

ON THE ANALYTICITY OF THE VALUE FUNCTION IN OPTIMAL INVESTMENT

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ABSTRACT. We study the analyticity of the value function in optimal investment with expected utility from terminal wealth. We identify both a class of utilities and a class of semi-martingale models for which we establish analyticity. Specifically, these utilities have completely monotonic inverse marginals, while the market models have a maximal element in the sense of infinite-order stochastic dominance. We construct two counterexamples, themselves of independent interest, which show that analyticity fails if either the utility or the market model does not belong to the respective special class. We also provide explicit formulas for the derivatives, of all orders, of the value functions as well as their optimizers.

1. INTRODUCTION

We study the analyticity and higher-order regularity of the value function (indirect utility), $u(x)$, in optimal investment problems in general semi-martingale markets,

$$(1.1) \quad u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0,$$

where $U : (0, \infty) \rightarrow \mathbb{R}$ is the utility function and $\mathcal{X}(x)$ the set of admissible wealth processes starting at x ; see section 2.1 for the precise definition. This foundational problem was analyzed in a number of seminal papers (see, among others, [KLS87], [KLSX91], and [KS99]).

Under rather general assumptions, the authors in [KS99, KS03] showed that if $U \in C^1((0, \infty))$, so is u . For higher order regularity, there are two types of results. If the utility function U is either power or logarithmic, the value function also inherits this form. This is a direct consequence of homotheticity and holds under minimal model assumptions.

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For utilities beyond homothetic ones, extra model conditions are needed even to obtain second-order differentiability of u when U is twice differentiable (see [KS06a]). To our knowledge, no other regularity results exist to date. We are then motivated to ask the following question:

Can we identify both a class of semi-martingale market models and a class of utility functions such that the value function in (1.1) retains the (highest possible) regularity of the utility function?

We propose i) the class of market models which possess a non-zero dual maximal element in the sense of infinite-degree stochastic dominance and ii) the class of utilities whose inverse marginal is a completely monotonic function (and thus analytic). We denote these classes by $\mathcal{SD}(\infty)$ and \mathcal{CMIM} , respectively.

We establish the following result: *If the market model is in $\mathcal{SD}(\infty)$ and the utility is in \mathcal{CMIM} , then the value function u is also in \mathcal{CMIM} and is, thus, analytic. In other words, we show that, for such market models and such analytic utilities, the associated indirect utility inherits the analyticity and, furthermore, remains in the same \mathcal{CMIM} class.*

We also examine the necessity of these classes of models and utilities. We provide two counterexamples, showing that the results fail outside the family of $\mathcal{SD}(\infty)$ models and/or the utility class \mathcal{CMIM} .

In the first counterexample, we construct a market model in $\mathcal{SD}(\infty)$ and an analytic, but not in \mathcal{CMIM} , utility, and show that the value function is not infinitely differentiable (and, thus, not analytic). In the second counterexample, we show that for any non-homothetic \mathcal{CMIM} utility (actually, the utility being two-times differentiable suffices), there exists a market model outside the $\mathcal{SD}(\infty)$ class, for which the value function is not even twice differentiable.

As mentioned above, under minimal model assumptions - well beyond the ones for the $\mathcal{SD}(\infty)$ class - homothetic utilities yield homothetic value functions. Such utilities belong to the \mathcal{CMIM} family and are analytic, and these properties are also inherited to their value functions.

The paper is organized as follows: in section 2, we specify the settings for problem (1.1). In section 3, we discuss the background notions on complete monotonicity and stochastic dominance and provide their characterizations. In section 4, we introduce the class of market models and utilities that we propose, followed by the main results on the analyticity of the value function together with the explicit expressions for the primal and dual optimizers and their derivatives of all orders, as well as other regularity results. Section 5 provides a counterexample for non- \mathcal{CMIM} utilities, while section 6 contains a counterexample for non- $\mathcal{SD}(\infty)$ market models.

2. THE OPTIMAL INVESTMENT PROBLEM

2.1. The market model. The market consists of a riskless asset, offering zero interest rate, and d traded stocks, whose price processes form a d -dimensional semi-martingale S on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$. Here $T \in (0, \infty)$ is the investment horizon.

A trading strategy H is a predictable and S -integrable process. It generates the wealth process $X := x + H \cdot S$, starting at $x > 0$, which, for the utilities considered herein, is taken to be non-negative. Using the notation of [KS99], we denote the set of admissible wealth processes,

$$(2.1) \quad \mathcal{X}(x) := \{X : X_t = x + H \cdot S_t \geq 0, t \in [0, T], \\ \text{for some } S\text{-integrable process } H\}, \quad x > 0.$$

Following [KK07], we say that a sequence $(X^n)_{n \in \mathbb{N}} \subset \mathcal{X}(1)$ generates an *unbounded profit with bounded risk (UPBR)*, if the family of the random variables $(X_T^n)_{n \in \mathbb{N}}$ is unbounded in probability, i.e., if

$$\limsup_{m \uparrow \infty} \mathbb{P}[X_T^n > m] > 0.$$

If no such sequence exists, the condition of *no unbounded profit with bounded risk (NUPBR)* is satisfied. A characterization of *NUPBR* is given via the dual feasible set, $\mathcal{Y}(y)$, introduced in [KS99],

$$(2.2) \quad \mathcal{Y}(y) := \{Y : Y_0 = y \text{ and } XY = (X_t Y_t)_{t \in [0, T]} \text{ is a supermartingale} \\ \text{for every } X \in \mathcal{X}(1)\}, \quad y > 0.$$

The elements of $\mathcal{Y}(1)$ are called super-martingale deflators, see [KK07]. It was established in [KK07] that *NUPBR* is equivalent to the existence of a strictly positive super-martingale deflator, namely,

$$(2.3) \quad \mathcal{Y}(1) \text{ contains a strictly positive element.}$$

In [TS14] and [KKS16], it was later proven that *NUPBR* is equivalent to the existence of a strictly positive local martingale deflator; see, also, [CCFM17]. Furthermore, it was shown in [KKS16] that *NUPBR* is equivalent to other no-arbitrage conditions, such as no arbitrage of the first kind and no asymptotic arbitrage of the first kind; we refer the reader to [KKS16, Lemma A.1] for further details.

2.2. Utility functions. We recall the standard class of utility functions $U : (0, \infty) \rightarrow \mathbb{R}$ which are strictly concave, strictly increasing, continuously differentiable and satisfy the Inada conditions

$$(2.4) \quad \lim_{x \downarrow 0} U'(x) = \infty \quad \text{and} \quad \lim_{x \uparrow \infty} U'(x) = 0.$$

To facilitate the upcoming exposition, we will denote the class of all such utility functions by $-\mathcal{C}$, in that

$$U \in -\mathcal{C} \iff -U \in \mathcal{C}.$$

2.3. Primal problem and the indirect utility. We recall the optimal investment problem from terminal wealth

$$(2.5) \quad u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0,$$

where $U \in -\mathcal{C}$ and $\mathcal{X}(x)$ as in (2.1).

2.4. Dual problem and the dual function. For any $U \in -\mathcal{C}$, its Legendre transform is given by

$$(2.6) \quad V(y) := \sup_{x > 0} (U(x) - xy), \quad y > 0,$$

and, by biconjugacy,

$$-\mathcal{C} \ni U \iff V \in \mathcal{C}.$$

In turn, we recall the dual value function,

$$(2.7) \quad v(y) := \sup_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \quad y > 0.$$

with V as in (2.6) and $\mathcal{Y}(y)$ as in (4.10).

It was shown in [KK07] that condition *NUPBR* is necessary for the non-degeneracy of problem (2.5) in that, if *NUPBR* does not hold, then, for any utility function U , (2.5) has either infinitely many solutions or no solution at all. Specifically, if $U(\infty) = \infty$, then $u(x) = \infty$, $x > 0$. Therefore, either there is no solution (when the supremum is not attained) or there are infinitely many solutions (when the supremum is attained). On the other hand, if $U(\infty) < \infty$, there is no solution.

If condition *NUPBR* holds, problem (2.5) has a solution under the weak assumption that the dual value function v in (2.7) is finite, i.e., $v(y) < \infty$, $y > 0$. In this case, all standard conclusions of the utility maximization theory hold; see, for example, [Kar13] and [Mos15] for details.

3. COMPLETE MONOTONICITY AND STOCHASTIC DOMINANCE

3.1. Complete monotonicity. Completely monotonic functions have been well-studied in the literature, see [Wid41] and [SSV10] for the historic overview of the development of the subject therein. A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called *completely monotonic*, denoted by $f \in \mathcal{CM}$, if it has derivatives of all orders and

$$(-1)^n f^{(n)}(x) \geq 0, \quad x > 0 \quad \text{and} \quad n = 0, 1, 2, \dots$$

Whenever needed, we extend f to $[0, \infty)$ setting $f(0) := \lim_{x \downarrow 0} f(x)$, where $f(0) \leq \infty$.

The celebrated Bernstein theorem (see [SSV10, Theorem 1.4] or [Wid41, Theorem 12b]) gives a characterization of completely monotonic functions, stating that $f \in \mathcal{CM}$ if and only if

$$f(x) = \int_0^\infty e^{-xz} d\mu(z),$$

where μ is a nonnegative sigma-finite measure on $[0, \infty)$ such that the integral converges for every $x > 0$.

Definition 3.1. We define \mathcal{D} to be the class of functions $W : [0, \infty) \rightarrow \mathbb{R}$, which satisfy

- (1) $-W' \in \mathcal{CM}$,
- (2) $W'(\infty) = 0$.

The reader should note that the definition above is related, but not the same, to what is called in literature a Bernstein functions, see, e.g., [SSV10, p. 15]. Bernstein functions would assume bounds on W , but no Inada-type conditions on W' . For a $W \in \mathcal{D}$, we have $W'(y) = -\int_0^\infty e^{-yz} d\mu(z)$ from the Bernstein representation characterization of completely monotonic functions. We then deduce that

$$W'(0+) := \lim_{y \downarrow 0} W'(y) = -\mu([0, \infty)) \quad \text{and} \quad W'(\infty) := \lim_{y \uparrow \infty} W'(y) = -\mu(\{0\}).$$

Therefore, the definition of \mathcal{D} dictates that the measure μ has no mass at $z = 0$, to satisfy $\mu(\{0\}) = -W'(\infty) = 0$. We note that the Inada-type condition $W'(0) = -\infty$ holds if and only if $\mu([0, \infty)) = \mu((0, \infty)) = \infty$, not assumed for $W \in \mathcal{D}$.

3.2. Monotonicity of finite order. A weaker notion of complete monotonicity is the monotonicity of finite order. We adopt the slightly more restrictive definition of monotonicity of order n in the paper [LN83] and not the somewhat weaker definitions in the earlier works [Wil56] and [Mul71].

A function $f : (0, \infty) \rightarrow \mathbb{R}$ is called monotonic of (finite) order n , denoted by $f \in \mathcal{CM}(n)$, if it has derivatives of order $k = 1, 2, \dots, n$ and

$$(-1)^k f^{(k)}(x) \geq 0, \quad x > 0 \quad \text{and} \quad k = 0, 1, 2, \dots, n.$$

As in the \mathcal{CM} case, whenever needed, we extend f to $[0, \infty)$ by $f(0) := \lim_{x \downarrow 0} f(x)$, where $f(0) \leq \infty$. In analogy to the class \mathcal{D} , we introduce the following definition.

Definition 3.2. For $n \geq 1$, we define $\mathcal{D}(n)$ to be the class of functions $W : [0, \infty) \rightarrow \mathbb{R}$, which satisfy

- (1) $-W' \in \mathcal{CM}(n-1)$,
- (2) $W'(\infty) = 0$.

We note that $W \in \mathcal{D}(n)$ is not necessarily strictly decreasing by definition. While we assume that $W'(\infty) = 0$ (both to make it similar to the $n = \infty$ case and to simplify the upcoming definition of stochastic dominance), we do not impose the condition $W'(0) = \infty$ for $W \in \mathcal{D}(n)$, nor assume that such a W is bounded below.

Proposition 3.3. *Fix $W \in \mathcal{D}(n)$, $n \in \{2, 3, \dots\}$. Then,*

$$(3.1) \quad W'(\infty) = W''(\infty) = \dots = W^{(n-1)}(\infty) = 0$$

and

$$0 \leq -W'(y_1) = \int_{y_1}^{\infty} \dots \int_{y_{n-1}}^{\infty} (-1)^n W^{(n)}(y_n) dy_n \dots dy_2 < \infty, \quad y_1 > 0.$$

Therefore, any $W \in \mathcal{D}(n)$ has the representation

$$(3.2) \quad W(y) = W(y_0) + \int_y^{y_0} \int_{y_1}^{\infty} \dots \int_{y_{n-1}}^{\infty} (-1)^n W^{(n)}(y_n) dy_n \dots dy_2 dy_1, \quad y > 0.$$

For each fixed $y_0 > 0$, the above representation holds.

Proof. As $W \in \mathcal{D}(n)$, we have $(-1)^k W^{(k)}(y) \geq 0$ for $k = 1, \dots, n$ and $W'(\infty) = 0$. Assume now that $W^{(k)}(\infty) = 0$ for some $k \leq n - 2$. Since $(-1)^{k+2} W^{(k+2)}(y) \geq 0$, we conclude that the function $y \rightarrow (-1)^{k+1} W^{(k+1)}(y) \geq 0$ is decreasing. Next, assume that $W^{(k+1)}(\infty) \neq 0$, so

$$(-1)^{k+1} W^{(k+1)}(\infty) > 0.$$

This, however, would contradict the monotonicity of $y \rightarrow (-1)^k W^{(k)}(y)$, which is decreasing, and the assumption that $W^{(k)}(\infty) = 0$. An inductive argument completes the proof. \square

3.3. Stochastic dominance of finite order. Let F and G be two cumulative distribution functions with supports on $\mathbb{R}_+ = [0, \infty)$. We recall that F *stochastically dominates* G in the first order if

$$F(y) \leq G(y), \quad y \geq 0.$$

To define stochastic dominance of higher orders, following, for example, [Thi93], we set

$$(3.3) \quad F_1 = F \quad \text{and} \quad F_i(y) = \int_0^y F_{i-1}(z) dz, \quad i = 2, 3, \dots$$

Since $0 \leq F \leq 1$, the integrals are well defined. The functions G_i are defined similarly. Next, we depart slightly from the definition customary in the literature, e.g., in [Thi93], see also [Zha01] and [Whi70]. On the one hand, we use a somewhat weaker definition, while, on the other, we can treat unbounded supports. More comments follow the definition.

Definition 3.4. For any $n \geq 1$, we say that F stochastically dominates G in the sense of the n -th order, and denote $F \succeq_n G$, if $F_n(y) \leq G_n(y)$, $y \geq 0$. For two random variables $\xi, \eta \geq 0$ we say that $\xi \succeq_n \eta$ if $F_\xi \succeq_n F_\eta$.

Remark 3.5. For $n \geq 3$, it is customary in the literature, in order to define $F \succeq_n G$, to both

- (1) assume that F and G are supported on a finite interval $[0, b]$,
- (2) have the additional condition $F_k(b) \leq G_k(b)$, $k = 1, \dots, n-1$.

Our definition by-passes both points above since we will only use a restrictive set of “test” functions, namely $\mathcal{D}(n)$. For such test functions, condition (3.1) ensures that we do not (even formally) need the extra assumption. Furthermore, our definition works well for $n \geq 3$ for measures fully supported on the $[0, \infty)$ that we need.

Proposition 3.6. *Consider two non-negative random variables ξ and η . Fix $n \geq 2$. Then, the following conditions are equivalent:*

- (1) $\xi \succeq_n \eta$,
- (2) $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}(n)$, such that $W(\infty) > -\infty$, (i.e., W is bounded from below),
- (3) $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}(n)$ such that $\mathbb{E}[W^-(\xi)] < \infty$ and $\mathbb{E}[W^-(\eta)] < \infty$.

Proof. If $W \in \mathcal{D}(n)$ is bounded below, we will suppose that $W(\infty) = 0$, without loss of generality. For $y_0 = \infty$, representation (3.2) becomes

$$(3.4) \quad \begin{aligned} W(y) &= \int_y^\infty \int_{y_1}^\infty \dots \int_{y_{n-1}}^\infty (-1)^n W^{(n)}(y_n) dy_n \dots dy_2 dy_1 \\ &= \int_{\mathbb{R}_+^n} 1_{\{y \leq y_1 \leq \dots \leq y_n\}} (-1)^n V^{(n)}(y_n) dy_n \dots dy_2 dy_1, \quad y > 0. \end{aligned}$$

Therefore, Fubini’s theorem yields

$$(3.5) \quad \begin{aligned} \mathbb{E}[W(\xi)] &= \mathbb{E} \left[\int_{\mathbb{R}_+^n} 1_{\{\xi \leq y_1 \leq \dots \leq y_n\}} (-1)^n W^{(n)}(y_n) dy_n \dots dy_2 dy_1 \right] \\ &= \int_0^\infty \left(\int_{\mathbb{R}_+^{n-1}} \mathbb{E} [1_{\{\xi \leq y_1 \leq \dots \leq y_n\}}] dy_1 \dots dy_{n-1} \right) (-1)^n W^{(n)}(y_n) dy_n. \end{aligned}$$

Fix y_n . Using the cdf F of ξ we can rewrite

$$\begin{aligned} \int_{\mathbb{R}_+^{n-1}} \mathbb{E} [1_{\{\xi \leq y_1 \leq \dots \leq y_n\}}] dy_1 \dots dy_{n-1} &= \int_0^{y_n} \dots \int_0^{y_2} \mathbb{P} [\xi \leq y_1] dy_1 \dots dy_{n-1} \\ &= \int_0^{y_n} \dots \int_0^{y_2} F(y_1) dy_1 \dots dy_{n-1} = F_n(y_n), \end{aligned}$$

where we have used that $\xi \geq 0$. Together with (3.5), we obtain

$$(3.6) \quad \mathbb{E}[W(\xi)] = \int_0^\infty W(y) dF(y) = \int_0^\infty (-1)^n W^{(n)}(y_n) F_n(y_n) dy_n.$$

This shows that (1) and (2) above are equivalent. To deal with the general case of $W \in \mathcal{D}(n)$ in (3), one has to use the representation (3.2), separately for values of ξ above and below y_0 . \square

3.4. Stochastic dominance of infinite degree. The infinite-order stochastic dominance is, intuitively, defined by letting $n \uparrow \infty$ in Definition 3.4. This, however, has to be done carefully. We again depart from [Thi93] for our definition.

To provide some intuition, we first note that, for each $z > 0$, the exponential function $W(y) = e^{-zy}$, $y > 0$, is in $\mathcal{D}(n)$, for every $z > 0$ and $n \geq 1$. For every $z > 0$, relation (3.6) reads

$$\mathbb{E}[e^{-z\xi}] = \int_0^\infty e^{-zy} dF(y) = \int_0^\infty z^n e^{-zy} F_n(y) dy, \quad n \geq 1.$$

Therefore, if for any n , no matter how large, we have $F \succeq_n G$, then the exponential moments of the two distributions compare, for all positive values of z . It thus appears to us that the weakest possible form of dominance, obtaining by letting $n \uparrow \infty$, is the one below.

Definition 3.7. Consider two cumulative distributions F and G on $[0, \infty)$. We say that F dominates G in *infinite degree stochastic dominance*, and denote by $F \succeq_\infty G$, if

$$\int_0^\infty e^{-zy} dF(y) \leq \int_0^\infty e^{-zy} dG(y), \quad z > 0.$$

For nonnegative random variables ξ and η , we say that ξ dominates η in *infinite-order stochastic dominance*, and denote $\xi \succeq_\infty \eta$ if $F_\xi \succeq_\infty F_\eta$, that is

$$\mathbb{E}[e^{-z\xi}] \leq \mathbb{E}[e^{-z\eta}], \quad z > 0.$$

Below, we provide a characterization of infinite-order stochastic dominance.

Proposition 3.8. *Consider two non-negative random variables ξ and η . Then, the following conditions are equivalent:*

- (1) $\xi \succeq_\infty \eta$,
- (2) $\mathbb{E}[W(\xi)] \leq \mathbb{E}[W(\eta)]$ for every function $W \in \mathcal{D}$ such that $\mathbb{E}[W^-(\xi)] < \infty$ and $\mathbb{E}[W^-(\eta)] < \infty$.

The proof goes along the lines of [BG87, Theorem 3], and is based on integrating the Bernstein formula in the spirit of (3.2). The case when W is unbounded can be treated by approximating W with similar bounded functions, truncating the corresponding Bernstein measure μ appropriately and using monotone convergence.

It is interesting that [BG87] does not use the specific name of infinite-order dominance. To the best of our knowledge, the name first appeared later in [Thi93] (but for a somewhat less precise definition).

4. MAIN RESULTS

4.1. The \mathcal{CMIM} and $\mathcal{CMIM}(n)$ utilities. As utility-based preferences are invariant under positive linear transformations of the form $U^*(x) = aU(x) + b$, $a > 0$, and in view of the importance of the marginal utility in many problems, it is natural to define a utility function through its derivative. Additionally, it has been observed that the most widely used utility functions have completely monotonic marginals, see [BG87]. In the present paper, we investigate a class of functions, whose *inverse marginals* are completely monotonic. This is particularly natural in view of the overall importance of the duality approach to the expected utility maximization.

We start with the following definition.

Definition 4.1. We define the \mathcal{CMIM} to be the class of utility functions $U \in -\mathcal{C}$ for which their inverse marginal $(U')^{-1} \in \mathcal{CM}$.

From Bernstein's theorem, we deduce that if $U \in \mathcal{CMIM}$, then we have the representation

$$(4.1) \quad (U')^{-1}(y) = \int_0^\infty e^{-yz} \mu(dz), \quad y > 0,$$

where μ is a nonnegative measure, such that the integral converges for every $y > 0$.

We stress that the Inada conditions (2.4) dictate that the underlying measure μ must satisfy $\mu(\{0\}) = 0$ and $\mu((0, \infty)) = \infty$. Indeed, $U'(0) = \infty$ yields $\mu(\{0\}) = 0$ while $U'(\infty) = 0$ yields $\mu((0, \infty)) = \infty$.

Example 4.2. Here we show that standard utilities are included.

(1) $U(x) = \log x$, $x > 0$. Then, $(U')^{-1}(y) = \frac{1}{y} \in \mathcal{CM}$ and we have

$$(U')^{-1}(y) = \int_0^\infty e^{-yz} dz, \quad y > 0.$$

(2) $U(x) = \frac{x^p}{p}$, $x > 0$, $p < 1$, $p \neq 0$. Then, $(U')^{-1}(y) = y^{-\frac{1}{1-p}} \in \mathcal{CM}$ and with $q = -\frac{p}{1-p}$ (i.e., such that $\frac{1}{p} + \frac{1}{q} = 1$), we have

$$(U')^{-1}(y) = \frac{1}{\Gamma(1-q)} \int_0^\infty e^{-yz} z^{-q} dz, \quad y > 0,$$

where Γ is the Gamma function, see [SSV10, p. viii, formula (2)].

Assuming less regularity on the utility function but keeping monotonic structure up to finite order leads to the following definition.

Definition 4.3. For $n \in \{2, 3, \dots\}$, we say that a utility function $U \in -\mathcal{C}$ is in the $\mathcal{CMIM}(n)$ class if its inverse marginal is completely monotonic of order $n - 1$, that is $(U')^{-1} \in \mathcal{CM}(n - 1)$.

Recalling Definitions 3.1 and 4.1, and denoting by V the convex conjugate of U in the sense of (2.6), we deduce that

$$U \in \mathcal{CMIM} \iff V \in \mathcal{D} \cap \mathcal{C}.$$

Likewise, from Definitions 3.2 and 4.3, we get

$$U \in \mathcal{CMIM}(n) \iff V \in \mathcal{D}(n) \cap \mathcal{C}.$$

4.2. The class $\mathcal{SD}(\infty)$ and $\mathcal{SD}(n)$ of market models. The stochastic dominance had to be formally defined separately for finite and infinite n (see Definitions 3.4 and 3.7). However, for the associated market models, now, we can give a unified definition (for both finite and infinite degree) below.

Definition 4.4. Fix $n \in \{2, 3, \dots\} \cup \{\infty\}$. We say that the financial model satisfies condition $\mathcal{SD}(n)$ if there exists $\hat{Y} \in \mathcal{Y}(1)$ such that $\hat{Y}_T \succeq_n Y_T$ for every $Y \in \mathcal{Y}(1)$.

In what follows, we will use the terminologies “market model in $\mathcal{SD}(n)$ class” and “ $\mathcal{SD}(n)$ -model” interchangeably. In view of Propositions 3.6 and 3.8, we have the following result, for *both* infinite and finite orders.

Lemma 4.5. Fix $n \in \{2, 3, \dots\} \cup \{\infty\}$. Assume that the model satisfies condition $\mathcal{SD}(n)$ and that $U \in \mathcal{CMIM}(n)$, thus $V \in \mathcal{D}(n) \cap \mathcal{C}$. Then, the dual value function has the representation

$$(4.2) \quad v(y) = \mathbb{E}[V(y\hat{Y}_T)], \quad y > 0.$$

This result yields the key property that, up to a multiplicative constant, the dual problem admits the *same optimizer*, \hat{Y} , for *any* initial $y > 0$. Representation (4.2) can be thought of as a *relaxation* of the notion of model *completeness*, in the following sense: while a market model in $\mathcal{SD}(n)$ is, in general, incomplete from the point of view of replication, it does behave like a complete one from the point of view of optimal investment, if the utility function $U \in \mathcal{CMIM}(n)$.

4.3. Main theorems. We will assume that

$$(4.3) \quad v(y) < \infty, \quad y > 0.$$

We recall that the above condition is the canonical integrability one on the dual value function that is necessary and sufficient for the standard assertions of the utility maximization theory to hold; see [KS03] (see, also, [Kar13] for the formulation without *NFLVR* and [Mos15] for the formulation with intermediate consumption and stochastic utility, where (4.3) is combined with the finiteness of the primal value function). We also recall that for $U \in -\mathcal{C}$, under (2.3) and (4.3), for every $x, y > 0$, there exist unique optimizers,

$\widehat{X}(x) \in \mathcal{X}(x)$ and $\widehat{Y}(y) \in \mathcal{Y}(y)$, for the primal (2.5) and dual (2.7) problems, respectively. This is a consequence of the abstract theorems in [Mos15].

Theorem 4.6. *Consider a financial model for which (2.3) holds, and which is in $\mathcal{SD}(\infty)$. For $U \in \mathcal{CMIM}$, consider the optimal investment problem (2.5) and assume that (4.3) holds. Then, the following assertions hold:*

- (1) *The value function $u \in \mathcal{CMIM}$ and is thus analytic.*
- (2) *The dual value function $v \in \mathcal{D} \cap \mathcal{C}$. Furthermore, for $n \in \{1, 2, \dots\}$, we have*

$$\begin{aligned}
 (-1)^n v^{(n)}(y) &= (-1)^n \mathbb{E} \left[V^{(n)}(\widehat{Y}_T(y)) \left(\frac{\widehat{Y}_T(y)}{y} \right)^n \right] \\
 (4.4) \quad &= (-1)^n \mathbb{E} \left[V^{(n)}(y \widehat{Y}_T) \left(\widehat{Y}_T \right)^n \right] \\
 &= \mathbb{E} \left[\int_0^\infty e^{-zy \widehat{Y}_T} z^{n-1} \widehat{Y}_T^n \mu(dz) \right] \in (0, \infty), \quad y > 0.
 \end{aligned}$$

- (3) *The \mathcal{CM} function $-v'$ admits the Bernstein representation*

$$(4.5) \quad v'(y) = - \int_0^\infty e^{-yz} \nu(dz), \quad y > 0,$$

for some sigma-finite measure ν supported in $(0, \infty)$, such that

$$\nu(\{0\}) = 0 \quad \text{and} \quad \nu((0, \infty)) = \infty,$$

and which satisfies

$$(4.6) \quad \lim_{n \uparrow \infty} \int_{(0, z]} (-1)^{n+1} v^{(n+1)} \left(\frac{n}{\rho} \right) \left(\frac{n}{\rho} \right)^{n+1} d\rho = \frac{\nu((0, z]) + \nu((0, z))}{2}, \quad z > 0.$$

- (4) *For $n \geq 2$ and $f(y) := -\frac{1}{v''(y)}$, $y > 0$, we have*

$$(4.7) \quad u^{(n)}(x) = \sum \frac{(n-2)!}{k_1! 1!^{k_1} \dots k_{n-2}! (n-2)!^{k_{n-2}}} f^{(k_1 + \dots + k_{n-2})}(u'(x)) \prod_{j=1}^{n-2} (u^{(j+1)}(x))^{k_j},$$

$x > 0,$

where the sum is over all n -tuples of nonnegative integers (k_1, \dots, k_{n-2}) satisfying $\sum_{i=1}^{n-2} i k_i = n - 2$.

The following theorem specifies the derivatives of optimizers of all orders for both primal and dual problems.

Theorem 4.7. *Under the conditions of Theorem 4.6, the following assertions hold:*

- (1) *The primal and dual optimizers are related via*

$$(4.8) \quad \widehat{X}_T(x) = \int_0^\infty e^{-zu'(x)\widehat{Y}_T} \nu(dz), \quad x > 0.$$

(2) Their derivatives are given by

$$(4.9) \quad \begin{aligned} \widehat{X}_T^{(1)}(x) &:= \lim_{h \rightarrow 0} \frac{\widehat{X}_T(x+h) - \widehat{X}_T(x)}{h} = -V''(u'(x)\widehat{Y}_T)u''(x)\widehat{Y}_T \\ &= \frac{u''(x)}{U''(\widehat{X}_T(x))}\widehat{Y}_T > 0, \end{aligned}$$

$$(4.10) \quad \widehat{Y}_T^{(1)}(y) := \lim_{h \rightarrow 0} \frac{\widehat{Y}_T(y+h) - \widehat{Y}_T(y)}{h} = \frac{\widehat{Y}_T(y)}{y} = \widehat{Y}_T.$$

(3) Recursively, for every $n \geq 2$, the higher-order derivatives are given by

$$\begin{aligned} \widehat{X}_T^{(n)}(x) &:= \lim_{h \rightarrow 0} \frac{\widehat{X}_T^{(n-1)}(x+h) - \widehat{X}_T^{(n-1)}(x)}{h} \\ &= \sum_{k_1!1!^{k_1}k_2!2!^{k_2}\dots k_n!n!^{k_n}} \frac{n!}{k_1!1!^{k_1}k_2!2!^{k_2}\dots k_n!n!^{k_n}} \left(-V^{(1+k_1+\dots+k_n)} \left(u'(x)\widehat{Y}_T \right) \right) \prod_{j=1}^n \left(u^{(j+1)}(x)\widehat{Y}_T \right)^{k_j}, \end{aligned}$$

where the sum is over all n -tuples of nonnegative integers (k_1, \dots, k_n) satisfying $\sum_{i=1}^n ik_i = n$. Trivially,

$$\widehat{Y}_T^{(n)}(y) := \lim_{h \rightarrow 0} \frac{\widehat{Y}_T^{(n-1)}(y+h) - \widehat{Y}_T^{(n-1)}(y)}{h} = 0.$$

The limits for the derivatives of the optimizers above are in probability. However, because of the multiplicative structure of the dual $\widehat{Y}(y) = y\widehat{Y}$, the limits above can be understood in the stronger sense: for every sequence $h_k \rightarrow 0$, the convergence holds for \mathbb{P} -a.e. $\omega \in \Omega$.

Analogous results (and even easier, in many ways) can be stated for the case $n < \infty$.

Proposition 4.8. Fix $n \in \{2, 3, \dots\}$. Consider a financial model for which (2.3) holds, and which is in $\mathcal{SD}(n)$. Let $U \in \mathcal{CMLM}(n)$ so $V = V_U \in \mathcal{D}(n) \cap \mathcal{C}$. Furthermore, assume (4.3) and that V satisfies the inequalities

$$(4.11) \quad 0 < c_k \leq -\frac{y V^{(k+1)}(y)}{V^{(k)}(y)} = \frac{(-y)^{k+1}V^{(k+1)}(y)}{(-y)^k V^{(k)}(y)} \leq d_k < \infty, \quad y > 0, \quad k = 1, \dots, n-1,$$

for some constants c_k, d_k , $k = 1, \dots, n-1$. Then, for the optimal investment problem (2.5), the following assertions hold:

- (1) The value function $u \in \mathcal{CMLM}(n)$.
- (2) Up to order n , the derivatives of the dual value function v are given by (4.4).
- (3) The dual value function v satisfies the bounds (4.11) with respect to the same constants c_k and d_k , for $k = 1, \dots, n-1$.

As mentioned earlier, stochastic dominance of order $n \in \{3, \dots\} \cup \{\infty\}$ is a weaker condition than second order stochastic dominance ($n = 2$). Therefore, the class of models considered here is more general than the market models with a maximal element in the

sense of second order dominance in [KS06b]. On the other hand, for $U \in \mathcal{CMIM}(n)$ and an $\mathcal{SD}(n)$ model, Lemma 4.5 herein and [KS06b, Theorem 5] show that *the risk tolerance wealth process* $R(x)$ exists, for every $x > 0$.

We briefly recall the definition of $R(x)$ in [KS06b] as the maximal wealth process $R(x) = (R_t(x))_{t \in [0, T]}$, such that $R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}$. The process $R(x)$ is equal, up to an initial value, to the first derivative of the primal optimizer:

$$\lim_{h \rightarrow 0} \frac{\widehat{X}_T(x+h) - \widehat{X}_T(x)}{h} = \frac{R_T(x)}{R_0(x)},$$

where the limit is in \mathbb{P} probability.

We also note that the existence of the risk tolerance wealth process is connected to asymptotic expansions of second order (see [KS06b], [KS07]) or [MS19]). For a formulation and asymptotics with consumption and their relationship to the risk tolerance wealth process, we refer to [HMK19].

Remark 4.9 (On an integrability convention). Recalling a common convention (see, e.g., [Mos15]), we set

$$(4.12) \quad \mathbb{E}[V(Y_T)] := \infty \quad \text{if} \quad \mathbb{E}[V^+(Y_T)] = \infty.$$

This is done to avoid issues related to $\mathbb{E}[V(Y_T)]$ not being well-defined.

However, the following argument shows that $\mathbb{E}[V(Y_T)]$ is well-defined for every $y > 0$ and $Y \in \mathcal{Y}(y)$. To see this, take an arbitrary $y > 0$ and $Y \in \mathcal{Y}(y)$. Then, from the conjugacy between U and V , we get

$$-V(Y_T) \leq Y_T - U(1) \leq Y_T + |U(1)|.$$

Therefore, by the supermartingale property of Y , we obtain

$$\mathbb{E}[V^-(Y_T)] \leq y + |U(1)| < \infty.$$

Therefore, $\mathbb{E}[V(Y_T)]$ is well-defined with or without convention (4.12).

In analogy for the primal problem, we set

$$(4.13) \quad \mathbb{E}[U(X_T)] := -\infty \quad \text{if} \quad \mathbb{E}[U^-(X_T)] = \infty.$$

Assume that

$$(4.14) \quad v(y) < \infty \quad \text{for some } y > 0,$$

which is even weaker than (4.3).

By the argument from the previous paragraph, we know that $\mathbb{E}[V(Y_T)] \in \mathbb{R}$. For an arbitrary $x > 0$ and $X \in \mathcal{X}(x)$, by conjugacy between U and V , we get

$$U(X_T) \leq V(Y_T) + X_T Y_T \leq V^+(Y_T) + V^-(Y_T) + X_T Y_T.$$

Therefore,

$$U^+(X_T) \leq V^+(Y_T) + V^-(Y_T) + X_T Y_T \in \mathbb{L}^1(\mathbb{P}).$$

Thus, $\mathbb{E}[U(X_T)]$ is well-defined for every $x > 0$, and $X \in \mathcal{X}(x)$ with or without the convention (4.13), under the minimal assumption (4.14).

Remark 4.10 (On the positivity of $\widehat{X}_T^{(1)}$). We note that, in general, the derivative of the primal optimizer with respect to the initial wealth does not have to be a positive random variable, see [KS06a, Example 4]. However, for models satisfying the assumptions of Theorem 4.6, (4.9) implies the strict positivity of $\widehat{X}_T^{(1)}$. This complements the results in [KS06b], see Theorem 4 there, which implies the positivity of $\widehat{X}_T^{(1)}$ in stochastically dominant models in the sense of the second order stochastic dominance.

Proof of Theorem 4.6. As $U \in \mathcal{CMIM}$, its convex conjugate $V \in \mathcal{D} \cap \mathcal{C}$. Then, Lemma 4.5 gives that

$$v(y) = \mathbb{E} \left[V(y\widehat{Y}_T) \right], \quad y > 0,$$

i.e., the dual minimizer is the same up to the multiplicative constant y : $\widehat{Y}(y) = y\widehat{Y}$, $y > 0$. The expectation above is well-defined, see the discussion in Remark 4.9. Note that (2.3), (4.3) and the structure of the utility function (the Inada conditions, together with the strict monotonicity, concavity, and smoothness) imply the strict concavity and continuous differentiability of both u and $-v$ on $(0, \infty)$; see [KS03], [Mos15], and [CCFM17]. For $n = 1$, (4.4) follows from the standard conclusions of the utility maximization theory, as

$$(4.15) \quad -v'(y)y = -\mathbb{E} \left[V'(\widehat{Y}_T(y))\widehat{Y}_T(y) \right] = -\mathbb{E} \left[V'(y\widehat{Y}_T)y\widehat{Y}_T \right] \in (0, \infty),$$

and therefore, $v'(y) = \mathbb{E} \left[V'(y\widehat{Y}_T)\widehat{Y}_T \right]$, for every $y > 0$.

Next, assume that (4.4) holds for $n = k$, i.e.,

$$v^{(k)}(y) = \mathbb{E} \left[V^{(k)}(y\widehat{Y}_T) \left(\widehat{Y}_T \right)^k \right] = (-1)^k \mathbb{E} \left[\int_0^\infty e^{-y\widehat{Y}_T z} z^{k-1} \mu(dz) \left(\widehat{Y}_T \right)^k \right], \quad y > 0,$$

where in the second equality we have used (4.1). Let us recall that (4.1) gives

$$V^{(k)}(y) = (-1)^k \int_0^\infty e^{-yz} z^{k-1} \mu(dz), \quad y > 0.$$

Then, let us consider

$$(4.16) \quad \begin{aligned} \frac{v^{(k)}(y+h) - v^{(k)}(y)}{h} &= \frac{1}{h} \mathbb{E} \left[V^{(k)}((y+h)\widehat{Y}_T)\widehat{Y}_T^k - V^{(k)}(y\widehat{Y}_T)\widehat{Y}_T^k \right] \\ &= (-1)^k \mathbb{E} \left[\int_0^\infty \frac{(\widehat{Y}_T z)^{k-1} \widehat{Y}_T}{h} \left(e^{-(y+h)\widehat{Y}_T z} - e^{-y\widehat{Y}_T z} \right) \mu(dz) \right], \end{aligned}$$

Let us fix $y > 0$. As for every $h \neq 0$, we have

$$0 \leq -\frac{1}{h} \left(e^{-(y+h)\widehat{Y}_T z} - e^{-y\widehat{Y}_T z} \right) \leq e^{-(y-|h|)\widehat{Y}_T z} z \widehat{Y}_T,$$

we deduce that, for a constant $h_0 \in (0, y)$, and every $h \in (-h_0, h_0)$, the following inequalities hold

$$(4.17) \quad 0 \leq -(\widehat{Y}_T z)^{k-1} \frac{1}{h} \left(e^{-(y+h)\widehat{Y}_T z} - e^{-y\widehat{Y}_T z} \right) \widehat{Y}_T \leq (\widehat{Y}_T z)^k e^{-\widehat{Y}_T(y-h_0)z} \widehat{Y}_T.$$

Furthermore, as there exists a constant M , such that

$$(4.18) \quad \bar{z}^k e^{-\bar{z}(y-h_0)} \leq M e^{-\frac{1}{2}\bar{z}(y-h_0)}, \quad \text{for every } \bar{z} \geq 0,$$

and since, by (4.4) for $n = 1$ (see also (4.15)), we have

$$(4.19) \quad \mathbb{E} \left[\int_0^\infty M \widehat{Y}_T e^{-\frac{1}{2}\widehat{Y}_T(y-h_0)z} \mu(dz) \right] = -M v' \left(\frac{1}{2}(y-h_0) \right) < \infty,$$

we deduce from (4.18) and (4.19) that the last expression in (4.17) is $\mathbb{P} \times \mu$ integrable. Therefore, in (4.16), one can pass to the limit as $h \rightarrow 0$ to deduce that (4.4) holds for $n = k + 1$. We conclude that (4.4) holds for every $n \in \mathbb{N}$. Now, the complete monotonicity of v follows from the complete monotonicity of V and (4.4). In turn, this implies the analyticity of v , see, e.g., [McH75]. Further, as $(-1)^n V^{(n)}$ do not vanish (see, e.g., [SSV10, Remark 1.5]), we deduce from (4.4) that $(-1)^n v^{(n)}$ are also strictly positive for every $n \in \mathbb{N}$. By [KS03, Theorem 4], $-v$ satisfies the Inada conditions, which imply that $\nu(\{0\}) = 0$ and $\nu((0, \infty)) = \infty$. Representation (4.5) follows, where (4.6) results from the inversion formula, see [Wid41, Chapter VII, Theorem 7a].

To obtain the properties of u , first, we observe that the biconjugacy relations between the value functions imply that u' exists at every $x > 0$, and it is the inverse of $-v'$. This, and since v' is strictly negative on $(0, \infty)$, imply the analyticity of u . In turn, (4.7) is the consequence of the Faà di Bruno formula, see [Por01, Section 4.3]. \square

Proof of Theorem 4.7. First, we observe that (4.8) is the consequence of (4.1) and standard assertions of the utility maximization theory. In turn, (4.10) follows from the optimality of $y\widehat{Y}$ for every $y > 0$, whereas (4.9) results from the relation

$$\widehat{X}_T(x) = -V'(y\widehat{Y}_T), \quad \text{for } y = u'(x).$$

The higher order derivatives of the dual and primal optimizers follow from the direct computations and an application of the Faà di Bruno formula. \square

Proof of Proposition 4.8. From Lemma 4.5, we have

$$v(y) = \mathbb{E}[V(y\widehat{Y}_T)], \quad y > 0.$$

By [KS03, Theorem 4], v is differentiable and we have

$$v'(y) = \mathbb{E}[\widehat{Y}_T V'(y\widehat{Y}_T)], \quad y > 0.$$

Therefore, it only remains to compute the higher order derivatives of v , recursively, up to order n , as in formula (4.4). The bounds (4.11) for v would follow immediately. In what

follows, we show that (4.4) holds up to order n . Assume that, for some $1 \leq k \leq n - 1$, we have

$$(4.20) \quad (-1)^{k-1}v^{(k-1)}(y) = \mathbb{E}[(-\widehat{Y}_T)^{k-1}V^{(k-1)}(y\widehat{Y}_T)] < \infty, \quad y > 0.$$

Using bounds (4.11) and following the proof of [KS06a, Lemma 3] we obtain that, for any $a > 1$, there exist some constants

$$\alpha_k < 1 < \beta_k < \infty$$

for which

$$(4.21) \quad \alpha_k(-1)^{k-1}V^{(k-1)}(y) \leq (-1)^{k-1}V^{(k-1)}(ay) \leq \beta_k(-1)^{k-1}V^{(k-1)}(y), \quad y > 0.$$

Fix $y > 0$. Then,

$$\begin{aligned} \frac{v^{(k-1)}(y+h) - v^{(k-1)}(y)}{h} &= \mathbb{E} \left[\frac{(\widehat{Y}_T)^{k-1} V^{(k-1)}((y+h)\widehat{Y}_T) - V^{(k-1)}(y\widehat{Y}_T)}{h} \right] \\ &= \mathbb{E} \left[(\widehat{Y}_T)^k V^{(k)}(\xi_h) \right], \end{aligned}$$

where ξ_h is a random variable taking values between $y\widehat{Y}_T$ and $(y+h)\widehat{Y}_T$.

Fix $a > 1$. Using the bounds (4.11) for $V^{(k)}$ in terms of $V^{(k-1)}$, together with (4.21), we conclude that there exists a finite constant C , such that for $|h|$ small enough so that

$$\frac{1}{a} \leq \frac{y-|h|}{y} \leq \frac{y+|h|}{y} \leq a,$$

we have

$$\left| (\widehat{Y}_T)^k V^{(k)}(\xi_h) \right| \leq C \left| (\widehat{Y}_T)^{k-1} V^{(k-1)}(y\widehat{Y}_T) \right|.$$

Since the right-hand side above is integrable, according to (4.20), and

$$\frac{V^{(k-1)}((y+h)\widehat{Y}_T) - V^{(k-1)}(y\widehat{Y}_T)}{h} \rightarrow \widehat{Y}_T V^{(k)}(y\widehat{Y}_T), \quad \mathbb{P} - a.s.,$$

we can use the Lebesgue dominated convergence theorem to conclude the assertions of part *ii*). The remaining assertions follow. \square

5. COUNTEREXAMPLE 1: $\mathcal{SD}(\infty)$ MARKET MODEL AND $U \notin \mathcal{CMIM}$

We show that the analyticity of the value function may fail if the utility is not \mathcal{CMIM} , even if it is analytic, and even if the market model is complete, and thus in the $\mathcal{SD}(n)$ class for every $n \in \{2, 3, \dots\} \cup \{\infty\}$. As the construction shows, we will be using completely monotonic functions of finite order. Working with this class allows to tailor the assumptions on the utility function so that differentiability holds up to order n , but fails at order $n + 1$, for *any* choice of $n \in \mathbb{N}$.

Proposition 5.1. *Fix $n \geq 1$. There exists a complete market model and an analytic utility function $U : (0, \infty) \rightarrow \mathbb{R}$ such that*

$$U \in \mathcal{CMIM}(n+1) \iff V \in \mathcal{D}(n+1) \cap \mathcal{C},$$

where the dual V satisfies the bounds (4.11) (up to order $k = n - 1$, but not up to order $k = n$), and for which the conjugate value functions u and v satisfy

$$u \in \mathcal{CMIM}(n) \iff v \in \mathcal{D}(n) \cap \mathcal{C},$$

together with identical bounds (4.11) up to order $k = n - 1$, but with

$$(-1)^{n+1}v^{(n+1)}(1) = \infty.$$

We will need the following Lemma.

Lemma 5.2. *There exists an analytic function $f : (0, \infty) \rightarrow \mathbb{R}$ with the following properties*

- (1) $1 \leq f \leq 2$,
- (2) $f' < 0$,
- (3) $-f'(i) \geq \frac{i^2}{C}$, $i = 1, 2, \dots$, for some constant $C > 0$.

Proof of Proposition 5.1. Consider the auxiliary dual value function $\bar{V}(y) := y^{-1}$. Consider now the new utility function V defined as

$$V(y) := \int_y^\infty \int_{y_1}^\infty \cdots \int_{y_{n-1}}^\infty (-1)^n f(y_n) \bar{V}^{(n)}(y_n) dy_n \cdots dy_2 dy_1 > 0.$$

Note that the intuition behind this definition comes from setting V at the level of the n -th order derivative,

$$V^{(n)}(y) := f(y) \bar{V}^{(n)}(y), \quad y > 0,$$

and then recovering V by integration.

Since $1 \leq f \leq 2$ and using the integral representations

$$\begin{aligned} (-1)^k \bar{V}^{(k)}(y_k) &= \int_{y_k}^\infty \cdots \int_{y_{n-1}}^\infty (-1)^n \bar{V}^{(n)}(y_n) dy_n \cdots dy_{k+1}, \quad y_k > 0, \\ (-1)^k V^{(k)}(y_k) &= \int_{y_k}^\infty \cdots \int_{y_{n-1}}^\infty (-1)^n f(y_n) \bar{V}^{(n)}(y_n) dy_n \cdots dy_{k+1}, \quad y_k > 0, \end{aligned}$$

we obtain bounds for derivatives of V in terms of derivatives of the same order of \bar{V} ,

$$(5.1) \quad (-1)^k \bar{V}^{(k)}(y) \leq (-1)^k V^{(k)}(y) \leq 2(-1)^k \bar{V}^{(k)}(y) \quad y > 0, \quad k = 0, 1, \dots, n.$$

Since

$$\bar{V}''(y) = 2y^{-3}, \bar{V}'''(y) = -6y^{-4}, \dots, \bar{V}^{(n+1)}(y) = (-1)^{n+1} C_{n+1} y^{-n-2}, \quad y > 0,$$

for some explicit positive constants C_k , \bar{V} satisfies some bounds on higher order risk tolerance coefficients of the type (4.11) up to order $k = n - 1$. Using the integral representations above, (5.1) yields similar bounds for risk tolerance type coefficients for V : for every $k = 1, \dots, n - 1$, there exist $0 < c_k < d_k < \infty$ such that

$$c_k \leq -\frac{yV^{(k+1)}(y)}{V^{(k)}(y)} = \frac{(-y)^{k+1}V^{(k+1)}(y)}{(-y)^kV^{(k)}(y)} \leq d_k, \quad y > 0, \quad k = 1, \dots, n - 1.$$

Since $-f'(i) \geq \frac{i^2}{C}$, we choose weights $q_i := \frac{1}{i^3} > 0$, such that

$$s_1 := \sum_{i=1}^{\infty} iq_i < \infty, \quad \sum_{i=1}^{\infty} -f'(i)q_i = \infty.$$

Denoting by

$$s_0 := \sum_{i=1}^{\infty} q_i < \sum_{i=1}^{\infty} iq_i = s_1 < \infty,$$

we define

$$\varepsilon := \frac{\frac{1}{2}}{\frac{s_1}{s_0} - \frac{1}{2}} \in (0, 1)$$

and consider a random variable Z such that

$$\mathbb{P}(Z = \frac{1}{2}) = 1 - \varepsilon, \quad \mathbb{P}(Z = i) = p_i := \varepsilon \frac{q_i}{s_0}, \quad i = 1, 2, 3, \dots$$

Then, we have

$$\mathbb{E}[Z] = \frac{1}{2}(1 - \varepsilon) + \frac{\varepsilon}{s_0}s_1 = \frac{1}{2} + \varepsilon \left(\frac{s_1}{s_0} - \frac{1}{2} \right) = 1.$$

Next, consider any market with the unique martingale measure with density Z . The dual value function is finite since Z is bounded from below. Therefore,

$$(5.2) \quad v(y) = \mathbb{E}[V(yZ)] < \infty, \quad y > 0,$$

i.e., (4.3) holds. Using Proposition 4.8, we obtain that

$$(5.3) \quad 0 < (-1)^k v^{(k)}(y) = (-1)^k \mathbb{E}[V^{(k)}(yZ)Z^k] < \infty, \quad y > 0, \quad k = 1, \dots, n.$$

Therefore, $v \in \mathcal{D}(n) \cap \mathcal{C}$ and, in particular, the n -th derivative is given by

$$(5.4) \quad v^{(n)}(y) = \mathbb{E}[V^{(n)}(yZ)Z^n], \quad y > 0.$$

The $n + 1$ derivative of V is

$$V^{(n+1)}(y) = f(y)\bar{V}^{(n+1)}(y) + f'(y)\bar{V}^{(n)}(y).$$

One should note that

$$(-1)^{n+1}V^{(n+1)}(y) = f(y)(-1)^{n+1}\bar{V}^{(n+1)}(y) + (-f'(y))(-1)^n\bar{V}^{(n)}(y) > 0,$$

and

$$(-y)^{(n+1)}V^{(n+1)}(y) > -C_n f'(y).$$

By construction, $\mathbb{E}[-f'(Z)] = \infty$, so $\mathbb{E}[(-Z)^{n+1}V^{(n+1)}(Z)] = \infty$. Finally, from (5.4) we have

$$(-1)^{n+1} \frac{v^{(n)}(y) - v^{(n)}(1)}{y - 1} = (-1)^{n+1} \frac{\mathbb{E}[V^{(n)}(yZ)Z^n - V^{(n)}(Z)Z^n]}{y - 1} = \mathbb{E}[V^{(n+1)}(\xi)(-Z)^{n+1}],$$

for some random variable ξ taking values between Z and yZ . Since

$$0 \leq V^{(n+1)}(\xi)(-Z)^{n+1} \rightarrow V^{(n+1)}(Z)(-Z)^{n+1},$$

we can now apply Fatou's lemma to obtain

$$(-1)^{n+1}v^{(n+1)}(1) = \infty.$$

□

Proof of Lemma 5.2. Consider the Gaussian densities (up to a multiplicative factor) with mean μ and standard deviation $\sigma > 0$, given by

$$g_{\mu,\sigma}(x) := \frac{1}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.$$

Then, we have

$$\int_{-\infty}^{\infty} g_{\mu,\sigma}(x) dx = \sqrt{2\pi}, \quad g_{\mu,\sigma^2}(\mu) = \frac{1}{\sigma}.$$

Next, let

$$g(z) := \sum_{i=1}^{\infty} \frac{1}{i^2} g_{i,i-4}(z), \quad z \in \mathbb{C}.$$

While not obvious, it is easy to see that the series converges uniformly on compacts (in the complex plane), so it is analytic (entire). In addition,

$$\int_{-\infty}^{\infty} g(x) dx \leq \sqrt{2\pi} \sum_{i=1}^{\infty} \frac{1}{i^2} =: C < \infty.$$

Furthermore, we have

$$g(i) \geq \frac{1}{i^2} g_{i,i-4}(i) = \frac{1}{i^2} i^4 = i^2.$$

Finally, we set

$$f(y) := 2 - \frac{1}{C} \int_0^y g(x) dx, \quad y > 0.$$

One can then see that f satisfies the desired properties. □

Remark 5.3. To obtain an example with a utility having positive third derivative $U''' > 0$, we just use Proposition 5.1 for $n = 2$. The fact that $-V''' = (-1)^{(n+1)}V^{(n+1)} > 0$ ensures, by duality, that the corresponding utility

$$U(x) := \inf_{y>0} (V(y) + xy), \quad x > 0,$$

satisfies the desired condition $U'''(x) > 0$ for all $x > 0$.

6. COUNTEREXAMPLE 2: NON-STOCHASTICALLY DOMINANT MODELS AND LACK OF DIFFERENTIABILITY

We show that for *any* non-homothetic utility $U \in -\mathcal{C}$ with $U \in C^2((0, \infty))$, we may construct a non- $\mathcal{SD}(\infty)$ market model such that, at some point $x > 0$, the two-times differentiability of u fails. We recall that standard results in utility maximization theory, in the form of Kramkov and Schachermayer [KS99], assert the continuous differentiability of the value functions. The result below demonstrates that differentiability might cease to exist at the very next order (even with a \mathcal{CMLM} utility).

We note that, due to the multiplicative structure $\widehat{Y}(y) = y\widehat{Y}$ under the assumptions of Theorem 4.6, we do not make any sigma-boundedness assumption, as in [KS06a]. Our counterexample is somewhat related to the sigma-boundedness counterexample from [KS06a], but it is stronger: we construct a (counterexample) model for *every* Inada utility function with non-constant relative risk aversion.

Let $U \in -\mathcal{C}$ with $U \in C^2((0, \infty))$, having non-constant relative risk aversion

$$(6.1) \quad A(x) := -\frac{U''(x)x}{U'(x)}, x > 0.$$

The assumption $U \in C^2((0, \infty))$ is without loss of generality. We may also choose $U \in \mathcal{CMLM}$.

Proposition 6.1. *For any non-homothetic¹ utility $U \in -\mathcal{C}$, with $U \in C^2((0, \infty))$, and thus, $U \in \mathcal{CMLM}(2)$, there exists a non- $\mathcal{SD}(\infty)$ market model such that the value function is not twice differentiable at some $x > 0$.*

Proof. We first assume that the risk aversion A satisfies $A(1/m) \neq A(1/k)$, for some m and k in \mathbb{N} . As we justify at the end of the proof, this is without loss of generality.

Let us suppose that the sample space $\Omega = \{\omega_0, \omega_1, \dots\}$, and consider a one-period model, where the market consists of a money market account with 0 interest rate and a stock, with $S_0 = 1$ and $S_1(\omega_0) = 2$, $S_1(\omega_n) = \frac{1}{n}$, $n \in \mathbb{N}$.

We are going to construct probabilities $p_n := \mathbb{P}[\omega_n] > 0$, $n \geq 0$, satisfying the following three properties

$$(6.2) \quad \mathbb{E}[-U''(S_1)] < \infty,$$

$$(6.3) \quad \mathbb{E}[U'(S_1)S_1] = \mathbb{E}[U'(S_1)] < \infty,$$

and

$$(6.4) \quad \mathbb{E}[U'(S_1)(1 - S_1)A(S_1)] \neq 0,$$

¹I.e., such that $A \neq \text{const}$, where A is defined in (6.1).

where A is defined in (6.1). Note that, relations (6.3) and (6.4) can hold together only if the function A is non-constant.

Direct computations show that (6.4) holds if and only if

$$(6.5) \quad \hat{\Delta} := -\frac{\mathbb{E}[U''(S_1)(S_1 - 1)]}{\mathbb{E}[U''(S_1)(S_1 - 1)^2]} \neq 1.$$

Furthermore, note that $\hat{\Delta} \in (-1, 2)$.

In addition to (6.2), (6.3), and (6.4), we will show that there exists $\bar{\varepsilon} \in (0, \frac{1}{2}]$, such that for a random variable defined as

$$\begin{aligned} G(\omega) := & \min_{\varepsilon \in [0, \bar{\varepsilon}]} U'' \left(S_1(\omega) + \varepsilon(1 + \hat{\Delta}(S_1(\omega) - 1)) \right) 1_{\{\hat{\Delta} < 1\}} \\ & + \min_{\varepsilon \in [-\bar{\varepsilon}, 0]} U'' \left(S_1(\omega) + \varepsilon(1 + \hat{\Delta}(S_1(\omega) - 1)) \right) 1_{\{\hat{\Delta} > 1\}}, \quad \omega \in \Omega, \end{aligned}$$

satisfies

$$(6.6) \quad G \in \mathbb{L}^1(\mathbb{P}).$$

Assuming for now that such probabilities indeed exist, we show that under (6.3) and stock as above, \mathbb{P} is not a martingale measure for S , and we have

$$(6.7) \quad 2 > \mathbb{E}[S_1] > 1.$$

Indeed, the monotonicity of U' yields

$$\begin{aligned} \mathbb{E}[1 - S_1] &= \mathbb{E}[(1 - S_1)1_{\{S_1 < 2\}}] - \mathbb{P}[\{S_1 = 2\}] \\ &< \mathbb{E} \left[\frac{U'(S_1)}{U'(2)} (1 - S_1) 1_{\{S_1 < 2\}} \right] - \mathbb{P}[\{S_1 = 2\}] \\ &= \frac{1}{U'(2)} (\mathbb{E}[U'(S_1)(1 - S_1)1_{\{S_1 < 2\}}] + U'(2)\mathbb{E}[(1 - S_1)1_{\{S_1=2\}}]) \\ &= \frac{1}{U'(2)} \mathbb{E}[U'(S_1)(1 - S_1)] = 0, \end{aligned}$$

where in the last equality we used (6.3). This implies (6.7), where the upper bound is also strict as $p_n > 0$, for every $n \geq 0$. Thus, \mathbb{P} is not a martingale measure for S . Therefore, the constant-valued process $Z \equiv 1$ is not an element of $\mathcal{Y}(1)$, and thus it is not the dual minimizer for $y = 1$.

Furthermore, we claim that (4.3) holds. This is rather clear by observing that, for n_0 large enough, one can choose a martingale measure \mathbb{Q} that changes the probabilities only for $\omega_0, \omega_1, \dots, \omega_{n_0}$ for some n_0 , but keeps the same probabilities for $\omega_n, n > n_0$. As the density $Z \in \mathcal{Y}(1)$ of such a martingale measure is bounded below away from 0, (4.3) holds.

Next, we construct appropriate probabilities p_n 's (such that (6.3), (6.6), and (6.5) hold; note that one might need to perturb finitely many of these p_n 's later such that (6.12) below holds too). For this, we set

$$(6.8) \quad p_n := \frac{1}{2^{n+1}} \frac{\min(1, U'(2))}{\max\left(1, U'\left(\frac{1}{n}\right) - \min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s)\right)}, \quad n \geq 2.$$

Note that

$$\min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s) \leq \min_{z \in [0, 1/3]} \min_{\Delta \in [-1, 1]} U''(1/n + z(1 + \Delta(1/n - 1)))$$

and

$$\min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s) \leq \min_{z \in [-1/3, 0]} \min_{\Delta \in [1, 2]} U''(1/n + z(1 + \Delta(1/n - 1))).$$

The intuition behind the exact form of the intervals above comes from taking $\bar{\varepsilon} = 1/3$ in the construction of G satisfying (6.6) when $\hat{\Delta}$ is not fixed yet.

Then, as $S_1 > 1$ only for ω_0 , we have

$$\begin{aligned} 0 &\leq \mathbb{E}[U'(S_1)S_1] \leq U'(2) + \mathbb{E}[U'(S_1)] \leq 2U'(2) + U'(1) \\ &+ \sum_{n \geq 2} \frac{1}{2^{n+1}} \frac{U'\left(\frac{1}{n}\right)}{\max\left(U'\left(\frac{1}{n}\right) - \min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s), 1\right)} < \infty, \end{aligned}$$

and, the finiteness in (6.3) holds (regardless of the choice of p_0 and p_1).

Now, with p_n , $n \geq 2$, given by (6.8), we show that we can simultaneously have (6.3) and (6.5). We define

$$(6.9) \quad \begin{aligned} p_0 &:= \frac{1}{U'(2)} \sum_{n \geq 2} p_n U'\left(\frac{1}{n}\right) \left(1 - \frac{1}{n}\right) \\ &= \frac{\min(U'(2), 1)}{U'(2)} \sum_{n \geq 2} \frac{1}{2^{n+1}} \frac{U'\left(\frac{1}{n}\right) \left(1 - \frac{1}{n}\right)}{\max\left(U'\left(\frac{1}{n}\right) - \min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s), 1\right)}. \end{aligned}$$

Then, using the above, we rewrite

$$2p_0 U'(2) + \sum_{n \geq 1} p_n U'\left(\frac{1}{n}\right) \frac{1}{n} = p_0 U'(2) + \sum_{n \geq 1} p_n U'\left(\frac{1}{n}\right),$$

and (6.3) follows. Thus, (6.3) holds with p_n , $n \geq 2$, given by (6.8) and p_0 specified by (6.9). Note that $p_0 \leq 1/4$, $\sum_{n \geq 2} p_n \leq 1/4$, and, therefore, $p_1 := 1 - (p_0 + \sum_{n \geq 2} p_n) \geq 1/2$.

To show (6.6), we observe that there exist constants a and a' , such that $0 < a < a'$ and, for an appropriate $\bar{\varepsilon}$, we have

$$0 \geq \mathbb{E}[G] \geq 2 \min_{s \in [a, a']} U''(s) + \sum_{n \geq 2} \frac{1}{2^{n+1}} \frac{\min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s)}{\max \left(U' \left(\frac{1}{n} \right) - \min_{s \in [\frac{2}{3n}, \frac{2}{3} + \frac{2}{3n}]} U''(s), 1 \right)} > -\infty.$$

Next, we show that (6.4) holds for the choice of p_n 's or for a slightly perturbed choice of p_n 's, where the distortion is such that the remaining assumptions of the example do not change. To this end, we rewrite (6.5) as

$$(6.10) \quad -U''(2)p_0 + \sum_{n \geq 2} U''\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)p_n \neq U''(2)p_0 + \sum_{n \geq 2} U''\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)^2 p_n.$$

Collecting terms and plugging the expression for p_0 from (6.9), we can rewrite (6.10) as

$$0 \neq -\frac{2U''(2)}{U'(2)} \sum_{n \geq 2} p_n U' \left(\frac{1}{n} \right) \left(1 - \frac{1}{n} \right) + \sum_{n \geq 2} U''\left(\frac{1}{n}\right)\left(1 - \frac{1}{n}\right)\frac{1}{n} p_n$$

or, in turn,

$$(6.11) \quad 0 \neq \sum_{n \geq 2} p_n U' \left(\frac{1}{n} \right) \left(1 - \frac{1}{n} \right) \left(A(2) - A\left(\frac{1}{n}\right) \right).$$

We note that, if $x \rightarrow A(x)$, $x > 0$, is strictly monotone², (6.5) holds, as all terms under the sum in (6.11) are non-zero and of the same sign.

When the relative risk aversion is not monotone, but also non-constant, and if (6.11) does not hold, it is enough to perturb finitely many of the p_n 's in a way to get simultaneously

$$(6.12) \quad \begin{aligned} \sum_{n \geq 0} p_n &= 1, \\ \sum_{n \geq 0} p_n U'(s_n)(1 - s_n) &= 0, \\ \sum_{n \geq 0} p_n U'(s_n)(1 - s_n)A(s_n) &\neq 0, \end{aligned}$$

while preserving the positivity of p_n 's (here $s_0 = 2$ and $s_n = 1/n$, $n \in \mathbb{N}$). As $A(1/m) \neq A(1/k)$, for some m and k , such a distortion of p_n 's exists. As we have only perturbed

²An example of an Inada utility function of class \mathcal{CMZM} , where the relative risk aversion is strictly monotone, is given via $-V'(y) = y^{-k} \frac{1}{y+1}$, $y > 0$, for some constant $k > 0$. Here $-V'$ is completely monotonic as a product of the completely monotonic functions $y \rightarrow y^{-k}$ and $y \rightarrow \frac{1}{y+1}$, $y > 0$, see [SSV10, Corollary 1.6]. Then, the relative risk tolerance at $x = -V'(y)$ is given by $B(y) = -\frac{V''(y)y}{V'(y)} = k + \frac{y}{y+1}$, which is a strictly monotone function of y on $(0, \infty)$. As $A(x)B(y) = 1$ for $y = U'(x)$, we deduce that A is also a strictly monotone function on $(0, \infty)$.

finitely many p_n 's, (6.6) still holds. This results in the choice of a probability measure, such that (6.3), (6.5), and (6.6) hold.

We show that $u''(1)$ does not exist. First, we will assume that in (6.5), the left-hand side is strictly less than 1, i.e.,

$$(6.13) \quad -\frac{\mathbb{E}[U''(S_1)(S_1 - 1)]}{\mathbb{E}[U''(S_1)(S_1 - 1)^2]} < 1.$$

As for every $x \geq 0$, $\{x + \Delta(S_1 - 1) : \Delta \in [-x, x]\}$ is the set of terminal values of the elements of $\mathcal{X}(x)$, we observe that buying the portfolio consisting of one share of stock is admissible for $x = 1$. Then, by conjugacy, we have

$$(6.14) \quad \mathbb{E}[U(S_1)] = \mathbb{E}[V(U'(S_1)) + U'(S_1)S_1].$$

Using (6.3) and with an arbitrary $\Delta \in [-1, 1]$, we can rewrite the latter expression as

$$(6.15) \quad \begin{aligned} \mathbb{E}[V(U'(S_1)) + U'(S_1)S_1] &= \mathbb{E}[V(U'(S_1)) + U'(S_1)] \\ &= \mathbb{E}[V(U'(S_1)) + U'(S_1)(1 + \Delta(S_1 - 1))] \geq \mathbb{E}[U(1 + \Delta(S_1 - 1))]. \end{aligned}$$

Combining (6.14) and (6.15), we get

$$\mathbb{E}[U(S_1)] \geq \mathbb{E}[U(1 + \Delta(S_1 - 1))], \quad \Delta \in [-1, 1],$$

which yields that $\widehat{X}_1(1) = S_1$.

In turn, by the relations between the primal and dual optimizers, we get

$$(6.16) \quad u'(1) = \mathbb{E}[S_1 U'(S_1)] = \mathbb{E}[U'(S_1)],$$

where the second equality follows from (6.3).

For ε being small, Taylor's expansion yields

$$(6.17) \quad U(\widehat{X}_1(1 + \varepsilon)) - U(S_1) = \varepsilon U'(S_1) \frac{\widehat{X}_1(1 + \varepsilon) - S_1}{\varepsilon} + \frac{\varepsilon^2}{2} U''(\eta(\varepsilon)) \frac{(\widehat{X}_1(1 + \varepsilon) - S_1)^2}{\varepsilon^2},$$

where $\eta(\varepsilon)$ is a random variable taking values between S_1 and $\widehat{X}_1(1 + \varepsilon)$. Therefore, from (6.16) and (6.17), we obtain

$$(6.18) \quad \begin{aligned} \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)) &= \frac{2}{\varepsilon} \mathbb{E} \left[U'(S_1) \left(\frac{\widehat{X}_1(1 + \varepsilon) - S_1}{\varepsilon} - 1 \right) \right] \\ &+ \mathbb{E} \left[U''(\eta(\varepsilon)) \frac{(\widehat{X}_1(1 + \varepsilon) - S_1)^2}{\varepsilon^2} \right]. \end{aligned}$$

Let us consider the first term in the right-hand side of (6.18),

$$(6.19) \quad \frac{2}{\varepsilon} \mathbb{E} \left[U'(S_1) \left(\frac{\widehat{X}_1(1 + \varepsilon) - S_1}{\varepsilon} - 1 \right) \right].$$

As $\widehat{X}_1(1 + \varepsilon) = 1 + \varepsilon + \widehat{\Delta}(1 + \varepsilon)(S_1 - 1)$, for some (fixed and nonrandom) $\widehat{\Delta}(1 + \varepsilon) \in [-1 - \varepsilon, 1 + \varepsilon]$, we can rewrite (6.19) as

$$\begin{aligned}
(6.20) \quad & \frac{2}{\varepsilon} \mathbb{E} \left[U'(S_1) \left(\frac{1 + \varepsilon + \widehat{\Delta}(1 + \varepsilon)(S_1 - 1) - S_1}{\varepsilon} - 1 \right) \right] \\
&= \frac{2}{\varepsilon} \mathbb{E} \left[U'(S_1) \left(\frac{1 + \varepsilon - S_1 - \varepsilon}{\varepsilon} \right) \right] + \frac{2}{\varepsilon^2} \widehat{\Delta}(1 + \varepsilon) \mathbb{E} [U'(S_1) (S_1 - 1)] \\
&= \frac{2}{\varepsilon^2} \mathbb{E} [U'(S_1) (1 - S_1)] + \frac{2}{\varepsilon^2} \widehat{\Delta}(1 + \varepsilon) \mathbb{E} [U'(S_1) (S_1 - 1)] \\
&= 0,
\end{aligned}$$

where in the last equality we used (6.3). Therefore, (6.18) becomes

$$(6.21) \quad \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)) = \mathbb{E} \left[U''(\eta(\varepsilon)) \frac{(\widehat{X}_1(1 + \varepsilon) - S_1)^2}{\varepsilon^2} \right].$$

Next, we look at

$$(6.22) \quad \limsup_{\varepsilon \uparrow 0} \mathbb{E} \left[U''(\eta(\varepsilon)) \frac{(\widehat{X}_1(1 + \varepsilon) - S_1)^2}{\varepsilon^2} \right].$$

Using that $u(1) < \infty$, [KS99, Lemma 3.6] together with the symmetry between the primal and dual problems (see the abstract theorems from [Mos15]) allow to conclude that $\widehat{X}_1(1 + \varepsilon) \rightarrow \widehat{X}_1(1) = S_1$, as $\varepsilon \rightarrow 0$, in probability. Consequently, since $\eta(\varepsilon)$ takes values between $\widehat{X}_1(1)$ and $\widehat{X}_1(1 + \varepsilon)$, we deduce that $\eta(\varepsilon) \rightarrow S_1$ in probability. Passing to the limit along a subsequence in (6.22), and applying Fatou's lemma, we get

$$(6.23) \quad \limsup_{\varepsilon \uparrow 0} \mathbb{E} \left[U''(\eta(\varepsilon)) \frac{(\widehat{X}_1(1 + \varepsilon) - S_1)^2}{\varepsilon^2} \right] \leq \sup_{\bar{\Delta} \geq 1} \mathbb{E} \left[U''(S_1) (\bar{\Delta} S_1 + 1 - \bar{\Delta})^2 \right],$$

where we used the representation $\frac{\widehat{X}_1(1 + \varepsilon) - S_1}{\varepsilon} = \bar{\Delta} S_1 + 1 - \bar{\Delta}$, for some constant $\bar{\Delta}$, which, for $\varepsilon \in (-1, 0)$, is bounded from below by 1, and the observation that, on $\bar{\Delta} > 2$, $U''(\eta(\varepsilon)) (\bar{\Delta} S_1 + 1 - \bar{\Delta})^2$ is monotone in $\bar{\Delta}$, for every sufficiently small and negative ε and for every ω .

Combining (6.21) and (6.23), we deduce that

$$(6.24) \quad \limsup_{\varepsilon \uparrow 0} \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)) \leq \sup_{\bar{\Delta} \geq 1} \mathbb{E} \left[U''(S_1) (\bar{\Delta} S_1 + 1 - \bar{\Delta})^2 \right].$$

On the other hand, there exists a constant $\varepsilon'_0 > 0$, such that for every $\bar{\Delta} \in [-1 - 2/\varepsilon'_0, 1]$, we have that

$$(6.25) \quad \mathcal{X}(1 + \varepsilon) \ni X^{\varepsilon, \bar{\Delta}} := 1 + \varepsilon + (1 + \bar{\Delta}\varepsilon)(S - 1), \quad \text{for every } \varepsilon \in (0, \varepsilon'_0].$$

In particular, for every $\bar{\Delta} < 1$, we can choose ε'_0 such that (6.25) holds. We then obtain

$$(6.26) \quad \begin{aligned} & \liminf_{\varepsilon \downarrow 0} \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)) \\ & \geq \liminf_{\varepsilon \downarrow 0} \frac{2}{\varepsilon^2} \left(\mathbb{E} \left[U \left(X_1^{\varepsilon, \bar{\Delta}} \right) \right] - u(1) - \varepsilon u'(1) \right), \end{aligned}$$

where $\mathbb{E} \left[U \left(X_1^{\varepsilon, \bar{\Delta}} \right) \right]$ is well-defined see the justification in Remark 4.9. Since

$$X^{\varepsilon, \bar{\Delta}} = S + (\bar{\Delta}\varepsilon)S + \varepsilon(1 - \bar{\Delta}),$$

applying Taylor's expansion once more in (6.26) gives

$$(6.27) \quad \begin{aligned} \frac{2}{\varepsilon^2} \left(\mathbb{E} \left[U \left(X_1^{\varepsilon, \bar{\Delta}} \right) \right] - u(1) - \varepsilon u'(1) \right) &= \frac{2}{\varepsilon^2} (\varepsilon(\bar{\Delta} - 1) \mathbb{E} [U'(S_1)(S_1 - 1)] \\ & \quad + \frac{1}{2} \mathbb{E} [U''(\tilde{\eta}(\varepsilon))(\bar{\Delta}\varepsilon S_1 + \varepsilon(1 - \bar{\Delta}))^2]) \\ &= \frac{1}{\varepsilon^2} \mathbb{E} [U''(\tilde{\eta}(\varepsilon))(\bar{\Delta}\varepsilon S_1 + \varepsilon(1 - \bar{\Delta}))^2], \end{aligned}$$

for some random variable $\tilde{\eta}(\varepsilon)$ taking values between S_1 and $X^{\varepsilon, \bar{\Delta}}$, and where in the last equality, we have used (6.3).

In particular, for $\bar{\Delta} = \hat{\Delta} = -\frac{\mathbb{E}[U''(S_1)(S_1-1)]}{\mathbb{E}[U''(S_1)(S_1-1)^2]}$, where by assumption (6.13), $\hat{\Delta} < 1$, we can rewrite the latter expression in (6.27) as

$$\mathbb{E} \left[U''(\tilde{\eta}(\varepsilon))(\hat{\Delta}S_1 + (1 - \hat{\Delta}))^2 \right].$$

Note that the function $f(\bar{\Delta}) := \mathbb{E} [U''(S_1)(\bar{\Delta}S_1 + (1 - \bar{\Delta}))^2]$, $\bar{\Delta} \in \mathbb{R}$, reaches its strict global maximum at $\hat{\Delta}$ defined above. Also, from (6.25), we deduce that $X^{\varepsilon, \bar{\Delta}} \rightarrow S_1$ as $\varepsilon \downarrow 0$. Combining (6.26) and (6.27) and using (6.6), for $\bar{\Delta} = \hat{\Delta}$, we deduce that

$$(6.28) \quad \liminf_{\varepsilon \downarrow 0} \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)) \geq \mathbb{E} \left[U''(S_1)(\hat{\Delta}S_1 + 1 - \hat{\Delta})^2 \right].$$

Therefore, from (6.23) and (6.28), we conclude that

$$\begin{aligned} & \liminf_{\varepsilon \downarrow 0} \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)) \geq \mathbb{E} \left[U''(S_1)(\hat{\Delta}S_1 + 1 - \hat{\Delta})^2 \right] \\ &= \sup_{\bar{\Delta} \in \mathbb{R}} \mathbb{E} \left[U''(S_1) (\bar{\Delta}S_1 + 1 - \bar{\Delta})^2 \right] > \sup_{\bar{\Delta} \geq 1} \mathbb{E} \left[U''(S_1) (\bar{\Delta}S_1 + 1 - \bar{\Delta})^2 \right] \\ &\geq \limsup_{\varepsilon \uparrow 0} \frac{2}{\varepsilon^2} (u(1 + \varepsilon) - u(1) - \varepsilon u'(1)), \end{aligned}$$

which shows that $u''(1)$ does not exist in the case when $\hat{\Delta} < 1$, where $\hat{\Delta}$ is defined in (6.5). The case when $\hat{\Delta} > 1$ can be handled similarly.

We conclude justifying why we may assume that, without loss of generality, A satisfies $A(1/m) \neq A(1/k)$, for some m and k in \mathbb{N} . Indeed, for a given utility function U , let us

consider the family $U_\lambda := U(\lambda \cdot)$, $\lambda > 0$. Then, for a given $\lambda > 0$ and every $x > 0$, we have

$$(6.29) \quad A_\lambda(x) := -\frac{U_\lambda''(x)x}{U_\lambda'(x)} = -\frac{\lambda^2 U''(\lambda x)x}{\lambda U'(\lambda x)} = A(\lambda x).$$

If $A(a) \neq A(b)$ for some $0 < a < b$, then, we have, by (6.29), that $A_\lambda(a/\lambda) \neq A_\lambda(b/\lambda)$. Therefore, by the choice of λ , we may assume that

$$A_\lambda(a/\lambda) \neq A_\lambda(1/m),$$

for some $m \in \mathbb{N}$ and where $a/\lambda \in (0, 1/m)$. If we add a/λ to the range of S_1 , and assign to this state a positive but small probability, the arguments above still go through, and imply that $u_\lambda''(1)$ does not exist, where

$$u_\lambda(x) := \sup_{X \in \mathcal{X}(1)} \mathbb{E}[U(\lambda x X_1)] = u(\lambda x).$$

Therefore, non-existence of $u_\lambda''(1)$ would imply that $u''(\lambda)$ does not exist either. \square

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