Forward Rank-Dependent Performance Criteria: Time-Consistent Investment Under Probability Distortion*

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Abstract

We introduce the concept of forward rank-dependent performance criteria, extending the original notion to forward criteria that incorporate probability distortions. A fundamental challenge is how to reconcile the time-consistent nature of forward performance criteria with the time-inconsistency stemming from probability distortions. For this, we first propose two distinct definitions, one based on the preservation of performance value and the other on the time-consistency of policies and, in turn, establish their equivalence. We then fully characterize the viable class of probability distortion processes and provide the following dichotomy: it is either the case that the probability distortions are degenerate in the sense that the investor would never invest in the risky assets, or the marginal probability distortion equals to a normalized power of the quantile function of the pricing kernel. We also characterize the optimal wealth process, whose structure motivates the introduction of a new, "distorted" measure and a related market. We then build a striking correspondence between the forward rank-dependent criteria in the original market and forward criteria without probability distortions in the auxiliary market. This connection also provides a direct construction method

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for forward rank-dependent criteria. Finally, our results on forward rank-dependent performance criteria motivate us to revisit the classical (backward) setting. We follow the so-called dynamic utility approach and derive conditions for existence and a construction of dynamic rank-dependent utility processes.

Keywords: forward performance criteria, rank-dependent utility, probability distortion, time-consistency, portfolio selection, inverse problems

1   Introduction

In the classical expected utility framework, there are three fundamental modeling ingredients, namely, the model, the trading horizon, and the risk preferences, and all are chosen at initial time. Furthermore, both the trading horizon and the risk preferences are set exogenously to the market. In most cases, the Dynamic Programming Principle (DPP) holds and provides a backward construction of the solution. This yields time-consistency of the optimal policies and an intuitively pleasing interpretation of the value function as the intermediate indirect utility. There are, however, several limitations with this setting.

It is rarely the case that the model is fully known at initial time. Model mis-specification occurs frequently, especially as the trading horizon increases. Even if a family of models, instead of a single model, is assumed and robust control criteria are used, the initial choice of this family of models could still turn out to be inaccurate. This is also the case when filtering is incorporated because it is based on the dynamics of some observation process which, however, can be wrongly pre-chosen. In addition, the trading horizon is almost never fixed and often not even fully known at the beginning of an investment period. It may change, depending on upcoming (even unforeseen) opportunities and/or changes of risk preferences. Finally, it might be difficult to justify that one knows his utility far ahead in the future. It is more natural to know how one feels towards uncertainty for the immediate future, rather than for instances in the distant one (see, for example, the old note of Fischer Black, Black (1968)).

Some of these limitations have been successfully addressed. For example, dynamic model correction is central in adaptive control where the model is revised as soon as new incoming information arrives

and, in turn, optimization starts anew for the remaining of the trading horizon. Flexibility with regards to trading horizons has been incorporated by allowing for rolling horizons. Risk preferences have been also considered in more complex settings, such as recursive utility, which is modeled through a "utility generator" that dictates a more sophisticated backward evolution structure.

Nevertheless, several questions related to genuinely dynamic revision of preferences and of the model, time-consistency across interlinked investment periods as well as under model revisions, endogenous versus exogenous specification of modeling ingredients, and others remain open. A complementary approach that seems to accommodate some of the above shortcomings is based on the so-called forward performance criteria. These criteria are progressively measurable processes that, compiled with the state processes along admissible controls, remain super-martingales and become martingales at candidate optimal policies. In essence, forward criteria are created by imposing the DPP forward in time, instead of backwards in time as in classical expected utility maximization problems. As a result, they adapt to the changing market conditions, do not rely on an a priori specification of the full model, and accommodate dynamically changing trading horizons. They produce endogenously a family of risk preferences that follow the market in "real-time" and, by construction, preserve time-consistency across all times.

Forward criteria were introduced by Musiela and Zariphopoulou (2006, 2008, 2009, 2010a,b, 2011) and further studied, among others, in Žitković (2009), Zariphopoulou and Žitković (2010), Nadtochiy and Zariphopoulou (2014), Shkolnikov et al. (2016), Liang and Zariphopoulou (2017), Nadtochiy and Tehranchi (2017) and Chong et al. (2018). Related notions were developed and studied in Henderson and Hobson (2007) and El Karoui and Mohamed (2013), see also Bernard and Kwak (2016) and Choulli and Ma (2017). More recently, they have also been considered in discrete-time by Angoshtari et al. (2020) and Strub and Zhou (2018), applied to problems arising in insurance by Chong (2018), extended to settings with model ambiguity in Källblad et al. (2018) and Chong and Liang (2018), and applied to optimal contract theory in Nadtochiy and Zariphopoulou (2018). The main challenge to study forward performance processes is that the associated stochastic optimization problems are, in general, ill-posed as one assigns an initial utility and not a terminal one. The solution is then propagated forward in time according to the required martingale/supermartingale properties. In Itô diffusion markets, infinitesimal calculus allows for deriving a stochastic partial differential equation

(SPDE) satisfied (at least formally) by the forward criterion. This SPDE, firstly derived in Musiela and Zariphopoulou (2010b), is a fully non-linear ill-posed equation with its volatility being an investor-specific input. Solving this SPDE is an open problem and specific cases have been studied to date. Among others, Nadtochiy and Zariphopoulou (2014), Nadtochiy and Tehranchi (2017) and Shkolnikov et al. (2016) studied cases with forward criteria represented as deterministic functions of stochastic factors. In Liang and Zariphopoulou (2017), a family of homothetic forward criteria were studied using ergodic control and ergodic BSDE. We note that discrete cases, developed in Angoshtari et al. (2020) and further studied in Strub and Zhou (2018) are particularly challenging as there is no infinitesimal stochastic calculus and, furthermore, there are no general results for the functional equations therein.

Despite the various technical difficulties, the concept of forward performance criteria is well defined for stochastic optimization settings whose classical, backward analogues satisfy the DPP, and thus the martingale/supermartingale properties as well as time-consistency hold. However, these fundamentally interlinked connections break down when the backward problems are time-inconsistent.

Time-inconsistency is an important feature that arises in a plethora of interesting problems in classical and behavioral finance. Among others, it is present in mean-variance optimization, hyperbolic discounting, and risk preferences involving probability distortions. Given, from the one hand, the recent developments in forward performance criteria and, from the other, the importance of time-inconsistent problems, an interesting question thus arises, namely, whether and how one can develop the concept of forward performance criteria for such settings. Herein, we study this question in the realm of *rank-dependent* utilities.

Rank-dependent utility (RDU) theory was developed by Quiggin (1982, 1993), see also Schmeidler (1989), and constitutes one of the most important alternative theories of choice under risk to the expected utility paradigm. It features two main components: a concave utility function that ranks outcomes and a probability distortion function. Rank-dependent utility theory is able to explain a number of empirical phenomena, such as the Allais paradox, low stock market participation (Polkovnichenko (2005)) and preference for skewness (Barberis and Huang (2008), Dimmock et al. (2018)).

Solving portfolio optimization problems under rank-dependent utility preferences is difficult because such problems are, due to the probability distortion, both *time-inconsistent* and *non-concave*.

The difficulty of non-concavity was overcome by the quantile approach developed in Jin and Zhou (2008), Carlier and Dana (2011), He and Zhou (2011) and Xu (2016). A general solution for a rank-dependent utility maximization problem in a complete market was derived in Xia and Zhou (2016) and its effects on optimal investment decisions were extensively studied in He and Zhou (2016) and He et al. (2017). On the other hand, it remains an open problem to solve portfolio optimization problems under rank-dependent utility in general incomplete markets, where one cannot apply the martingale approach and, thus, time-inconsistency becomes a real challenge.

The difficulties in developing forward rank-dependent criteria are both conceptual and technical. Conceptually, it is not clear what could replace the martingality/supermartingality requirements given that, in the backward setting, the DPP fails. Furthermore, there is not even a notion of a (super)martingale under probability distortion. In addition, it is not clear how time-consistency could be incorporated, if at all. From the technical point of view, challenges arise due to the fact that probability distortions are not amenable to infinitesimal stochastic calculus, which plays a key role in deriving the forward stochastic PDE.

We address these difficulties by first proposing two *distinct* definitions for a pair of processes, $\left((u_t(\cdot))_{t \geqslant 0}, (w_{s,t}(\cdot))_{0 \leqslant s < t} \right)$, to be a forward rank-dependent criterion. The first component, $u_t(x)$, $x \geqslant 0$, is the utility process while the second, $w_{s,t}(p)$, $p \in [0,1]$, plays the role of the probability distortion. The first definition imitates the martingale/supermartingale properties by requiring, for all times, analogous conditions but under the distorted conditional probabilities (cf. Definition 3.2). It is based on the preservation of value along optimal policies and its loss along suboptimal ones. On the other hand, the second definition is related to time-consistency. It uses a continuum of optimization problems under RDU with the candidate pair $(u_t(x), w_{s,t}(p))$ and requires time-consistency of the candidate policies across any sub-horizon (cf. Definition 3.4).

We note that in both definitions, the utility process $u_t(x)$ is defined for all times $t \geqslant 0$ while the probability distortion $w_{s,t}(p)$ is defined for all intermediate times $0 \leqslant s < t$. We also note that we consider deterministic processes $(u_t(x), w_{s,t}(p))$. This is important conceptually because it is not clear how to evaluate a stochastic utility function in the presence of probability distortion. Considering deterministic processes also allows us to develop a direct connection to deterministic, time-monotone forward criteria without probability distortion, as we show herein.

Naturally, in the absence of probability distortion, i.e., in the degenerate case $w_{s,t}(p) \equiv p$, $0 \leqslant$

$s < t$, both definitions reduce to the existing one of the forward performance criterion. Surprisingly, it turns out that the two definitions we propose are also equivalent even in the non-degenerate case, as we establish in Proposition 3.5. This is one of the first novel features in our analysis as it relates probability distortions, which so far have been yielding time-inconsistent policies, to time-consistent optimal behavior. This connection is also used to establish new results for the so-called *dynamic utilities* in the traditional, backward setting, as we explain later on.

The second main result is the derivation of necessary conditions for a distortion process $w_{s,t}$ to belong to a forward rank-dependent pair. We establish a dichotomy; specifically, it is either the case in which

$$(1.1) \qquad w_{s,t}(p) = \frac{1}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]} \int_0^p \left(\left(F_{s,t}^\rho\right)^{-1}(q)\right)^{1-\gamma} dq, \quad p \in [0,1],$$

for some $\gamma > 0$ and for *all* $0 \leqslant s < t$, where $\rho_{s,t}$ is the pricing kernel and $F_{s,t}^\rho$ is its cumulative distribution function, or the case in which the inequality

$$(1.2) \qquad w_{s,t}(p) \leqslant \mathbb{E}\left[\rho_{s,t}\mathbf{1}_{\left\{\rho_{s,t} \leqslant \left(F_{s,t}^\rho\right)^{-1}(p)\right\}}\right], \quad p \in [0,1],$$

holds for all $0 \leqslant s < t$. These are the only two viable cases and they imply rather distinct behavior with regards to the optimal portfolio, with the latter family yielding zero allocation in the risky assets at all times. We will be referring to γ as the *(investor-specific) distortion parameter*.

For the market considered herein, the probability distortion process (1.2) reduces to

$$(1.3) \qquad w_{s,t}(p) = \Phi\left(\Phi^{-1}(p) + (\gamma - 1)\sqrt{\int_s^t \|\lambda_r\|^2 dr}\right),$$

where Φ is the cumulative normal distribution function and λ is the market price of risk. This is a rather interesting formula because it connects $w_{s,t}$ with a probability distortion function that was proposed by Wang (2000) and became popular in the literature, which is of the form $\Phi\left(\Phi^{-1}(\cdot) + a\right)$ for some displacement parameter a. In the forward setting, however, $w_{s,t}$ is affected not only by the distortion parameter γ, which is chosen by the investor, but also by the current market behavior, as manifested by the *market-specific input* $A_{s,t} := \sqrt{\int_s^t \|\lambda_r\|^2 dr}$. This is intuitively pleasing because forward criteria

are expected to follow the market in "real-time". Furthermore, the multiplicative coefficient $(\gamma - 1)$ combines in a very transparent way the market condition with the investor's attitude. The latter can be interpreted as *objective* $(\gamma = 1)$, *pessimistic* $(\gamma < 1)$ or *optimistic* $(\gamma > 1)$.

As a corollary to the above results, we obtain that the distortion process of any forward rank-dependent performance process satisfies a monotonicity condition proposed by Jin and Zhou (2008) in their study of portfolio selection; see Assumption 4.1 and the discussion in Section 6.2 therein. An interesting analogy is a result of Xia and Zhou (2016) showing that the above monotonicity condition is also automatically satisfied for a representative agent in an Arrow-Debreu economy. In other words, we have that the monotonicity condition of Jin and Zhou (2008) is satisfied if the market is exogenously given and the preferences are endogenously determined through the framework of forward criteria, or if the preferences are exogenously given and the pricing kernel is endogenously determined through an equilibrium condition.

The third main result is the *construction* of forward rank-dependent criteria. In the degenerate case (1.3), it follows easily that $u_t(x) = u_0(x)$, $t \geqslant 0$, because the optimal investment strategy never invests in the risky assets and thus the wealth remains unchanged over time (with zero interest rate as assumed for simplicity in the present paper). In the non-degenerate case (1.2), we establish a direct equivalence with deterministic, time-monotone forward criteria in the absence of probability distortion. Specifically, for a given γ, we introduce a new probability measure, which we call the γ-*distorted probability measure*, and a related *distorted market* with modified risk premium $\tilde{\lambda}_t := \gamma \lambda_t$ (see Subsection 5.1). As we will explain later on, the motivation for considering these measure and market variations comes from the form of the optimal wealth process for the non-degenerate case.

In the distorted market, we in turn recall the standard (no probability distortion) time-monotone forward criterion, denoted by $U_t(x)$. As established in Musiela and Zariphopoulou (2010a), it is given by $U_t(x) = v(x, \int_s^t \|\tilde{\lambda}_r\|^2 dr)$, with the function $v(x,t)$ satisfying $v_t = \frac{1}{2} \frac{v_x^2}{v_{xx}}$. Herein, we establish that

$$(1.4) \qquad u_t(x) = U_t(x) = v\left(x, \int_s^t \gamma^2 \|\lambda_r\|^2 dr\right).$$

In other words, the utility process $u_t(x)$ of the forward rank-dependent criterion in the original market corresponds to a deterministic, time-monotone forward criterion in a pseudo-market with modified risk premia and vice-versa. If the investor is objective $(\gamma = 1)$, there is no probability distortion and, as

a result, the two markets become identical and the two criteria coincide. For optimistic investors ($\gamma > 1$), the time-monotonicity of the function $v(x, t)$ results in a more pronounced effect on how the forward rank-dependent utility decays with time. Specifically, the higher the optimism (higher γ), the larger the time-decay in the utility criterion, reflecting a larger loss of subjectively viewed better opportunities. The opposite behavior is observed for pessimistic investors ($\gamma < 1$), where the time-decay is slower because the market opportunities look subjectively worse. Finally, the limiting case $\gamma = 0$ corresponds to a subjectively worthless market.

In addition to the construction approach, the equivalence established in Theorem 5.1 yields explicit formulae for the optimal wealth and investment policies under forward probability distortions by using the analogous formulae under deterministic, time-monotone forward criteria.

As mentioned earlier, our construction of time-consistent criteria in the presence of probability distortion prompts us to revisit the RDU maximization in the classical, backward setting and investigate if and how our findings can be used to build time-consistent policies therein. The *dynamic utility approach* developed in Karnam et al. (2017) seems suitable to this end. Their approach builds on the observation that the time-inconsistency of stochastic optimization problems is partially due to the following restriction: The utility functional determining the objective at an intermediate time is essentially the same as the utility functional at initial time modulo conditioning on the filtration. The dynamic utility approach relaxes this restriction and allows the intermediate utility functional to vary more freely so that the DPP holds. In a recent work closely related to this paper, Ma et al. (2018) introduce a *dynamic distortion function*. This leads to a distorted conditional expectation which is time-consistent in the sense that the tower-property holds. In their setting, an Itô process is given and fixed, and the dynamic distortion function is then constructed for this specific process. Because the construction depends on the drift and volatility parameters of the Itô process, their results are not directly applicable to our setting, where we consider an investment problem and thus the state process is not a priori given, but is instead controlled by the investment policy. Herein, we extend the construction of dynamic distortion functions to controlled processes for the problem of RDU maximization in a financial market with determinstic coefficients. We find that constructing a dynamic utility which is restricted to remain in the class of RDU preference functionals is possible if and only if the initial probability distortion function belongs to the family introduced in Wang (2000).

Studying time-inconsistency induced by distorting probabilities is one of the remaining open chal-

lenges for the psychology of tail events identified in the review article Barberis (2013). We contribute to this research direction by developing a new class of risk preferences and showing that investment under probability distortion can be time-consistent. Furthermore, we fully characterize the conditions under which this is possible, namely if and only if the marginal probability distortion equals to a normalized power of the quantile function of the pricing kernel.

The paper is organized as follows. In Section 2, we describe the model and review the main results for RDU maximization in the classical, backward setting. In Section 3, we introduce the definitions of the forward rank-dependent performance criteria and establish their equivalence. We continue in Section 4, where we derive the necessary conditions for a probability distortion process to belong to a forward rank-dependent pair. In Section 5, we establish the connection with the deterministic, time-monotone forward criteria and the form of the optimal wealth and portfolio processes, and then provide examples. In Section 6, we combine our results with the dynamic utility approach to show that constructing a dynamic utility restricted to remain within the class of RDU preference functionals is possible if and only if the initial distortion function belongs to the class introduced in Wang (2000). We conclude in Section 7. To ease the presentation, we delegate all proofs in the Appendix.

2   The investment model and background results

We start with the description of the market model and a review of the main concepts and results for rank-dependent utilities.

2.1   The financial market

The financial market consists of one risk-free asset offering zero interest rate and N risky assets. The price of the ith risky asset solves

$$(2.1) \qquad dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{N} \sigma_t^{ij} dW_t^j \right), \quad t \geqslant 0,$$

with $S_0^i = s_0^i > 0$, $i = 1, \ldots, N$. The process $W = (W_t)_{t \geqslant 0}$ is an N-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions, where $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ is the completed filtration generated by W.

The drift and volatility coefficients are assumed to be deterministic functions that satisfy $\int_0^t |\mu_s^i| ds < \infty$ and $\int_0^t \left(\sigma_s^{ij}\right)^2 ds < \infty$, $t \geqslant 0$ and $i,j = 1, \ldots, N$. We denote the volatility matrix by $\sigma_t := \left(\sigma_t^{ij}\right)_{N \times N}$. We assume that σ_t is invertible for all $t \geqslant 0$ to ensure that the market is arbitrage free and complete. We also define the *market price of risk* process

$$(2.2) \qquad \lambda_t := \sigma_t^{-1} \mu_t$$

and assume that $\left\{t \geqslant 0 | \lambda_t = 0 \in \mathbb{R}^N\right\}$ is a Lebesgue null set.

For each $t > 0$, we consider the (unique) pricing kernel

$$(2.3) \qquad \rho_t = \exp\left(-\int_0^t \frac{1}{2} \|\lambda_r\|^2 dr - \int_0^t \lambda_r^\top dW_r\right).$$

Here and hereafter, we denote the transpose of a matrix A by $A^\top$. For $0 < s \leqslant t$, we further define

$$(2.4) \qquad \rho_{s,t} := \frac{\rho_t}{\rho_s} = \exp\left(-\int_s^t \frac{1}{2} \|\lambda_r\|^2 dr - \int_s^t \lambda_r^\top dW_r\right).$$

We also denote the cumulative distribution function of ρ_t and $\rho_{s,t}$ by F_t^ρ and $F_{s,t}^\rho$ respectively. Note that F_t^ρ and $F_{s,t}^\rho$ correspond to log-normal distributions because λ_t is deterministic and nonzero.

The agent starts at $t = 0$ and trades the risk-free and the risky assets using a self-financing trading policy $(\pi_t^0, \pi_t)_{t \geqslant 0}$, where $\pi_t^0 = \pi_t^0(\omega; x)$ denotes the allocation in the risk-free asset and the vector $\pi_t := \left(\pi_t^1, \pi_t^2, \ldots, \pi_t^N\right)$, with $\pi_t^i = \pi_t^i(\omega; x)$, $i = 1, \ldots, N$, representing the amount invested, at time t, in the risky asset i. Strategies are allowed to depend on the initial wealth $x > 0$ and the state of the world $\omega \in \Omega$. We usually drop the ω and x argument whenever the context is clear. In turn, the wealth process $X = (X_t^{x,\pi})_{t \geqslant 0}$ solves the stochastic differential equation

$$(2.5) \qquad dX_t^{x,\pi} = \pi_t^\top \mu_t dt + \pi_t^\top \sigma_t dW_t, \quad t \geqslant 0,$$

with $X_0^{x,\pi} = x$.

The set of admissible strategies is defined as

$$
(2.6) \qquad \mathcal{A} := \Bigg\{ \pi \Big| \pi_t \text{ is } \mathbb{F}\text{-progressively measurable,}
$$
$$
\text{with } \int_0^t \|\pi_s\|^2 ds < \infty \text{ and } X_t^{x,\pi} \geqslant 0, \text{ for } t \geqslant 0, \ x > 0 \Bigg\}.
$$

For a given time, t_0, and an admissible policy, $\widetilde{\pi}$, we also introduce

$$
(2.7) \qquad \mathcal{A}(\widetilde{\pi}, t_0) := \{ \pi \in \mathcal{A} | \pi_s \equiv \widetilde{\pi}_s, s \in [0, t_0] \},
$$

namely, the set of admissible strategies which *coincide* with this specific policy in $[0, t_0]$.

2.2 Rank-dependent utility theory

To ease the presentation and build motivation for the upcoming analysis, we start with a brief overview of RDU and the main results on portfolio optimization under RDU preferences for the market considered herein.

The rank-dependent utility value of a prospect X is defined as

$$
(2.8) \qquad V(X) := \int_0^\infty u(\xi) d\left(-w \left(1 - F_X(\xi) \right) \right),
$$

where F_X denotes the cumulative distribution function of X, u is a *utility function* and w is a *probability distortion function*, cf. Quiggin (1982, 1993).

We assume that u and w belong to the sets $\mathcal{U}$ and $\mathcal{W}$, which are introduced next.

Definition 2.1. *Let $\mathcal{U}$ be the set of all utility functions $u : [0, \infty) \to \mathbb{R}$, with u being strictly increasing, strictly concave, twice continuously differentiable in $(0, \infty)$, and satisfying the Inada conditions $\lim_{x \downarrow 0} u'(x) = \infty$ and $\lim_{x \uparrow \infty} u'(x) = 0$.*

Let $\mathcal{W}$ be the set of probability distortion functions $w : [0, 1] \to [0, 1]$ that are continuous and strictly increasing on $[0, 1]$, continuously differentiable on $(0, 1)$ and satisfies $w(0) = 0$ and $w(1) = 1$.

At initial time $t = 0$, an agent chooses her investment horizon $T > 0$, the dynamics in (2.1) for

$[0, T]$, together with $u \in \mathcal{U}$ and $w \in \mathcal{W}$. She then solves the portfolio optimization problem

$$
(2.9) \qquad v(x, 0) = \sup_{\pi \in \mathcal{A}_T} V(X_T^{x,\pi}),
$$

where $X_s^{x,\pi}, s \in [0, T]$ solves (2.5) and satisfies $X_0^{x,\pi} = x$, V is given in (2.8), and $\mathcal{A}_T$ is defined similarly to $\mathcal{A}$ in (2.6) up to horizon T.

Problem (2.9) has been studied by various authors; see, among others, Carlier and Dana (2011), Xia and Zhou (2016), and Xu (2016). Fundamental difficulties arise from the time-inconsistency which stems from the probability distortion. Consequently, key elements in stochastic optimization, like the Dynamic Programming Principle, the Hamilton-Jacobi-Bellman (HJB) equation, the martingality of the value function process along an optimum process and others, are lost. The analysis of problem (2.9) has been carried out using a well-known reformulation to a static problem and the quantile method developed in Jin and Zhou (2008), Carlier and Dana (2011), He and Zhou (2011) and Xu (2016). Specifically, because the market is complete, any $\mathcal{F}_T$-measurable prospect X that satisfies the *budget constraint* $\mathbb{E}[\rho_T X] = x$ can be replicated by a self-financing policy. In turn, problem (2.9) reduces to

$$
(2.10) \qquad \sup_X V(X) \quad \text{with} \quad \mathbb{E}[\rho_T X] \leqslant x, \ X \geqslant 0, \ X \in \mathcal{F}_T.
$$

The above problem is difficult to solve because the objective function $V(X)$ is nonconcave in X due to the presence of probability distortion. This difficulty is overcome by first solving the optimal quantile function of X and then finding the most economic way to attain the optimal quantile. The complete solution is obtained by Xia and Zhou (2016) and Xu (2016), which is summarized as follows:

Theorem 2.2 (Xia and Zhou (2016); Xu (2016)). *Let $u \in \mathcal{U}$ and $w \in \mathcal{W}$. If there exists an optimal wealth to (2.10), it is given by*

$$
(2.11) \qquad X_T^* = (u')^{-1} \left(\eta^* \hat{N}' \left(1 - w \left(F_T^\rho(\rho_T) \right) \right) \right),
$$

where $\hat{N}$ is the concave envelope of

$$(2.12) \qquad N(z) := -\int_0^{w^{-1}(1-z)} (F_T^\rho)^{-1}(t)dt, \quad z \in [0,1],$$

and the Lagrangian multiplier $\eta^* > 0$ is determined by $\mathbb{E}[\rho_T X_T^*] = x$.

We also recall that if, in addition, the so called *Jin-Zhou monotonicity condition* holds, namely, if the function $f : [0,1] \to \mathbb{R}^+$, defined as

$$(2.13) \qquad f(p) := \frac{(F_T^\rho)^{-1}(p)}{w'(p)}$$

is nondecreasing, then equality (2.11) simplifies to

$$(2.14) \qquad X_T^* = \left(u'\right)^{-1}\left(\eta^* \frac{\rho_T}{w'\left(F_T^\rho(\rho_T)\right)}\right),$$

see Jin and Zhou (2008) or Remark 3.4 in Xia and Zhou (2016).

Further results for problem (2.10) were derived in Xia and Zhou (2016), where it was shown that if, for each $\eta > 0$, the inequality

$$\mathbb{E}\left[\rho_T \left(u'\right)^{-1}\left(\eta \hat{N}'\left(1 - w\left(F_T^\rho(\rho_T)\right)\right)\right)\right] < \infty$$

holds, with $\hat{N}'$ as in (2.12), then an optimal solution exists and is of form (2.11). In addition, Xu (2016) showed that the existence of a non-degenerate optimal investment policy is *equivalent* to the existence of a Lagrangian multiplier η^* and that, in this case, the terminal optimal wealth is as the one given in Theorem 2.2.

3   Forward rank-dependent performance criteria

We introduce the concept of forward performance criteria in the framework of rank-dependent preferences. To this end, we first review the definition of the forward performance criterion (slightly modified for the setting and notation herein); see, among others, Musiela and Zariphopoulou (2006, 2009, 2010a). We then discuss the various difficulties in extending this concept when probability

distortions are incorporated.

Forward criteria allow for substantial flexibility in terms of model choice, horizon and risk preferences specification. Indeed, their definition does not require to pre-specify the model nor to pre-determine a specific horizon. Furthermore, the utility is given as an initial input and not a terminal one. In turn, forward performance criteria are constructed infinitesimally and forward in time, as the market evolves. Naturally, they are dynamically and endogenously constructed. Given both this flexibility and endogeneity, it is an intriguing question to investigate whether similar concepts could be developed for more complex risk preferences, and in particular when probability distortions are allowed. As the analysis herein shows, it is far more complex to understand how both the utility and the probability distortion processes will evolve, together with the market, as there is a intricate interplay between the utility process at a given time and the probability distortion process up to that time. This dependence itself might also evolve with time and new incoming information, which makes the problem considerably harder.

Definition 3.1. *An $\mathbb{F}$-adapted process $(U_t)_{t \geqslant 0}$ is a forward performance criterion if*

   i) for any $t \geqslant 0$ and fixed $\omega \in \Omega$, $U_t \in \mathcal{U}$,

   ii) for any $\pi \in \mathcal{A}$, $0 \leqslant s \leqslant t$ and $x > 0$

$$(3.1) \qquad \mathbb{E}\left[U_t\left(X_t^{x,\pi}\right) \middle| \mathcal{F}_s\right] \leqslant U_s\left(X_s^{x,\pi}\right),$$

   iii) there exists $\pi^* \in \mathcal{A}$ such that, for any $0 \leqslant s \leqslant t$, and $x > 0$,

$$(3.2) \qquad \mathbb{E}\left[U_t\left(X_t^{x,\pi^*}\right) \middle| \mathcal{F}_s\right] = U_s\left(X_s^{x,\pi^*}\right).$$

The above definition was directly motivated by the DPP, a key feature in the classical stochastic optimization, which yields the above supermartingality and martingality properties, (3.1) and (3.2), of the value function process along an admissible and an optimal policy, respectively. Furthermore, directly embedded in this fundamental connection between DPP and the supermartingality and martingality properties is the time-consistency of the optimal policies.

Once, however, probability distortions are incorporated, none of these features exist in the clas-

sical rank-dependent case. Indeed, the DPP does not hold and, naturally, time-inconsistency arises. Furthermore, there is no general notion of supermartingality and martingality under probability distortions, so it is not clear what the analogues of (3.1) and (3.2) are.[1] In other words, we lack the deep connection among the DPP, the martingality/supermartingality of the value function process, and the time-consistency of the optimal policies, which is the cornerstone in the expected utility paradigm. Thus, it is not at all clear how to define the forward rank-dependent performance criteria. We address these difficulties in two steps.

We first propose a definition of forward rank-dependent criteria by directly imitating requirements (3.1) and (3.2). Specifically, we propose (3.3) and (3.4), respectively, where we use (conditional) distorted probabilities instead of the regular ones. This definition is a natural, direct analogue to Definition 3.1 because it is built on the preservation of value along an optimal policy and its decay along a suboptimal one. The novel element in Definition 3.2 is that we seek a *pair* of processes $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$, corresponding to a dynamic "forward utility" u_t that is defined for *each time*, $t \geqslant 0$, and a dynamic "forward probability distortion" $w_{s,t}$ that is defined for *all intermediate* times $s \in [0, t)$. In other words, while u_t is parametrized solely by t, the second component $w_{s,t}$ is parametrized by *both* the starting and the end points, s and t. Note that this definition is not empty. Indeed, if we choose $w_{s,t}(p) \equiv p$, for all $t \geqslant 0$ and $0 \leqslant s < t$, then Definition 3.2 reduces to Definition 3.1 above; see Proposition 3.3 below.

Definition 3.2, however, does not give any insights about the time-consistency of the optimal policies. As a matter of fact, it is not even clear whether we should even seek such a property given that, after all, time-consistency does not hold in the classical rank-dependent setting. Surprisingly, it turns out that we can actually build a direct connection between time-consistency and forward rank-dependent criteria. For this, we first introduce the concept of *time-consistent* rank-dependent processes and, subsequently, a subclass of this family that preserve the forward performance value along an optimal policy; see parts (i) and (ii) in Definition 3.4, respectively. In turn, we show in Proposition 3.5 that Definitions 3.2 and 3.4 are equivalent. In other words, we establish an equivalence between forward rank-dependent performance criteria and the time-consistent ones that also preserve the performance value.

Finally, we note that herein we work exclusively with *deterministic* processes for both u_t and $w_{s,t}$. We do this for various reasons. Firstly, it is assumed that the coefficients of the risky assets are

deterministic (cf. (2.1)) and, therefore, it is natural to first explore the class of deterministic forward rank-dependent criteria. Secondly, working with deterministic criteria enables us to build a direct connection with time-monotone analogues that are also deterministic. Thirdly, it is not yet clear how to define non-deterministic criteria even for the market herein. We recall that in the standard forward case, stochasticity arises both from the market dynamics and the forward volatility process, which is an investor-specific input. It is conceptually unclear how to evaluate a stochastic preference functional when probability distortions are incorporated.

Next, we introduce some notation and provide the relevant definitions and results. To this end, we denote by $\mathbb{P}\left[\cdot\,|\,\mathcal{G}\right](\omega)$ the conditional probability given a sigma-algebra $\mathcal{G}\subseteq\mathcal{F}$. For a random variable $X\in\mathcal{F}$, we denote by $F_{X|\mathcal{G}}(\cdot\,;\omega)=\mathbb{P}\left[X\leqslant\cdot\,|\,\mathcal{G}\right](\omega)$ the regular conditional distribution function of X given $\mathcal{G}$. We will also follow the standard convention to set $\int_0^\infty u(\xi)d\left(-w\left(1-F_{X|\mathcal{G}}(\xi)\right)\right)=-\infty$ whenever $\int_0^\infty \max\left(0,-u(\xi)\right)d\left(-w\left(1-F_{X|\mathcal{G}}(\xi)\right)\right)=\int_0^\infty \max\left(0,u(\xi)\right)d\left(-w\left(1-F_{X|\mathcal{G}}(\xi)\right)\right)=\infty$, for $u\in\mathcal{U}$ and $w\in\mathcal{W}$.

Definition 3.2. *A pair of deterministic processes $\left((u_t)_{t\geqslant 0},(w_{s,t})_{0\leqslant s<t}\right)$ is a forward rank-dependent performance criterion if the following properties hold:*

    i) for any $t\geqslant 0$, $u_t(\cdot)\in\mathcal{U}$ and, for any $0\leqslant s<t$, $w_{s,t}(\cdot)\in\mathcal{W}$.

    ii) for any $\pi\in\mathcal{A}$, $0\leqslant s<t$ and $x>0$,

$$\text{(3.3)}\qquad \int_0^\infty u_t(\xi)d\left(-w_{s,t}\left(1-F_{X_t^{x,\pi}|\mathcal{F}_s}(\xi)\right)\right)\leqslant u_s\left(X_s^{x,\pi}\right).$$

    iii) there exists $\pi^*\in\mathcal{A}$, such that, for any $0\leqslant s<t$ and $x>0$,

$$\text{(3.4)}\qquad \int_0^\infty u_t(\xi)d\left(-w_{s,t}\left(1-F_{X_t^{x,\pi^*}|\mathcal{F}_s}(\xi)\right)\right)=u_s\left(X_s^{x,\pi^*}\right).$$

Naturally, if there is no probability distortion, the above definition should reduce to Definition 3.1. We prove this formally in the following proposition.

Proposition 3.3.   *i) Let $\left((u_t)_{t\geqslant 0},(w_{s,t})_{0\leqslant s<t}\right)$ be a forward rank-dependent performance criterion with $w_{s,t}(p)\equiv p$, $p\in[0,1]$ and $0\leqslant s<t$. Then $(u_t)_{t\geqslant 0}$ is a forward performance process.*

ii) Conversely, if $(u_t)_{t\geqslant 0}$ is a deterministic forward performance process, then the pair $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s < t}\right)$, with $w_{s,t}(p) \equiv p$, $p \in [0,1]$ for all $0 \leqslant s < t$, is a forward rank-dependent performance criterion.

We now present an alternative definition.

Definition 3.4. *Let $(u_t)_{t\geqslant 0}$ and $(w_{s,t})_{0\leqslant s < t}$ be deterministic processes with $u_t \in \mathcal{U}$ and $w_{s,t} \in \mathcal{W}$.*

i) A pair $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s < t}\right)$ is called a time-consistent rank-dependent performance criterion, if there exists $\pi^* \in \mathcal{A}$ such that π^* solves the optimization problem

$$(3.5) \qquad \max_{\pi \in \mathcal{A}(\pi^*, s)} \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi}|\mathcal{F}_s}(\xi)\right)\right)$$

with $dX_r^{x,\pi} = \pi_r^\top \mu_r dr + \pi_r^\top \sigma_r dW_r$, $r \in [0,t]$ and $X_0^{x,\pi} = x$, for any $0 \leqslant s < t$ and $x > 0$.

ii) A pair $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s < t}\right)$ satisfying (i) is called a time-consistent rank-dependent performance criterion preserving the performance value if, for any optimal policy π^* as in (i), we have

$$(3.6) \qquad \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi^*}|\mathcal{F}_s}(\xi)\right)\right) = u_s\left(X_s^{x,\pi^*}\right),$$

for any $0 \leqslant s < t$ and $x > 0$.

Definition 3.4 above is built around the time-consistency of optimal policies (assuming that at least one such policy exists). It is also in direct alignment with the DPP as we require that, for any investment horizon $t \geqslant 0$ and intermediate time $s \in [0,t]$, the investment policy $\pi^* = (\pi_r^*)_{0\leqslant r \leqslant t}$ optimizing the rank-dependent utility value determined by $(u_t, w_{0,t})$, remains optimal over the investment interval $[s,t]$ with respect to the rank-dependent utility described by u_t and $w_{s,t}$.

The following Proposition states that Definitions 3.2 and 3.4 are actually equivalent.

Proposition 3.5. *A pair of deterministic functions $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s < t}\right)$ is a forward rank-dependent performance criterion if and only if it is a time-consistent rank-dependent performance criterion preserving the performance value.*

In the absence of probability distortion, properties (3.1) and (3.2), together with stochastic calculus, yield a stochastic PDE that the forward performance process $U_t(x)$ is expected to satisfy. This stochastic PDE plays the role of the Hamilton-Jacobi-Bellman (HJB) equation that arises in the backward setting. In the rank-dependent case, however, these concepts and tools do not exist, which makes the analysis considerably harder. Besides, we need to characterize a pair of processes—the utility and the probability distortion processes—instead of one process only. The methodology developed herein starts with a complete characterization of all viable probability distortion functions.

4 Characterization of the forward probability distortion functions

We present one of the main results herein, deriving *necessary conditions* for a deterministic probability distortion process, $w_{s,t}$, to belong to a forward rank-dependent pair $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$. We first state all pertinent results and then provide a discussion at the end of this section.

Theorem 4.1. *Let the pair* $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$ *be a deterministic forward rank-dependent performance criterion. Then there exist only the following two cases:*

i) For each $t \geqslant 0$ and $0 \leqslant s < t$, $w_{s,t}$ has the representation

$$(4.1) \qquad w_{s,t}(p) = \frac{1}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]} \int_0^p \left(\left(F_{s,t}^\rho\right)^{-1}(q)\right)^{1-\gamma} dq, \quad p \in [0,1],$$

for some $\gamma > 0$ which is independent of $0 \leqslant s < t$.

ii) For each $t \geqslant 0$ and $0 \leqslant s < t$, $w_{s,t}$ satisfies

$$(4.2) \qquad w_{s,t}(p) \leqslant \mathbb{E}\left[\rho_{s,t} \mathbf{1}_{\left\{\rho_{s,t} \leqslant \left(F_{s,t}^\rho\right)^{-1}(p)\right\}}\right], \quad p \in [0,1].$$

We remark that the *distortion parameter* γ appearing in case i) of Theorem 4.1 is neither unique nor depends on the underlying market or the process $(u_t)_{t \geqslant 0}$.

The following result follows directly from (4.1) and (2.4).

Corollary 4.2. *If the forward probability distortion is given by (4.1), then*

$$(4.3) \qquad w_{s,t}(p) = \Phi\left(\Phi^{-1}(p) + (\gamma - 1)\sqrt{\int_s^t \|\lambda_r\|^2 dr}\right), \quad p \in [0,1],$$

where Φ denotes the standard normal cumulative distribution function and $(\lambda_t)_{t \geqslant 0}$ is the market price of risk (cf. (2.2)).

The following result yields that it is necessary to allow the family of probability distortion functions of a forward rank-dependent performance process to depend on both the initial and terminal time. Otherwise, forward rank-dependent criteria reduce to the case without any probability distortion or to the case of no investment in risky assets.

Corollary 4.3. *Let $\left((u_t)_{0 \leqslant t}, (w_{s,t})_{0 \leqslant s < t}\right)$ be a forward rank-dependent performance criterion such that, for each $t \geqslant 0$, $w_{s,t}(p) = w_{r,t}(p)$ for all $0 \leqslant s, r < t$ and $p \in [0,1]$. Then, it is either the case $w_{s,t}(p) = p$, for all $t \geqslant 0$, $0 \leqslant s < t$, and $p \in [0,1]$ or the case $w_{s,t}(p) \leqslant \Phi\left(\Phi^{-1}(p) - \sqrt{\int_0^t \|\lambda_r\|^2 dr}\right)$, for all $t \geqslant 0$, $0 \leqslant s < t$, and $p \in [0,1]$, and the latter case implies no investment in risky assets.*

The following result yields the optimal wealth processes under the two cases (4.1) and (4.2).

Proposition 4.4. *Let the pair $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$ be a forward rank-dependent performance criterion. The following assertions hold:*

i) If $w_{s,t}$, $t \geqslant 0$ and $0 \leqslant s < t$, satisfies (4.1), then the corresponding optimal wealth process $X^* = (X_t^*)_{t \geqslant 0}$ is given by

$$(4.4) \qquad X_t^* = (u_t')^{-1}\left(\eta_{s,t}^* \mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right] \rho_{s,t}^\gamma\right) = (u_t')^{-1}\left(u_0'(x)\mathbb{E}\left[\rho_t^{1-\gamma}\right] \rho_t^\gamma\right), \quad 0 \leqslant s < t,$$

where $\eta_{s,t}^*$ is the Lagrangian multiplier corresponding to the budget constraint $\mathbb{E}\left[\rho_{s,t} X_t^* | \mathcal{F}_s\right] = X_s^*$ and satisfies $\eta_{s,t}^* = u_s'(X_s^*)$. The optimal investment policy is given by

$$(4.5) \qquad\qquad \pi_t^* = -\gamma \sigma_t^{-1} \lambda_t \frac{u_t'(X_t^*)}{u_t''(X_t^*)}.$$

ii) If $w_{s,t}$, $t \geqslant 0$ and $0 \leqslant s < t$, satisfies (4.2), then $X_t^* = x$, $t \geqslant 0$ and the optimal policy is $\pi_t^* = 0$, $t \geqslant 0$.

The above results give several valuable insights on the nature of the candidate probability distortion processes. Firstly, we see that forward probability distortions satisfy a *dichotomy* in that it is either the case in which equality (4.1) holds for all times and all $p \in [0,1]$ or the case in which inequality (4.2) holds throughout. The latter is a degenerate case, as it induces zero optimal investment in all risky assets and for all times.

The non-degenerate case, given by (4.1) or equivalently by (4.3), has striking similarities with a popular distortion function used in the insurance literature. Specifically, it resembles the distortion $w(p) = \Phi\left(\Phi^{-1}(p) + a\right), p \in [0,1]$, for some displacement parameter $a \in \mathbb{R}$, which was proposed by Wang (2000). However, in (4.3) the analogous displacement term $(\gamma - 1)\sqrt{\int_s^t \|\lambda_r\|^2 dr}$ is neither exogenous to the market nor static. Rather, it depends on both the investor's probability distortion parameter γ and the market performance, as measured by the term $\sqrt{\int_s^t \|\lambda_r\|^2 dr}$ with λ being the market price of risk process. This is intuitively pleasing as forward performance criteria are expected to follow the market changes in "real-time", and we see that $w_{s,t}$ does exactly this. We also see that the dynamic displacement $(\gamma - 1)\sqrt{\int_s^t \|\lambda_r\|^2 dr}$ is positive (negative) if $\gamma > 1$ ($\gamma < 1$), while the case $\gamma = 1$ corresponds to no probability distortion.

To give an economic interpretation of the distortion parameter γ we first recall the notion of *pessimism* introduced in Quiggin (1993). For a RDU representation V as in (2.8), with utility function u, and distortion function w, one can define for a prospect X its *certainty equivalent* CE(X) by CE(X) := $u^{-1}(V(X))$ and its *risk-premium* $\Delta(X)$ by $\Delta(X) := \mathbb{E}[X] - \text{CE}(X)$. Under RDU, the risk premium of X can be decomposed into the *pessimism premium* $\Delta_w(X)$, defined by

$$\Delta_w(X) := \mathbb{E}[X] - \int_0^\infty \xi d\left(-w(1 - F_X(\xi))\right),$$

and the *outcome premium* $\Delta_{u,w}(X)$ of X, defined by

$$\Delta_{u,w}(X) := \int_0^\infty \xi d\left(-w(1 - F_X(\xi))\right) - CE(X).$$

Indeed, one clearly has that $\Delta(X) = \Delta_w(X) + \Delta_{u,w}(X)$.

Definition 4.5. *Let V, V_1, V_2 be RDU representations as in (2.8) with utility functions u, u_1, u_2 and distortion functions w, w_1, w_2, respectively. Then,*

i) V is said to be pessimistic if for any X with bounded support, $\Delta_w(X) \geq 0$.

ii) V_1 is said to be more pessimistic than V_2 if for any X with bounded support, $\Delta_{w_1}(X) \geqslant \Delta_{w_2}(X)$.

The following proposition shows how the distortion parameter γ reflects the investor's attitude as objective $(\gamma = 1)$, pessimistic $(\gamma < 1)$ or optimistic $(\gamma > 1)$.

Proposition 4.6 (Quiggin (1993))**.** *Let* V, V_1, V_2 *be RDU representations as in (2.8) with utility functions* u, u_1, u_2 *and distortion functions* w, w_1, w_2 *given by (4.1) with distortion parameters* $\gamma, \gamma_1, \gamma_2$ *respectively. Then the following holds:*

i) V *is pessimistic if and only if* $\gamma \leqslant 1$.

ii) V_1 *is more pessimistic than* V_2 *if and only if* $\gamma_1 \leqslant \gamma_2$.

We refer to Ghossoub and He (2019) for an extended discussion and further results on comparative risk aversion for RDU preferences.

We remark that while most commonly used probability distortion functions, such as the ones introduced in Tversky and Kahneman (1992), Tversky and Fox (1995) or Prelec (1998), do not satisfy the Jin-Zhou monotonicity condition when paired with a lognormal pricing kernel, the proof of Theorem 4.1 shows that an endogenously determined probability distortion function of a forward rank-dependent performance criterion automatically satisfies this condition. In other words, while general classical, backward RDU optimization problems are typically hard to solve and rely on concavification techniques, the endogenous determination of the probability distortion by means of forward criteria provides additional structure in terms of the Jin-Zhou monotonicity. This, in turn, leads to a simpler expression for the optimal wealth process as described in Proposition 4.4. Interestingly, Xia and Zhou (2016) also find that the Jin-Zhou monotonicity condition is automatically satisfied for a representative agent when the pricing kernel is endogenously determined through an equilibrium condition of an Arrow-Debreu economy.

Finally, we comment on the form of the optimal wealth process as it plays a pivotal role in developing the upcoming construction approach. In the degenerate case, we easily deduce that $X_t^* = x$, $t \geqslant 0$. Note that the no participation effect occurs even if $\lambda_t \neq 0$, $t \geqslant 0$, as assumed herein. This is consistent with the finding that probability distortion can imply "first-order" risk aversion and thus lead to non-participation in the stock market; see for instance Polkovnichenko (2005).[2]

For the non-degenerate case, we observe that X_t^* takes a form that resembles the one in the classical setting but under a *different* probability measure, as manifested by the term $\mathbb{E}\left[\rho_t^{1-\gamma}\right]\rho_t^{\gamma}$ in the second equality in (4.4). This motivated us to introduce a new probability measure, which in turn guided us to develop a connection with the existing deterministic, time-monotone forward criteria in an auxiliary market. We present these results in the next section.

5   Construction of forward rank-dependent criteria

This section contains the main result herein. It provides a direct connection between forward rank-dependent performance criteria $\left((u_t)_{t\geqslant 0},(w_{s,t})_{0\leqslant s<t}\right)$ in the original market and deterministic, time-monotone forward criteria $(U_t)_{t\geqslant 0}$ in an auxiliary market.

5.1   The auxiliary market

For a fixed number $\gamma \geqslant 0$, we let $\mathbb{P}_\gamma$ be the unique probability measure on $(\Omega,\mathcal{F})$ satisfying, for each $t \geqslant 0$,

$$(5.1) \qquad \left.\frac{d\mathbb{P}_\gamma}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \frac{\rho_t^{1-\gamma}}{\mathbb{E}[\rho_t^{1-\gamma}]}.$$

Such a probability measure exists and is unique and equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F}_t)$, for each $t \geqslant 0$; see for instance Karatzas and Shreve (1998). We call $\mathbb{P}_\gamma$ the *γ-distorted probability measure*; see also Ma et al. (2018). In turn, the price processes (cf. (2.1)) solve

$$(5.2) \qquad dS_t^i = S_t^i\left(\gamma\mu_t^i dt + \sum_{j=1}^{N}\sigma_t^{ij}dW_{\gamma,t}^j\right), \quad t \geqslant 0,$$

for $i = 1,\dots,N$, where

$$(5.3) \qquad W_{\gamma,t}^i := (1-\gamma)\int_0^t \lambda_s^i ds + W_t^i, \quad t \geqslant 0$$

for $i = 1,\dots,N$ is a Brownian motion under $\mathbb{P}_\gamma$. Note that the market price of risk in this market is given by $\lambda_{\gamma,t} := \gamma\lambda_t$.

We now consider an auxiliary market consisting of the risk-free asset (with zero interest rate) and

N stocks whose prices evolve as in (5.2) above. We will refer to this as the γ-*distorted market.* It is complete and its pricing kernel, denoted by $\rho_{\gamma,t}$, is given by

$$(5.4) \qquad \rho_{\gamma,t} = \rho_t \frac{d\mathbb{P}}{d\mathbb{P}_\gamma} = \rho_t^\gamma \mathbb{E}[\rho_t^{1-\gamma}].$$

In this auxiliary market, we recall the associated time-monotone forward performance criteria, denoted by $U_t(x)$. It is given by

$$(5.5) \qquad U_t(x) = v(x, A_{\gamma,t}) \quad \text{with} \quad A_{\gamma,t} := \int_0^t \|\lambda_{\gamma,s}\|^2 ds.$$

The function $v(x,t)$ solves, for $x \geqslant 0$, $t \geqslant 0$,

$$v_t = \frac{1}{2} \frac{v_x^2}{v_{xx}},$$

and $v(x,0)$ must be of the form $(v')^{-1}(x,0) = \int_{0+}^\infty x^{-y} \mu(dy)$, where μ is a positive finite Borel measure. It also holds that

$$(5.6) \qquad v(x,t) = -\frac{1}{2} \int_0^t e^{-h^{-1}(x,s)+\frac{s}{2}} ds + \int_0^x e^{-h^{-1}(z,0)} dz,$$

where $h(z,t)$, $z \in \mathbb{R}$, $t \geqslant 0$ is given by

$$(5.7) \qquad h(z,t) := \int_{0+}^\infty e^{zy - \frac{1}{2}y^2 t} \mu(dy).$$

We refer the reader to Musiela and Zariphopoulou (2010a) for an extensive exposition of these results as well as detailed assumptions on the underlying measure μ.

We are now ready to present the main result herein, which connects the forward rank-dependent criteria in the original market with the deterministic, time-monotone forward criteria in the γ-distorted market and provides a *construction* method for forward rank-dependent performance criteria.

Theorem 5.1. *Let $\gamma \geq 0$. If $(u_t)_{t\geq 0}$ is a deterministic, time-monotone forward performance criterion in the γ-distorted market and the family of probability distortions $(w_{s,t})_{0\leq s<t}$ is defined by (4.1), then*

$\left((u_t)_{t \geq 0}, (w_{s,t})_{0 \leq s < t} \right)$ *is a forward rank-dependent performance criterion.*

Conversely, let $\left((u_t)_{t \geq 0}, (w_{s,t})_{0 \leq s < t} \right)$ *be a forward rank-dependent performance criterion. Then, there exists* $\gamma \geq 0$ *such that* $(u_t)_{t \geq 0}$ *is a deterministic, time-monotone forward performance criterion in the* γ-*distorted market.*

Forward rank-dependent performance criteria can thus be constructed as follows. Let $\gamma \geqslant 0$ *and consider an initial datum of the form* $(u')^{-1}(x,0) = \int_{0^+}^{\infty} x^{-y} \mu(dy)$ *for an appropriate measure* μ. *Let* $h(z,t)$ *and* $v(x,t)$ *be given by (5.7) and (5.6), respectively. Then, the pair* $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t} \right)$ *with*

$$(5.8) \qquad u_t(x) := v\left(x, \gamma^2 \int_0^t \|\lambda_s\|^2 ds \right) \quad \text{and} \quad w_{s,t}(p) := \frac{1}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma} \right]} \int_0^p \left(\left(F_{s,t}^{\rho} \right)^{-1}(q) \right)^{1-\gamma} dq$$

is a forward rank-dependent performance criterion.

We stress that, while the necessary conditions on the probability distortion function in Theorem 4.1 have been established independently of the utility function process $(u_t)_{t \geqslant 0}$, both processes $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t} \right)$ depend on the distortion parameter γ as (5.8) indicates. Indeed, $\gamma > 0$ manifests itself *both* as a parameter in the probability distortion function and in the rescaled time argument for the utility function, through the process $A_{\gamma,t}$. As $\gamma \downarrow 0$, $\lim_{\gamma \downarrow 0} A_{\gamma,t} = 0$, for all $t \geqslant 0$, and in turn $u_t(x) = u_0(x) = v(x,0)$. This is expected because when $\gamma = 0$, the risky asset prices in the $\gamma-$distorted market become martingales (cf. (5.2)) and thus no participation is expected. Indeed, the γ-distorted measure $\mathbb{P}_{\gamma}$ coincides with the risk-neutral measure when $\gamma = 0$.

Proposition 5.2. *i) Let* $\gamma > 0$ *be the investor's distortion parameter and* $h(z,t)$ *as in (5.7). Then, the associated optimal wealth,* $(X_t^*)_{t \geqslant 0}$, *and investment policy* $(\pi_t^*)_{t \geqslant 0}$ *corresponding to the forward rank-dependent performance criterion as constructed in (5.8) are given, respectively, by*

$$(5.9) \qquad X_t^* = h\left(h^{-1}(x,0) + \gamma \int_0^t \|\lambda_s\|^2 ds + \gamma \int_0^t \lambda_s dW_s, \gamma^2 \int_0^t \|\lambda_s\|^2 ds \right)$$

and

$$(5.10) \qquad \pi_t^* = \gamma \sigma_t^{-1} \lambda_t h_x \left(h^{-1}(x,0) + \gamma \int_0^t \|\lambda_s\|^2 ds + \gamma \int_0^t \lambda_s dW_s, \gamma^2 \int_0^t \|\lambda_s\|^2 ds \right).$$

ii) Let $\gamma = 0$. Then, $X_t^* = x$ and $\pi_t^* = 0$, for all $t \geqslant 0$.

We remind the reader that (5.9) and (5.10) offer an alternative expression for the optimal wealth and policy that have already been derived with different arguments in (4.4) and (4.5).

Next, we provide examples where the underlying measure μ is a single Dirac or sum of two Dirac functions.

Example 5.3. *Fix $\alpha > 0$ and let $u_0(x) = \frac{1}{1-\alpha}x^{1-\alpha}, x > 0$ for $\alpha \neq 1$ and $u_0(x) = \log x$ for $\alpha = 1$. This is equivalent to setting $\mu(dy) = \delta_{1/\alpha}$. In turn,*

$$h(x,t) = e^{\frac{x}{\alpha} - \frac{1}{2\alpha^2}t}, \quad v(x,t) = \begin{cases} \frac{1}{1-\alpha}x^{1-\alpha}e^{\frac{1}{2}(1-\frac{1}{\alpha})t}, & \alpha \neq 1, \\ \log x - \frac{1}{2}, & \alpha = 1. \end{cases}$$

Let $\gamma > 0$. Then, the pair $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$ defined, for $t \geqslant 0$ and $0 \leqslant s < t$, as

$$u_t(x) = \begin{cases} \frac{1}{1-\alpha}x^{1-\alpha}e^{\frac{1}{2}(1-\frac{1}{\alpha})\gamma^2 \int_0^t \|\lambda_s\|^2 ds}, & \alpha \neq 1, \\ \log x - \frac{1}{2}\gamma^2 \int_0^t \|\lambda_s\|^2 ds, & \alpha = 1, \end{cases} \quad w_{s,t}(p) := \frac{1}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]} \int_0^p \left(\left(F_{s,t}^\rho\right)^{-1}(q)\right)^{1-\gamma} dq$$

is a forward rank-dependent criterion. Furthermore, from (5.9) and (5.10) we deduce that

$$(5.11) \qquad X_t^* = x e^{\frac{\gamma}{\alpha}\left(1-\frac{\gamma}{2\alpha}\right)\int_0^t \|\lambda_s\|^2 ds + \frac{\gamma}{\alpha}\int_0^t \lambda_s dW_s} \qquad and \qquad \pi_t^* = \frac{\gamma}{\alpha}\sigma_t^{-1}\lambda_t X_t^*.$$

Direct calculations also yield that, for $t \geqslant 0$,

$$(5.12) \qquad X_t^* = x \frac{\rho_t^{-\frac{\gamma}{\alpha}}}{\mathbb{E}\left[\rho_t^{1-\frac{\gamma}{\alpha}}\right]},$$

with the pricing kernel given in (2.3).

Example 5.4. *Fix $0 < \theta < 1$ and let $u_0(x) = \frac{2^{-\theta}}{\theta(1+\theta)}\left(\sqrt{4x+1} - 1\right)^\theta \left(\theta\sqrt{4x+1} + 1\right)$, which is equivalent to setting $\mu(dy) = \delta_{\frac{1}{1-\theta}} + \delta_{\frac{2}{1-\theta}}$. Then,*

$$h(x,t) = e^{\frac{1}{1-\theta}x - \frac{1}{2}\frac{1}{(1-\theta)^2}t} + e^{\frac{2}{1-\theta}x - \frac{2}{(1-\theta)^2}t}$$

and

$$v(x,t) = \frac{2^{-\theta}}{\theta(1+\theta)} e^{\frac{t}{2}\left(1-\frac{3}{1-\theta}+\frac{2}{(1-\theta)^2}\right)} \left(\sqrt{4xe^{-\frac{t}{(1-\theta)^2}}+1}-1\right)^{\theta} \left(\theta\sqrt{4xe^{-\frac{t}{(1-\theta)^2}}+1}+1\right).$$

Let $\gamma > 0$. Then, the pair $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s<t}\right)$ defined, for $t \geqslant 0$ and $0 \leqslant s < t$, by

$$u_t(x) = \frac{2^{-\theta}}{\theta(1+\theta)} e^{\frac{\gamma^2}{2}\left(1-\frac{3}{1-\theta}+\frac{2}{(1-\theta)^2}\right)\int_0^t \|\lambda_s\|^2 ds}$$
$$\times \left(\sqrt{4xe^{-\frac{\gamma^2}{(1-\theta)^2}\int_0^t \|\lambda_s\|^2 ds}+1}-1\right)^{\theta} \left(\theta\sqrt{4xe^{-\frac{\gamma^2}{(1-\theta)^2}\int_0^t \|\lambda_s\|^2 ds}+1}+1\right)$$

and

$$w_{s,t}(p) := \frac{1}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]} \int_0^p \left(\left(F_{s,t}^{\rho}\right)^{-1}(q)\right)^{1-\gamma} dq$$

is a forward rank-dependent criterion. Furthermore, from (5.9) and (5.10) we deduce that

$$(5.13) \qquad \begin{aligned} X_t^* &= \frac{1}{2}\left(\sqrt{4x+1}-1\right) e^{\frac{\gamma}{1-\theta}\left(1-\frac{\gamma}{2(1-\theta)}\right)\int_0^t \|\lambda_s\|^2 ds + \frac{\gamma}{1-\theta}\int_0^t \lambda_s dW_s} \\ &+ \frac{1}{4}\left(\sqrt{4x+1}-1\right)^2 e^{\frac{2\gamma}{1-\theta}\left(1-\frac{\gamma}{1-\theta}\right)\int_0^t \|\lambda_s\|^2 ds + \frac{2\gamma}{1-\theta}\int_0^t \lambda_s dW_s} \end{aligned}$$

and

$$(5.14) \qquad \begin{aligned} \pi_t^* &= \frac{\gamma}{2(1-\theta)}\sigma_t^{-1}\lambda_t\Bigg(\left(\sqrt{4x+1}-1\right) e^{\frac{\gamma}{1-\theta}\left(1-\frac{\gamma}{2(1-\theta)}\right)\int_0^t \|\lambda_s\|^2 ds + \frac{\gamma}{1-\theta}\int_0^t \lambda_s dW_s} \\ &+ \left(\sqrt{4x+1}-1\right)^2 e^{\frac{2\gamma}{1-\theta}\left(1-\frac{\gamma}{1-\theta}\right)\int_0^t \|\lambda_s\|^2 ds + \frac{2\gamma}{1-\theta}\int_0^t \lambda_s dW_s}\Bigg). \end{aligned}$$

6 Relations with the dynamic utility approach

Motivated by the construction of time-consistent rank-dependent criteria we here revisit the classical, backward RDU optimization problem. Specifically, we follow the *dynamic utility approach* developed in Karnam et al. (2017) and explore whether it is possible to derive a family of dynamic RDU optimization problems under which the initial investment policy remains optimal over time. In a recent work related to this paper, Ma et al. (2018) employ this approach to derive a time-consistent conditional expectation under probability distortion.

We emphasize that, for this section only, we deviate from the theme of forward criteria in that we consider a classical RDU maximization problem of the form

$$(6.1) \qquad \max_{\pi \in \mathcal{A}} \quad \int_0^\infty u_{0,T}(\xi) d\left(-w_{0,T}\left(1 - F_{X_T^{x,\pi}}(\xi)\right)\right)$$

with $dX_r^{x,\pi} = \pi_r^\top \mu_r dr + \pi_r^\top \sigma_r dW_r$, $r \in [0,T]$ and $X_0^{x,\pi} = x > 0$ for a fixed time horizon $T > 0$, utility function $u_{0,T} \in \mathcal{U}$ and probability distortion function $w_{0,T} \in \mathcal{W}$. We assume that there exists an optimal policy π^* to problem (6.1), which are unique due to the strict concavity of $u_{0,T}$ and the non-atomic distribution of the pricing kernel $\rho_{0,T}$, and make the following definition of *dynamic rank-dependent utility processes*.

Definition 6.1. *A family of utility functions $u_{t,T} \in \mathcal{U}$, $t \in (0,T)$ and probability distortion functions $w_{t,T} \in \mathcal{W}$, $t \in (0,T)$ is called a dynamic rank-dependent utility process for $u_{0,T} \in \mathcal{U}$ and $w_{0,T} \in \mathcal{W}$ over the time horizon $T > 0$ if $\lim_{t \searrow 0} u_{t,T} = u_{0,T}$, $\lim_{t \searrow 0} w_{t,T} = w_{0,T}$ pointwise and the ==(unique)== optimal policy π^* for (6.1) also solves, for any $t \in (0,T)$,*

$$(6.2) \qquad \max_{\pi \in \mathcal{A}(\pi^*,t)} \quad \int_0^\infty u_{t,T}(\xi) d\left(-w_{t,T}\left(1 - F_{X_T^{x,\pi}|\mathcal{F}_t}(\xi)\right)\right)$$

with $dX_r^{x,\pi} = \pi_r^\top \mu_r dr + \pi_r^\top \sigma_r dW_r$, $r \in [0,T]$ and $X_0^{x,\pi} = x > 0$.

Our definition of dynamic rank-dependent utility processes relies on the existence of an optimal policy for the initial problem (6.1). Karnam et al. (2017) do not rely on this assumption, and being able to determine the value of a time-inconsistent stochastic control problem without the assumption of an optimal control is indeed one of their main contributions. However, for our specific problem of rank-dependent utility maximization in a complete, continuous-time financial market with deterministic coefficients, conditions for the existence of an optimal policy are well known and not restrictive; see for instance Xia and Zhou (2016) and Xu (2016).

We also deviate from Karnam et al. (2017) in that we restrict the objective of the dynamic family of problems (6.2) to belong to the same class of preferences as the initial problem (6.1), namely rank-dependent utility preferences. In Karnam et al. (2017) on the other hand, the objective is allowed to vary more freely. However, we believe that within the dynamic utility approach, it is an interesting mathematical and economic question whether one is able to construct a dynamic utility belonging to

the same class of preference functionals as the initial preferences, and this is exactly the question we want to address. There have been some positive results in this regard for time-inconsistent mean-risk portfolio optimization problems; see Cui et al. (2012) for the mean-variance and Strub et al. (2019) for the mean-CVaR problem.

Ma et al. (2018) introduce the notion of a *dynamic distortion function* under which the distorted, nonlinear conditional expectation is time-consistent in the sense that the tower-property holds. Specifically, they consider an Itô process described by the stochastic differential equation

$$(6.3) \qquad Y_t = y_0 + \int_0^t b(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dW_s, \quad 0 \leqslant t \leqslant T,$$

and a given family of probability distortion functions $(w_{0,t})_{0 \leqslant t \leqslant T}$, where $w_{0,t}$ applies over $[0, t]$. Under technical conditions, they are able to derive a family of (random) probability distortion functions $(w_{s,t})_{0 \leqslant s \leqslant t \leqslant T}$ such that $w_{s,t}$ is $\mathcal{F}_s$-measurable, for any $0 \leqslant s \leqslant t \leqslant T$, and the tower-property

$$\mathcal{E}_{r,t}[g(Y_t)] = \mathcal{E}_{r,s} \left[\mathcal{E}_{s,t} \left[g(Y_t) \right] \right], \quad 0 \leqslant r \leqslant s \leqslant t \leqslant T,$$

holds for any continuous, bounded, increasing and nonnegative function g, where $\mathcal{E}_{s,t}$ denotes the nonlinear conditional expectation

$$\mathcal{E}_{s,t}[\xi] = \int_0^\infty w_{s,t} \left(\mathbb{P} \left[\xi \geqslant x | \mathcal{F}_s \right] \right) dx.$$

We remark that the tower-property plays an important role in the theory on dynamic risk measures; see for instance Bielecki et al. (2017) for a survey and refer to Ma et al. (2018) for further applications and discussions.

The important difference between the portfolio optimization problem we study herein and the setting of Ma et al. (2018) is that the Itô process described through the stochastic differential equation (6.3) is given and fixed. In particular, there is no control or investment policy. Moreover, the construction of the family of probability distortion functions $(w_{s,t})_{0 \leqslant s \leqslant t \leqslant T}$ in Ma et al. (2018) depends on the drift b and volatility σ in (6.3), cf. Theorem 5.2 therein. In our setting, on the other hand, the drift and volatility of the wealth process are controlled by the investment policy π.

As a corollary to our results on forward rank-dependent performance processes, it follows that,

when the utility function $u_{t,T} = u_{0,T}$ for any $t \in (0, T)$ and the optimal policy is not degenerate, we can construct a dynamic rank-dependent utility process if and only if the probability distortion function belongs to the class of Wang (2000). The following theorem shows that this result remains valid even if one allows both the utility function $u_{t,T}$ and probability distortion function $w_{t,T}$ to depend on the initial time of the investment.

Theorem 6.2. *Consider a fixed time-horizon T, utility function $u_{0,T} \in \mathcal{U}$ and probability distortion function $w_{0,T} \in \mathcal{W}$, and suppose that the optimal solution to (6.1) exists. Depending on the probability distortion function $w_{0,T}$, we have the following two cases:*

i) If

$$(6.4) \qquad w_{0,T}(p) \leqslant \mathbb{E}\left[\rho_{0,T} \mathbf{1}_{\left\{\rho_{0,T} \leqslant \left(F^{\rho}_{0,T}\right)^{-1}(p)\right\}}\right], \quad p \in [0,1],$$

then a family of utility functions $u_{t,T} \in \mathcal{U}$, $t \in (0,T)$, together with a family of probability distortion functions $w_{t,T} \in \mathcal{W}$, $t \in (0,T)$, is a dynamic rank-dependent utility process for $u_{0,T}$ and $w_{0,T}$ if and only if the family of probability distortion functions satisfies

$$w_{t,T}(p) \leqslant \mathbb{E}\left[\rho_{t,T} \mathbf{1}_{\left\{\rho_{t,T} \leqslant \left(F^{\rho}_{t,T}\right)^{-1}(p)\right\}}\right], \quad p \in [0,1],$$

for any $t \in [0,T)$.

ii) If (6.4) does not hold, then a family of utility functions $u_{t,T} \in \mathcal{U}$, $t \in (0,T)$ and probability distortion functions $w_{t,T} \in \mathcal{W}$, $t \in (0,T)$ with $\lim_{t \searrow 0} u_{t,T} = u_{0,T}$ and $\lim_{t \searrow 0} w_{t,T} = w_{0,T}$ is a dynamic rank-dependent utility process for $u_{0,T}$ and $w_{0,T}$ if and only if there exists a deterministic process $\gamma_t \geqslant 0$, $t \in [0,T)$ that is continuous at zero and such that

$$(6.5) \quad w_{t,T}(p) = \frac{1}{\mathbb{E}\left[\rho_{t,T}^{1-\gamma_t}\right]} \int_0^p \left(\left(F^{\rho}_{t,T}\right)^{-1}(q)\right)^{1-\gamma_t} dq = \Phi\left(\Phi^{-1}(p) + (\gamma_t - 1)\sqrt{\int_t^T \|\lambda_r\|^2 dr}\right)$$

for any $t \in [0,T)$, and the absolute risk aversion process of the dynamic utility function satisfies

$$(6.6) \qquad -\frac{1}{\gamma_t}\frac{u''_{t,T}(x)}{u'_{t,T}(x)} = -\frac{1}{\gamma_0}\frac{u''_{0,T}(x)}{u'_{0,T}(x)}$$

for any $t \in [0, T)$ and $x > 0$.

Theorem 6.2 shows in particular that, when there is some non-zero investment in the risky asset, an extension of the construction of dynamic distortion functions of Ma et al. (2018) to controlled processes is in general only possible if the initial probability distortion function belongs to the family introduced in Wang (2000). Moreover, in order to maintain time-consistency, the utility function and probability distortion function must be coordinated with each other at different times through the relationship (6.6). This dynamic constraint connects the risk aversion of the dynamic utility function with the dynamic parameter of the probability distortion function. Recall from Proposition 4.6 that the distortion process γ_t reflects the investor's attitude as objective ($\gamma_t = 1$), pessimistic ($\gamma_t < 1$) or optimistic ($\gamma_t > 1$). The dynamic constraint (6.6) can thus be interpreted as follows: In order to be time-consistent, the investor's utility function must have a higher degree of risk aversion if she becomes less pessimistic (i.e., γ_t increases) and a lower degree of risk aversion if she becomes more pessimistic (i.e., γ_t decreases). Moreover, the relationship between the risk aversion and pessimism as reflected in the parameter γ_t is linear.[3]

Note that, if the risk aversion of the utility function is time-invariant, then $\gamma_t = \gamma_0$, $t \in (0, T)$. This implies that the effect of probability distortion (measured by the time parameter t) thus must decay over time at the order of $\sqrt{T-t}$ (assuming that $(\|\lambda_s\|)_{s \geq 0}$ is bounded away from 0), as it follows from (6.5). In particular, the probability distortion effect should disappear when the remaining time approaches zero. On the other hand, if the probability distortion function is time-invariant, we must have $\gamma_t = 1 + (\gamma_0 - 1)\sqrt{\frac{\int_0^T \|\lambda_s\|^2 ds}{\int_t^T \|\lambda_s\|^2 ds}}$, $t \in (0, T)$. Therefore, the measure of absolute risk aversion of the utility function must increase at the order of $1/\sqrt{T-t}$ as t approaches T.

7 Conclusions

We introduced the concept of forward rank-dependent performance processes and thereby extended the study of forward performance process to settings involving probability distortions. Forward rank-dependent performance criteria are herein taken to be deterministic. This made the problem tractable but also guided us in building a fundamental connection with deterministic, time-monotone forward criteria in an auxiliary market.

We provided two alternative definitions, in terms of time-consistency and performance value preser-

vation, respectively. We, then, provided a complete characterization of the viable probability distortion functions. Specifically, we showed that for the non-degenerate case (non-zero risky allocation) the probability distortion function resembles the one introduced by Wang (2000) but modified appropriately to capture the market evolution. We also showed that it satisfies the Jin-Zhou monotonicity condition.

We further derived the optimal wealth process, which motivated the introduction of the distorted probability measure. We in turn established the key result, namely, a one-to-one correspondence between forward rank-dependent performance processes and deterministic, time-monotone forward performance criteria in an auxiliary market (under the distorted measure). Subsequently, this result allowed to build on earlier findings of Musiela and Zariphopoulou (2009, 2010a) to characterize forward rank-dependent performance criteria and their optimal processes.

Finally, we related our results to the dynamic utility approach of Karnam et al. (2017) and, specifically, the dynamic distortion function of Ma et al. (2018). While Ma et al. (2018) are able to construct a dynamic distortion function that is time-consistent in the sense that the tower-property holds for a general class of initial probability distortion functions and given and fixed state process, our results show that, when the wealth processes is controlled by the investment policy and there is investment in the risky asset, time-consistent investment under probability distortion is possible if and only if the probability distortion belongs to the class of Wang (2000).

Extending the deterministic case to the stochastic one is by no means trivial, as there are both conceptual and technical challenges. Indeed, if one would allow u_t and/or $w_{s,t}$ to be $\mathcal{F}_t$-measurable, in direct analogy to the forward performance case, then the value of a prospect as specified in (2.8) would be a random variable. Simply taking the expectation of this random value of the prospect seems ad hoc. In particular, it seems unreasonable that an agent distorting probabilities to evaluate a prospect would subsequently apply a mere linear expectation to average the resulting value of the prospect.

There are a number of possible directions for future research. First, one might consider forward cumulative prospect theory performance criteria, which incorporate two further behavioral phenomena, namely reference dependence and loss aversion. There exists a rich and active literature on how the reference point evolves in time, and the results herein indicate that the framework of forward preferences seems suitable to derive conditions under which a time-varying reference point does not lead to time-inconsistent investment policies.

A second possible direction is to consider discrete-time rank-dependent forward criteria. Indeed, much of the research in behavioral finance and economics assumes a discrete-time setting. Furthermore, considering discrete-time forward criteria already leads to valuable insights if there is no probability distortion.

Finally, it would be interesting to extend the framework of forward rank-dependent performance criteria beyond problems of portfolio selection. Possible problems could for example come from the areas of pricing and hedging, insurance, optimal contracting, real world options or in situations where there is competition among different players.

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Notes

[1]In a recent work by Ma et al. (2018) the concept of nonlinear expectation and time-consistency was studied in a specific setting. We refer to these results in Section 6 herein, where we also provide some new results in this direction.

[2]The term first-order risk aversion was coined in Segal and Spivak (1990) and means that, for a prospect X with $\mathbb{E}[X] = 0$, the risk premium of ϵX is proportional to ϵ as ϵ goes to zero.

[3]To the best of our knowledge, there is no time series analysis of the joint estimation of risk aversion and pessimism under rank-dependent utility preferences in the existing literature. It is thus an open question whether the relationship between pessimism and risk aversion that is required by time-consistency can be substantiated empirically.

A   Appendix. Proofs

A.1   Proof of Proposition 3.3

If $w(p) = p, p \in [0, 1]$, and $u \in \mathcal{U}$, then for any prospect X,

$$\int_0^\infty u(\xi) d\left(-w\left(1 - F_{X|\mathcal{F}_t}(\xi)\right)\right) = \mathbb{E}\left[u(X)|\mathcal{F}_t\right].$$

This completes the proof. $\qquad\square$

A.2   Proof of Proposition 3.5

Suppose that $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$ is a forward rank-dependent performance criterion. We argue by contradiction, assuming that π^* is not optimal for problem (3.5) for some $0 \leqslant s < t$ and $x > 0$. Then, recalling $\mathcal{A}$ and $\mathcal{A}(\pi^*, s)$ as in (2.6) and (2.7), respectively, we can find some policy $\widehat{\pi} \in \mathcal{A}(\pi^*, s)$ such that

$$\int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\widehat{\pi}}|\mathcal{F}_s}(\xi)\right)\right) > \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi^*}|\mathcal{F}_s}(\xi)\right)\right)$$

$$= u_s\left(X_s^{x,\pi^*}\right) = u_s\left(X_s^{x,\widehat{\pi}}\right),$$

on a set $A_s \in \mathcal{F}_s$ with $\mathbb{P}\left[A_s\right] > 0$. This, however, contradicts ii) of Definition 3.2. Thus, π^* is optimal for problem (3.5) for any $0 \leqslant s < t$ and $x > 0$, and it then follows that the optimal value of (3.5) is given by $u_s\left(X_s^{x,\pi^*}\right)$.

Next, suppose that the pair $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$ is a time-consistent rank-dependent performance criterion preserving the performance value. We only have to argue that for any admissible policy π, any $t \geqslant 0$ and $0 \leqslant s < t$, and $x > 0$, the inequality

$$\int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\pi}|\mathcal{F}_s}(\xi)\right)\right) \leqslant u_s\left(X_s^{x,\pi}\right)$$

holds. We argue by contradiction. Suppose that there exists policy $\bar{\pi} \in \mathcal{A}$, time $t \geqslant 0$ and an s with

$0 \leqslant s < t$, an $x > 0$, a set $\tilde{A}_s \in \mathcal{F}_s$ with $\mathbb{P}\left[\tilde{A}_s\right] > 0$ and $\varepsilon > 0$ such that

$$\int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^{x,\bar{\pi}}|\mathcal{F}_s}(\xi;\omega)\right)\right) > u_s\left(X_s^{x,\bar{\pi}}(\omega)\right) + \varepsilon, \quad \omega \in \tilde{A}_s.$$

Using results on classical rank-dependent utility maximization (cf. Theorem 2.2), the wealth process corresponding to the policy π^* is given by

$$(\text{A.1}) \qquad X_t^{y,\pi^*} = \left(u_t'\right)^{-1}\left(\eta_t^*(y)\hat{N}_{0,t}'\left(1 - w_{0,t}\left(F_{0,t}^\rho(\rho_{0,t})\right)\right)\right),$$

for any initial wealth $y > 0$ and any $t > 0$, where $\hat{N}_{0,t}$ is the concave envelope of

$$N_{0,t}(z) := -\int_0^{w_{0,t}^{-1}(1-z)} (F_{0,t}^\rho)^{-1}(r)dr, \quad z \in [0,1],$$

and $\eta_t^*(y)$ is the one such that $\mathbb{E}\left[\rho_t X_t^{y,\pi^*}\right] = y$.

Note that the range of $\hat{N}_{0,t}'\left(1 - w_{0,t}\left(F_{0,t}^\rho(\rho_{0,t})\right)\right)$ does not depend on the initial wealth $y > 0$. Furthermore, using results from Jin and Zhou (2008) or He and Zhou (2011, 2016), we obtain that $\eta_{0,t}^*(y) = u_0'(y)$, which has range $(0,\infty)$ due to the Inada condition.

Therefore, for any $\delta > 0$, there exist $a, y_0 > 0$ such that the event

$$A_s := \left\{X_s^{x,\bar{\pi}} \in \left[a, a + \frac{\delta}{2}\right], X_s^{y_0,\pi^*} \in \left[a + \frac{\delta}{2}, a + \delta\right]\right\} \cap \tilde{A}_s$$

satisfies $A_s \in \mathcal{F}_s$ and $\mathbb{P}[A_s] > 0$. Indeed, we fix $\delta > 0$ and note that because

$$0 < \mathbb{P}\left[\tilde{A}_s\right] = \mathbb{P}\left[\left(\bigcup_{a \in \mathbb{Q}^+}\left\{X_s^{x,\bar{\pi}} \in \left[a, a + \frac{\delta}{2}\right]\right\}\right) \cap \tilde{A}_s\right] \leqslant \sum_{a \in \mathbb{Q}^+} \mathbb{P}\left[\left\{X_s^{x,\bar{\pi}} \in \left[a, a + \frac{\delta}{2}\right]\right\} \cap \tilde{A}_s\right],$$

there exists $a > 0$ such that $\mathbb{P}\left[\left\{X_s^{x,\bar{\pi}} \in \left[a, a + \frac{\delta}{2}\right]\right\} \cap \tilde{A}_s\right] > 0$. We fix such an $a > 0$. Following a similar argument, we obtain that there exists $b > 0$ such that

$$\mathbb{P}\left[\left\{\hat{N}_{0,s}'\left(1 - w_{0,s}\left(F_{0,s}^\rho(\rho_{0,s})\right)\right) \in \left[b, \frac{u_s'(a + \delta/2)}{u_s'(a + \delta)}b\right]\right\} \cap \left\{X_s^{x,\bar{\pi}} \in \left[a, a + \frac{\delta}{2}\right]\right\} \cap \tilde{A}_s\right] > 0.$$

Choosing $y_0 = (u_0')^{-1}\left(\frac{u_s'(a+\delta)}{b}\right)$, we obtain $\eta_s^*(y_0) = u_0'(y_0) = u_s'(a+\delta)/b$. Consequently, we conclude from (A.1) (with time index t replaced by s) that $\hat{N}_{0,s}'\left(1 - w_{0,s}\left(F_{0,s}^\rho(\rho_{0,s})\right)\right) \in \left[b, \frac{u_s'(a+\delta/2)}{u_s'(a+\delta)}b\right]$ implies $X_s^{y_0,\pi^*} \in \left[a + \frac{\delta}{2}, a + \delta\right]$. Thus, with this $a, y_0 > 0$, we have that $\mathbb{P}[A_s] > 0$.

Next, we introduce the policy

$$\vartheta_r(\omega; y) := \pi_r^*(\omega; y) + (\bar{\pi}_r(\omega; x) - \pi_r^*(\omega; y))\left(\mathbf{1}_{[s,\infty)}(r) \times \mathbf{1}_{A_s}(\omega) \times \mathbf{1}_{y=y_0}\right).$$

We then have that $\vartheta \in \mathcal{A}(\pi^*, s)$ because $X_s^{y_0,\pi^*} \geqslant X_s^{x,\bar{\pi}}$ on A_s. In turn, for $\omega \in A_s$ we obtain

$$
\begin{aligned}
\int_0^\infty u_t(\xi)d\left(-w_{s,t}\left(1 - F_{X_t^{y_0,\vartheta}|\mathcal{F}_s}(\xi;\omega)\right)\right) &\geqslant \int_0^\infty u_t(\xi)d\left(-w_{s,t}\left(1 - F_{X_t^{x,\bar{\pi}}|\mathcal{F}_s}(\xi;\omega)\right)\right) \\
&> u_s\left(X_s^{x,\bar{\pi}}(\omega)\right) + \varepsilon \\
&\geqslant u_s\left(X_s^{y_0,\pi^*}(\omega) - \delta\right) + \varepsilon \\
&\geqslant u_s\left(X_s^{y_0,\pi^*}(\omega)\right) \\
&= \int_0^\infty u_t(\xi)d\left(-w_{s,t}\left(1 - F_{X_t^{y_0,\pi^*}|\mathcal{F}_s}(\xi;\omega)\right)\right),
\end{aligned}
$$

where the first inequality is the case because $X_s^{y_0,\pi^*} \geqslant X_s^{x,\bar{\pi}}$ on A_s and $\vartheta_r(\omega; y_0) = \bar{\pi}_r(\omega; x), r \in [s,\infty)$ for $\omega \in A_s$, the third inequality is the case because $X_s^{y_0,\pi^*} \leq X_s^{x,\bar{\pi}} + \delta$ on A_s, and the last inequality holds for sufficiently small δ and all $\omega \in A_s$ because u_s is uniformly continuous on the compact interval $[a - \delta/2, a]$. This, however, contradicts the optimality of π^* and we conclude. $\qquad\square$

A.3 Proof of Theorem 4.1

From Proposition 3.5 we have that $\left((u_t)_{t\geqslant 0}, (w_{s,t})_{0\leqslant s<t}\right)$ is a time-consistent rank-dependent performance criterion preserving the performance value. We first show that the family of probability distortion functions either satisfies the Jin-Zhou monotonicity condition, namely that the function $\frac{(F_{s,t}^\rho)^{-1}(\cdot)}{w_{s,t}'(\cdot)}$ is nondecreasing for any $0 \leqslant s < t$, or inequality (4.2) holds.

Because the market is complete, π^* is optimal for (3.5) if and only if the corresponding wealth

process X^* solves

$$
(A.2) \qquad \max_X \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X|\mathcal{F}_s}(\xi)\right)\right)
$$

with $\mathbb{E}\left[\rho_{s,t} X \,|\, \mathcal{F}_s\right] = X_s^*$, $X \geqslant 0$, and X being $\mathcal{F}_t$-measurable, for any $0 \leqslant s < t$.

Note that (A.2) is a family of random optimization problems with different initial and terminal times, s and t, and state-dependent initial state and objective function. However, this does not impose any difficulty since both the initial state and objective are known at time $s \geqslant 0$; see for example Chapter 4, Section 3 in Yong and Zhou (1999).

Recall that, according to Theorem 2.2, the optimal wealth for (A.2) is given by

$$
X_t^{s,t} = (u_t')^{-1} \left(\eta_{s,t}^*(X_s^*) \hat{N}_{s,t}' \left(1 - w_{s,t}\left(F_{s,t}^\rho(\rho_{s,t})\right)\right) \right),
$$

where $\hat{N}_{s,t}$ is the concave envelope of

$$
(A.3) \qquad N_{s,t}(z) := -\int_0^{w_{s,t}^{-1}(1-z)} (F_{s,t}^\rho)^{-1}(r)\,dr, \quad z \in [0,1].
$$

and $\eta_{s,t}^*(y) > 0$ is given by the implicit function theorem as the unique continuously differentiable, strictly decreasing function satisfying

$$
\mathbb{E}\left[\rho_{s,t}(u_t')^{-1}\left(\eta_{s,t}^*(y)\hat{N}_{s,t}'\left(1 - w_{s,t}\left(F_{s,t}^\rho(\rho_{s,t})\right)\right)\right)\right] = y.
$$

Due to the independence of X_s^* and $\rho_{s,t}$ we then in particular have that $\mathbb{E}\left[\rho_{s,t} X_t^{s,t} \,\middle|\, \mathcal{F}_s\right] = X_s^*$.

The time-consistency property is satisfied if and only if $X_t^{s,t}\left(X_s^*(x)\right) = X_t^*(x)$ for any $0 \leqslant s < t$ and $x > 0$. By the above expression for the optimal wealth process we therefore have that

$$
\begin{aligned}
(A.4) \qquad & \eta_{s,t}^*\left(\left(u_s'\right)^{-1}\left(\eta_{0,s}^*(x)\hat{N}_{0,s}'\left(1 - w_{0,s}\left(F_{0,s}^\rho(\rho_{0,s})\right)\right)\right)\right) \hat{N}_{s,t}'\left(1 - w_{s,t}\left(F_{s,t}^\rho(\rho_{s,t})\right)\right) \\
& = \eta_{0,t}^*(x)\hat{N}_{0,t}'\left(1 - w_{0,t}\left(F_{0,t}^\rho(\rho_{0,t})\right)\right).
\end{aligned}
$$

Next, we define the auxiliary functions $h_{s,t}^{1,x}, h_{s,t}^2, h_{s,t}^{3,x} : (0, \infty) \to (0, \infty)$ by

$$h_{s,t}^{1,x}(y) = \eta_{s,t}^* \left((u_s')^{-1} \left(\eta_{0,s}^*(x) \hat{N}_{0,s}' \left(1 - w_{0,s} \left(F_{0,s}^\rho(y) \right) \right) \right) \right),$$

$$h_{s,t}^2(y) = \hat{N}_{s,t}' \left(1 - w_{s,t} \left(F_{s,t}^\rho(y) \right) \right),$$

$$h_{s,t}^{3,x}(y) = \eta_{0,t}^*(x) \hat{N}_{0,t}' \left(1 - w_{0,t} \left(F_{0,t}^\rho(y) \right) \right).$$

Because $\rho_{0,s}$ and $\rho_{s,t}$ are independent, $\rho_{0,s} \rho_{s,t} = \rho_{0,t}$, and supported on $(0, \infty)$ for $0 \leqslant s < t$, we deduce firstly for Lebesgue almost everywhere, and then by continuity of $h_{s,t}^{1,x}, h_{s,t}^2, h_{s,t}^{3,x}$ for any $y, z > 0$, that

(A.5) $$h_{s,t}^{1,x}(y) h_{s,t}^2(z) = h_{s,t}^{3,x}(yz).$$

Next, suppose that there exist y_1, y_2 with $0 < y_1 < y_2$ and such that $h_{s,t}^{1,x}(y_1) = h_{s,t}^{1,x}(y_2)$. Then, equality (A.5) together with the monotonicity of $\hat{N}_{0,t}'$ imply that $h_{s,t}^{3,x}$, and in turn $h_{s,t}^{1,x}$ and $h_{s,t}^2$, are constant. Hence, $\hat{N}_{s,t}'(z)$ must be constant in z and, furthermore, this constant must be one because $\hat{N}_{s,t}(0) = N_{s,t}(0) = -1$ and $\hat{N}_{s,t}(1) = N_{s,t}(1) = 0$. This occurs if and only if

$$N_{s,t}(z) \leqslant z - 1$$

for all $z \in [0, 1]$. Using the definition of $N_{s,t}$ in (A.3) and substituting $x = w_{s,t}^{-1}(1 - z)$ yield that the above inequality is equivalent to (4.2). The same argument can be made if there exist z_1, z_2 with $0 < z_1 < z_2$ and such that $h_{s,t}^2(z_1) = h_{s,t}^2(z_2)$. Similarly, if there exist ξ_1, ξ_2 with $0 < \xi_1 < \xi_2$ and $h_{s,t}^{3,x}(\xi_1) = h_{s,t}^{3,x}(\xi_2)$, then $h_{s,t}^{1,x}(\xi_1) h_{s,t}^2(1) = h_{s,t}^{3,x}(\xi_1) = h_{s,t}^{3,x}(\xi_2) = h_{s,t}^{1,x}(\xi_2) h_{s,t}^2(1)$ and the statement follows.

Note that $h_{s,t}^{1,x}, h_{s,t}^2$ and $h_{s,t}^{3,x}$ are nondecreasing because $\eta_{s,t}^*$ and $(u_s')^{-1}$ are strictly decreasing and $\hat{N}_{0,s}, \hat{N}_{s,t}$, and $\hat{N}_{0,t}$ are concave. The above argument then shows that there are two cases: either the family of probability distortion functions satisfies (4.2) or $h_{s,t}^{1,x}, h_{s,t}^2$, and $h_{s,t}^{3,x}$ are strictly increasing. On the other hand, the latter case implies that $N_{s,t}(\cdot)$ is concave, which is equivalent to the probability distortion $w_{s,t}$ satisfying the Jin-Zhou monotonicity condition, namely, that the function $\frac{\left(F_{s,t}^\rho \right)^{-1}(\cdot)}{w_{s,t}'(\cdot)}$ is nondecreasing.

Next, we focus on the case in which $h_{s,t}^{1,x}, h_{s,t}^2$, and $h_{s,t}^{3,x}$ are strictly increasing. Then $N_{0,s}, N_{s,t}$, and $N_{0,t}$ are concave and, thus, coincide with $\hat{N}_{0,s}, \hat{N}_{s,t}$, and $\hat{N}_{0,t}$, respectively. Then, a straightforward

computation shows that the functions $h_{s,t}^{1,x}$, $h_{s,t}^2$ and $h_{s,t}^{3,x}$ simplify to

$$h_{s,t}^{1,x}(y) = \eta_{s,t}^* \left((u_s')^{-1} \left(\eta_{0,s}^*(x) \frac{y}{w_{0,s}' \left(F_{0,s}^\rho (y) \right)} \right) \right),$$

$$h_{s,t}^2(y) = \frac{y}{w_{s,t}' \left(F_{s,t}^\rho (y) \right)} \quad \text{and} \quad h_{s,t}^{3,x}(y) = \eta_{0,t}^*(x) \frac{y}{w_{0,t}' \left(F_{0,t}^\rho (y) \right)}.$$

Taking $y = 1$ in (A.5) yields $h_{s,t}^{3,x}(z) = h_{s,t}^{1,x}(1) h_{s,t}^2(z)$ while taking $z = 1$ gives $h_{s,t}^{3,x}(y) = h_{s,t}^{1,x}(y) h_{s,t}^2(1)$. Combining the two gives

$$h_{s,t}^{3,x}(yz) = h_{s,t}^{1,x}(y) h_{s,t}^2(z) = \frac{h_{s,t}^{3,x}(y) h_{s,t}^{3,x}(z)}{h_{s,t}^{1,x}(1) h_{s,t}^2(1)} = \frac{h_{s,t}^{3,x}(y) h_{s,t}^{3,x}(z)}{h_{s,t}^{3,x}(1)}.$$

Next, we define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) := \log \left(h_{s,t}^{3,x}(e^z) \right) - \log h_{s,t}^{3,x}(1)$. We deduce that g satisfies Cauchy's functional equation $g(y + z) = g(y) + g(z)$. Indeed,

$$
\begin{aligned}
g(y + z) &= \log \left(h_{s,t}^{3,x} \left(e^{y+z} \right) \right) - \log h_{s,t}^{3,x}(1) \\
&= \log \left(\frac{h_{s,t}^{3,x}(e^y) h_{s,t}^{3,x}(e^z)}{h_{s,t}^{3,x}(1)} \right) - \log h_{s,t}^{3,x}(1) \\
&= \log \left(h_{s,t}^{3,x}(e^y) \right) + \log \left(h_{s,t}^{3,x}(e^z) \right) - \log \left(h_{s,t}^{3,x}(1) \right) - \log h_{s,t}^{3,x}(1) \\
&= g(y) + g(z),
\end{aligned}
$$

for any $y, z \in \mathbb{R}$. Because g is continuous, there must be a $\gamma \in \mathbb{R}$ such that $g(z) = \gamma z$, $z \in \mathbb{R}$. This, in turn, yields that for $z > 0$,

$$h_{s,t}^{3,x}(z) = h_{s,t}^{3,x}(1) z^\gamma.$$

Next, we note that γ must be positive as $h_{s,t}^{3,x}$ is strictly increasing. We, therefore, obtain that, for $p \in [0, 1]$,

$$w_{s,t}'(p) = \frac{\eta_{s,t}^*(x)}{h_{s,t}^{3,x}(1)} \left(\left(F_{s,t}^\rho \right)^{-1} (p) \right)^{1-\gamma}.$$

Furthermore, because

$$1 = \int_0^1 w'_{s,t}(p)dp = \int_0^1 \frac{\eta^*_{s,t}(x)}{h^{3,x}_{s,t}(1)} \left(\left(F^\rho_{s,t} \right)^{-1} (p) \right)^{1-\gamma} dp = \frac{\eta^*_{s,t}(x)}{h^{3,x}_{s,t}(1)} \mathbb{E} \left[\rho^{1-\gamma}_{s,t} \right],$$

we have that $h^{3,x}_{s,t}(1) = \eta^*_{s,t}(x)\mathbb{E}\left[\rho^{1-\gamma}_{s,t}\right]$ and, in turn, that

$$w_{s,t}(p) = \frac{1}{\mathbb{E}\left[\rho^{1-\gamma}_{s,t}\right]} \int_0^p \left(\left(F^\rho_{s,t} \right)^{-1} (q) \right)^{1-\gamma} dq.$$

Finally, because $h^2_{s,t}(z) = \frac{h^{3,x}_{s,t}(z)}{h^{1,x}_{s,t}(1)}$, using the same arguments as above, we conclude that (4.1) holds for any $0 \leqslant s < t$. $\qquad\square$

A.4   Proof of Corollary 4.3

For the first case of Theorem 4.1, the only $\gamma > 0$ for which the probability distortion

$$w_{s,t}(p) = \frac{1}{\mathbb{E}\left[\rho^{1-\gamma}_{s,t}\right]} \int_0^p \left(\left(F^\rho_{s,t} \right)^{-1} (q) \right)^{1-\gamma} dq$$

is independent of s, for every $s \in [0, t)$, is when $\gamma = 1$ and the assertion follows.

For the second case, recalling condition (4.2) and noting that

$$\mathbb{E}\left[\rho_{s,t}\mathbf{1}_{\left\{\rho_{s,t} \leqslant \left(F^\rho_{s,t}\right)^{-1}(p)\right\}}\right] = \Phi\left(\Phi^{-1}(p) - \sqrt{\int_s^t \|\lambda_r\|^2 dr}\right)$$

and that the right-hand side of the above is strictly increasing in $s \in [0, t]$, we conclude.

$\qquad\square$

A.5   Proof of Proposition 4.4

Case ii) follows immediately from the proof of Theorem 4.1 and thus we only need to show case i). We first show that the Lagrangian multiplier satisfies $\eta^*_{s,t}(X^*_s) = u'_s(X^*_s)$. Indeed, from Theorem 4.1

and the functional relation (A.4), we deduce that

$$\eta_{s,t}^* \left(\left(u_s' \right)^{-1} \left(\eta_{0,s}^*(x) \mathbb{E} \left[\rho_{0,s}^{1-\gamma} \right] \rho_{0,s}^\gamma \right) \right) \mathbb{E} \left[\rho_{s,t}^{1-\gamma} \right] \rho_{s,t}^\gamma = \eta_{0,t}^*(x) \mathbb{E} \left[\rho_{0,t}^{1-\gamma} \right] \rho_{0,t}^\gamma.$$

Therefore, for any $0 \leqslant s < t$,

$$\eta_{s,t}^* \left(\left(u_s' \right)^{-1} \left(u_0'(x) \mathbb{E} \left[\rho_{0,s}^{1-\gamma} \right] \rho_{0,s}^\gamma \right) \right) = u_0'(x) \mathbb{E} \left[\rho_{0,s}^{1-\gamma} \right] \rho_{0,s}^\gamma.$$

From this, we conclude that $\eta_{s,t}^*(x) = u_s'(x)$, firstly for Lebesgue almost everywhere and, then, by continuity for all $x > 0$.

Next, we prove that the optimal wealth process is given by (4.4). Because the distortion functions in (4.1) satisfy the Jin-Zhou monotonicity condition, the optimal wealth process is given by

$$X_t^* = \left(u_t' \right)^{-1} \left(\eta_{0,t}^*(x) \frac{\rho_t}{w_{0,t}'(F_{0,t}^\rho(\rho_t))} \right) = \left(u_t' \right)^{-1} \left(u_0'(x) \mathbb{E} \left[\rho_t^{1-\gamma} \right] \rho_t^\gamma \right), \quad t \geqslant 0,$$

where we used the form of the Lagrangian multiplier determined above and the form of $w_{0,t}$ in (4.1). From this, we deduce that

$$X_t^* = \left(u_t' \right)^{-1} \left(\eta_{s,t}^* \left(X_s^* \right) \mathbb{E} \left[\rho_{s,t}^{1-\gamma} \right] \rho_{s,t}^\gamma \right) = \left(u_t' \right)^{-1} \left(u_0'(x) \mathbb{E} \left[\rho_t^{1-\gamma} \right] \rho_t^\gamma \right), \quad 0 \leqslant s < t. \quad \square$$

A.6 Proof of Proposition 4.6

Note that $w(p) \leqslant p$, $p \in [0,1]$, if and only if $\gamma \leqslant 1$ and that $w_1(p) \leq w_2(p)$, $p \in [0,1]$, if and only if $\gamma_1 \leq \gamma_2$. Thus, the assertion follows by Theorem 3.1 and Corollary 3.1 in Ghossoub and He (2019). We remark that in the formulation of RDU therein, the probability distortion function is applied to the cumulative distribution of a random payoff. Herein, on the other hand, the probability distortion function is applied to the decumulative distribution, and this conjugate relation therefore flips a dominating probability weighting function to a dominated one. $\square$

A.7 Proof of Theorem 5.1

We first prove the correspondence between forward rank-dependent performance processes and deterministic, time-monotone forward performance processes under the distorted probability measure. We

start with the converse direction.

Let $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t}\right)$ be a forward rank-dependent performance criterion. If (4.2) holds and, consequently, the optimal wealth process $X^* = (X_t^*)_{t \geqslant 0}$ is constant, we set $\gamma = 0$. Then, the risky assets in the γ-distorted market become martingales and do not offer any excess return under the distorted measure. It is thus optimal to invest everything into the risk-free asset. Thus, $u_t(x) = u_0(x) = v(x, 0)$ is a deterministic, time-monotone forward performance criterion in the γ-distorted market by Proposition 3.3 and Proposition 3.5.

Now suppose that we are in case (4.1). Let $\mathbb{P}_\gamma$ be given by (5.1) and denote the expectation under it by $\mathbb{E}_\gamma$. According to Proposition 4.4, the wealth process X^* also solves the family of expected utility maximization problems generated by the optimal policy, say π^*,

$$(A.6) \qquad \max_X \mathbb{E}_\gamma \left[u_t(X) \right]$$

with $\mathbb{E}_\gamma \left[\rho_{\gamma,s,t} X \mid \mathcal{F}_s \right] = X_s^*, \; X \geqslant 0, \; X \in \mathcal{F}_t$, where $\rho_{\gamma,s,t} := \frac{\rho_{\gamma,t}}{\rho_{\gamma,s}} = \rho_{s,t}^\gamma \mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]$. Moreover, we have

$$
\begin{aligned}
u_s(X_s^*) &= \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(1 - F_{X_t^* \mid \mathcal{F}_s}(\xi)\right)\right) \\
&= \int_0^\infty u_t(\xi) d\left(-w_{s,t}\left(\mathbb{P}\left[\rho_{s,t} \leqslant \left(\frac{u_t'(\xi)}{u_s'(X_s^*)\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]}\right)^{1/\gamma} \Bigg| \mathcal{F}_s\right]\right)\right) \\
(A.7) \qquad &= \int_0^\infty u_t\left((u_t')^{-1}\left(u_s'(X_s^*)\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right] y^\gamma\right)\right) dw_{s,t}\left(F_{s,t}^\rho(y)\right) \\
&= \mathbb{E}\left[u_t\left((u_t')^{-1}\left(u_s'(X_s^*)\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right] \rho_{s,t}^\gamma\right)\right) w_{s,t}'\left(F_{s,t}^\rho(\rho_{s,t})\right) \Big| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[u_t\left((u_t')^{-1}\left(u_s'(X_s^*) \rho_{\gamma,s,t}\right)\right) \frac{\rho_{s,t}^{1-\gamma}}{\mathbb{E}\left[\rho_{s,t}^{1-\gamma}\right]} \Bigg| \mathcal{F}_s\right] \\
&= \mathbb{E}_\gamma\left[u_t\left((u_t')^{-1}\left(u_s'(X_s^*) \rho_{\gamma,s,t}\right)\right) \Big| \mathcal{F}_s\right] \\
&= \mathbb{E}_\gamma\left[u_t(X_t^*) \mid \mathcal{F}_s\right].
\end{aligned}
$$

Therefore, for each fixed $t \geqslant 0$, we have that, from the one hand, u_s corresponds to the value function of the expected utility maximization problem (A.6) under the distorted measure $\mathbb{P}_\gamma$ with time horizon t and utility function u_t, and, from the other, this policy π^* is optimal. Hence, $\mathbb{E}_\gamma \left[u_t(X_t^\pi) \mid \mathcal{F}_s \right] \leqslant u_s(X_s^\pi)$ for any admissible policy π with the same argument as in Proposition 3.5. Thus, $(u_t)_{t \geqslant 0}$ is a

forward performance criterion in the γ-distorted market.

To establish the reverse direction we work as follows. Let $\gamma \geqslant 0$ and let $(u_t)_{t \geqslant 0}$ be a deterministic, time-monotone forward performance process in the γ-distorted market. Together with $\tilde{w}_{s,t}(p) \equiv p$, $p \in [0,1]$ for all $0 \leqslant s < t$, the pair $\left((u_t)_{t \geqslant 0}, (\tilde{w}_{s,t})_{0 \leqslant s < t} \right)$ is a forward rank-dependent performance process in the γ-distorted market, as it follows from Proposition 3.3.

According to Proposition 4.4, the corresponding optimal wealth process is given by

$$X_t^* = \left(u_t' \right)^{-1} \left(u_0'(x) \rho_{\gamma,t} \right) = \left(u_t' \right)^{-1} \left(u_0'(x) \mathbb{E} \left[\rho_t^{1-\gamma} \right] \rho_t^{\gamma} \right), \quad 0 \geqslant t.$$

Next, we define the family of probability distortions $(w_{s,t})_{0 \leqslant s < t}$ by (4.1) and note that the optimal wealth process X^* solves (A.2) for any $0 \leqslant s < t$. Therefore, the pair $\left((u_t)_{t \geqslant 0}, (w_{s,t})_{0 \leqslant s < t} \right)$ is a time-consistent rank-dependent performance criterion and, because of (A.7), it also preserves the performance value. Hence, it is a forward rank-dependent performance criterion according to Proposition 3.5.

Finally, we deduce the construction method for forward rank-dependent performance criteria. Using Girsanov's theorem, it is straightforward to compute that the market price of risk under the distorted probability measure $\mathbb{P}_\gamma$ is given by $\gamma \lambda$. The statement then follows as a direct consequence of the results on time-monotone forward criteria in Musiela and Zariphopoulou (2010a) and the correspondence shown above. $\qquad \square$

A.8 Proof of Proposition 5.2

Following the results in Musiela and Zariphopoulou (2010a), cf. in particular Theorem 4 therein, we deduce that the optimal wealth under the time-monotone forward performance criteria in the γ-distorted market is given by

$$X_t^* = h \left(h^{-1}(0,t) + \int_0^t \|\lambda_{\gamma,s}\|^2 ds + \int_0^t \lambda_{\gamma,s} dW_{\gamma,s}, \int_0^t \|\lambda_{\gamma,s}\|^2 ds \right),$$

with h as in (5.7). Using that $\lambda_{\gamma,t} = \gamma \lambda_t$ and (5.3) we conclude. We similarly deduce (5.10). $\qquad \square$

A.9   Proof of Theorem 6.2

Given initial wealth $x > 0$ at initial time 0, denote by X^* the optimal wealth at time T that maximizes the RDU value evaluated at the initial time. From Theorem 2.2, we have that for $0 \leqslant t < T$, the optimal wealth process for

$$\max_X \quad \int_0^\infty u_{t,T}(\xi) d\left(-w_{t,T}\left(1 - F_{X|\mathcal{F}_t}(\xi)\right)\right)$$

$$\text{s.t.} \quad \mathbb{E}\left[\rho_{t,T} X | \mathcal{F}_t\right] = \mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t\right], \quad X \geqslant 0, \quad X \text{ is } \mathcal{F}_T - \text{measurable}.$$

is given by

$$X^{*,t} = (u'_{t,T})^{-1}\left(\eta^*_{t,T}\left(\mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t\right]\right) \hat{N}'_{t,T}\left(1 - w_{t,T}\left(F^\rho_{t,T}(\rho_{t,T})\right)\right)\right),$$

where $\hat{N}_{t,T}$ is the concave envelope of (A.3), with s, t therein replaced by t, T, and $\eta^*_{t,T}(y) > 0$ is given by the implicit function theorem as the unique continuously differentiable, strictly decreasing function satisfying

$$\mathbb{E}\left[\rho_{t,T}(u'_T)^{-1}\left(\eta^*_{t,T}(y)\hat{N}'_{t,T}\left(1 - w_{t,T}\left(F^\rho_{t,T}(\rho_{t,T})\right)\right)\right)\right] = y.$$

Thus, the optimality of the initial optimal solution $X^* = X^{*,0}$ is maintained at time t if and only if

(A.8)
$$\eta^*_{t,T}\left(\mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t\right]\right) \hat{N}'_{t,T}\left(1 - w_{t,T}\left(F^\rho_{t,T}(\rho_{t,T})\right)\right)$$
$$= u'_{t,T}\left((u'_{0,T})^{-1}\left(\eta^*_{0,T}(x) \hat{N}'_{0,T}\left(1 - w_{0,T}\left(F^\rho_{0,T}(\rho_{0,T})\right)\right)\right)\right).$$

Similar to the proof of Theorem 4.1, we define functions $g_t^{1,x}, g_t^2, g_t^{3,x} : (0, \infty) \to (0, \infty)$ by

(A.9)
$$g_t^{1,x}(y) = \eta^*_{t,T}\left(\mathbb{E}\left[\rho_{t,T}(u'_{0,T})^{-1}\left(\eta^*_{0,T}(x) \hat{N}'_{0,T}\left(1 - w_{0,T}\left(F^\rho_{0,T}(y\rho_{t,T})\right)\right)\right)\right]\right),$$
$$g_t^2(y) = \hat{N}'_{t,T}\left(1 - w_{t,T}\left(F^\rho_{t,T}(y)\right)\right),$$
$$g_t^{3,x}(y) = u'_{t,T}\left((u'_{0,T})^{-1}\left(\eta^*_{0,T}(x) \hat{N}'_{0,T}\left(1 - w_{0,T}\left(F^\rho_{0,T}(y)\right)\right)\right)\right).$$

Because $\rho_{0,t}$ and $\rho_{t,T}$ are independent and supported on $(0, \infty)$ for $0 \leqslant t < T$, and because $\rho_{0,t}\rho_{t,T} = \rho_{0,T}$, we have that (A.8) holds if and only if

$$\text{(A.10)} \qquad\qquad g_t^{1,x}(y)g_t^2(z) = g_t^{3,x}(yz),$$

firstly for Lebesgue almost everywhere and, then, for all $y, z > 0$ by the continuity of $g_t^{1,x}, g_t^2, g_t^{3,x}$. Since $u_{t,T}'$ is strictly decreasing for any $t \in [0, T)$, we can follow the same argument as in the proof of Theorem 4.1 to conclude that (A.8) holds if and only if it is either the case where

$$w_{t,T}(p) \geqslant \mathbb{E}\left[\rho_{t,T}\mathbf{1}_{\left\{\rho_{t,T} \leqslant \left(F_{t,T}^\rho\right)^{-1}(p)\right\}}\right], \quad p \in [0, 1],$$

for any $t \in [0, T)$ or the case that $g_t^{1,x}, g_t^2,$ and $g_t^{3,x}$ are strictly increasing. If the first case holds, the first part of the theorem follows.

If the latter case holds, then $N_{0,T}$ and $N_{t,T}$ are concave, which is equivalent to the probability distortion functions satisfying the Jin-Zhou monotonicity condition. In this case, $g_t^{1,x}, g_t^2, g_t^{3,x}$ simplify to

$$g_t^{1,x}(y) = \eta_{t,T}^*\left(\mathbb{E}\left[\rho_{t,T}(u_{0,T}')^{-1}\left(\eta_{0,T}^*(x)\frac{y\rho_{t,T}}{w_{0,T}'\left(F_{0,T}^\rho(y\rho_{t,T})\right)}\right)\right]\right),$$

$$g_t^2(y) = \frac{y}{w_{t,T}'\left(F_{t,T}^\rho(y)\right)},$$

$$g_t^{3,x}(y) = u_{t,T}'\left((u_{0,T}')^{-1}\left(\eta_{0,T}^*(x)\frac{y}{w_{0,T}'\left(F_{0,T}^\rho(y)\right)}\right)\right).$$

As in the proof of Theorem 4.1, we can show that $g_t^{3,x}$ satisfies Cauchy's functional equation and conclude that there exists $\gamma_t > 0$ such that $g_t^{3,x}(y) = g_t^{3,x}(1)y^{\gamma_t}$, for all $y > 0$. Thus, from (A.10), we deduce that

$$\frac{g_t^{1,x}(y)}{y^{\gamma_t}} = g_t^{3,x}(1)\frac{z^{\gamma_t}}{g_t^2(z)} = C_{x,t}.$$

for some constant $C_{x,t}$. From the definition of g_t^2 we obtain that, for any $t \in (0, T)$,

$$w_{t,T}(p) = \frac{1}{\mathbb{E}\left[\rho_{t,T}^{1-\gamma_t}\right]} \int_0^p \left(\left(F_{t,T}^\rho\right)^{-1}(q)\right)^{1-\gamma_t} dq.$$

By continuity of $w_{t,T}$ in t at zero,

(A.11)
$$w_{0,T}(p) = \frac{1}{\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]} \int_0^p \left(\left(F_{0,T}^\rho\right)^{-1}(q)\right)^{1-\gamma_0} dq$$

where $\gamma_0 = \lim_{t \searrow 0} \gamma_t$. Thus, the equality $g_t^{3,x}(y) = g_t^{3,x}(1)y^{\gamma_t}$ yields

$$u_{t,T}'\left((u_{0,T}')^{-1}\left(\eta_{0,T}^*(x)\,\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]y^{\gamma_0}\right)\right) = g_t^{3,x}(1)y^{\gamma_t}.$$

With the substitution $z = \left(u_{0,T}'\right)^{-1}\left(\eta_{0,T}^*(x)\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]y^{\gamma_0}\right)$ the above becomes

(A.12)
$$u_{t,T}'(z) = \frac{g_t^{3,x}(1)}{\left(\eta_{0,T}^*(x)\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]\right)^{\gamma_t/\gamma_0}}\left(u_{0,T}'(z)\right)^{\gamma_t/\gamma_0}.$$

Differentiating (A.12) with respect to z yields

(A.13)
$$u_{t,T}''(z) = \frac{g_t^{3,x}(1)}{\left(\eta_{0,T}^*(x)\mathbb{E}\left[\rho_{0,T}^{1-\gamma_0}\right]\right)^{\gamma_t/\gamma_0}}\frac{\gamma_t}{\gamma_0}\left(u_{0,T}'(z)\right)^{\gamma_t/\gamma_0-1}u_{0,T}''(z).$$

Dividing (A.13) by (A.12) gives (6.6).

==It remains to prove the if-direction in part ii) of the theorem. To this end,== we fix $t \in (0, T)$ and first note that (6.6) is equivalent to

$$\frac{d}{dz}\log\left(u_{t,T}'(z)\right) = \frac{\gamma_t}{\gamma_0}\frac{d}{dz}\log\left(u_{0,T}'(z)\right)$$

and thus $u_{t,T}'(z) = \widetilde{C}\left(u_{0,T}'(z)\right)^{\gamma_t/\gamma_0}$ for some constant $\widetilde{C} > 0$. From this we derive

(A.14)
$$\left(u_{t,T}'\right)^{-1}\left(\widetilde{C}z^{\gamma_t}\right) = \left(u_{0,T}'\right)^{-1}\left(z^{\gamma_0}\right),$$

$z \in (0, \infty)$. Moreover, (6.5) and the continuity of $w_{t,T}$ at $t = 0$ imply that $w_{0,T}$ is given by (A.11). Furthermore, because we assume that (6.4) does not hold, γ_0 must be positive.

According to Theorem 2.2, the optimal solution to (6.2) is given by

$$(A.15) \qquad X^{*,t} = (u'_{t,T})^{-1} \left(\eta^*_{t,T} \left(\mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t \right] \right) \mathbb{E}\left[\rho^{1-\gamma_t}_{t,T} \right] \rho^{\gamma_t}_{t,T} \right),$$

where $\eta^*_{t,T}(y) > 0$ is given by the implicit function theorem as the unique continuously differentiable, strictly decreasing function satisfying $\mathbb{E}\left[\rho_{t,T}(u'_{t,T})^{-1} \left(\eta^*_{t,T}(y) \mathbb{E}\left[\rho^{1-\gamma_t}_{t,T} \right] \rho^{\gamma_t}_{t,T} \right) \right] = y$. In particular, $\mathbb{E}\left[\rho_{t,T} X^{*,t} | \mathcal{F}_t \right] = \mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t \right]$.

To complete the proof, we need to show that $X^* = X^{*,t}$. Recalling that X^* is the initial optimal solution, i.e., $X^* = X^{*,0}$, and that $w_{0,T}$ is given by (A.11), we obtain

$$X^* = (u'_{0,T})^{-1} \left(\eta^*_{0,T}(x) \mathbb{E}\left[\rho^{1-\gamma_0}_{0,T} \right] \rho^{\gamma_0}_{0,T} \right),$$

where $\eta^*_{0,T}(x) > 0$ is the Lagrange multiplier such that $\mathbb{E}[\rho_{0,T} X^*] = x$. Using the relation (A.14), we obtain $X^* = (u'_{t,T})^{-1} \left(\Lambda \rho^{\gamma_t}_{t,T} \right)$, where

$$\Lambda := \widetilde{C} \left(\eta^*_{0,T}(x) \mathbb{E}\left[\rho^{1-\gamma_0}_{0,T} \right] \right)^{\gamma_t/\gamma_0} \rho^{\gamma_t}_{0,t}.$$

Comparing X^* in the above and $X^{*,t}$ in (A.15), noting that Λ is $\mathcal{F}_t$-measurable, and recalling that $\mathbb{E}\left[\rho_{t,T} X^{*,t} | \mathcal{F}_t \right] = \mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t \right]$, we conclude that $\eta^*_{t,T} \left(\mathbb{E}\left[\rho_{t,T} X^* | \mathcal{F}_t \right] \right) \mathbb{E}[\rho^{1-\gamma_t}_{t,T}] = \Lambda$ and, thus, $X^{*,t} = X^*$.

$\square$