# Mean field games with unbounded controlled common noise in portfolio management with relative performance criteria *

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**Abstract**

Motivated by optimal allocation models with relative performance criteria, we introduce a mean field game in which the terminal expected utility of the representative agent depends on her own state as well as the average of her peers. We derive the master equation, which, in view of the presence of controls in the volatility, needs to be coupled with a compatibility condition for the mean field optimal feedback control. We concentrate on the class of separable payoffs under both general utilities and couplings. We derive a solution to the master equation and find the associated optimal feedback control expressed via the value function in the absence of competition and a dynamic coupling function solving a non-local quasilinear equation. In turn, we construct the related optimal state and control processes, and give representative examples. Projecting the mean field solutions on finite dimensions, we recover the solution of the $N-$game for linear couplings and arbitrary utilities, and, we study the proximity of these approximations to their $N-$player game counterparts.

## 1 Introduction

We introduce a mean field game (MFG) arising in optimal investment models with relative performance concerns. In such models, each player is concerned with both her own and the performance of her peers at the end of the optimization horizon. This interdependence is modeled via expected terminal payoffs

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that depend on the individual as well as the average population state, which creates a stochastic coupling among all agents.

There is a rich literature in financial economics on optimal allocation/fund management problems with relative performance concerns as fund management is always performed in relation to a benchmark (index, returns of competitors, clustered financial targets, etc.). The prevailing way to classify these models is based on whether agents compete while investing in a common market (asset diversification) or, more generally, also include individual assets inaccessible to their competitors (asset specialization). For both categories, the existing applied papers primarily consider only two player games, single period models, and linear or quadratic criteria (see, among others, [1], [2], [3], [8], [9], [13], [14], [28], [31] and [36]).

The literature for continuous time is relatively recent. An $N-$player asset diversification game for players with common exponential utility and linear competition functions was introduced in [15]. This work was extended by Lacker and the second author in [22] who provided the first MFG formulation under asset specialization but, still, under linear competition and exponential utilities; they also studied the case of power utilities and geometric competition function to accommodate non-negative state constraints, which is also rather restrictive and can be solved similarly. The mean field game in [22] was defined probabilistically, and static (random) equilibria were constructed when both common noise (common assets) and individual noise (specialized assets) were included in a log normal market.

The work in [22] was extended in a number of papers, allowing, among others, for intermediate consumption, external habit formation, systemic risk, non-linear price impact, forward utilities, relaxed controls and learning (see, for example, [4], [5], [6], [7], [16], [17], [21], [30] and [37]. In all these works, the MFG were defined probabilistically, directly following the definition in [22], and considered either exponential or quadratic payoffs with linear coupling, or geometric mean/power type couplings with power utilities or recursive Epstein-Zin utilities. An asymptotic result for utilities close to CARA with power type coupling can be found in [35]. As in [22], in the works to date the combination of homothetic payoffs and couplings, together with the linearity of state dynamics in the control variables, lead to dimensionality reduction and considerable tractability.

Herein, we allow for both general utilities and general couplings, which to the best of our knowledge has not been done before. We, also, depart from the probabilistic definition and introduce the MFG directly through the stochastic PDE game system and its master equation (see (2), (3) and (4)).

To ease the presentation, we only focus on unconstrained problems, allowing for the state process to be in the entire space. The case of general utilities and arbitrary couplings in the half space is being currently investigated by the authors in [33].

We consider a model in which the players control both the drift and volatility of their state, with controls appearing linearly therein. We make this more precise next but without laying out the technical details which are presented later.

The MFG is the natural limit of the following $N-$player game. Consider $N$ agents, labeled by $i=1,...,N$, with respective individual state $\left(X^{\pi_i}_{i,s}\right)_{0\leq t\leq s\leq T}$ solving the controlled SDE

$$dX^{\pi_i}_{i,s}=b\pi_{i,s}ds+\sigma\pi_{i,s}dW_s,\quad X^{\pi_i}_t=x_i,$$

with $x_1,...,x_N\in\mathbb{R}^N$, and having value function

$$\begin{aligned}&v^i(x_1,...,x_N,t)\\&\quad=\sup_{\pi_i\in\mathcal{A}}\mathbb{E}\big[J\big(X^{\pi_i}_{i,T},\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}X^{\pi_j}_{j,T}\big)\big|X^1_t=x_1,...,X^N_t=x_N\big].\end{aligned}\tag{1}$$

The set of admissible policies $\mathcal{A}$ and the terminal payoff function $J$ are common across players. For the generic player $i$, her expected terminal payoff depends on both the individual state $X^{\pi_i}_{i,T}$ and the average $\frac{1}{N-1}\sum\limits_{j=1,j\neq i}^{N}X^{\pi_j}_{j,T}$ of the rest of the population. This is, in general, a non-tractable problem and, for this, we consider its limit as $N\to\infty$.

To study the emerging mean field game, we first derive the associated master equation together with a compatibility condition for the candidate MFG optimal control. The resulting equation is new and outside the reach of the current theory. It is given, for $(x,m,t)\in(\mathbb{R},\mathcal{P},[0,T])$, where $\mathcal{P}$ denotes the set of probability measures on $\mathbb{R}$, by

$$\begin{aligned}&U_t(x,m,t)+\frac{1}{2}\sigma^2\left(\pi^*\left(x,m,t\right)\right)^2U_{xx}\left(x,m,t\right)+b\pi^*\left(x,m,t\right)U_x\left(x,m,t\right)\\&+\sigma^2\pi^*\left(x,m,t\right)\int\pi^*\left(z,m,t\right)U_{xm}\left(x,m,z,t\right)dm\left(z\right)\\&+\frac{1}{2}\sigma^2\int\int\pi^*\left(z,m,t\right)\pi^*(y,m,t)U_{mm}\left(x,m,z,y,t\right)dm\left(z\right)dm\left(y\right)\\&+b\int\pi^*(z,m,t)U_m\left(x,m,z,t\right)dm\left(z\right)=0,\end{aligned}\tag{2}$$

with terminal condition

$$U\left(x,m,T\right)=J\left(x,m\right).\tag{3}$$

The optimality condition for the mean field feedback control $\pi^*:\mathbb{R}\times\mathcal{P}\times[0,T]\to\mathbb{R}$ is given, for $\lambda=\frac{b}{\sigma}$, by

$$\begin{aligned}&\pi^*\left(x,m,t\right)U_{xx}\left(x,m,t\right)\\&\qquad+\int\pi^*(z,m,t)U_{xm}(x,m,z,t)dm\left(z\right)=-\frac{\lambda}{\sigma}U_x\left(x,m,t\right).\end{aligned}\tag{4}$$

In (2) and (4), the $m-$subscripts in $U_m, U_{xm}$, and $U_{mm}$ stand for the so-called Lions derivative; (see [11] for its definition).

We refer to the coupled equations (2) and (4) as the "master system".

The presence of the control $\pi^*$ in (2) in front of the second derivatives of $U$ and the compatibility condition (4) puts the problem outside the existing theory of the master equation developed in [11] and the subsequent references.

To gain tractability and further investigate the MFG, we consider the general class of separable payoffs $J(x,m)$ given by

$$J(x,m) = G\big(x - F\left(\int xdm\,(x)\right)\big), \tag{5}$$

where $G$ represents the utility of the representative agent and $F$ models the coupling with the population average. We introduce minimal assumptions on $G$ and $F$, thus substantially extending the rather restrictive cases of exponential $G$ and linear $F$ that have been studied so far.

The first step in our analysis is to produce closed form solutions for the master system (2) and (4). Specifically, we show that a solution $U$ to (2) is given by

$$U(x,m,t) = v\big(x - f\big(\int xdm(x), t\big), t\big), \tag{6}$$

where $v$ and $f$ are two auxiliary functions representing respectively the value of the game in the absence of competition and the "backward in time" evolution of the coupling interaction.

Indeed we show that $v = v(x,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$ is given by (1) for $N = 1$ and $F \equiv 0$, and satisfies the well known equation (see, for example, [27])

$$v_t - \frac{1}{2}\lambda^2 \frac{v_x^2}{v_{xx}} = 0 \quad \text{in } \ \mathbb{R} \times [0,T) \quad \text{and} \quad v(x,T) = G(x). \tag{7}$$

The function $f = f(z,t;m) : \mathbb{R} \times [0,T] \times \mathcal{P} \to \mathbb{R}$ satisfies the terminal value problem

$$\begin{aligned} & f_t + \frac{1}{2}\lambda^2 \frac{\left(\int r\,(y-f,t)\,dm\,(y)\right)^2}{\left(1-f_z\right)^2} f_{zz} = 0 \ \text{ in } \ \mathbb{R} \times [0,T), \\ & f(z,T;m) = F\,(z)\,, \end{aligned} \tag{8}$$

where $r\,(x,t) = -\dfrac{v_x\,(x,t)}{v_{xx}\,(x,t)}$.

In turn, (4) yields that the optimal feedback control is given by

$$\begin{aligned} & \pi^*(x,m,t) = \frac{\lambda}{\sigma} r\big(x - f\big(\int xdm(x), t\big), t\big) \\ & + \frac{\lambda}{\sigma}\frac{f_z\left(\int xdm(x), t\right)}{1 - f_z\left(\int xdm(x), t\right)} \int r\big(y - f\big(\int xdm(x), t\big), t\big) dm\,(y)\,. \end{aligned} \tag{9}$$

Using this feedback control, we construct explicitly the optimal mean field state and control processes, $(X_t^*)_{t\in[0,T]}$ and $(\pi_t^*)_{t\in[0,T]}$, and study their properties.

Among others, we show that the former is given by

$$X_t^* = x_t^{*,x-f(\tilde{x}_0,0)} + f\Big(\int x dm_t^*(x), t\Big),$$

where $x_t^{*,x-f(\tilde{x}_0,0)}$ is the optimal state process of the single-agent optimization problem starting at $x - f(\tilde{x}_0, 0)$, $\tilde{x}_0 = \int x dm_0(x)$ with $m_0$ being the initial population distribution, and $m_t^*$ is the conditional on $\mathcal{F}_t^W$ law of $X_t^*$.

We, in turn, provide representative examples for the general class of arbitrary utilities and linear couplings, and for exponential utilities and general couplings. The intersection of these two families provides the only unconstrained case that has been so far studied (exponential utility and linear coupling).

Finally, we investigate how the mean field solution $U$ and $\pi^*$ approximate their counterparts in the $N-$player game.

For this, we consider, for each $i = 1, ..., N$, the measure

$$\mu^{N,i} = \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \delta_{x_j}, \tag{10}$$

and compare $U\left(x_i, \mu^{N,i}, t\right)$ to $v^i\left(x_1, ..., x_N, t\right)$, and $\pi^*(x_i, \mu^{N,i}, t)$ to the optimal feedback control $\pi_{N,i}^*(x_1, ..., x_N, t)$, of the $i^{th}$ player.

When the coupling is linear, that is, $F(z) = \theta z$ for some $\theta \in (0, 1)$, we find that the associated values coincide and the controls differ by order $\frac{1}{N-1}$. In particular we show that, in $\mathbb{R}^N \times [0, T]$ and for $i = 1, \ldots, N$,

$$v^i(x_1, ..., x_N, t) = U(x_i, \mu^{N,i}, t) = v\Big(x_i - \frac{\theta}{N-1} \sum_{j=1, j\neq i}^{N} x_j, t\Big),$$

and, for a universal constant $C$,

$$\begin{aligned} &\left|\pi_{N,i}^*(x_1, ..., x_N, t) - \pi^*(x_i, \mu^{N,i}, t)\right| \\ &\leq K \frac{\lambda}{\sigma} \frac{\theta}{1-\theta} \Big(1 + \frac{1-\theta}{N} \sum_{j=1}^{N} |x_j| + \frac{1}{N} \sum_{j=1}^{N} |x_i - x_j|\Big). \end{aligned}$$

We conclude the introduction with a brief summary about MFG. The MFG theory were introduced by Lasry and Lions in [23], [24] and [25] and, at the same time, by Caines, Huang, and Malhamé in [18] for a particular setting. In the presence of both idiosyncratic and common noise, the stochastic MFG system was first investigated by Cardaliaguet, Delarue, Lasry and Lions in [11]. This reference considered games with the space being the torus, and showed the existence and uniqueness of a strong solution for strictly monotone coupling functions and non-degenarate diffusions. The results of [11] were extended to $\mathbb{R}^d$ by Carmona and Delarue in the monograph [12].

An alternative analytic approach to study MFG equilibria with a common noise is the master equation, introduced by Lasry and Lions and presented by Lions in [26]. The master equation is a deterministic nonlinear nonlocal transport equation in the space of measures, which encompasses all the information about the game and provides suitable approximations to Nash equilibria of the finite player games. The existence and the uniqueness of a classical solution to this equation was first established in [11] (see also [12]).

By now, the MFG literature has expanded with many important contributions. Listing references is beyond the scope of this paper.

Most of the existing theory about the master equation so far applies to dynamics with uncontrolled and homogeneous noises. Extending the theory to inhomogeneous and, especially, unbounded controlled noises remains as one of the main open problems.

The master equation studied in this paper is new and no general theory exists for its analysis. In addition, several standard existing assumptions, like Lipschitz regularity of the criterion and compactness of the control policies, are lacking. The monotonicity assumption in terms of the measure in the final data is also lacking, as one may consider both the competitive and the homophilous case of interaction.

The paper is organized as follows. In section 2, we introduce the finite population game and the related MFG, and derive formally the master system (master equation and the compatibility condition). In section 3, we focus on payoffs of form (5) and construct a solution to the master system. In section 4, we produce the associated optimal allocation and state processes while in section 5 we give representative examples. In section 6, we examine the approximation of the mean field solution and policies to their finite game counterparts.

# 2 The N-player game, the mean field game and the master system

## 2.1 The $N-$player game

We consider a game of $N$ players with controlled state processes $(X_s^i)_{0\le t\le s\le T}$ evolving, for each $i=1,...,N$, according to the SDE

$$dX_{i,s}^{\pi_i} = b\pi_{i,s}ds + \sigma\pi_{i,s}dW_s \ \text{ in } \ (t,T) \qquad X_{i,t}=x_i,\ x_i\in\mathbb{R}, \tag{11}$$

with $b,\sigma>0$ fixed constants and $(W_t)_{t\ge0}$ a standard Brownian motion on $(\Omega,\mathcal{F},\mathbb{P})$ with natural filtration $\mathbb{F}=\left\{\mathcal{F}_t^W\right\}_{t\ge0}$.

Each player $i$ controls the evolution of her state using $\pi_i \in \mathcal{A}$, with

$$\mathcal{A} = \{\pi : \ \pi_t \in \mathcal{F}_t^W \ \text{ and } \mathbb{E}\int_0^T \pi_s^2 ds < \infty\}, \tag{12}$$

and has value function

$$v^i(x_1, ..., x_N, t) = \sup_{\pi_i \in \mathcal{A}} \mathbb{E}\big[J\big(X_{i,T}^{\pi_i}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} X_{j,T}^{\pi_j}\big)|X_t^1 = x_1, ..., X_t^N = x_N\big], \tag{13}$$

The payoff function $J : \mathbb{R}^2 \to \mathbb{R}$ is common across players and depends on both their terminal individual state and the average performance of their piers.

**Remark 1** *The state controlled dynamics (11) may be extended to be*

$$dX_{i,s}^{\pi} = \Sigma_{j=1}^M b_j \pi_{j,s} ds + \Sigma_{j=1}^M \sigma_j \pi_{j,s} dW_s^j \quad X_{i,t} = x_i,$$

*with* $(W_t)_{t\geq 0} = (W_t^1, ..., W_t^M)_{t\geq 0}$ *being a standard* $M$*-dimensional Brownian motion on* $(\Omega, \mathcal{F}, \mathbb{P})$. *All results herein are readily modified but we choose to work with (11) for simplicity.*

**Remark 2** *Criterion (13) may be generalized by allowing intermediate payoffs. Then, the dynamics (11) take the form*

$$dX_{i,s}^{\pi_i} = b\pi_{i,s} ds - C_{i,s} ds + \sigma \pi_{i,s} dW_s^i \qquad X_{i,t} = x_i,$$

*and, in turn, for* $i = 1, \ldots, N$,

$$v^i(x,t) = \sup_{C_i, \pi_i} \mathbb{E}\big[\int_t^T J_1(C_{i,s}) ds + J_2\big(X_{i,T}^{\pi_i}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} X_{j,T}^{\pi_j}\big)|X_t^{\pi} = x\big],$$

*with* $x = (x_1, \ldots, x_N)$ *and* $X_t^{\pi} = (X_{1,t}^{\pi}, \ldots, X_{N,t}^{\pi})$ *and common, across players, payoff functions* $J_1$ *and* $J_2$. *This case is being currently studied in [32].*

We recall that a strategy $(\pi_{1,s}^*, ..., \pi_{N,s}^*)_{s\in[0,T]}$ is a *Nash equilibrium* of the game, if, for each $i = 1, \ldots, N$ and all $\pi_i \in \mathcal{A}$,

$$\mathbb{E}\big[J\big(X_{i,T}^{\pi_i}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} X_{j,T}^{\pi_j^*}\big)|X_{1,t}^{\pi_1^*} = x_1, ..., X_{i,t}^{\pi_i} = x_i, ..., X_{N,t}^{\pi_N^*} = x_N\big]$$

$$\leq \mathbb{E}\big[J\big(X_{i,T}^{\pi_i^*}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} X_{j,T}^{\pi_j^*}\big)|X_{1,t}^{\pi_1^*} = x_1, ..., X_{i,t}^{\pi_i^*} = x_i, ..., X_{N,t}^{\pi_N^*} = x_N\big].$$

Next, we assume that there exist Nash equilibrium control processes $(\pi_{1,s}^*, ..., \pi_{N,s}^*)_{0\leq t\leq s\leq T}$ in the feedback form

$$\pi_{i,s}^* = \pi_{N,i}^*\left(X_{1,s}^*, ..., X_{N,s}^*, s\right),$$

for each $i = 1, \dots, N$ and for some functions $\pi^*_{N,i} : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ and with the $(X^*_{i,s})_{0 \le t \le s \le T, i=1,\dots,N}$, solving (11) with the processes $\pi^*_{i,s}$ as controls.

If the value functions $v^i\big(x_1, ..., x_N, t\big)$ for $i = 1, \dots, N$ are smooth, they are expected to satisfy, for each $i = 1, \dots, N$ and in $\mathbb{R} \times [t, T)$, the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} & v^i_t + \max_{\pi_{N,i}} \big( \frac{1}{2}\sigma^2 \pi^2_{N,i} v^i_{x_i x_i} + \pi_{N,i} \big( b v^i_{x_i} + \sigma^2 \sum_{j=1, j\neq i}^{N} \pi^*_{N,j} v^i_{x_i x_j} \big)\big) \\ & + \frac{1}{2}\sigma^2 \sum_{j=1, j\neq i}^{N} \sum_{k=1, k\neq i}^{N} \pi^*_{N,j}\pi^*_{N,k} v^i_{x_j x_k} + b \sum_{j=1, j\neq i}^{N} \pi^*_{N,j} v^i_{x_j} = 0, \end{aligned} \tag{14}$$

with terminal condition

$$v^i\big(x_1, ..., x_N, T\big) = J\big(x_i, \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} x_j\big). \tag{15}$$

Furthermore, also if the maximum in (14) is well defined in each respective HJB equation, we deduce that the optimal feedback functions $\big(\pi^*_{N,1}, ..., \pi^*_{N,N}\big)$ must satisfy the linear system

$$\sum_{j=1}^{N} \pi^*_{N,j} v^i_{x_i x_j} = -\frac{\lambda}{\sigma} v^i_{x_i} \quad \text{for} \quad i = 1, ..., N \quad \text{and} \quad \lambda = b/\sigma. \tag{16}$$

Although it appears simple, this linear system is *not* tractable due to the interlinked dependence of the $\pi^{*\prime}_{N,i}s$ and the coefficients $v^i_{x_i}, v^i_{x_i x_i}$ and $v^i_{x_i x_j}$. We also note that, in general, it is not even known, although it is very likely, that the value functions $v^{i\prime}s$ are smooth enough for the latter partial derivatives to be well defined, except for very specific cases on which we comment in section 5

Motivated by the intractability and complications of the $N-$player game, we introduce next a related mean field game.

## 2.2 The mean field game

The representative agent's state $(X_s)_{0 \le t \le s \le T}$ solves, for $\pi \in \mathcal{A}$, the continuum analogue of SDE (11), namely,

$$dX^\pi_s = b\pi_s ds + \sigma \pi_s dW_s \ \text{ in } \ (t, T] \ \text{ and } \ X^\pi_t = x.$$

For each $m \in \mathcal{C}([t,T]; \mathcal{P})$, the value function of the representative player is defined as

$$u(x, t; m) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[J(X^\pi_T, m_T) | \mathcal{F}^{W_T}].$$

We say that the game is at equilibrium, if there exists $m \in \mathcal{C}([t,T];\mathcal{P})$ which is the conditional law of the dynamics of the player associated with the optimal $\pi$ in the above value function.

The definition above is translated to saying that the game has a value if there exists a triplet of $\mathcal{F}_t^W$-progressively measurable processes $(u, V, m)$ satisfying, in $\mathbb{R} \times (t,T)$, the backward-forward system of stochastic PDE

$$\begin{aligned} du(x,s) &= -\max_{\pi}\Big(\frac{1}{2}\sigma^2\pi^2 u_{xx}(x,s) + \pi\big(bu_x(x,s)+\sigma^2 V_x(x,s)\big)\Big)ds \\ &\quad + \sigma V\left(x,s\right)dW_s, \\ dm(x,s) &= -b\partial_x\big(\pi^*(x,m,s)m(x,s)\big)ds \\ &+ \frac{1}{2}\sigma^2\partial_{xx}\big(\big(\pi^*(x,m,s)\big)^2 m(x,s)\big)ds - \sigma\partial_x\left(\pi^*\left(x,m,s\right)m\left(x,s\right)\right)dW_s, \\ u(x,T) &= J(x,m_T) \quad \text{and} \quad m(x,t) = m_0, \end{aligned} \tag{17}$$

where

$$\pi^*\left(x,m,s\right) = \arg\max_{\pi}\Big(\frac{1}{2}\sigma^2\pi^2 u_{xx}\left(x,s\right) + \pi\left(bu_x(x,s)+\sigma^2 V_x\left(x,t\right)\right)\Big).$$

Solving system (17) is rather complicated. When there is no control in front of the common noise, the analogous system was studied in [11]. It is not known, however, if similar arguments can be employed here.

### 2.2.1 The master system

We proceed with the derivation of a master system consisting of the master equation (2) and (3) coupled with the optimality condition (4) .

We begin with a formal argument passing to the limit at the HJB equation (14) of the $N-$player game and, then, we briefly present a possible rigorous argument about how to go directly to the master system from the stochastic PDE (17).

To this end, we revert to (14) and observe that, at the optimum $\pi^*_{N,i}$, we must have, for each $i = 1, \ldots, N$,

$$\sigma^2\big(\pi^*_{N,i}v^i_{x_i x_i} + \sum_{j=1,j\neq i}^{N}\pi^*_{N,j}v^i_{x_i x_j}\big) = -bv^i_{x_i}. \tag{18}$$

Next, for large $N$, we suppose that the feedback control functions $\pi^*_{N,i}\left(x_1,...,x_N,t\right)$ can be written as

$$\pi^*_{N,i}\left(x_1,...,x_N,t\right) \;\simeq\; \pi^*_N\left(x_i,\mu^{N,i},t\right).$$

We assume then that, as $N\to\infty$, $\mu^{N,i}$ converges weakly to a measure $m$ and, furthermore,

$$\pi^*_N\left(x_i,\mu^{N,i},t\right) \;\to\; \pi^*\left(x,m,t\right) \quad \text{and} \quad v^i\left(x_1,...,x_N,t\right) \to\; U\left(x,m,t\right), \tag{19}$$

where $\pi^*(x,m,t)$ will be the associated mean field equilibrium optimal feedback function and $U(x,m,t)$ a solution to the master equation.

From (18) and (19), we would then expect that $\pi^*(x,m,t)$ satisfies the optimality/compatibility condition

$$\begin{aligned}&\pi^*(x,m,t)\,U_{xx}(x,m,t)\\&+\int \pi^*(z,m,t)U_{xm}(x,m,z,t)dm(z)=-\frac{\lambda}{\sigma}U_x(x,m,t)\,.\end{aligned} \tag{20}$$

Analogously, we expect that, as $N\to\infty$,

$$\begin{aligned}&\sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N}\pi^*_{N,j}\pi^*_{N,k}v^i_{x_jx_k}\\&\to\int\int\pi^*(z,m,t)\pi^*(y,m,t)U_{mm}(x,m,z,yt)\,dm(z)\,dm(y)\,,\end{aligned}$$

and

$$\sum_{j=1,j\neq i}^{N}\pi^*_{N,j}v^i_{x_j}\quad\to\quad\int\pi^*(z,m,t)U_m(x,m,z,t)\,dm(z)\,.$$

Combining the above, the formal limit, as $N\to\infty$, of the HJB equation (14) is (2), which is rewritten below for the reader's convenience,

$$\begin{aligned}&U_t(x,m,t)+\frac{1}{2}\sigma^2\big(\pi^*(x,m,t)\big)^2U_{xx}(x,m,t)+b\pi^*(x,m,t)U_x(x,m,t)\\&\quad+\sigma^2\pi^*(x,m,t)\int\pi^*(z,m,t)\,U_{xm}(x,m,z,t)\,dm(z)\\&+\frac{1}{2}\sigma^2\int\int\pi^*(z,m,t)\pi^*(y,m,t)U_{mm}(x,m,z,y,t)dm(z)dm(y)\\&\quad+b\int\pi^*(z,m,t)U_m(x,m,z,t)dm(z)=0\ \text{ in }\ R\times\mathcal{P}(\mathbb{R})\times[0,T),\end{aligned} \tag{21}$$

and

$$U(x,m,T)=J(x,m). \tag{22}$$

*Remark*: Let

$$\begin{aligned}&\mathcal{H}(U_m,U_{mm},\pi^*)\\&=\frac{1}{2}\sigma^2\int\int\pi^*(z,m,t)\,\pi^*(y,m,t)U_{mm}(x,m,z,y,t)\,dm(z)\,dm(y)\\&\quad+b\int\pi^*(z,m,t)U_m(x,m,z,t)\,dm(z)\,.\end{aligned}$$

Then, combining (20) and (21), we may rewrite the master equation as

$$
\begin{aligned}
U_t(x,m,t) + \max_{\pi\in\mathbb{R}}\Big(\frac{1}{2}\sigma^2\pi^2 U_{xx}(x,m,t) \\
+ \pi(\sigma^2 \int \pi^*(z,m,t)U_{xm}(x,m,z,t)dm(z) + bU_x(x,m,t))\Big) \\
+ \mathcal{H}\left(U_m, U_{mm}, \pi^*\right) = 0,
\end{aligned}
$$

and, alternatively, as

$$
U_t\left(x,m,t\right) - \frac{1}{2}\sigma^2\left(\pi^*\left(x,m,t\right)\right)^2 U_{xx}\left(x,m,t\right) + \mathcal{H}\left(U_m, U_{mm}, \pi^*\right) = 0. \quad (23)
$$

Returning to the derivation of the master equation assuming that the forward-backward stochastic PDE system (17) has a classical solution, we argue following the program outlined in [11] about the derivation of the master equation with uncontrolled and homogeneous common noise. However, since we do not know the existence of classical solutions to (17), here we only show the beginning of the argument and spare the reader of tedious calculations.

The main idea is to turn the forward Fokker-Planck equation to an ordinary PDE with random coefficients and, at same time, identify the $V$ part of the solution to (17) and turn the backward SPDE for $u$ to a stochastic PDE with a deterministic part plus a martingale.

We describe here the first step of this program. Fix $t_0 \in [0,T)$ and let $(Y_t)_{0\le t_0\le t\le T}$ be defined by

$$
dY_s = \pi^*(Y_s, m(Y_s,s),s)dW_s \text{ in } (t_0,T] \text{ and } Y_{t_0} = x.
$$

The transformation we need to use is

$$
\tilde{m}(x,t) = \left(\exp\int_{t_0}^{t}\pi_x^*(Y_s, m(Y_s,s),s)dW_s\right)m(Y_t,t),
$$

and

$$
\tilde{u}(x,t) = u(Y_t, m(Y_t,t),t).
$$

Making use of the Ito-Wentzell formula and the above transformation we can now proceed.

# 3 A solvable MFG class

The rest of the paper is dedicated to the analysis of the mean field game (20), (21) for payoffs of the form

$$
J(x,m) = G\big(x - F\big(\int x dm(x)\big)\big), \quad (24)
$$

where $G$ is the representative agent's utility and $F$ a coupling function modeling the relative performance effects from the average of the continuum of the population.

As mentioned earlier in the introduction, the only case that has been examined so far in unbounded domains is when $G, F : \mathbb{R} \to \mathbb{R}$ are of the $G(x) = -e^{-\delta x}$ and $F(z) = \theta z$, for some $\delta > 0$ and $\theta \in (0, 1)$. We recall that in these works, the definition of the MFG was probabilistic and not through the MFG system (20) and (21); we revisit this case in section 5.

## 3.1 Assumptions

For the utility function $G : \mathbb{R} \to \mathbb{R}$, we assume that

$$\begin{aligned} &i)\ G \in \mathcal{C}^4(\mathbb{R}) \ \text{ is strictly concave and strictly increasing, and} \\ &\lim_{x\to-\infty} G'(x) = \infty \text{ and } \lim_{x\to\infty} G'(x) = 0, \\ &ii)\ \text{for some } \delta, K > 0, \text{ the map } R = -\frac{G'}{G''} : \mathbb{R} \to \mathbb{R}^+ \ \text{ satisfies} \\ &0 < \delta \le R \ \text{ and } |R'| \le K. \end{aligned} \tag{25}$$

Note that (25) immediately yields that, for some $K_1, K_2 > 0$ and $x \in \mathbb{R}$,

$$0 < \delta \le R(x) \le K_1 |x| + K_2. \tag{26}$$

The above properties are satisfied by a large class of utility functions. Among others, popular examples include the exponential case $G(x) = -e^{-\delta x}$ with $\delta > 0$ for which $R = \delta$, and the so-called SAHARA utilities, introduced in [29] (see, also, [38]) which are modeled indirectly through the parametric family

$$R(x) = -\frac{G'(x)}{G''(x)} = \sqrt{\alpha x^2 + \delta} \ \text{ with } \ \alpha \ge 0 \text{ and } \ \delta > 0. \tag{27}$$

For the coupling function $F : \mathbb{R} \to \mathbb{R}$, we assume that

$$\begin{aligned} &F \in \mathcal{C}^2(\mathbb{R}), \quad F(0) = 0 \ \text{ and, for some } k_1, k_2, K, L > 0 \text{ and all } z \in \mathbb{R}, \\ &0 < k_1 < 1 - F'(z) < k_2. \end{aligned} \tag{28}$$

## 3.2 Solving the master system

We are seeking smooth functions $v, f : \mathbb{R} \times [0, T] \to \mathbb{R}$, such that (21) has a solution of the form

$$U(x, m, t) = v(x - f(\tilde{x}, t), t), \tag{29}$$

with

$$\tilde{x} = \int x \, m(dx). \tag{30}$$

We proceed with formal calculations which will be made rigorous afterwards. Differentiating the candidate solution in (29) gives

$$\begin{aligned}
U_t\left(x,m,t\right) &= -f_t\left(\tilde{x},t\right)v_x\left(x-f\left(\tilde{x},t\right),t\right)+v_t\left(x-f\left(\tilde{x},t\right),t\right),\\
U_x\left(x,m,t\right) &= v_x\left(x-f\left(\tilde{x},t\right),t\right),\\
U_{xx}\left(x,m,t\right) &= v_{xx}\left(x-f\left(\tilde{x},t\right),t\right)\\
U_m\left(x,m,t\right) &= -f_z\left(\tilde{x},t\right)v_x\left(x-f\left(\tilde{x},t\right),t\right),\\
U_{xm}\left(x,m,z,t\right) &= -f_z\left(\tilde{x},t\right)v_{xx}\left(x-f\left(\tilde{x},t\right),t\right),\\
U_{mm}\left(x,m,z,t\right) &= -f_{zz}\left(\tilde{x},t\right)v_x\left(x-f\left(\tilde{x},t\right),t\right)\\
&\quad+f_z^2\left(\tilde{x},t\right)v_{xx}\left(x-f\left(\tilde{x},t\right),t\right),
\end{aligned} \tag{31}$$

while the optimality condition (20) becomes

$$\begin{aligned}
&\sigma^2\pi^*(x,m,t)v_{xx}(x-f(\tilde{x},t),t)\\
&=\sigma^2\Big(\int\pi^*(z,m,t)m(dz)\Big)f_z(\tilde{x},t)v_{xx}(x-f(\tilde{x},t),t)-bv_x(x-f(\tilde{x},t),t).
\end{aligned} \tag{32}$$

Inserting these expressions in (21) and after some calculations, we find that we must have

$$\begin{aligned}
&v_t+\sigma^2\Big(\frac{1}{2}(\pi^*)^2-\pi^*\int\pi^*(z,m,t)dm(z)f_z\Big)v_{xx}+b\pi^*v_x\\
&+\frac{\sigma^2}{2}\Big(\int\pi^*(z,m,t)dm(z)\Big)^2f_z^2v_{xx}-b\Big(\int\pi^*(z,m,t)dm(z)\Big)f_zv_x\\
&-\Big(f_t+\frac{\sigma^2}{2}\Big(\int\pi^*(z,m,t)dm(z)\Big)^2f_{zz}\Big)v_x=0,
\end{aligned} \tag{33}$$

where to simplify the notation we omitted (except in the integrals) the dependence of $v_t$, $v_x$ and $v_{xx}$ on $x-f(\tilde{x},t)$ and $t$, of $f_t$, $f_z$ and $f_{zz}$ on $\tilde{x}$ and $t$, and of $\pi^*$ on $x,m$ and $t$.

Combining terms in (33) yields

$$\begin{aligned}
&v_t+\frac{1}{2}\sigma^2\Big(\pi^*-\int\pi^*(z,m,t)dm(z)f_z\Big)^2v_{xx}\\
&+b\Big(\pi^*-\int\pi^*(z,m,t)dm(z)f_z\Big)v_x\\
&-\Big(f_t+\frac{\sigma^2}{2}\Big(\int\pi^*(z,m,t)dm(z)\Big)^2f_{zz}\Big)v_x=0,
\end{aligned} \tag{34}$$

and, after using (32) and recalling that $\lambda=b/\sigma$,

$$v_t-\frac{1}{2}\lambda^2\frac{v_x^2}{v_{xx}}-\Big(f_t+\frac{\sigma^2}{2}\Big(\int\pi^*(z,m,t)dm(z)\Big)^2f_{zz}\Big)v_x=0. \tag{35}$$

We conclude that, for $U$ defined by (29) to satisfy (21), it suffices for the functions $v$ and $f$ to solve respectively

$$v_t - \frac{1}{2}\lambda^2 \frac{v_x^2}{v_{xx}} = 0 \ \text{ in } \ \mathbb{R} \times [0,T) \ \text{ and } \ v(x,T) = G(x), \tag{36}$$

which is the equation appearing in the model for a single player in the absence of coupling, and

$$f_t + \frac{\sigma^2}{2}\Big(\int \pi^*(z,m,t)dm(z)\Big)^2 f_{zz} = 0 \ \text{ in } \ \mathbb{R}\times[0,T) \ \text{ and } \ f(z,T) = F(z). \tag{37}$$

At this point, we also observe that, in view of (32), equation (37) can be written as

$$f_t + \frac{1}{2}\lambda^2 \frac{\left(\int r\left(y-f,t\right)dm\left(y\right)\right)^2}{\left(1-f_z\right)^2} f_{zz} = 0 \ \text{ in } \ \mathbb{R}\times[0,T) \ \text{ and } \ f(z,T) = F(z), \tag{38}$$

where $r : \mathbb{R}\times [0,T] \to \mathbb{R}_+$ is defined by

$$r(x,t) = -\frac{v_x(x,t)}{v_{xx}(x,t)}. \tag{39}$$

Indeed, using $r$, we can rewrite (32) as

$$\pi^*(x,m,t) = \frac{\lambda}{\sigma} r\left(x - f(\tilde{x},t),t\right) + f_z(\tilde{x},t)\int \pi^*(y,m,t)dm(y). \tag{40}$$

Integrating (40) yields

$$\left(1 - f_z\left(\tilde{x},t\right)\right)\int \pi^*\left(x,m,t\right)dm\left(y\right) = \frac{\lambda}{\sigma}\int r\left(x - f\left(\tilde{x},t\right),t\right)dm\left(x\right)$$

and, thus,

$$\int \pi^*\left(x,m,t\right)dm\left(y\right) = \frac{\lambda}{\sigma}\frac{1}{1-f_z\left(\tilde{x},t\right)}\int r\left(x - f\left(\tilde{x},t\right),t\right)dm\left(x\right), \tag{41}$$

which implies (38).

The next proposition, which is proved in the following subsection, establishes the existence of the auxiliary functions $v$ and $f$.

**Proposition 3** *Assume that $G$ and $F$ satisfy* (25) *and* (28)*. Then, there exists a unique solution $v \in \mathcal{C}^{4,1}\left(\mathbb{R}\times[0,T]\right)$ to* (36) *which, for each $t \in [0,T]$, is strictly increasing and strictly concave in $x$. Moreover,* (38) *has a unique solution $f \in \mathcal{C}^{2,1}\left(\mathbb{R}\times[0,T]\right)$.*

With Proposition 3 and the previous formal computations, which are now rigorous, we have shown the following theorem.

**Theorem 4** *Assume* (25) *and* (28), *let* $J$ *be given by* (24), *and consider the solution* $v \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T])$ *of* (36), *which is the value function of a single player model in the absence of coupling solving* (36), *and* $f \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T])$ *to* (38) *with* $r = -\frac{v'}{v''}$.

*Then,* $U : \mathbb{R} \times \mathcal{P} \times [0,T] \to \mathbb{R}$ *given by*

$$U(x,m,t) = v(x - f(\tilde{x},t),t),$$

*and* $\pi^* : \mathbb{R} \times \mathcal{P} \times [0,T] \to \mathbb{R}$ *given by*

$$\pi^*(x,m,t) = \frac{\lambda}{\sigma}\big(r(x - f(\tilde{x},t),t) + \frac{f_z(\tilde{x},t)}{1 - f_z(\tilde{x},t)} \int r(y - f\,(\tilde{x},t),t)dm(y)),$$

*are classical solutions to the master system* (20), (21) *and* (22).

## 3.3 The auxiliary functions $v$, $r$ and $f$ and the proof of Proposition 3

We provide here the proof of Proposition 3 which, to ease the presentation, is divided in several parts stated separately.

**Proposition 5** *Assume* (25). *The function* $v : \mathbb{R}\times [0,T] \to \mathbb{R}$ *given by*

$$v\,(x,t) = \sup_{a \in \mathcal{A}} \mathbb{E}\left[ G\,(x_T) \right| x_t = x] \,, \tag{42}$$

*with* $\mathcal{A}$ *as in (12) and*

$$dx_s = ba_s ds + \sigma a_s dW_s \quad in \quad (t,T] \quad and \quad x_t = x \in \mathbb{R}, \tag{43}$$

*is the unique strictly increasing and strictly concave* $\mathcal{C}^{4,1}\,(\mathbb{R} \times [0,T])$ *solution of the HJB equation*

$$v_t + \max_a \big(\frac{1}{2}\sigma^2 a^2 v_{xx} + bav_x\big) = 0 \quad in \quad \mathbb{R} \times [0,T) \quad and \quad v(x,T) = G(x). \tag{44}$$

*The optimal feedback control* $a^*\,(x,t)$ *is given by*

$$a^*\,(x,t) = -\frac{\lambda}{\sigma}\frac{v_x\,(x,t)}{v_{xx}\,(x,t)} = \frac{\lambda}{\sigma} r\,(x,t)\,, \tag{45}$$

*and the optimal policy process* $(\alpha^*_s)_{s \in [t,T]}$ *by*

$$\alpha^*_s = \frac{\lambda}{\sigma} r\,(x^*_s, s)\,,$$

*with* $(x^*_s)_{s \in [t,T]}$ *solving (43) with* $\alpha_s = \alpha^*_s$.

The function $v$, which is the single agent ($N = 1$) value function (13) in the absence of interaction ($F \equiv 0$), was introduced in [27] and has been extensively studied since then; the assertions of the theorem are well known results.

The auxiliary function $r$ which represents the so-called dynamic risk tolerance function is also well studied. The following result can be found in Kallblad and Zariphopoulou [20].

**Proposition 6** *Assume* (25). *The function $r$ is the unique $\mathcal{C}^{2,1}(\mathbb{R} \times [0,T])$ solution to*

$$r_t + \frac{1}{2}\lambda^2 r^2 r_{xx} = 0 \quad in \quad \mathbb{R} \times [0,T) \quad and \quad r(x,T) = R(x). \tag{46}$$

*Moreover, $r$ is positive, strictly increasing in $x$, and Lipschitz continuous in $x$ uniformly in time, that is, for some $K > 0$ and all $x, y \in \mathbb{R}$ and $t \in [0,T]$,*

$$|r(x,t) - r(y,t)| \leq K|x - y|. \tag{47}$$

*Finally, the function $h : \mathbb{R} \times [0,T] \to \mathbb{R}$ defined by*

$$r\left(h\left(x,t\right),t\right) = h_x\left(x,t\right) \tag{48}$$

*solves*

$$h_t + \frac{1}{2}\lambda^2 h_{xx} = 0 \quad in \quad \mathbb{R} \times [0,T) \quad and \quad h(x,T) = \left(G'\right)^{(-1)}\left(e^{-x}\right). \tag{49}$$

The Lipschitz continuity and positivity of $r$ imply the existence of constants $K_1, K_2$ such that, for all $x \in \mathbb{R}$ and $t \in [0,T]$,

$$r\left(x,t\right) \leq K_1\left|x\right| + K_2. \tag{50}$$

Next we investigate the solvability of (38). To ease the notation, we introduce the function $H : \mathbb{R} \times [0,T] \to \mathbb{R}_+$ given, for $m \in \mathcal{P}$ and $r$ as in (39), by

$$H\left(p,t;m\right) = \int r\left(y-p,t\right) dm\left(y\right). \tag{51}$$

**Lemma 7** *Let $R$ as in (25) and assume that either $R \equiv \delta$ or $R \geq \delta$. Then, for each $m \in \mathcal{P}$ and all $(x,t)$ in $\mathbb{R} \times [0,T]$,*

$$either \quad H\left(x,t\right) = \delta \quad or \quad H\left(x,t\right) \geq \delta. \tag{52}$$

**Proof.** If $R\left(x\right) = \delta$, (46) has the unique solution $r\left(x,t\right) = \delta$. If $R\left(x\right) \geq \delta$, then the comparison result for (46) yields $r \geq \delta$. ∎

**Proposition 8** *Assume* (25) *and* (28)*, and let $r$ be given by* (39)*. Then,* (38) *has a unique solution $f \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T])$ satisfying*

$$0 < \varepsilon \leq f_z\left(z,t\right) < 1. \tag{53}$$

*Furthermore,*

$$f\left(z,t\right)=z-g^{(-1)}\left(z,t\right), \tag{54}$$

*where* $g:\mathbb{R}\times[0,T]\to\mathbb{R}$ *is the unique* $\mathcal{C}^{2,1}\left(\mathbb{R}\times[0,T]\right)$ *solution of*

$$\begin{aligned} &g_t+\frac{1}{2}\lambda^2H^2\big(z-g^{(-1)},t\big)g_{zz}=0 \quad \textit{in} \quad \mathbb{R}\times[0,T] \quad \textit{and}\\ &g(z,T)=(z-F\left(z\right))^{(-1)}. \end{aligned} \tag{55}$$

**Proof.** We write (38) as

$$f_t+\frac{1}{2}\lambda^2\frac{H^2(f(z,t),t)}{(1-f_z)^2}f_{zz}=0 \;\text{ in }\; \mathbb{R}\times[0,T) \;\text{ and }\; f(z,T)=F(z). \tag{56}$$

A straightforward application of the maximum principle yields that any smooth solution of (56) will satisfy for some $K>0$ the bound (53).

Lemma 7 and (25) also imply that (55) is uniformly elliptic. Moreover, (50) and (51) yield that $H$ is Lipschitz continuous and has at most linear growth in $f$. Hence, $H^2$ grows at most quadratically in $f$ and, in view of (53), at most quadratically in $z$.

We may now apply the standard theory of parabolic PDE to obtain the existence of a smooth $f$. The rest of the proof follows easily. ■

# 4 Optimal mean field equilibrium processes

We construct the optimal processes generated by the mean field feedback control $\pi^*\left(x,m,t\right)$ given in (4). Specifically, we seek $\left(X_t^*\right)_{t\in[0,T]}$ and $\left(\pi_t^*\right)_{t\in[0,T]}$ solving

$$dX_t^*=b\pi_t^*dt+\sigma\pi_t^*dW_t \;\text{ in }\; (0,T) \;\text{ and }\; X_0^*=x, \tag{57}$$

with

$$\begin{aligned} \pi_t^*=\pi^*\big(X_t^*,m_t^*,t\big)=\frac{\lambda}{\sigma}\Big(&r(X_t^*-f(\tilde{X}_t^*,t),t)\\ &+\frac{f_z(\tilde{X}_t^*,t)}{1-f_z(\tilde{X}_t^*,t)}\int r(y-f(\tilde{X}_t^*,t),t)dm_t^*(y)\Big). \end{aligned} \tag{58}$$

Herein, $\left(m_t^*\right)_{t\in[0,T]}$ is the conditional on $\mathcal{F}_t^W$ law of $X_t^*$ and $\big(\tilde{X}_t^*\big)_{t\in[0,T]}$ is the related conditional average

$$\tilde{X}_t^*=\int y\,dm_t^*\left(y\right) \quad\text{and}\quad \tilde{X}_0^*=\tilde{x}_0=\int x\,dm_0(x). \tag{59}$$

We start with some auxiliary results.

**Proposition 9** *Let $r$ be as in (39). Then, the SDE*

$$dy_t = \lambda^2 r\left(y_t, t\right) dt + \lambda r\left(y_t, t\right) dW_t \quad \text{in} \quad (0, T] \quad \text{and} \quad y_0 = y \in \mathbb{R} \tag{60}$$

*admits a unique strong solution given by*

$$y_t = h(h^{(-1)}\left(y, 0\right) + N_t, t), \tag{61}$$

*where $h$ solves (49) and, for $t \in [0, T]$,*

$$N_t = \lambda^2 t + \lambda W_t. \tag{62}$$

*Furthermore,*

$$r\left(y_t, t\right) = h_x(h^{(-1)}\left(y, 0\right) + N_t, t)\,. \tag{63}$$

**Proof.** The uniqueness follows from the uniform in time Lipschitz continuity of $r$ in space.

For the existence, we show that the process in (61) satisfies (60). To this end, let $z_t = h^{(-1)}\left(y, 0\right) + N_t$. Then, using Ito's formula and (49), we find

$$dy_t = dh(z_t, t) = \lambda^2 h_x(z_t, t)dt + \lambda h_x(z_t, t)dW_t$$
$$+ \left(h_t(z_t, t) + \frac{1}{2}\lambda^2 h_{xx}(z_t, t)\right)dt = \lambda^2 h_x\left(z_t, t\right) dt + \lambda h_x\left(z_t, t\right) dW_t.$$

On the other hand, (48) yields that $h_x\left(z_t, t\right) = r\left(h\left(z_t, t\right), t\right)$ and, thus,

$$dy_t = dh\left(z_t, t\right) = \lambda^2 r\left(h\left(z_t, t\right), t\right) dt + \lambda r\left(h\left(z_t, t\right), t\right) dW_t.$$

To establish (63), we observe that, by the definition of $h$, $r\left(h\left(y_t, t\right), t\right) = h_x\left(y_t, t\right)$ and, in turn,

$$r\left(y_t, t\right) = h_x\left(h^{(-1)}\left(y_t, t\right), t\right) = h_x\left(h^{(-1)}\left(y, 0\right) + N_t, t\right).$$

■

Next, we present the optimal processes $\left(x_t^{*,x}\right)_{t\in[0,T]}$ and $\left(a_t^{*,x}\right)_{t\in[0,T]}$; for convenience, we have added in the notation their dependence on the initial condition.

**Proposition 10** *The optimal processes $\left(x_t^{*,x}\right)_{t\in[0,T]}$ and $\left(a_t^{*,x}\right)_{t\in[0,T]}$ for problem (42) (single agent and no coupling) are given by*

$$x_t^{*,x} = h\left(h^{(-1)}\left(x, 0\right) + N_t, t\right) \quad \text{and} \quad a_t^{*,x} = \frac{\lambda}{\sigma} h_x\left(h^{(-1)}\left(x, 0\right) + N_t, t\right). \tag{64}$$

**Proof.** Using the optimal feedback policy (45), we find that $x_t^{*,x}$ solves (60), that is,

$$dx_t^{*,x} = \lambda^2 r\left(x_t^{*,x}, t\right) dt + \lambda r\left(x_t^{*,x}, t\right) dW_t \quad \text{and} \quad x_0^{*,x} = x \in \mathbb{R}.$$

Then, the first equality in (64) follows. To show the second equality, we use that

$$
\begin{aligned}
a_t^{*,x} &= \frac{\lambda}{\sigma} r\left(x_t^{*,x}, t\right) = \frac{\lambda}{\sigma} r\left(h\left(h^{(-1)}(x,0) + N_t, t\right), t\right) \\
&= \frac{\lambda}{\sigma} h_x\left(h^{(-1)}(x,0) + N_t, t\right).
\end{aligned}
$$

■

With $h$ and $f$ solving (49) and (38), $\tilde{x}_0$ as in (59), $N_t$ given in (62) and for each initial population distribution $m_0 \in \mathcal{P}$, we introduce the auxiliary processes $(Y_t)_{t\in[0,T]}$ and $(\tilde{Y}_t)_{t\in[0,T]}$

$$
Y_t = h\left(h^{(-1)}\left(x - f\left(\tilde{x}_0, 0\right), 0\right) + N_t, t\right), \tag{65}
$$

and

$$
\tilde{Y}_t = \int h\left(h^{(-1)}\left(x - f\left(\tilde{x}_0, 0\right), 0\right) + N_t, t\right) dm_0(x). \tag{66}
$$

We are now ready to present the main result in this section.

**Proposition 11** *Let $f$ be the solution to (38), $g(z,t) = (z - f(z,t))^{(-1)}$, and processes $(Y_t)_{t\in[0,T]}$, $(\tilde{X}_t^*)_{t\in[0,T]}$ and $(\tilde{Y}_t^*)_{t\in[0,T]}$ as in (65), (59) and (66). Then, the optimal mean field process $\left(X_t^{*,x}\right)_{t\in[0,T]}$ satisfies*

$$
X_t^{*,x} = Y_t + f\left(\tilde{X}_t^*, t\right). \tag{67}
$$

*Furthermore,*

$$
\tilde{X}_t^* = g\left(\tilde{Y}_t, t\right), \tag{68}
$$

*and, in turn,*

$$
\begin{aligned}
X_t^{*,x} &= h\big(h^{(-1)}\big(x - f\big(\tilde{x}_0, 0\big), 0\big) + N_t, t\big) \\
&+ f\big(g\big(\int h\big(h^{(-1)}\big(x - f\big(\tilde{x}_0, 0\big), 0\big) + N_t, t\big) dm_0(x), t\big), t\big).
\end{aligned} \tag{69}
$$

**Proof.** From (57) and (58) we have

$$
\begin{aligned}
dX_t^* = \lambda^2 \Big( & r(X_t^* - f(\tilde{X}_t^*, t), t) \\
&+ \frac{f_z(\tilde{X}_t^*, t)}{1 - f_z(\tilde{X}_t^*, t)} \int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y) dt \Big) \\
&+ \lambda \Big( r(X_t^* - f(\tilde{X}_t^*, t), t) \\
&+ \frac{f_z(\tilde{X}_t^*, t)}{1 - f_z(\tilde{X}_t^*, t)} \int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y) dW_t \Big).
\end{aligned} \tag{70}
$$

Therefore,

$$
\begin{aligned}
d\tilde{X}_t^* = \lambda^2\Big( &\int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y) \\
&+ \frac{f_z(\tilde{X}_t^*, t)}{1 - f_z(\tilde{X}_t^*, t)} \int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y)\Big) dt \\
&+ \lambda\Big( \int r\left(y - f\left(\tilde{X}_t^*, t\right), t\right) dm_t^*(y) \\
&+ \frac{f_z(\tilde{X}_t^*, t)}{1 - f_z(\tilde{X}_t^*, t)} \int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y)\Big) dW_t\Big) \\
&= \lambda^2 \frac{\int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y)}{1 - f_z(\tilde{X}_t^{*,x}, t)} dt \\
&+ \lambda \frac{\int r(y - f(\tilde{X}_t^{*,x}, t), t) dm_t^*(y)}{1 - f_z(\tilde{X}_t^*, t)} dW_t.
\end{aligned}
\tag{71}
$$

If $f$ satisfies (38), Ito's formula gives

$$
\begin{aligned}
df\left(\tilde{X}_t^*, t\right) = \big(f_t(\tilde{X}_t^*, t) &+ \frac{1}{2}\lambda^2 \frac{\left(\int r(y - f(\tilde{X}_t^*, t), t) dm_t^*(y)\right)^2}{\left(1 - f_z(\tilde{X}_t^*, t)\right)^2} f_{zz}\left(\tilde{X}_t^*, t\right)\big) dt \\
&+ \lambda^2 \frac{f_z\left(\tilde{X}_t^*, t\right)}{1 - f_z\left(\tilde{X}_t^{*,x}, t\right)} \left(\int r\left(y - f\left(\tilde{X}_t^*, t\right), t\right) dm_t^*(y)\right) dt \\
&+ \lambda \frac{f_z\left(\tilde{X}_t^*, t\right)}{1 - f_z\left(\tilde{X}_t^{*,x}, t\right)} \left(\int r\left(y - f\left(\tilde{X}_t^*, t\right), t\right) dm_t^*(y)\right) dW_t \\
&= \lambda^2 \frac{f_z\left(\tilde{X}_t^*, t\right)}{1 - f_z\left(\tilde{X}_t^{*,x}, t\right)} \left(\int r\left(y - f\left(\tilde{X}_t^*, t\right), t\right) dm_t^*(y)\right) dt \\
&+ \lambda \frac{f_z\left(\tilde{X}_t^*, t\right)}{1 - f_z\left(\tilde{X}_t^{*,x}, t\right)} \left(\int r\left(y - f\left(\tilde{X}_t^*, t\right), t\right) dm_t^*(y)\right) dW_t.
\end{aligned}
$$

Therefore, the process $\hat{X}_t = X_t^{*,x} - f(\tilde{X}_t^*, t)$ satisfies the autonomous SDE

$$
d\hat{X}_t = \lambda^2 r(\hat{X}_t, t) dt + \lambda r(\hat{X}_t, t) dW_t \quad \text{in} \quad (0, T) \quad \text{and} \quad \hat{X}_0 = x - f\left(\tilde{x}_0, 0\right).
$$

Then, from Proposition 9 and (65), we find that

$$
X_t^{*,x} - f\left(\tilde{X}_t^*, t\right) = h\left(h^{(-1)}\left(x - f\left(\tilde{x}_0, 0\right), 0\right) + N_t, t\right),
$$

and obtain (67).

The rest of the proof follows. ■

The following proposition follows directly from (58) and (67).

**Proposition 12** *The optimal mean field equilibrium policy* $(\pi_t^*)_{t\in[0,T]}$ *is given by*

$$\pi_t^* = \frac{\lambda}{\sigma}\Big(r\left(Y_t,t\right) + \frac{f_z\left(\tilde{X}_t^*,t\right)}{1-f_z\left(\tilde{X}_t^*,t\right)}\int r\left(y - f\left(\tilde{X}_t^*,t\right),t\right)dm_t^*\left(y\right)\Big)$$

*with* $\tilde{X}_t^*$ *as in* (71).

# 5 Examples

We provide two families of representative examples. In the first, we allow for general coupling and exponential utility while in the second we assume general utility and linear coupling. To the best of our knowledge, these cases have not been examined before.

## 5.1 General coupling and exponential utility

Assume that the coupling function $F$ satisfies (28) and the utility $G:\mathbb{R}\to\mathbb{R}$ is given, for some $\delta>0$, by

$$G\left(x\right) = -e^{-x/\delta}. \tag{72}$$

Direct calculations in (44) give that

$$v\left(x,t\right) = -\exp\left(-\frac{1}{\delta}x - \frac{1}{2}\lambda^2\left(T-t\right)\right).$$

Then, (30) implies that

$$U(x,m,t) = -\exp\left(-\frac{1}{\delta}\left(x - f\left(\int xdm(x),t\right)\right) - \frac{1}{2}\lambda^2\left(T-t\right)\right). \tag{73}$$

Furthermore, from (39) we get that $r\left(x,t\right)=\delta$ and, thus, $H\left(x,m,t\right)=\delta$. Then, (38) reduces to

$$f_t + \frac{1}{2}\frac{\lambda^2\delta^2}{\left(1-f_z\right)^2}f_{zz} = 0 \quad \text{in} \quad \mathbb{R}\times[0,T) \ \text{ and } \ f(x,T) = F\left(x\right). \tag{74}$$

Therefore, the auxiliary function $g$ in (54) solves the heat equation

$$g_t + \frac{1}{2}\lambda^2\delta^2 g_{zz} = 0 \quad \text{in} \quad \mathbb{R}\times[0,T) \ \text{ and } \ g\left(z,T\right) = \left(z - F\left(z\right)\right)^{(-1)}. \tag{75}$$

It also has the probabilistic representation

$$g(z,t) = \mathbb{E}'\left[ \left(w_T - F(w_T)\right)^{(-1)} \middle| w_t = z \right],$$

where $dw_s = \lambda\delta dW'_s$ and $w_t = z$, with $\left(W'_t\right)_{t\geq 0}$ being a standard Brownian motion under measure $\mathbb{P}'$, and, therefore,

$$f(z,t) = z - \left(\mathbb{E}'\left[ \left(w_T - F(w_T)\right)^{(-1)} \middle| w_t = z \right]\right)^{(-1)}. \tag{76}$$

From (9), we deduce that the optimal feedback policy $\pi^*(x,m,t)$ takes the form

$$\pi^*(x,m,t) = \frac{\lambda}{\sigma} \frac{\delta}{1 - f_z\left(\int x dm(x), t\right)}.$$

Thus, the optimal process $(X^*_t)_{t\in[0,T]}$ satisfies

$$dX^*_t = \frac{\lambda^2\delta}{1 - f_z\left(\tilde{X}^*_t, t\right)} dt + \frac{\lambda\delta}{1 - f_z\left(\tilde{X}^*_t, t\right)} dW_t \quad \text{and} \quad X^*_t = x_0,$$

and, therefore,

$$d\tilde{X}^*_t = \frac{\lambda^2\delta}{1 - f_z\left(\tilde{X}^*_t, t\right)} dt + \frac{\lambda\delta}{1 - f_z\left(\tilde{X}^*_t, t\right)} dW_t \quad \text{and} \quad \tilde{X}^*_0 = \tilde{x}_0,$$

which implies that $X^*_t = x + \tilde{X}^*_t - \tilde{x}_0$.

With $G$ as in (72), we deduce from the terminal condition in (49) that

$$h(x,T) = \delta x - \delta \ln \delta.$$

It follows from (49) that $h(x,t) = \delta x - \delta \ln \delta$ and, therefore, $h^{(-1)}(x,t) = \frac{1}{\delta}x + \ln \delta$.

In turn, $Y_t = x - f(\tilde{x}_0, 0) + \delta N_t$ and $\tilde{Y}_t = \tilde{x}_0 - f(\tilde{x}_0, 0) + \delta N_t$, and then (67) implies that

$$X^*_t = x - f(\tilde{x}_0, 0) + \delta N_t + f\left(g\left(\tilde{x}_0 - f(\tilde{x}_0, 0) + \delta N_t, t\right), t\right).$$

Moreover,

$$\tilde{X}^*_t = \tilde{x}_0 - f(\tilde{x}_0, 0) + \delta N_t + f\left(g\left(\tilde{x}_0 - f(\tilde{x}_0, 0) + \delta N_t, t\right), t\right) \text{ and } \pi^*_t = \frac{\lambda}{\sigma} \frac{\delta}{1 - f_z\left(\tilde{X}^*_t, t\right)}.$$

## 5.2 General utility and linear coupling

Let the utility function $G$ satisfy (25) and the coupling given by

$$F(z) = \theta z, \quad z \in \mathbb{R}, \ \theta \in (0,1). \tag{77}$$

Then (38) yields

$$f(z,t) = \theta z \quad \text{in } \mathbb{R}\times [0,T]. \tag{78}$$

Therefore, the solution to the master system (20) and (21) is given by

$$\pi^*(x,m,t) = \frac{\lambda}{\sigma}\Big(r\big(x-\theta\tilde{x},t\big) + \frac{\theta}{1-\theta}\int r\big(y-\theta\tilde{x},t\big)dm(y)\Big) \tag{79}$$

and

$$U(x,m,t) = v(x-\theta\tilde{x},t), \tag{80}$$

where $\tilde{x} = \int x dm(x)$.

The optimal process $(X_t^*)_{t\in[0,T]}$ is given by

$$\begin{aligned} X_t^* &= Y_t + \frac{\theta}{1-\theta}\tilde{Y}_t^* \\ &= h\left(h^{-1}(x-\theta\tilde{x}_0,0) + N_t, t\right) + \frac{\theta}{1-\theta}\int h\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right)dm_0(x). \end{aligned}$$

Therefore,

$$\tilde{X}_t^* = \frac{1}{1-\theta}\int h\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right)dm_0(x),$$

and, thus,

$$X_t^* - \theta\tilde{X}_t^* = h\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right). \tag{81}$$

The optimal policy $(\pi_t^*)_{t\in[0,T]}$ is, in turn, given by

$$\pi_t^* = \frac{\lambda}{\sigma}\left(r\left(X_t^*-\theta\tilde{X}_t^*,t\right) + \frac{\theta}{1-\theta}\int r\left(y-\theta\tilde{X}_t^*,t\right)dm_t^*(y)\right).$$

From (48) and (81) we obtain that

$$\begin{aligned} r\left(X_t^*-\theta\tilde{X}_t^*,t\right) &= r\left(h\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right),t\right) \\ &= h_x\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right), \end{aligned}$$

and, then,

$$\int r\left(y-\theta\tilde{X}_t^*,t\right)dm_t^*(y) = \int h_x\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right)dm_0(x).$$

Therefore,

$$\begin{aligned} \pi_t^* =& h_x\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right) \\ &+ \frac{\theta}{1-\theta}\int h_x\left(h^{-1}(x-\theta\tilde{x}_0,0)+N_t,t\right)dm_0(x). \end{aligned} \tag{82}$$

Comparing (67) and (82) to (64), we see that the optimal state and control processes can be written as

$$X_t^* = x_t^{*,x-\theta\tilde{x}_0} + \frac{\theta}{1-\theta}\int x_t^{*,x-\theta\tilde{x}_0} dm_0\left(x\right),$$

and

$$\pi_t^* = a_t^{*,x-\theta\tilde{x}_0} + \frac{\theta}{1-\theta}\int a_t^{*,x-\theta\tilde{x}_0} dm_0\left(x\right).$$

In other words, the optimal process $X_t^*$, starting at $x$ is represented as the sum of the optimal state process in the absence of coupling, but with modified initial condition $x - \theta\int x dm_0\left(x\right)$, and the average of all such optimal processes $\int x_t^{*,x-\theta\tilde{x}_0} dm_0\left(x\right)$ with respect to the initial measure $m_0$, multiplied by factor $\frac{\theta}{1-\theta}$.

The optimal policy $\pi_t^*$ is, similarly, decomposed as the sum of the optimal policy $a_t^{*,x-\theta\tilde{x}_0}$ that generates $x_t^{*,x-\theta\tilde{x}_0}$ and the average $\frac{\theta}{1-\theta}\int a_t^{*,x-\theta\tilde{x}_0} dm_0\left(x\right)$ of all such processes, which generates $\frac{\theta}{1-\theta}\int x_t^{*,x-\theta\tilde{x}_0} dm_0\left(x\right)$.

We stress that this interpretation is universal, independently of the type of utility function.

#### 5.2.1 Asymptotically linear risk tolerance functions

This class of risk tolerance functions was first introduced in [29]; see also [38], and were also considered in [10] under the name SAHARA (see, also, [34]).

They are represented by the two-parameter family

$$R(x) = \sqrt{\alpha x^2 + \delta} \quad \text{for} \quad x \in \mathbb{R},\ \alpha \geq 0,\ \delta > 0. \tag{83}$$

The corresponding utility function is given by

$$G(x) = \begin{cases} -C_1 e^{-\frac{x}{\sqrt{\delta}}} + C_2 \quad \text{if} \quad \alpha = 0, \\ C_1\left(\frac{x}{x+\sqrt{x^2+\delta}} + \log\left(x + \sqrt{x^2+\delta}\right)\right) + C_2 \quad \text{if} \quad \alpha = 1, \\ C_1\left(\frac{x}{\left(x+\sqrt{x^2+\delta}\right)^{\frac{1}{\sqrt{\alpha}}}} + \frac{1}{\sqrt{\alpha}-1}\left(x + \sqrt{x^2+\delta}\right)^{1-\frac{1}{\sqrt{\alpha}}}\right) + C_2 \text{ otherwise,} \end{cases}$$

where $C_1 > 0$, $C_2 \in \mathbb{R}$ are two generic constants.

To ease the presentation, we only assume $\alpha = 1$, since the case $\alpha > 0$ follows similarly and the exponential utility, $\alpha = 0$, is presented afterwards.

To this end, without loss of generality we choose the constants $C_1 = \sqrt{\delta}/2$ and $C_2 = 0$. From (48), we deduce that $h(x,T) = \sqrt{\delta}\sinh x$ and, in turn, $h(x,t) = \sqrt{\delta}e^{\frac{1}{2}\lambda^2(T-t)}\sinh x$ and $h^{(-1)}(x,0) = \operatorname{arcsinh}\left(\sqrt{\delta}e^{\frac{1}{2}\lambda^2 T}x\right)$.

Therefore, $x_t^{*,x} = \sqrt{\delta}e^{\frac{1}{2}\lambda^2(T-t)}\sinh(\text{arcsinh}(\sqrt{\delta}e^{\frac{1}{2}\lambda^2 T}x) + N_t)$ and, thus, the optimal mean field state process is given by

$$X_t^* = \sqrt{\delta}e^{\frac{1}{2}\lambda^2(T-t)}\sinh\left(\text{arcsinh}\left(\sqrt{\delta}e^{\frac{1}{2}\lambda^2 T}(x-\theta\tilde{x}_0)+N_t\right)\right)$$

$$+\frac{\theta}{1-\theta}\sqrt{\delta}e^{\frac{1}{2}\lambda^2(T-t)}\int\sinh\left(\text{arcsinh}\left(\sqrt{\delta}e^{\frac{1}{2}\lambda^2 T}\left(x-\theta\tilde{x}_0\right)\right)+N_t\right)dm_0\left(x\right).$$

Furthermore, the related optimal policy process is given by

$$\pi_t^* = \sqrt{\delta}e^{\frac{1}{2}\lambda^2(T-t)}\cosh\left(\text{arcsinh}\left(\sqrt{\delta}e^{\frac{1}{2}\lambda^2 T}(x-\theta\tilde{x}_0\right)+N_t\right)$$

$$+\frac{\theta}{1-\theta}\sqrt{\delta}e^{\frac{1}{2}\lambda^2(T-t)}\int\cosh\left(\text{arcsinh}\left(\sqrt{\delta}e^{\frac{1}{2}\lambda^2 T}\left(x-\theta\tilde{x}_0\right)+N_t\right)\right)dm_0(x).$$

We also deduce that

$$r\left(x,t\right)=\sqrt{x^2+\delta e^{\lambda^2(T-t)}}\quad\text{in } \mathbb{R}^N\times[0,T],$$

which can be, in turn, used to recover (integrating twice) the value function $v$ and, in turn, $U$.

### 5.3 Exponential utility and linear coupling

The intersection of the aforementioned families is when

$$G(x) = -e^{-\frac{1}{\delta}x}\quad\text{and}\quad F(z)=\theta z.$$

Using either (73) for $F(z)=\theta z$ or (80) for $G\left(x\right)=-e^{-\frac{1}{\delta}x}$, we deduce that

$$U(x,m,t) = -\exp\left(-\frac{1}{\delta}(x-\theta\tilde{x}_0)-\frac{1}{2}\lambda^2(T-t)\right)\quad\text{and}\quad\pi^*\left(x,m,t\right)=\frac{\lambda}{\sigma}\frac{\delta}{1-\theta}.$$

Furthermore, we easily obtain that

$$X_t^* = x+\frac{\delta}{1-\theta}N_t.$$

Similar quantities were, also, derived in [22] for the asset diversification case (see Corollary 11 when $\nu=0$).

## 6 The MFG approximation to the $N-$player game

Having produced a solution $U\left(x,m,t\right)$ and $\pi^*\left(x,m,t\right)$ to the mean field system (21) and (20), we examine how well it approximates the analogous solutions of the $N-$player game.

For this, we let, for $i = 1, ..., N$,

$$\hat{x}^{-i} = \frac{1}{N-1} \sum_{j=1, j \neq i}^{N} x_j \quad \text{and} \quad \mu^{N,i} = \frac{1}{N-1} \sum_{j=1, j \neq i}^{N} \delta_{x_j},$$

and compare $U(x_i, \mu^{N,i}, t)$ to $v_i(x_1, ..., x_N, t)$ (the value function of the $i^{th}$ player) and $\pi^*(x_i, \mu^{N,i}, t)$ to $\pi^*_{N,i}(x_1, ..., x_N, t)$ (the optimal feedback control of the $i^{th}$ player).

For $i = 1, ..., N$, let $f^i : \mathbb{R} \times [0, T] \to \mathbb{R}$ be the solution of (38) evaluated at $\mu^{N,i}$, which satisfies in $\mathbb{R} \times [0, T)$,

$$f_t^i\left(\hat{x}^{-i}, t\right) + \frac{1}{2}\lambda^2 \Big(\frac{1}{N-1} \sum_{j=1, j \neq i}^{N} r\left(x_j - f^i\left(\hat{x}^{-i}, t\right), t\right)\Big)^2 \frac{f_{zz}^i\left(\hat{x}^{-i}, t\right)}{\left(1 - f_z^i\left(\hat{x}^{-i}, t\right)\right)^2} = 0 \tag{84}$$

and $f^i\left(\hat{x}^{-i}, T\right) = F\left(\hat{x}^{-i}\right)$.

It follows from Theorem 4 that

$$U(x_i, \mu^{N,i}, t) = v\left(x_i - f^i\left(\hat{x}^{-i}, t\right), t\right), \tag{85}$$

and

$$\begin{aligned} \pi^*(x_i, \mu^{N,i}, t) &= \frac{\lambda}{\sigma} \Big( r\left(x_i - f^i\left(\hat{x}^{-i}, t\right), t\right) \\ &+ \frac{f_z^i\left(\hat{x}^{-i}, t\right)}{1 - f_z^i\left(\hat{x}^{-i}, t\right)} \frac{1}{N-1} \sum_{j=1, j \neq i}^{N} r\left(x_j - f^i\left(\hat{x}^{-i}, t\right), t\right) \Big). \end{aligned} \tag{86}$$

## 6.1 General utility functions and linear coupling

If $F(z) = \theta z$ with $\theta \in (0, 1)$, then $f^i(z, t) = \theta z$, and, thus,

$$U(x_i, \mu^{N,i}, t) = v\left(x_i - \theta \hat{x}^{-i}, t\right),$$

and

$$\pi^*(x_i, \mu^{N,i}, t) = \frac{\lambda}{\sigma} \Big( r(x_i - \theta \hat{x}^{-i}, t) + \frac{\theta}{1-\theta} \frac{1}{N-1} \sum_{j=1, j \neq i}^{N} r(x_j - \theta \hat{x}^{-i}, t) \Big).$$

### 6.1.1 Solution of the $N-$player game

We present the following result, which is of independent interest and yields a solution to the $N-$player game.

**Proposition 13** *The $N-$player game has a solution $v^i$ given, for $i = 1, ..., N$, by*

$$v^i(x_1, ..., x_N, t) = v\left(x_i - \theta \hat{x}^{-i}, t\right), \tag{87}$$

*and optimal strategies, given, for each $i = 1, ..., N$, by*

$$\pi^*_{N,i}(x_1, ..., x_N, t) = \frac{\lambda}{\sigma}\frac{N-1}{N-1+\theta} r\left(x_i - \theta\hat{x}^{-i}, t\right) + \frac{\lambda}{\sigma}\frac{1}{N-1+\theta}\frac{\theta}{1-\theta}\sum_{j=1}^{N} r(x_j - \theta\hat{x}^{-j}, t). \tag{88}$$

**Proof.** For the $v^i$'s to be a solution to (14), it is necessary to find $\pi^*_{N,j}(x_1, ..., x_N, t)$'s which satisfy the optimality condition (16) and to show that, for each $i = 1, \dots, N$, the $v^i$ satisfies the $i^{\text{th}}$ HJB equation in (14).

We begin with the latter claim observing that, given $\alpha_j : \mathbb{R}^N \times [0, T]$, each $v^i$ defined by (87) solves

$$v^i_t + \max_{\pi}\big(\frac{1}{2}\sigma^2\pi^2 v^i_{x_i x_i} + \pi\big(b v^i_{x_i} + \sigma^2\sum_{j=1, j\neq i}^{N}\alpha_j v^i_{x_i x_j}\big) + \frac{1}{2}\sigma^2\sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N}\alpha_j\alpha_k v^i_{x_j x_k} + b\sum_{j=1,j\neq i}^{N}\alpha_j v^i_{x_j} = 0, \tag{89}$$

provided there exists $\pi^*_i$ such that

$$\pi^*_i v^i_{x_i x_i} - \sum_{j=1,j\neq i}^{N}\alpha_j v^i_{x_i x_j} = \frac{\lambda}{\sigma} v^i_{x_i}. \tag{90}$$

We show next that the above holds for the particular $v^i$'s and $\pi^*_{N,j}$'s defined by (87) and (88) respectively.

A straightforward computation shows that, for each $v^i$ given by (87), we have

$$\begin{aligned} &v^i_t + \max_{\pi}\big(\frac{1}{2}\sigma^2\pi^2 v^i_{x_i x_i} + \pi\big(b v^i_{x_i} + \sigma^2\sum_{j=1,j\neq i}^{N}\alpha_j v^i_{x_i x_j}\big) \\ &+ \frac{1}{2}\sigma^2\sum_{j=1,j\neq i}^{N}\sum_{k=1,k\neq i}^{N}\alpha_j\alpha_k v^i_{x_j x_k} b\sum_{j=1,\ j\neq i}^{N}\alpha_j v^i_{x_j} \\ &= v_t + \max_{\pi}\Big(\frac{1}{2}\sigma^2\big(\pi - \sum_{j=1,j\neq i}^{N}\alpha_j\big)^2 v_{xx} + b(\pi - \sum_{j=1,j\neq i}^{N}\alpha_j)v_x\Big), \end{aligned} \tag{91}$$

with the second equation above evaluated at $(x_i - \theta\hat{x}^{-i}, t)$.

In view of the choice of $v$, the right hand side of (91) equals 0 provided the maximization is happening at some $\alpha_i$ which is related to $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \dots, \alpha_N$ by

$$\alpha_i - \frac{\theta}{N-1}\sum_{j=1,j\neq i}^{N}\alpha_j = \frac{\lambda}{\sigma} r(x_i - \theta\hat{x}^{-i}, t), \tag{92}$$

which is the form that (91) has for the particular $v^i$ we use here.

We continue showing that it is possible to find such $\alpha_1, \dots, \alpha_N$. Specifically, we prove that they are the claimed $\pi^*_{N,i}$'s given in (88).

This amounts to solving for, each $i = 1, \dots, N$, the system

$$\pi^*_{N,i} - \frac{\theta}{N-1} \sum_{j=1,\ j\neq i}^{N} \pi^*_{N,j} = \frac{\lambda}{\sigma} r(x_i - \theta \hat{x}^{-i}, t). \tag{93}$$

We fix such $i$ and observe that, by rewriting (93) as

$$(1 + \frac{\theta}{N-1})\pi^*_{N,i} - \frac{\theta}{N-1} \sum_{j=1}^{N} \pi^*_{N,j} = \frac{\lambda}{\sigma} r(x_i - \theta \hat{x}^{-i}, t), \tag{94}$$

and summing over all $i$'s we get

$$(1-\theta) \sum_{j=1}^{N} \pi^*_{N,j} = \frac{\lambda}{\sigma} \sum_{j=1}^{N} r(x_j - \theta \hat{x}^{-j}, t). \tag{95}$$

Using (94) in (95) leads to

$$(1 + \frac{\theta}{N-1})\pi^*_{N,i} = \frac{\lambda}{\sigma}\Big( r(x_i - \theta \hat{x}^{-i}, t) + \frac{\theta N}{N-1} \sum_{j=1}^{N} r(x_j - \theta \hat{x}^{-j}, t) \Big),$$

and we may easily conclude.

■

We are now ready to provide the following approximation result. Since its proof is an immediate consequence of (87) and (88) and the upper bound on $r$, we omit it.

**Proposition 14** *For each* $i = 1, ..., N$,

$$v_i\,(x_1, ..., x_N, t) = U(x_i, \mu^{N,i}, t), \tag{96}$$

*and, for some independent of* $N$, $K > 0$,

$$\begin{aligned} &\left|\pi^*_{N,i}(x_1, ..., x_N, t) - \pi^*(x_i, \mu^{N,i}, t)\right| \\ &\leq K \frac{\lambda}{\sigma} \frac{\theta}{1-\theta} \Big( 1 + \frac{1-\theta}{N} \sum_{j=1}^{N} |x_j| + \frac{1}{N} \sum_{j=1}^{N} |x_i - x_j| \Big). \end{aligned} \tag{97}$$

We conclude with a model where the mean field game solution is also a solution of the $N-$player game.

**Corollary 15** *If* $G(x) = -e^{-x/\delta}$ *for some* $\delta > 0$, *then*

$$U(x_i, \mu^{N,i}, t) = v_i\left(x_1, ..., x_N, t\right) = -\exp\left(-\frac{1}{\delta}\left(x_i - \theta\hat{x}^{-i}\right) - \frac{1}{2}\lambda^2\left(T - t\right)\right)$$

*and*

$$\pi^*(x_i, \mu^{N,i}, t) = \pi^*_{N,i}(x_1, ..., x_N, t) = \frac{\lambda}{\sigma}\frac{\delta}{1-\theta}.$$

We stress that, for each $i = 1, ..., N$, the approximation $U(x_i, \mu^{N,i}, t)$ and the value function $v_i\left(x_1, ..., x_N, t\right)$ coincide even though they solve different HJB equations with the same terminal condition.

Indeed, it follows from the master equation (21) that if we let

$$w_i\left(x_1, ..., x_N, t\right) = U(x_i, \mu^{N,i}, t),$$

then

$$w^i_t + \max_{a_i}\left(\frac{1}{2}\sigma^2 a_i^2 w^i_{x_i x_i} + a_i\big(bV^i_{x_i} + \sigma^2 \sum_{j=1,\ j\neq i}^{N} \pi^*(x_j, \mu^{N,i}, t) w^i_{x_i x_j}\big)\right)$$
$$+\frac{1}{2}\sigma^2 \sum_{j=1, j\neq i}^{N} \sum_{k=1, k\neq i}^{N} \pi^*(x_j, \mu^{N,i}, t)\pi^*(x_k, \mu^{N,i}, t) w^i_{x_j x_k}$$
$$+ b\sum_{j=1,\ j\neq i}^{N} \pi^*(x_j, \mu^{N,i}, t) w^i_{x_j} = 0,$$

while $v_i\left(x_1, ..., x_N, t\right)$ solves

$$v^i_t + \max_{\pi_i}\left(\frac{1}{2}\sigma^2 \pi_i^2 v^i_{x_i x_i} + \pi_i\big(bv^i_{x_i} + \sigma^2 \sum_{j=1, j\neq i}^{N} \sum_{k=1, k\neq i}^{N} \pi^*_{N,j}(x_1, ..., x_N, t) v^i_{x_i x_j}\big)\right)$$
$$+\frac{1}{2}\sigma^2 \sum_{j=1, j\neq i}^{N} \sum_{k=1, k\neq i}^{N} \pi^*_{N,j}(x_1, ..., x_N, t)\pi^*_{N,k}(x_1, ..., x_N, t) v^i_{x_j x_k}$$
$$+ b\sum_{j=1,\ j\neq i}^{N} \pi^*_{N,j}(x_1, ..., x_N, t) v^i_{x_j} = 0.$$

On the other hand, as we have shown above, the respective control coefficients do not coincide, that is,

$$\pi^*(x_i, \mu^{N,i}, t) \neq \pi^*_i\left(x_1, ..., x_N, t\right).$$

Note, however, that in the case of exponential utilities, both $\pi^*(x_i, \mu^{N,i}, t)$ and $\pi^*_i\left(x_1, ..., x_N, t\right)$ are constants and coincide, and the above HJB equations are identical.

# 7 Conclusions

We introduced a mean field game arising in optimal allocation models with relative performance concerns. The terminal expected utility of the representative agent depends on both her own state and the average state of her peers. We derived the master equation together with a compatibility condition for the mean field optimal feedback control. For the class of separable payoffs under both general utilities and couplings, we derived a solution to the master equation, expressed via the value function in the absence of competition and a dynamic coupling function solving a non-local quasilinear equation. We also constructed the associated optimal feedback control as well as the related optimal state and control processes. Evaluating the mean filed solutions on $\left(x_i, \frac{1}{N-1}\Sigma_{j\neq i}^N \delta_{x_j}, t\right)$ we constructed the solution of the $N-$player game for linear couplings and arbitrary utilities, and we studied the proximity of these approximations to their $N-$player game counterparts.

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