{-(m+1)} c\, y); \;\; y > 0. \tag{6.4}$$

(iii) *In parts (i) and (ii), the corresponding I_1 satisfies the uniqueness condition(s) of Lemma 6.2 and, moreover, $I_1 \in \mathcal{I}$; i.e., I_1 preserves the inverse marginal properties.*

(iv) *The function I_1 in parts (i) and (ii), respectively, is the only positive solution of (5.4). It is also the only inverse marginal function that solves (5.4).*

Proof. See Appendix C. ∎

Now, we apply the above results to the case when the initial utility is a power function. The following example provides results complementary to the ones in Example 6.1, where uniqueness is lacking since the conditions of Lemma 6.2 are not satisfied.

COROLLARY 6.4. *Let $U_0(x) = \left(1 - \frac{1}{\theta}\right)^{-1} x^{1-\frac{1}{\theta}}$, $x > 0$, and assume that $1 \neq \theta > 0$, $\theta \neq -\log_a b$, with $a, b, c > 0$ given by (5.5). Then, the following assertions hold:*

(i) *The unique inverse marginal function that satisfies the functional equation (5.4) with initial $I_0(y) = y^{-\theta}$ is given by*

$$I_1(y) = \delta y^{-\theta}; \;\; y > 0, \tag{6.5}$$

where $\delta = \frac{1+b}{c^\theta(a^{-\theta}+b)}$.

(ii) *The unique utility function U_1 that satisfies the inverse investment problem (5.2) is given by*

$$U_1(x) = \delta^{\frac{1}{\theta}} \left(1 - \frac{1}{\theta}\right)^{-1} x^{1-\frac{1}{\theta}} = \delta^{\frac{1}{\theta}} U_0\,(x)\,; \;\; x > 0.$$

(iii) *The corresponding optimal allocation is given by*

$$\pi^*(x) = \frac{\delta(p/q)^\theta - 1}{u-1}\, x; \;\; x > 0.$$

Therefore, if we start with an initial power utility U_0, then the forward utility at $t = 1$ is a multiple of the initial datum, with the constant given by $\delta^{\frac{1}{\theta}}$. Note that δ incorporates both the preference parameter θ and the market parameters a, b, and c at the *beginning* of the trading period $t = 0$. Proceeding iteratively, the utilities for all future periods remain power functions. In other words, in the binomial setting, the (predictable) power utility preferences are preserved throughout.

We conclude this section by summarizing the findings for existence and uniqueness of solutions. If (5.4) does not admit a solution, then it follows from Theorem 5.1 that there will be no utility function U_1 satisfying (5.2). Hence there will be no predictable forward performance process starting from the initial marginal utility function I_0. On the other hand, (5.4) may have more than one solution and, in particular, more than one solution that are inverse marginal functions. In this case, problem (5.2) has multiple solutions as well. An open question is which of these solutions can be chosen to be the "correct" forward utility. Lemma 6.2 suggests that uniqueness follows from imposing certain decay conditions for large or small wealth; this is in accordance with the well-known elasticity condition in the classical setting.

7. Construction of the predictable forward performance process. We are now ready to present the general algorithm for the construction of forward performance processes, as well as the associated optimal investment strategies and their wealth processes. We stress that one of the main strengths of our approach is that for every given trading period, say $[t_n, t_{n+1}]$, we do not have to update the model parameters $(u_{n+1}, d_{n+1}, p_{n+1})$ for this period until time t_n arrives. Thus, we take full advantage of the incoming information up to time t_n. This is in contrast with the classical setting where, as we mentioned earlier, these parameters have to be prespecified at initial time.

The algorithm is based on repeatedly applying, conditionally on the new "real-time" information, the following result on the single-period inverse investment problem (5.2).

THEOREM 7.1. *For the inverse investment problem* (5.2), *assume that the initial inverse marginal* $I_0 = (U_0')^{-1}$ *satisfies condition* (i) *(resp., condition* (ii)*) in Theorem* 6.3, *and define* I_1 *by* (6.3) *(resp.,* (6.4)*). Then, the unique solution to* (5.2) *is given by*

$$U_1(x) = U_0(1) + E_{\mathbb{P}}\left[\int_{I_1(\rho_1 U_0'(1))}^{x} I_1^{-1}(\xi)d\xi\right]; \quad x > 0,$$

where ρ_1 *is as in* (5.1). *Moreover, the optimal wealth* $X_1^*(x)$ *and the associated optimal investment allocation* $\pi^*(x)$ *are given, respectively, by*

$$X_1^*(x) = I_1(\rho_1 U_0'(x)) = X_1^{*,u}(x)\mathbf{1}_{\{R_1=u\}} + X_1^{*,d}(x)\mathbf{1}_{\{R_1=d\}}$$

and

$$\pi^*(x) = \frac{X_1^{*,u}(x) - X_1^{*,d}(x)}{u-d},$$

where

$$X_1^{*,u}(x) := I_1\left(\frac{q}{p}U_0'(x)\right) \qquad and \qquad X_1^{*,d}(x) := I_1\left(\frac{1-q}{1-p}U_0'(x)\right).$$

Proof. The results follow directly from Theorem 6.3 and Theorem 5.2. ∎

Given an initial performance function U_0 and initial wealth X_0, the following algorithm provides the predictable forward performance process $\{U_1, U_2, \dots\}$ along with the associated optimal portfolio process $\{\pi_1^*, \pi_2^*, \dots\}$ and the wealth process $\{X_1^*, X_2^*, \dots\}$ in the binomial market model.

- At $t = 0$: Assess the market parameters (u_1, d_1, p_1) for the first investment period, $[0, t_1)$. Compute

$$q_1 = \frac{1-d_1}{u_1-d_1}, \quad a_1 = \frac{q_1(1-p_1)}{p_1(1-q_1)}, \quad b_1 = \frac{1-q_1}{q_1}, \quad \text{and} \quad c_1 = \frac{1-p_1}{1-q_1},$$

and

$$\rho_1^u = \frac{q_1}{p_1} \qquad \text{and} \quad \rho_1^d = \frac{1-q_1}{1-p_1}.$$

Using (a_1, b_1, c_1), check the conditions in part (i) (resp., (ii)) of Theorem 6.3, and obtain the inverse marginal function I_1 from (6.3) (resp., (6.4)). Then, apply Theorem 7.1 to compute

$$U_1(x) = U_0(1) + p_1 \int_{I_1(\rho_1^u U_0'(1))}^{x} I_1^{-1}(\xi)d\xi + (1-p_1) \int_{I_1(\rho_1^d U_0'(1))}^{x} I_1^{-1}(\xi)d\xi; \;\; x > 0,$$

$$\pi_1^* = \frac{X_1^{*,u}(X_0) - X_1^{*,d}(X_0)}{u-d},$$

and

$$X_1^* = X_0 + \pi_1^* \left(R_1 - 1\right),$$

where

$$X_1^{*,u}(x) = I_1\left(\frac{q_1}{p_1} U_0'(x)\right) \qquad \text{and} \qquad X_1^{*,d}(x) = I_1\left(\frac{1-q_1}{1-p_1} U_0'(x)\right); \;\; x > 0.$$

- At $t = t_n$ $(n = 1, 2, \dots)$: We have already obtained $\{U_1, \dots, U_n; I_1, \dots, I_n\}$, $\{\pi_1^*, \dots, \pi_n^*\}$, and $\{X_1^*, \dots, X_n^*\}$.
  Estimate the market parameters $(u_{n+1}, d_{n+1}, p_{n+1})$ for the upcoming investment period $[t_n, t_{n+1})$. Let

$$q_{n+1} = \frac{1-d_{n+1}}{u_{n+1}-d_{n+1}}, \quad a_{n+1} = \frac{q_{n+1}(1-p_{n+1})}{p_{n+1}(1-q_{n+1})}, \quad b_{n+1} = \frac{1-q_{n+1}}{q_{n+1}},$$
$$c_{n+1} = \frac{1-p_{n+1}}{1-q_{n+1}}, \quad \rho_{n+1}^u = \frac{q_{n+1}}{p_{n+1}}, \quad \text{and} \quad \rho_{n+1}^d = \frac{1-q_{n+1}}{1-p_{n+1}}.$$

  Check the conditions in part (i) (resp., (ii)) in Theorem 6.3, using $(a_{n+1}, b_{n+1}, c_{n+1})$ (instead of (a, b, c)) and I_n instead of I_0, and obtain I_{n+1} from (6.3) (resp., (6.4)).[5]

[5]If both conditions in part (i) and (ii) do not hold, then the functional equation (5.4) may not have a solution, or the solution may not be unique. For the case of initial power utility $U_0(x) = \frac{x^{1-1/\theta}}{1-1/\theta}$, $\theta > 0$, Example 6.1 and Corollary 6.4 show that both conditions fail at t_n if and only if $\theta = -\log_a b > 0$, in which case the solution exists but is not unique. This case is pathological, but to solve it remains a technically interesting question.

Compute

$$U_{n+1}(x) = U_n(1) + p_{n+1} \int_{I_{n+1}(\rho^u_{n+1} U'_n(1))}^{x} I^{-1}_{n+1}(\xi) d\xi + (1 - p_{n+1}) \int_{I_{n+1}(\rho^d_{n+1} U'_n(1))}^{x} I^{-1}_{n+1}(\xi) d\xi; \quad x > 0, \tag{7.1}$$

$$\pi^*_{n+1} = \frac{X^{*,u}_{n+1}\left(X^*_n\right) - X^{*,d}_{n+1}\left(X^*_n\right)}{R^u_{n+1} - R^d_{n+1}},$$

and

$$X^*_{n+1} = X^*_n + \pi^*_{n+1}\left(R_{n+1} - 1\right) = X_0 + \sum_{i=1}^{n+1} \pi^*_i \left(R_i - 1\right),$$

where

$$X^{*,u}_{n+1}(x) = I_{n+1}\left(\frac{q_{n+1}}{p_{n+1}} U'_n(x)\right)$$

and

$$X^{*,d}_{n+1}(x) = I_{n+1}\left(\frac{1 - q_{n+1}}{1 - p_{n+1}} U'_n(x)\right); \quad x > 0.$$

In summary, starting with an initial datum U_0, we have constructed for (the end of) each trading period, say $(t_n, t_{n+1}]$, $n = 1, 2, \ldots$, a performance criterion U_{n+1} at t_{n+1} that is indeed $\mathcal{F}_n$-measurable. This measurability is inherited by the same measurability of the inverse marginal I_{n+1} that enters the lower part of the integration in (7.1). Moreover, as expected, the optimal wealth X^*_{n+1} is $\mathcal{F}_{n+1}$-measurable, given that the pricing kernel ρ_{n+1} is $\mathcal{F}_{n+1}$-measurable. The optimal portfolio π^*_{n+1} is $\mathcal{F}_n$-measurable, chosen at the beginning of the period $[t_n, t_{n+1})$.

8. Conclusions. We have introduced a discrete-time analogue of the continuous-time forward performance processes, focusing on the predictability of such criteria. Specifically, at the beginning of each evaluation period, the investor assesses the market parameters only for this period (during which trading may take place once or many times, in both discrete or continuous fashion). Then, the investor solves an inverse single-period inverse investment model which yields the utility at the end of the period, given the one at the beginning. The martingality and supermartingality requirements of the forward performance process ensure that this construction, "period-by-period forward in time" and adapted to the new market information, yields time-consistent policies.

We have implemented this new approach in a binomial model with random, dynamically updated parameters, including both the probabilities and the levels of the stock returns. We have then discussed in detail how the construction of predictable forward performance processes essentially reduces to a single-period inverse investment problem. We have in turn shown that the latter is equivalent to solving a functional equation involving the inverse marginal functions at the beginning and the end of the trading period, and we have established conditions for the existence and uniqueness of solutions in the class of inverse marginal functions.

Finally, we have provided an explicit algorithm that yields the forward performance processes as well as their optimal portfolio and associated optimal wealth processes.

There are a number of possible future research directions. First, one may depart from the binomial model to study general discrete-time models, while allowing for trading to be discrete or continuous. Such models are inherently incomplete, and additional difficulties are expected to arise with regard to the derivation of the functional equation for the inverse marginals, as well as the existence and uniqueness of its solutions among suitable classes of functions.

A second direction is to enrich the predictable framework by incorporating model ambiguity. This will allow for the specification of all possible market models only one evaluation period ahead, thus offering substantial flexibility to narrow down the most realistic models period-by-period as the market evolves.

From the theoretical point of view, an interesting question is to investigate whether discrete predictable forward performance processes converge to their continuous-time counterparts. While this is naturally and intuitively expected, conditions on the appropriate convergence scaling need to be imposed, which might be quite challenging due to the ill-posedness of the problem. Such results may also shed light on deeper questions on the construction of continuous-time forward performance criteria related to the appropriate choice of their volatility, finite-dimensional approximations, and Markovian or path-dependent cases, among others.

Appendix A. Proof of Theorem 5.2. We start with the following auxiliary result, showing that the expected utility problem (5.2) is equivalent to

$$U_0\left(I_0(y)\right) = E_{\mathbb{P}}\left(U_1\left(I_1(\rho_1\, y)\right)\right); \quad y > 0. \tag{A.1}$$

LEMMA A.1. *Suppose that $U_0, U_1 \in \mathcal{U}$, and let I_0 and I_1 be, respectively, their inverse marginals. Then,* (5.2) *holds if and only if* (A.1) *holds.*

Proof. We first show that (5.2) implies (A.1). Indeed, standard results in expected utility maximization yield that (5.2) implies

$$U_0(x) = E_{\mathbb{P}}\Big[U_1\Big(I_1\big(\rho_1 U_0'(x)\big)\Big)\Big]; \quad x > 0,$$

and (A.1) is then obtained by the change of variable $y = U_0'(x)$.

Next, we show that (A.1) yields (5.2). Define the value function $\tilde{U}$ by

$$\tilde{U}(x) := \sup_{\mathcal{A}(x)} E_{\mathbb{P}}\left[U_1\left(X\right)\right]; \quad x > 0.$$

We claim that $\tilde{U} \equiv U_0$. Let $\tilde{I}$ be the inverse marginal of $\tilde{U}$. By (i), one must then have

$$\tilde{U}\big(\tilde{I}(y)\big) = E_{\mathbb{P}}\Big[U_1\big(I_1(\rho_1\, y)\big)\Big]; \quad y > 0,$$

and it follows that $\tilde{U}\big(\tilde{I}(y)\big) = U_0\big(I_0(y)\big)$ for $y > 0$.

Differentiating with respect to y yields $\tilde{I}' \equiv I_0'$. Therefore, $\tilde{I}(y) = I_0(y) + C$, $y > 0$, for some constant C. Taking the limit as $y \to \infty$ and using the Inada condition $\tilde{I}(\infty) = I_0(\infty) = 0$, we deduce that $C = 0$. Therefore, we obtain $\tilde{I} \equiv I_0$, which implies $\tilde{U}'(x) = U_0'(x)$ for all $x > 0$. Finally, we obtain

$$\tilde{U}(x) = E_{\mathbb{P}}\Big[U_1\big(I_1(\rho \tilde{U}'(x))\big)\Big] = E_{\mathbb{P}}\Big[U_1\big(I_1(\rho U_0'(x))\big)\Big] = U_0(x); \quad x > 0. \qquad \square$$

Proof of Theorem 5.2. (i) From (5.9) it follows that

$$U_1(x) := U_0(1) + p\int_{x_u(1)}^{x} I_1^{-1}(\xi)d\xi + (1-p)\int_{x_d(1)}^{x} I_1^{-1}(\xi)d\xi; \quad x>0,$$

where $x_u(\cdot)$ and $x_d(\cdot)$ are given by

$$x_i(c) = I_1\big(\rho^i\, U_0'(c)\big); \quad c>0,\ i=u,d. \tag{A.2}$$

Thus,

$$U_1'(x) = p\, I_1^{-1}(x) + (1-p)\, I_1^{-1}(x) = I_1^{-1}(x); \quad x>0.$$

It then follows that I_1 is the inverse marginal of U_1 and that U_1 is a utility function.

(ii) Define the function F by

$$F(x,c) := U_0(c)+p\int_{x_u(c)}^{x} I_1^{-1}(\xi)d\xi+(1-p)\int_{x_d(c)}^{x} I_1^{-1}(\xi)d\xi; \quad (x,c)\in\mathbb{R}^+\times\mathbb{R}^+, \tag{A.3}$$

with $x_u(c)$ and $x_d(c)$ as in (A.2). We claim that

$$\frac{\partial F}{\partial c}(x,c) = 0; \quad x,c>0.$$

Indeed, differentiating (A.3) with respect to c and then using the fact that $I_1^{-1}\big(x_i(c)\big) = \rho^i\, U_0'(c)$ for $c>0$, we have

$$\begin{aligned}\frac{\partial F}{\partial c}(x,c) &= U_0'(c) - p x_u'(c) G\big(x_u(c)\big) - (1-p)x_d'(c)G\big(x_d(c)\big)\\ &= U_0'(c) - p x_u'(c)\rho^u\, U_0'(c) - (1-p)x_d'(c)\rho^d\, U_0'(c)\\ &= U_0'(c)\Big(1-p\rho^u x_u'(c) - (1-p)\rho^d x_d'(c)\Big) = 0.\end{aligned}$$

To obtain the last equation, note that (5.4) is equivalent to

$$I_0(y) = p\rho^u I_1(y\,\rho_u) + (1-p)\rho^d I_1(y\,\rho_d); \quad y>0.$$

Therefore, substituting $y = U_0(c)$ and differentiating with respect to c yield

$$\begin{aligned}1 &= \frac{d}{dc}\Big(I_0'\Big(U_0'(c)\Big)\Big) = \frac{d}{dc}\Big(p\rho^u I_1\Big(\rho_u\, U_0'(c)\Big) + (1-p)\rho^d I_1\Big(\rho_d\, U_0'(c)\Big)\Big)\\ &= p(\rho^u)^2 I_1'\Big(\rho_u\, U_0'(c)\Big)U_0''(c) + p(\rho^d)^2 I_1'\Big(\rho_d\, U_0'(c)\Big)U_0''(c)\\ &= p\rho^u x_u'(c) + (1-p)\rho^d x_d'(c).\end{aligned}$$

Note that, by definition, $U_1(x) = F(x,1)$. Since we have shown that $\frac{\partial F}{\partial c}\equiv 0$, we must have $U_1(x) = F(x,c)$ for all $x>0$ and $c>0$. In other words, for all $x,c\in\mathbb{R}^+$, U_1 satisfies

$$U_1(x) = U_0(c) + p\int_{x_u(c)}^{x} I_1^{-1}(\xi)d\xi + (1-p)\int_{x_d(c)}^{x} I_1^{-1}(\xi)d\xi.$$

On the other hand, as shown in (i), $U_1' \equiv I_1^{-1}$. Therefore, for all $x>0$ and $c>0$,

$$U_1(x) = U_0(c) + p\Big(U_1(x) - U_1\big(x_u(c)\big)\Big) + (1-p)\Big(U_1(x) - U_1\big(x_d(c)\big)\Big),$$

which in turn yields that

$$U_0(c) = pU_1\big(x_u(c)\big) + (1-p)U_1\big(x_d(c)\big) = E_{\mathbb{P}}\Big[U_1\big(I_1(\rho_1\, U_0'(c))\big)\Big]; \quad c > 0.$$

This is equivalent to (A.1). Hence, (ii) follows from Lemma A.1.

(iii) This part follows easily from existing results in the classical expected utility problems, if we view (5.2) as a terminal expected utility problem with U_1 now given and U_0 being its value function. ∎

Appendix B. Proof of Lemma 6.2. Let F_1 and F_2 be two solutions of (5.4) that both satisfy either condition given in the lemma. We show that their difference $w := F_1 - F_2 \equiv 0$.

The function w satisfies the homogeneous equation $w(ay) = -bw(y)$, $y > 0$. Therefore, for $k = 1, 2, \ldots$,

$$w(y) = \frac{w(ay)}{-b} = \frac{w(a^2 y)}{(-b)^2} = \cdots = \frac{w(a^k y)}{(-b)^k}$$

and

$$w(y) = -bw\left(\frac{y}{a}\right) = (-b)^2 w\left(\frac{y}{a^2}\right) = \cdots = (-b)^k w\left(\frac{y}{a^k}\right).$$

It then follows that for $k = \pm 1, \pm 2, \ldots$ and $y > 0$,

$$\begin{aligned} |w(y)| &= b^k \left| w\left(\frac{y}{a^k}\right)\right| = y^{\log_a b}\left(\frac{y}{a^k}\right)^{-\log_a b}\left| w\left(\frac{y}{a^k}\right)\right| \\ &\le y^{\log_a b}\left(\frac{y}{a^k}\right)^{-\log_a b}\left(\left|F_1\left(\frac{y}{a^k}\right)\right| + \left| F_2\left(\frac{y}{a^k}\right)\right|\right). \end{aligned}$$

The right side vanishes as either $k \to \infty$ or $k \to -\infty$, and we conclude.

Appendix C. Proof of Theorem 6.3. We only show part (i) and the corresponding statements in parts (iii) and (iv), since (ii) follows from similar arguments.

(i) Direct substitution shows that if the infinite series in (6.3) converges, then I_1 satisfies (5.4). Thus, to show (i), it only remains to establish that the series converges. Note that (6.3) can be written, for $y > 0$, as

$$I_1(y) = \frac{b}{1+b} y^{\log_a b} \sum_{m=0}^{\infty} (-1)^m \Psi_0(a^m\, y), \tag{C.1}$$

which, by the Leibniz test for alternating series, converges if $\lim_{m\to\infty} \Psi_0(a^m\, y) = 0$ monotonically. The fact that $\lim_{m\to\infty} \Psi_0(a^m\, y) = 0$ follows directly from either of the conditions in (i) on a and Ψ_0. To show that the convergence is monotonic, note that (6.2) yields

$$\Psi_0(a^{m+1}\, y) - \Psi_0(a^m\, y) = b^{-m-1} y^{-\log_a b}\Phi_0(a^m\, y); \quad y > 0, \;\; m = 0, 1, \ldots. \tag{C.2}$$

On the other hand, since Φ_0 is increasing and $\lim_{y\to\infty} \Phi_0(y) = \lim_{y\to\infty} \big(I_0(a\, c\, y) - b\, I_0(c\, y)\big) = 0$, by Inada's condition, we must have $\Phi_0(y) < 0$ for $y > 0$. Thus, by (C.2), we deduce that $\Psi_0(a^m\, y) > \Psi_0(a^{m+1}\, y)$ and $\lim_{m\to\infty} \Psi_0(a^m\, y) = 0$ monotonically.

(iii) First, we prove that I_1 is strictly decreasing. Indeed, (C.1) and (C.2) yield

$$I_1(y) = \frac{b}{1+b} y^{\log_a b} \sum_{m=0}^{\infty}\Big(\Psi_0(a^{2m}\, y) - \Psi_0(a^{2m+1}\, y)\Big) = -\frac{1}{1+b}\sum_{m=0}^{\infty} b^{-2m}\Phi_0(a^{2m}\, y).$$

It then follows that, for $y < y'$,

$$I_1(y') - I_1(y) = \frac{1}{1+b}\sum_{m=0}^{\infty} b^{-2m}\Big(\Phi_0(a^{2m}\,y) - \Phi_0(a^{2m}\,y')\Big) < 0,$$

where the inequality holds because Φ_0 is strictly increasing.

Using (5.4), the facts that $a, b, c > 0$ and $\lim_{y\to\infty} I_0(y) = 0$, and the monotonicity of I_1, we deduce that $\lim_{y\to\infty} I_1(y) = 0$, and hence $I_1(y) > 0$, for $y > 0$. Similarly, the fact that $\lim_{y\to 0^+} I_0(y) = \infty$ yields that $\lim_{y\to 0^+} I_1(y) = \infty$. Thus, we have shown that $I_1 \in \mathcal{I}$.

Finally, the conditions in Lemma 6.2 follow from $\Psi_0(y) \to 0$, as either $y \to 0^+$ or $y \to \infty$, and from the inequalities

$$0 < y^{\log_a b} I_1(y) = \frac{I_1(y)}{I_0(c\,y)}\Psi_0(y) < \frac{b+1}{b}\Psi_0(y); \quad y > 0,$$

where we used (5.4) and the fact that $I_1(y) > 0$ to obtain

$$\frac{I_1(y)}{I_0(c\,y)} = \frac{(1+b)I_1(y)}{I_1(a\,y) + b\,I_1(y)} < \frac{1+b}{b}.$$

(iv) Repeating the last part of the arguments in part (iii) for any solution $\tilde{I} > 0$ yields that $\tilde{I}$ satisfies the same uniqueness condition for (5.4) as I_1. The result then follows directly from Lemma 6.2.

Appendix D. Proof of Corollary 6.4. Assertion (ii) follows from (i) and Theorem 5.2. Also, one can easily check that I_1 given by (6.5) is thus an inverse marginal satisfying (5.4).

It only remains to show the uniqueness of solutions that are inverse marginals. To this end, it suffices to check that the conditions of Theorem 6.3 hold for all possible values of the parameters. Setting $G(y) = y^{-\theta}$, $y > 0$, in (6.2) yields

$$\Phi_0(y) = (a^{-\theta} - b)c^{-\theta}y^{-\theta} \quad \text{and} \quad \Psi_0(y) = y^{-(\theta+\log_a b)}.$$

Since $\theta \neq -\log_a b$ and $a \neq 1$, we have the following dichotomy:

(a) Either $\theta < -\log_a b$ and $a < 1$ or $\theta > -\log_a b$ and $a > 1$. Then, one can show that condition (i) of Theorem 6.3 hold.

(b) Either $\theta < -\log_a b$ and $a > 1$ or $\theta > -\log_a b$ and $a < 1$. Then, one can show that condition (ii) of Theorem 6.3 hold.

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REFERENCES

[1] F. Black, *Individual investment and consumption under uncertainty*, in Portfolio Insurance: A Guide to Dynamic Hedging, D. L. Luskin, ed., John Wiley and Sons, New York, 1988, pp. 207–225.

[2] N. El Karoui and M. Mrad, *An exact connection between two solvable SDEs and a nonlinear utility stochastic PDE*, SIAM J. Financial Math,, 4 (2013), pp. 697–736, https://doi.org/10.1137/10081143X.

[3] M. Kuczma, B. Choczewski, and R. Ger, *Iterative Functional Equations*, Encyclopedia Math. Appl., Cambridge University Press, 1990.

[4] M. Musiela and T. Zariphopoulou, *Investments and Forward Utilities*, Technical report, 2006, https://www.ma.utexas.edu/users/zariphop/pdfs/TZ-TechnicalReport-4.pdf.

[5] M. Musiela and T. Zariphopoulou, *Portfolio choice under dynamic investment performance criteria*, Quant. Finance, 9 (2009), pp. 161–170.

[6] M. Musiela and T. Zariphopoulou, *Portfolio choice under space-time monotone performance criteria*, SIAM J. Financial Math., 1 (2010), pp. 326–365, https://doi.org/10.1137/080745250.

[7] M. Musiela and T. Zariphopoulou, *Stochastic partial differential equations in portfolio choice*, in Contemporary Quantitative Finance, C. Chiarella and A. Novikov, eds., Springer-Verlag, 2010, pp. 195–216.

[8] M. Musiela and T. Zariphopoulou, *Initial investment choice and optimal future allocations under time-monotone performance criteria*, Int. J. Theoret. Appl. Finance, 14 (2011), pp. 61–81.

[9] S. Nadtochiy and M. Tehranchi, *Optimal investment for all time horizons and Martin boundary of space-time diffusions*, Math. Finance, 27 (2017), pp. 438–470.

[10] S. Nadtochiy and T. Zariphopoulou, *A class of homothetic forward investment performance processes with non-zero volatility*, in Inspired by Finance: The Musiela Festschrift, Y. Kabanov, M. Rutkowski, and T. Zariphopoulou, eds., Springer, 2014, pp. 475–504.

[11] A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations: Exact Solutions*, Faktorial Moscow, 1998.

[12] M. Shkolnikov, R. Sircar, and T. Zariphopoulou, *Asymptotic analysis of forward performance processes in incomplete markets and their ill-posed HJB equations*, SIAM J. Financial Math., 7 (2016), pp. 588–618, https://doi.org/10.1137/15M1016059.