Representation of forward performance criteria with random endowment via FBSDE and application to forward optimized certainty equivalent*

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Abstract

We extend the notion of forward performance criteria to settings with random endowment in incomplete markets. Building on these results, we introduce and develop the novel concept of *forward optimized certainty equivalent (forward OCE)*, which offers a genuinely dynamic valuation mechanism that accommodates progressively adaptive market model updates, stochastic risk preferences, and incoming claims with arbitrary maturities.

In parallel, we develop a new methodology to analyze the emerging stochastic optimization problems by directly studying the candidate optimal control processes for both the primal and dual problems. Specifically, we derive two new systems of forward-backward stochastic differential equations (FBSDEs) and establish necessary and sufficient conditions for optimality, and various equivalences between the two problems. This new approach is general and complements the existing one based on backward stochastic partial differential equations (backward SPDEs) for the related value functions. We, also, consider representative examples for both forward performance criteria with random endowment and forward OCE, and for the case of exponential criteria, we investigate the connection between forward OCE and forward entropic risk measures.

Keywords: Forward performance criteria, random endowment, convex dual, FBSDE, forward OCE, backward SPDE, forward entropic risk measures.

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1 Introduction

This work contributes to the theory of forward performance criteria in incomplete markets. It studies forward performance processes in the presence of random endowment and, building on this, introduces the novel concept of *forward optimized certainty equivalent (forward OCE)*. In parallel, it develops a new methodological approach to study the emerging forward stochastic optimization problems through new, interesting on their own right, *forward-backward stochastic differential equations (FBSDEs)* satisfied by the optimal control processes of the primal and dual problems.

Random endowments are a central object of study in expected utility maximization and, furthermore, play an important role in indifference valuation/certainty equivalent, where they model the payoffs/liabilities to be priced and hedged. The aim herein is to develop a general framework to incorporate them within the broader class of forward performance processes, analyze the related optimal control processes and investigate how they can be used to generalize the widely-used notion of optimized certainty equivalent.

The motivation for the plan of study herein stems from various shortcomings of the classical (backward) setting, as highlighted next and further explained later in the paper. Recalling the standard paradigm in the random endowment literature, one pre-specifies at initial time $t = 0$ a quadruple consisting of i) the (longest) horizon $[0, T]$ within which random endowments will arrive, ii) the utility function U at the end of this horizon, iii) the underlying market model $\mathcal{M}_{[0,T]}$ and iv) the upcoming random endowments. In other words, these modeling ingredients are chosen statically, once and for all, at initiation. However, more flexibility is frequently needed as new random endowments might arrive at times not known before, the market model might be revised and, furthermore, the risk preferences themselves could be modified. Indeed, let us consider the following simple representative case: for simplicity, it is assumed that there is a random endowment, given by a random variable P_T specified at $t = 0$. However, at some future time τ, with $0 < \tau < T$, the utility maximizer learns that an additional payoff P_{T_1} is expected at time $T_1 < T$. One now sees that the solution of the problem in $[0, \tau]$ has not considered this updated information, yielding *a posteriori* time-inconsistent solutions. The situation becomes even more complex if this new random endowment actually arrives at time $T_1 > T$, for the utility U was pre-defined only at T, and not beyond this time. Therefore, a modified utility maximization problem in $[T, T_1]$ needs first to be defined in order to accommodate the new random endowment. Additional considerations arise if at an intermediate time, say $\tau' < T$, the market model is updated to $M'_{[\tau',T]}$, which will also yield *a posteriori* time-inconsistent solutions, even if the assumptions on the random endowments remain the same.

Naturally, these limitations also have undesirable consequences on indifference prices. Indeed, the expected utility maximization framework does not allow to price in a time-consistent manner claims arriving at times not known at initiation, and especially when these new claims mature at instances beyond the pre-specified horizon, in which case the underlying model is not even well defined. It, also, fails to price, in a time-consistent manner, claims when the market model is being dynamically revised, or when the risk preferences themselves evolve stochastically. These limitations were one of the main motivations for the third author and Musiela to develop the theory of forward performance criteria in the early 2000s (see, among others, [32, 33, 34, 35, 36]).

Since then, the effort in the forward approach has mainly focused on, from the one hand, developing a probabilistic characterization of forward performance processes and, from the other, studying forward indifference prices for the special class of exponential criteria; see, for example, the stochastic partial differential equation (SPDE) approach initiated in [36], and also studied in [16, 17, 18, 19]. When the forward performance process is homothetic, like the exponential case, a new class of ergodic backward stochastic differential equations (BSDEs) have been proposed to characterize this class (see [13, 31]). When the forward performance process is time-monotonic, Widder's theorem has been applied for their characterization ([4, 37]). In addition, the corresponding discrete-time theory has been recently explored extensively ([1, 2, 30, 38]). The applications of forward performance processes have extended to various domains, such as relative performance criteria ([3]), general semimartingale models ([9]), insurance ([12]), duality theory ([14, 40]), behavioral finance ([20]), regime-switching models ([23]), intertemporal consumption ([24]), model uncertainty ([25]), and maturity-independent risk measures ([41]).

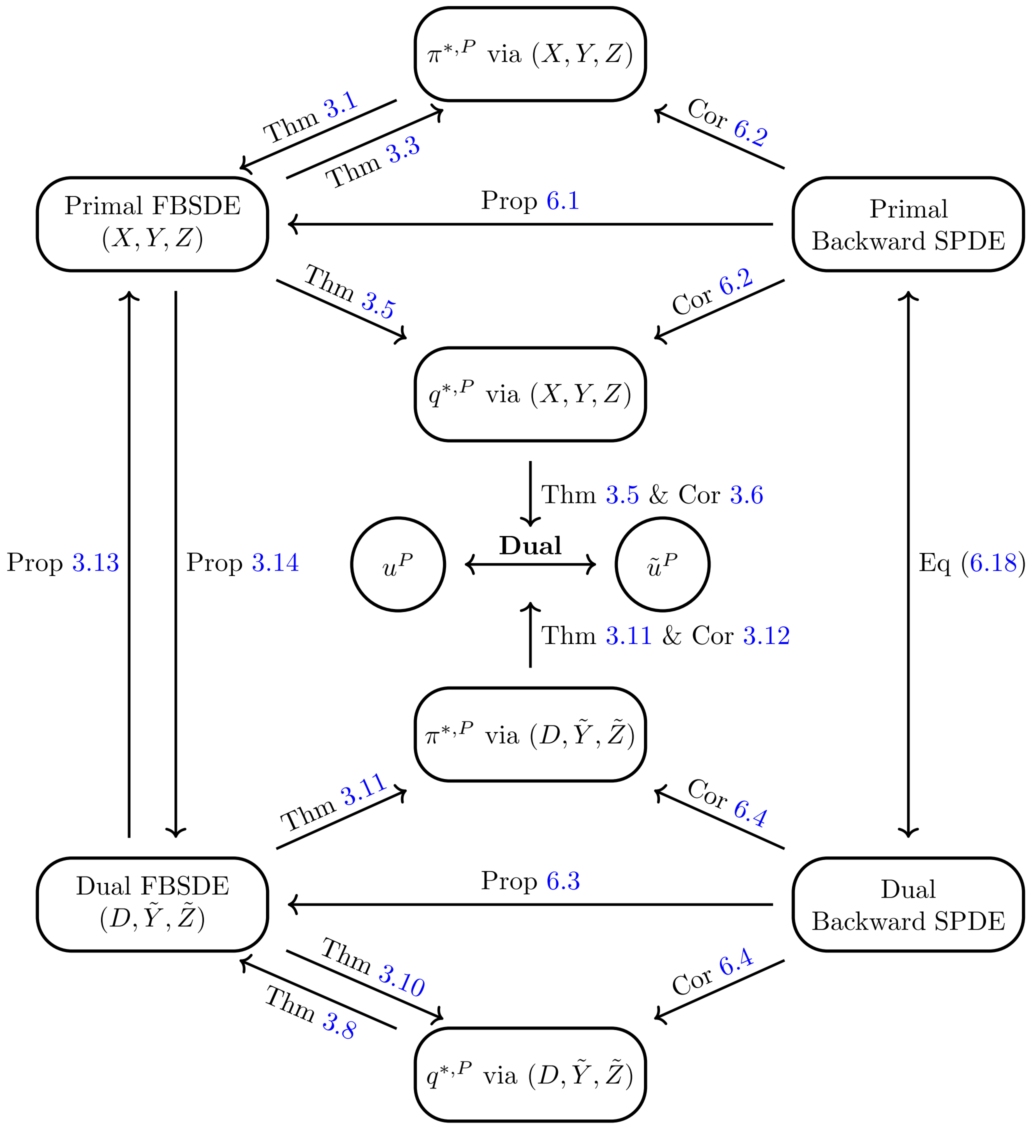
The only studies of the forward performance process with random endowment were the ones for exponential criteria in the context of exponential forward indifference prices (see, among others, [28, 32, 33]). The most general result can be found in Chong et al. [13] who built on the work of forward entropic risk measures, initially developed in [41], and proposed a BSDE, coupled with an ergodic BSDE, representation of exponential forward indifference prices. However, how this BSDE/ergodic BSDE approach may be extended beyond the exponential case remains an open problem.

Herein, we depart from both the specific class of exponential forward performance processes and the SPDE approach for general forward performance processes without random endowment. We take an entirely different approach by working directly with the optimal policies and the corresponding optimal state price density processes via the solutions of two FBSDEs, one referred to as the *primal FBSDE (3.5)* and the other as the *dual FBSDE (3.14)*. These two FBSDEs form a convex dual relationship akin to the primal and the dual problems of the forward performance process with random endowment. We demonstrate that both FBSDEs offer necessary and sufficient conditions for the forward performance process with random endowment and its convex dual.

We focus on introducing the new methodology, establish necessary and sufficient optimality conditions via the new FBSDEs and explore equivalences between the primal and dual problems. We do not present general results on existence and uniqueness of their solutions; such questions are being currently investigated by the authors in the companion paper [29] and require a rather involved analysis and suitable assumptions on the forward volatility process, the families of viable claims, etc. which are beyond the scope of this paper. On the other hand, in this work, we present two representative examples, one for complete markets and general forward performance criteria, and another for an incomplete market (with a single stochastic factor) and exponential forward performance criteria.

The derivation of the primal and dual FBSDEs draws inspiration from Horst et al. [21], who considered the FBSDE characterization of utility maximization with random endowment but only from the primal perspective. However, our work differs from [21] as the static utility functions therein are being replaced by general forward performance processes. Their stochastic nature combined with their inherent martingale optimality leads to simpler and more interpretable terms in the obtained FBSDEs. Furthermore, we explore the convex dual of the primal FBSDE, presenting a novel approach to studying the convex duality in the forward setting. The results herein also provide a new perspective for forward performance processes in the absence of random endowment, yielding the self-generation property, first studied in [40] for the exponential case.

For the reader's convenience, we next provide a road map of the FBSDE approach and the underlying equivalences and dual relationships. For completeness, we also include information about the formal derivation of the corresponding primal and dual backward SPDEs discussed in Section 6.



Glossary of notation.

P	random endowment
u^P	value function of the primal problem
$\tilde{u}^P$	value function of the dual problem
$\pi^{*,P}$	optimal control of the primal problem
$q^{*,P}$	optimal control of the dual problem

Figure 1: The FBSDE approach road map of main results.

Building on the results on forward performance processes with random endowment, we introduce the novel concept, the *forward optimized certainty equivalent (forward OCE)*. Its static counterpart, introduced by Ben-Tal and Teboulle in [5] and [6], is a decision-making criterion based on expected utility theory and represents the outcome of an optimal fund allocation where an investor can choose to allocate a portion of the money from the random endowment to spend. However, due to the static nature of the utility function, extending the existing OCE notion to a dynamic setting while maintaining time consistency poses conceptual challenges. Recent studies by Backhoff-Veraguas et al. [7] and [8] began with the convex dual representation of OCE and generalized it to a dynamic version by introducing an additional variable to ensure time consistency. While [7] and [8] generalize the static case of [5] and [6], the dynamic OCE therein is directly tied down to the pre-chosen horizon T even though they are dynamically time-consistent within $[0, T]$. This is a direct consequence of the fact that the underlying utility function is by nature tied down to T. Similar horizon dependence is also observed in the classical indifference prices.

One then poses the question if there is a way to construct an OCE-type valuation mechanism that is *horizon invariant*, or *maturity independent*, since we typically align the horizon with the longest maturity. We prefer to use the terminology maturity independent to align it with the studies in [41] and [13].

We stress that we do not expect that the OCE of a given claim should not depend on the maturity of it, an obviously wrong result. Rather we seek a valuation mechanism, like the classical conditional expectation or the exponential forward indifference pricing, that does not depend *per se* on a specific horizon or maturity. This is what we develop herein within the OCE framework.

Next, we provide the key idea that helps us build the forward OCE notion. To this end, we present a novel perspective on the static OCE by interpreting it as the value function of the convex dual for a utility maximization problem with the random endowment within an auxiliary financial market orthogonal to the random endowment. We, then, replace the static utility function with the forward performance process within this auxiliary financial market and, in turn, define a forward OCE as the convex dual of the forward performance process with the random endowment. This forward OCE represents the result of an optimal dynamic allocation of funds, where an investor faces the choice of saving a portion of their current wealth alongside the random endowment to maximize the forward performance process. The auxiliary variable introduced in [7] and [8] has a natural interpretation as a stochastic deflator, converting nominal values into real values. The forward OCE mirrors the optimal balance between maximizing the forward performance process from the saving alongside the random endowment and reducing spending due to this saving choice.

Unlike the framework proposed in [7] and [8], the newly defined forward OCE is both time consistent and maturity independent. Furthermore, in the forward OCE framework, we introduce an auxiliary financial market where the underlying asset can be utilized to partially hedge the risk associated with the random endowment. In contrast, in the dynamic OCE framework presented in [7] and [8], this auxiliary market is assumed to be orthogonal to the random endowment, resulting in null hedging effects. Finally, we apply the new definition and the related results to the exponential case and we demonstrate that the forward OCE aligns with the negative of the forward entropic risk measure proposed in [13].

The paper is organized as follows. In section 2, we introduce the market and recall the notion of forward performance criteria and their convex dual. In section 3, we present the primal and dual problems in the presence of random endowment, and the main results on the new FBSDEs, while in section 4, we introduce the notion of forward OCE. In section 5, we discuss the family of exponential forward performance criteria, construct the exponential forward OCE and explore its connection with forward entropic risk measures. In section 6, we discuss the backward SPDE approach and compare it with the FBSDE one proposed herein. Finally, section 7 concludes. All proofs are deferred to the appendix.

2 Background results

We start with the description of the incomplete market and review fundamental concepts and results in forward performance processes. Specifically, we review their definition and the (ill-posed) SPDE that governs their evolution. We also recall their convex dual and derive the corresponding SPDE.

2.1 The incomplete market

Let $W = (W^1, W^2)$ be a two-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with the Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The market consists of one riskless asset (taken to be numeraire) with zero interest rate and one stock whose price solves

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t^1, \quad t \geq 0,\ S_0 = S > 0.$$

The coefficients $\mu \in \mathbb{R}$ and $\sigma > 0$ are $\mathbb{F}$-progressively measurable processes and the market price of risk

$$\theta := \frac{\mu}{\sigma}$$

is assumed to be uniformly bounded. The market is considered incomplete because the dimension of noise is higher than the number of stocks. Models with such incompleteness can be readily extended to multidimensional cases (see [21] and [22]), but for simplicity, we keep a low dimensionality.

In this market, let $\tilde{\pi}$ be the amount invested in the stock and $\pi := \tilde{\pi}\sigma$. Then, the self-financing condition yields the wealth SDE,

$$dX_t^\pi = \pi_t \left(\theta_t dt + dW_t^1\right), \quad t \geq 0,\ X_0^\pi = x \in \mathbb{R}. \tag{2.1}$$

For $0 \leq t \leq T < \infty$, we denote by $\mathcal{A}_{[t,T]}$ the set of admissible trading strategies in horizon $[t, T]$, defined as

$$\mathcal{A}_{[t,T]} := \left\{\pi \in \mathbb{L}^2_{BMO}\left[t, T\right], \pi_s \in \mathbb{R} \text{ for } t \leq s \leq T\right\},$$

where

$$\mathbb{L}^2_{BMO}\left[t, T\right] := \Bigg\{ (\pi_s)_{s\in[t,T]} : \pi \text{ is } \mathbb{F}\text{-progressively measurable and } \mathbb{E}\left[\int_\tau^T |\pi_u|^2\, du \middle| \mathcal{F}_\tau\right] \leq C,\ \mathbb{P}\text{-a.s.}, \\ \text{for some constant } C \text{ and all } \mathbb{F}\text{-stopping times } \tau \in [t, T] \Bigg\},$$

and denote by $\mathcal{A} := \cup_{t\geq 0}\mathcal{A}_{[0,t]}$ the set of admissible trading strategies for all $t \geq 0$.

The market model admits multiple state price densities (or equivalent martingale measures), which we parameterize by q on $[t, T]$ for $0 \leq t \leq T < \infty$, namely,

$$Z_s^{t,q} = \exp\left(-\int_t^s \left(\theta_u dW_u^1 + q_u dW_u^2\right) - \frac{1}{2}\int_t^s \left(|\theta_u|^2 + |q_u|^2\right) du\right), \quad t \leq s \leq T,$$

where $q \in \mathcal{Q}_{[t,T]}$ with

$$\mathcal{Q}_{[t,T]} := \left\{q \in \mathbb{L}^2_{BMO}\left[t, T\right], q_s \in \mathbb{R} \text{ for } t \leq s \leq T\right\}.$$

Therefore, for $q \in \mathcal{Q}_{[t,T]}$, the state price density process satisfies the SDE

$$dZ_s^{t,q} = -Z_s^{t,q}\left(\theta_s dW_s^1 + q_s dW_s^2\right), \quad t \leq s \leq T, \tag{2.2}$$

with initial condition $Z_t^{t,q} = 1$; moreover, $Z_s^{t,q}$ is a true martingale since θ is uniformly bounded and $q \in \mathcal{Q}_{[t,T]}$ (see [27]). Furthermore, we also have the following martingale property.

Lemma 2.1 *Let $T > 0$. Then for any $0 \leq t \leq T$, $\pi \in \mathcal{A}_{[t,T]}$ and $q \in \mathcal{Q}_{[t,T]}$, the process*

$$M_s^t := \int_t^s \pi_u \left(\theta_u du + dW_u^1 \right) \cdot Z_s^{t,q}$$

is a true martingale on $[t, T]$.

2.2 Forward performance processes and their convex dual representation

We recall the definition of forward performance processes and their SPDE representation introduced by Musiela and Zariphopoulou in [32, 33] and established in [36], respectively. We also recall their convex dual representation introduced by El Karoui and Mrad in [18].

For $t \geq 0$, let $L^0(\mathcal{F}_t)$ denote the space of $\mathcal{F}_t$-measurable random variables, $L^p(\mathcal{F}_t)$, $p \geq 1$, the space of L^p-integrable $\mathcal{F}_t$-measurable random variables, and $L^\infty(\mathcal{F}_t)$ the space of bounded $\mathcal{F}_t$-measurable random variables.

Definition 2.2 *A process $U(t,x)$, $(t,x) \in [0,\infty) \times \mathbb{R}$, is called a forward performance process if*

(i) for each $x \in \mathbb{R}$, $U(t,x)$ is $\mathbb{F}$-progressively measurable;

(ii) for each $t \geq 0$, $x \longmapsto U(t,x)$ is strictly increasing and strictly concave;

(iii) for any $0 \leq t \leq T < \infty$ and any $\xi \in L^0(\mathcal{F}_t)$,

$$U(t,\xi) = \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_u \left(\theta_u du + dW_u^1 \right) \right) \middle| \mathcal{F}_t \right]. \tag{2.3}$$

We will frequently refer to (2.3) as the self-generation property (see [40]). In [36], the forward performance process U was shown to be associated to the forward performance SPDE

$$dU(t,x) = \beta(t,x)\, dt + \alpha^\top(t,x)\, dW_t, \quad t \geq 0 \text{ and } x \in \mathbb{R}, \tag{2.4}$$

with the drift β given by

$$\beta(t,x) = \frac{1}{2} \frac{\left| U_x(t,x)\, \theta_t + \alpha_x^1(t,x) \right|^2}{U_{xx}(t,x)},$$

and the volatility $\alpha = (\alpha^1, \alpha^2)$ being an $\mathbb{R}^2$-valued $\mathbb{F}$-progressively measurable process. We recall that, contrary to the classical case, the volatility coefficient serves as a model input.

The optimal control process is, in turn, expressed in terms of the solution to the above SPDE. Specifically, if there exists a strong solution to

$$dX_s = -\frac{U_x(s,X_s)\, \theta_s + \alpha_x^1(s,X_s)}{U_{xx}(s,X_s)} \left(\theta_s ds + dW_s^1 \right), \quad s \geq 0, \tag{2.5}$$

and the feedback control

$$\pi_s^* := -\frac{U_x(s,X_s)\, \theta_s + \alpha_x^1(s,X_s)}{U_{xx}(s,X_s)}, \quad s \geq 0, \tag{2.6}$$

is in the admissible set $\mathcal{A}$, then π^* is optimal.

Assumption 2.3 *The forward performance process $U(t,x)$ satisfies the following conditions:*

(i) for each $t \geq 0$, $U(t,x) \in C^3(\mathbb{R})$, $x \in \mathbb{R}$;

(ii) $U(t,x)$ satisfies SPDE (2.4) for some volatility process $\alpha(t,x)$ with $\alpha(t,x) \in C^2(\mathbb{R})$ for each $t \geq 0$;

(iii) $U(t,x)$ is regular enough such that differential rule

$$dU_x\left(t,x\right) = \beta_x\left(t,x\right)dt + \alpha_x^{\top}\left(t,x\right)dW_t, \quad t \geq 0 \text{ and } x \in \mathbb{R}, \tag{2.7}$$

holds, where

$$\begin{aligned}\beta_x\left(t,x\right) = &\left(U_x\left(t,x\right)\theta_t + \alpha_x^1\left(t,x\right)\right)\theta_t + \frac{\left(U_x\left(t,x\right)\theta_t + \alpha_x^1\left(t,x\right)\right)\alpha_{xx}^1\left(t,x\right)}{U_{xx}\left(t,x\right)} \\ &- \frac{1}{2}\frac{\left|U_x\left(t,x\right)\theta_t + \alpha_x^1\left(t,x\right)\right|^2 U_{xxx}\left(t,x\right)}{\left|U_{xx}\left(t,x\right)\right|^2},\end{aligned} \tag{2.8}$$

and, furthermore, the Itô-Ventzell formula can be applied to both $U(t,x)$ and $U_x(t,x)$;

(iv) there exist positive constants C_l, C_u and C_α such that

$$C_l \leq -\frac{U_x\left(t,x\right)}{U_{xx}\left(t,x\right)} \leq C_u \quad \text{and} \quad \left|\frac{\alpha_x^i\left(t,x\right)}{U_{xx}\left(t,x\right)}\right| \leq C_\alpha, \quad i = 1,2.$$

Note that C_u provides an upper bound for the Arrow-Pratt measures of risk tolerance $-U_x/U_{xx}$ and $1/C_l$ provides an upper bound for the Arrow-Pratt measure of risk aversion $-U_{xx}/U_x$.

Next, we recall the convex dual of U, introduced in [18]. For a given forward performance process $U(t,x)$, its convex conjugate is defined as the Fenchel-Legendre transform of $-U(t,-x)$, namely,

$$\tilde{U}\left(t,z\right) := \sup_{x\in\mathbb{R}}\left(U\left(t,x\right) - xz\right), \quad t \geq 0 \text{ and } z > 0. \tag{2.9}$$

One may readily verify that $\tilde{U}(t,z)$ satisfies:

(i) for each $t \geq 0$, $\tilde{U}(t,z) \in C^3(\mathbb{R}^+)$ and is strictly convex in z;

(ii) the inverse of the marginal utility U_x is the negative of the marginal of the conjugate utility $\tilde{U}_z$,

$$U_x\left(t, -\tilde{U}_z\left(t,z\right)\right) = z \quad \text{and} \quad -\tilde{U}_z\left(t, U_x\left(t,x\right)\right) = x. \tag{2.10}$$

(iii) for $t \geq 0$ and $x \in \mathbb{R}$, the bidual relation

$$U\left(t,x\right) = \inf_{z>0}\left(\tilde{U}\left(t,z\right) + xz\right) \tag{2.11}$$

holds.

The following relations will be useful in the upcoming analysis. For $t \geq 0$ and $x \in \mathbb{R}$, we have

$$U\left(t,x\right) = \tilde{U}\left(t, U_x\left(t,x\right)\right) + xU_x\left(t,x\right), \tag{2.12}$$

$$\tilde{U}_{zz}\left(t, U_x\left(t,x\right)\right) = -\frac{1}{U_{xx}\left(t,x\right)}, \tag{2.13}$$

and

$$\tilde{U}_{zzz}\left(t, U_x\left(t, x\right)\right) = \frac{U_{xxx}\left(t, x\right)}{\left(U_{xx}\left(t, x\right)\right)^3}. \tag{2.14}$$

Furthermore, for $t \geq 0$ and $z > 0$,

$$\tilde{U}\left(t, z\right) = U\left(t, -\tilde{U}_z\left(t, z\right)\right) + z\tilde{U}_z\left(t, z\right), \tag{2.15}$$

$$U_{xx}\left(t, -\tilde{U}_z\left(t, z\right)\right) = -\frac{1}{\tilde{U}_{zz}\left(t, z\right)}, \tag{2.16}$$

and

$$U_{xxx}\left(t, -\tilde{U}_z\left(t, z\right)\right) = -\frac{\tilde{U}_{zzz}\left(t, z\right)}{\left(\tilde{U}_{zz}\left(t, z\right)\right)^3}. \tag{2.17}$$

Finally, we derive the dynamics of the marginal of the conjugate utility $\tilde{U}_z$, which is a key quantity in establishing the FBSDE representation in Section 3.1. To this end, assume that $\tilde{U}_z$ admits the Itô decomposition

$$d\tilde{U}_z\left(t, z\right) = \tilde{\beta}_z\left(t, z\right) dt + \tilde{\alpha}_z^\top\left(t, z\right) dW_t. \tag{2.18}$$

By (2.10), we know that $dU_x(t, -\tilde{U}_z(t, z)) = 0$. Applying the Itô-Ventzell formula to the processes $U_x(t, x)$ (cf. (2.7)) and $-\tilde{U}_z(t, z)$ yields

$$\begin{aligned}
&\beta_x\left(t, -\tilde{U}_z\left(t, z\right)\right) dt + \alpha_x^\top\left(t, -\tilde{U}_z\left(t, z\right)\right) dW_t - U_{xx}\left(t, -\tilde{U}_z\left(t, z\right)\right)\left(\tilde{\beta}_z\left(t, z\right) dt + \tilde{\alpha}_z^\top\left(t, z\right) dW_t\right)\\
&+\frac{1}{2}U_{xxx}\left(t, -\tilde{U}_z\left(t, z\right)\right)\left|\tilde{\alpha}_z\left(t, z\right)\right|^2 dt - \alpha_{xx}^\top\left(t, -\tilde{U}_z\left(t, z\right)\right)\tilde{\alpha}_z\left(t, z\right) dt = 0,
\end{aligned}$$

which, in view of (2.18), implies that

$$\tilde{\beta}_z\left(t, z\right) = \frac{\beta_x\left(t, -\tilde{U}_z\left(t, z\right)\right) + \frac{1}{2}U_{xxx}\left(t, -\tilde{U}_z\left(t, z\right)\right)\left|\tilde{\alpha}_z\left(t, z\right)\right|^2 - \alpha_{xx}^\top\left(t, -\tilde{U}_z\left(t, z\right)\right)\tilde{\alpha}_z\left(t, z\right)}{U_{xx}\left(t, -\tilde{U}_z\left(t, z\right)\right)} \tag{2.19}$$

and

$$\tilde{\alpha}_z\left(t, z\right) = \frac{\alpha_x\left(t, -\tilde{U}_z\left(t, z\right)\right)}{U_{xx}\left(t, -\tilde{U}_z\left(t, z\right)\right)}. \tag{2.20}$$

On the other hand, we have by (2.16) that

$$\tilde{\alpha}_z\left(t, z\right) = -\tilde{U}_{zz}\left(t, z\right)\alpha_x\left(t, -\tilde{U}_z\left(t, z\right)\right), \tag{2.21}$$

and, thus,

$$\tilde{\alpha}_{zz}\left(t, z\right) = -\tilde{U}_{zzz}\left(t, z\right)\alpha_x\left(t, -\tilde{U}_z\left(t, z\right)\right) + \left|\tilde{U}_{zz}\left(t, z\right)\right|^2\alpha_{xx}\left(t, -\tilde{U}_z\left(t, z\right)\right). \tag{2.22}$$

Therefore,

$$\alpha_x\left(t, -\tilde{U}_z\left(t, z\right)\right) = -\frac{\tilde{\alpha}_z\left(t, z\right)}{\tilde{U}_{zz}\left(t, z\right)}, \tag{2.23}$$

$$\alpha_{xx}\left(t, -\tilde{U}_z\left(t, z\right)\right) = \frac{\tilde{\alpha}_{zz}\left(t, z\right)}{\left|\tilde{U}_{zz}\left(t, z\right)\right|^2} - \frac{\tilde{U}_{zzz}\left(t, z\right)\tilde{\alpha}_z\left(t, z\right)}{\left(\tilde{U}_{zz}\left(t, z\right)\right)^3}. \tag{2.24}$$

Substituting (2.8), (2.16), (2.17), (2.23) and (2.24) into (2.19) yields that

$$\tilde{\beta}_z(t,z) = -\tilde{U}_{zz}(t,z)\,z\,|\theta_t|^2 + \tilde{\alpha}^1_{zz}(t,z)\,z\theta_t + \tilde{\alpha}^1_z(t,z)\,\theta_t - \frac{1}{2}\tilde{U}_{zzz}(t,z)\,|z\theta_t|^2 - \frac{1}{2}\frac{\tilde{U}_{zzz}(t,z)}{\left|\tilde{U}_{zz}(t,z)\right|^2}\left|\tilde{\alpha}^2_z(t,z)\right|^2 + \frac{\tilde{\alpha}^2_z(t,z)\,\tilde{\alpha}^2_{zz}(t,z)}{\tilde{U}_{zz}(t,z)}. \tag{2.25}$$

We conclude by mentioning that it is not clear whether the convex conjugate $\tilde{U}$ satisfies a property similar to (2.3). This was studied in [40]. We will re-establish this property using the FBSDE approach in Remark 3.7.

3 Forward performance process with random endowment: an FBSDE approach

We start by choosing an arbitrary time $T>0$ at which we introduce a random endowment, denoted by $P \in L^\infty(\mathcal{F}_T)$. We then consider the *primal* problem

$$u^P(t,\xi;T) = \operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T,\xi+\int_t^T \pi_u\left(\theta_u du + dW^1_u\right)+P\right)\middle|\mathcal{F}_t\right], \quad 0\le t\le T \text{ and } \xi\in L^0(\mathcal{F}_t), \tag{3.1}$$

where $U(t,x)$ is a forward performance process governed by SPDE (2.4) with volatility process $\alpha = (\alpha^1,\alpha^2)$. The associated *dual* problem is

$$\tilde{u}^P(t,\eta;T) = \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T,\eta Z^{t,q}_T\right)+\eta Z^{t,q}_T P\middle|\mathcal{F}_t\right], \quad 0\le t\le T \text{ and } \eta\in L^{0,+}(\mathcal{F}_t), \tag{3.2}$$

where $\tilde{U}$ is the convex conjugate of U given in (2.9), and $L^{0,+}(\mathcal{F}_t)$ denotes the space of the positive $\mathcal{F}_t$-measurable random variables.

A conventional approach to solving the above primal and dual problems is to first characterize their value functions, which are expected to solve related backward SPDEs due to the non-Markovian nature of the model. The corresponding optimal controls for (3.1) and (3.2) are then expressed in terms of the solutions to these backward SPDEs. However, as we demonstrate in Section 6, the derivation of the two backward SPDEs for (3.1) and (3.2) is formal, and their solvability is far from clear.

In this section, we take a different approach by directly characterizing the optimal control processes for both the primal and dual problems using an FBSDE approach. By substituting the optimal controls into (3.1) and (3.2), we can in turn obtain the value functions. Our main contributions are the derivation of two FBSDEs: the primal FBSDE (3.5) and the dual FBSDE (3.14), both of which provide a means to characterize the solutions for both the primal and dual problems. These two FBSDEs are closely connected, forming a convex dual relationship, similar to what (3.1) and (3.2) represent.

3.1 Optimal policy characterization using the primal FBSDE

The first main result is the derivation of both necessary and sufficient conditions for the optimal control process for the primal problem (3.1) through a primal FBSDE.

Theorem 3.1 *Let $T>0$, $P\in L^{\infty}(\mathcal{F}_T)$, and $\pi^{*,P}\in\mathcal{A}_{[t,T]}$ be an optimal control process of the primal problem (3.1) with $\xi\in L^0(\mathcal{F}_t)$, $0\le t\le T$. Let X^* be the solution of (2.1) with $\pi^{*,P}$ being used,*

$$X_s^* := \xi + \int_t^s \pi_u^{*,P}\left(\theta_u du + dW_u^1\right), \quad t\le s\le T. \tag{3.3}$$

Assume that i) $\mathbb{E}[|U_x(T,X_T^*+P)|^p]<\infty$, for some $p>1$, and ii) for any $\pi\in\mathcal{A}_{[t,T]}$,

$$\frac{1}{\varepsilon}\left|U\left(T,\xi+\int_t^T\left(\pi_s^{*,P}+\varepsilon\pi_s\right)\left(\theta_s ds+dW_s^1\right)+P\right)-U\left(T,X_T^*+P\right)\right|$$

is uniformly integrable in $\varepsilon\in(0,1)$. Then, there exists a process $(Y_s)_{s\in[t,T]}$ with $Y_T=P$ such that

$$\pi_s^{*,P} = -\frac{U_x\left(s,X_s^*+Y_s\right)\theta_s+\alpha_x^1\left(s,X_s^*+Y_s\right)}{U_{xx}\left(s,X_s^*+Y_s\right)} - Z_s^1, \quad t\le s\le T,$$

where $Z_s^1 := \frac{d\langle Y,W^1\rangle_s}{ds}$. More precisely, $\pi^{*,P}$ is given by

$$\pi_s^{*,P} = -\frac{U_x\left(s,X_s+Y_s\right)\theta_s+\alpha_x^1\left(s,X_s+Y_s\right)}{U_{xx}\left(s,X_s+Y_s\right)} - Z_s^1, \quad t\le s\le T, \tag{3.4}$$

where (X,Y,Z) satisfies on $[t,T]$ the FBSDE

$$\begin{cases} X_s = \xi - \displaystyle\int_t^s\left(\frac{U_x\left(u,X_u+Y_u\right)\theta_u+\alpha_x^1\left(u,X_u+Y_u\right)}{U_{xx}\left(u,X_u+Y_u\right)}+Z_u^1\right)\left(\theta_u du+dW_u^1\right), \\ Y_s = P + \displaystyle\int_s^T\left(-Z_u^1\theta_u+\frac{1}{2}\frac{U_{xxx}\left(u,X_u+Y_u\right)}{U_{xx}\left(u,X_u+Y_u\right)}\left|Z_u^2\right|^2+\frac{\alpha_{xx}^2\left(u,X_u+Y_u\right)}{U_{xx}\left(u,X_u+Y_u\right)}Z_u^2\right)du \\ \qquad - \displaystyle\int_s^T Z_u^{\top}dW_u, \end{cases} \tag{3.5}$$

with initial-terminal condition (ξ,P), and $X=X^*$ on $[t,T]$.

Remark 3.2 *Using the above FBSDE and the dynamics of U_x in (2.7), we obtain*

$$\begin{aligned} &dU_x\left(s,X_s+Y_s\right) \\ &= -U_x\left(s,X_s+Y_s\right)\left(\theta_s dW_s^1-\frac{U_{xx}\left(s,X_s+Y_s\right)Z_s^2+\alpha_x^2\left(s,X_s+Y_s\right)}{U_x\left(s,X_s+Y_s\right)}dW_s^2\right). \end{aligned}$$

In turn, introducing $q^{*,P}$, defined as the feedback control process

$$q_s^{*,P} := -\frac{U_{xx}\left(s,X_s+Y_s\right)Z_s^2+\alpha_x^2\left(s,X_s+Y_s\right)}{U_x\left(s,X_s+Y_s\right)}, \quad t\le s\le T, \tag{3.6}$$

we obtain

$$dU_x\left(s,X_s+Y_s\right) = -U_x\left(s,X_s+Y_s\right)\left(\theta_s dW_s^1+q_s^{*,P}dW_s^2\right).$$

Comparing with the SDE for the state price density in (2.2), we have thus identified a candidate process for the optimal state price density, namely,

$$Z_s^{t,q^{*,P}} = \frac{U_x\left(s,X_s+Y_s\right)}{U_x\left(t,\xi+Y_t\right)}, \quad t\le s\le T.$$

If the solution component Z^2 of FBSDE (3.5) belongs to $\mathbb{L}^2_{BMO}[t,T]$, then, using Assumption 2.3 (iv), we deduce that $q^{*,P}\in\mathcal{Q}_{[t,T]}$ and, thus, $Z^{t,q^{*,P}}$ is a true martingale. This point is used next to verify that FBSDE (3.5) also serves as a sufficient condition for both the primal problem (3.1) in Theorem 3.3 and the dual problem (3.2) in Theorem 3.5.

Theorem 3.3 *Let $T>0$, $P\in L^{\infty}(\mathcal{F}_T)$ and $\xi\in L^0(\mathcal{F}_t)$, $0\le t\le T$. Let, also, (X,Y,Z) be a solution to FBSDE (3.5) on $[t,T]$ with initial-terminal condition (ξ,P) satisfying $Z^i\in\mathbb{L}^2_{BMO}[t,T]$, $i=1,2$. Then, the control process*

$$\pi^{*,P}_s := -\frac{U_x\left(s,X_s+Y_s\right)\theta_s+\alpha^1_x\left(s,X_s+Y_s\right)}{U_{xx}\left(s,X_s+Y_s\right)}-Z^1_s,\quad t\le s\le T, \tag{3.7}$$

is optimal for the primal problem (3.1), namely, $\pi^{*,P}\in\mathcal{A}_{[t,T]}$ and

$$\begin{aligned} u^P\left(t,\xi;T\right) &= \mathbb{E}\left[U\left(T,\xi+\int_t^T\pi^{*,P}_u\left(\theta_u du+dW^1_u\right)+P\right)\middle|\mathcal{F}_t\right]\\ &=\mathbb{E}\left[\left.U\left(T,X_T+P\right)\right|\mathcal{F}_t\right],\end{aligned}$$

with $X=X^*$.

Remark 3.4 *When $P=0$, we obtain*

$$\begin{aligned} u^0\left(t,\xi;T\right) &= \operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T]}}\mathbb{E}\left[U\left(T,\xi+\int_t^T\pi_u\left(\theta_u du+dW^1_u\right)\right)\middle|\mathcal{F}_t\right]\\ &= U\left(t,\xi\right), \end{aligned} \tag{3.8}$$

due to the self-generation condition. It then follows directly that the triplet $(X,Y,Z)=(X,0,0)$, with X satisfying

$$X_s=\xi-\int_t^s\frac{U_x\left(u,X_u\right)\theta_u+\alpha^1_x\left(u,X_u\right)}{U_{xx}\left(u,X_u\right)}\left(\theta_u du+dW^1_u\right),$$

solves FBSDE (3.5). Thus, Theorem 3.3 yields that the control process

$$\pi^*_s := -\frac{U_x\left(s,X_s\right)\theta_s+\alpha^1_x\left(s,X_s\right)}{U_{xx}\left(s,X_s\right)},\quad t\le s\le T,$$

is optimal and $X=X^{\pi^*}$, which aligns with (2.6).

3.2 Primal FBSDE and the dual problem

We demonstrate that the primal FBSDE (3.5) also provides a solution to the dual problem (3.2).

Theorem 3.5 *Let $T>0$, $P\in L^{\infty}(\mathcal{F}_T)$ and $\xi\in L^0(\mathcal{F}_t)$, $0\le t\le T$. Let, also, (X,Y,Z) be a solution to FBSDE (3.5) on $[t,T]$ with initial-terminal condition (ξ,P), satisfying $Z^i\in\mathbb{L}^2_{BMO}[t,T]$, $i=1,2$. Let*

$$\hat{\eta}:=U_x\left(t,\xi+Y_t\right)\in L^{0,+}(\mathcal{F}_t). \tag{3.9}$$

Then,

(i) the control process $q^{*,P}$ defined in (3.6) is optimal for the dual problem (3.2), namely, $q^{*,P}\in\mathcal{Q}_{[t,T]}$ and

$$\begin{aligned}\tilde{u}^P\left(t,\hat{\eta};T\right) &= \mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}Z^{t,q^{*,P}}_T\right)+\hat{\eta}Z^{t,q^{*,P}}_T P\middle|\mathcal{F}_t\right]\\ &=\mathbb{E}\left[\tilde{U}\left(T,U_x\left(T,X_T+P\right)\right)+U_x\left(T,X_T+P\right)P\middle|\mathcal{F}_t\right],\end{aligned}$$

(ii) the bidual relation

$$u^P(t,\xi;T) = \operatorname*{essinf}_{\eta\in L^{0,+}(\mathcal{F}_t)} \left(\tilde{u}^P(t,\eta;T) + \xi\eta\right) = \tilde{u}^P(t,\hat{\eta};T) + \xi\hat{\eta}$$

holds.

The converse side of the above Theorem 3.5 is when the initial state η of the density process is arbitrary, but the initial wealth $\hat{\xi}$ is given as a particular class of initial wealth states depending on η.

Corollary 3.6 *Let $T>0$, $P\in L^\infty(\mathcal{F}_T)$ and $\eta\in L^{0,+}(\mathcal{F}_t)$, $0\le t\le T$. Let, also, (X,Y,Z) be a solution to FBSDE (3.5) on $[t,T]$ with initial-terminal condition $(\hat{\xi},P)$, satisfying $Z^i\in\mathbb{L}^2_{BMO}[t,T]$, $i=1,2$ and*

$$\hat{\xi} = -\tilde{U}_z(t,\eta) - Y_t \in L^0(\mathcal{F}_t). \tag{3.10}$$

Then,

(i) the control process $q^{*,P}$ defined in (3.6) is an optimal control process for the dual problem (3.2), namely, $q^{*,P}\in\mathcal{Q}_{[t,T]}$ and

$$\begin{aligned}\tilde{u}^P(t,\eta;T) &= \mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q^{*,P}}\right) + \eta Z_T^{t,q^{*,P}} P \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\tilde{U}\left(T,U_x(T,X_T+P)\right) + U_x(T,X_T+P)P \,\middle|\, \mathcal{F}_t\right],\end{aligned}$$

(ii) the bidual relation

$$\tilde{u}^P(t,\eta;T) = \operatorname*{esssup}_{\xi\in L^0(\mathcal{F}_t)} \left(u^P(t,\xi;T) - \xi\eta\right) = u^P\left(t,\hat{\xi};T\right) - \hat{\xi}\eta$$

holds.

Remark 3.7 *We establish the self-generation property of $\tilde{U}$ using the above primal FBSDE characterization. When $P=0$, we know that $(X,Y,Z)=(X,0,0)$, with X satisfying*

$$X_s = \hat{\xi} - \int_t^s \frac{U_x(u,X_u)\theta_u + \alpha_x^1(u,X_u)}{U_{xx}(u,X_u)} \left(\theta_u du + dW_u^1\right),$$

solves FBSDE (3.5) on $[t,T]$ with initial-terminal condition $(\hat{\xi},0)$ with $\hat{\xi} = -\tilde{U}_z(t,\eta)$, $\eta\in L^{0,+}(\mathcal{F}_t)$ (see Remark 3.4). Thus, by Corollary 3.6, we have

$$q_s^* := -\frac{\alpha_x^2(s,X_s)}{U_x(s,X_s)}, \quad t\le s\le T. \tag{3.11}$$

This quantity, which is bounded by Assumption 2.3 (iv), is optimal for $\tilde{u}^0(t,\eta;T)$, i.e.,

$$\tilde{u}^0(t,\eta;T) = \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q}\right)\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q^*}\right)\,\middle|\,\mathcal{F}_t\right].$$

Moreover,

$$\tilde{u}^0(t,\eta;T) = \operatorname*{esssup}_{\xi\in L^0(\mathcal{F}_t)} \left(u^0(t,\xi;T) - \xi\eta\right) = u^0\left(t,\hat{\xi};T\right) - \hat{\xi}\eta.$$

Note that $u^0(t,\xi;T) = U(t,\xi)$ for any $\xi \in L^0(\mathcal{F}_t)$ according to Remark 3.4, which implies that

$$\begin{aligned}
\tilde{u}^0(t,\eta;T) &= \underset{\xi\in L^0(\mathcal{F}_t)}{\operatorname{esssup}} \left(u^0(t,\xi;T) - \xi\eta\right) \\
&= \underset{\xi\in L^0(\mathcal{F}_t)}{\operatorname{esssup}} \left(U(t,\xi) - \xi\eta\right) \\
&= \tilde{U}(t,\eta).
\end{aligned}$$

It then follows that

$$\tilde{U}(t,\eta) = \underset{q\in\mathcal{Q}_{[t,T]}}{\operatorname{essinf}} \mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q}\right)\middle|\mathcal{F}_t\right], \tag{3.12}$$

and, for q^* in (3.11),

$$\tilde{U}(t,\eta) = \mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q^*}\right)\middle|\mathcal{F}_t\right]. \tag{3.13}$$

Hence, we have re-established the self-generation property of $\tilde{U}$, first proven in [40].

3.3 Dual FBSDE and its relation to primal FBSDE

Having formulated FBSDE (3.5) for the primal problem, we revert to the dual problem and derive an analogous FBSDE, given in (3.14) below. With a slight abuse of terminology, we will be calling (3.14) the *dual FBSDE*.

3.3.1 Optimal state price density characterization using the dual FBSDE

We characterize the optimal control process of the dual problem (3.2) using the dual FBSDE (3.14). We begin by deriving a necessary condition for the optimal density process through this new FBSDE and subsequently, we demonstrate that it also yields a sufficient condition for optimality.

Theorem 3.8 *Let $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and $q^{*,P} \in \mathcal{Q}_{[t,T]}$ be an optimal control process for the dual problem (3.2) with $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \le t \le T$. Define $Z^* := Z^{t,q^{*,P}}$ and assume that i) $\mathbb{E}[|(\tilde{U}_z(T,\eta Z_T^*)+P)\eta Z_T^*|^p] < \infty$, for some $p > 1$, and ii) for any $q \in \mathcal{Q}_{[t,T]}$,*

$$\frac{1}{\varepsilon}\left|\tilde{U}\left(T,\eta Z_T^{t,q^{*,P}+\varepsilon q}\right) - \tilde{U}(T,\eta Z_T^*)\right|$$

is uniformly integrable in $\varepsilon \in (0,1)$. Then, there exists a process $(\tilde{Y}_s)_{s\in[t,T]}$ with $\tilde{Y}_T = P$ such that

$$q_s^{*,P} = \frac{\tilde{Z}_s^2 + \tilde{\alpha}_z^2(s,\eta Z_s^*)}{\eta Z_s^* \tilde{U}_{zz}(s,\eta Z_s^*)}, \quad t \le s \le T,$$

where $\tilde{Z}_s^2 := \frac{d\langle\tilde{Y},W^2\rangle_s}{ds}$. More precisely, $q^{*,P}$ is given by

$$q_s^{*,P} = \frac{\tilde{Z}_s^2 + \tilde{\alpha}_z^2(s,D_s)}{D_s \tilde{U}_{zz}(s,D_s)}, \quad t \le s \le T,$$

where $(D, \tilde{Y}, \tilde{Z})$ satisfies on $[t,T]$ the FBSDE

$$\begin{cases} D_s = \eta - \int_t^s \left(D_u \theta_u dW_u^1 + \dfrac{\tilde{Z}_u^2 + \tilde{\alpha}_z^2(u, D_u)}{\tilde{U}_{zz}(u, D_u)} dW_u^2 \right), \\ \tilde{Y}_s = P + \int_s^T \left(\dfrac{1}{2} \dfrac{\tilde{U}_{zzz}(u, D_u) \left| \tilde{Z}_u^2 \right|^2}{\left| \tilde{U}_{zz}(u, D_u) \right|^2} - \tilde{Z}_u^1 \theta_u - \tilde{Z}_u^2 \left(\dfrac{\tilde{\alpha}_{zz}^2(u, D_u)}{\tilde{U}_{zz}(u, D_u)} - \dfrac{\tilde{U}_{zzz}(u, D_u) \tilde{\alpha}_z^2(u, D_u)}{\left| \tilde{U}_{zz}(u, D_u) \right|^2} \right) \right) du \\ \qquad - \int_s^T \tilde{Z}_u^\top dW_u, \end{cases} \tag{3.14}$$

with initial-terminal condition (η, P), and $D = \eta Z^*$ on $[t,T]$.

Remark 3.9 *For $(D, \tilde{Y}, \tilde{Z})$ satisfying FBSDE (3.14), we use the dynamics of $\tilde{U}_z$ in (2.18) to obtain*

$$\begin{aligned} d\left(\tilde{U}_z(s, D_s) + \tilde{Y}_s \right) &= \left(\tilde{\alpha}_z^1(s, D_s) + \tilde{Z}_s^1 - \tilde{U}_{zz}(s, D_s) D_s \theta_s \right) \left(\theta_s ds + dW_s^1 \right) \\ &= -\pi_s^{*,P} \left(\theta_s ds + dW_s^1 \right), \end{aligned} \tag{3.15}$$

where

$$\pi_s^{*,P} := \tilde{U}_{zz}(s, D_s) D_s \theta_s - \tilde{\alpha}_z^1(s, D_s) - \tilde{Z}_s^1, \quad t \le s \le T. \tag{3.16}$$

Using (2.10), (2.16) and (2.20), we may rewrite $\pi^{*,P}$ as a feedback control of density D,

$$\pi_s^{*,P} = -\frac{\alpha_x^1\left(s, -\tilde{U}_z(s, D_s) \right)}{U_{xx}\left(s, -\tilde{U}_z(s, D_s) \right)} - \frac{U_x\left(s, -\tilde{U}_z(s, D_s) \right)}{U_{xx}\left(s, -\tilde{U}_z(s, D_s) \right)} \theta_s - \tilde{Z}_s^1, \quad t \le s \le T.$$

Hence, if the solution components $\tilde{Z}^i$, $i = 1, 2$, belong to $\mathbb{L}^2_{BMO}[t,T]$, then by Assumption 2.3 (iv), we have that $\pi^{*,P} \in \mathcal{A}_{[t,T]}$. Consequently, $\pi^{*,P}$ is a candidate optimal control process for the primal problem.

We next demonstrate that FBSDE (3.14) also serves as a sufficient condition for the optimality of the density process.

Theorem 3.10 *Let $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \le t \le T$. Let, also, $(D, \tilde{Y}, \tilde{Z})$ be a solution to FBSDE (3.14) on $[t,T]$ with initial-terminal condition (η, P), satisfying $\tilde{Z}^i \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1, 2$. Then, the control process*

$$q_s^{*,P} := \frac{\tilde{Z}_s^2 + \tilde{\alpha}_z^2(s, D_s)}{D_s \tilde{U}_{zz}(s, D_s)}, \quad t \le s \le T,$$

is optimal for the dual problem (3.2), namely, $q^{*,P} \in \mathcal{Q}_{[t,T]}$ and

$$\begin{aligned} \tilde{u}^P(t, \eta; T) &= \mathbb{E}\left[\tilde{U}\left(T, \eta Z_T^{t,q^{*,P}} \right) + \eta Z_T^{t,q^{*,P}} P \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}\left[\tilde{U}(T, D_T) + D_T P \middle| \mathcal{F}_t \right], \end{aligned}$$

with $D = \eta Z^*$.

3.3.2 Dual FBSDE and the primal problem

The dual FBSDE (3.14) can be also used to characterize the solution of the primal problem (3.1).

Theorem 3.11 *Let $T > 0, P \in L^\infty(\mathcal{F}_T)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \le t \le T$. Let, also, $(D, \tilde{Y}, \tilde{Z})$ be a solution to FBSDE (3.14) on $[t,T]$ with initial-terminal condition (η, P) satisfying $\tilde{Z}^i \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$. Define*

$$\hat{\xi} := -\tilde{U}_z(t, \eta) - \tilde{Y}_t \in L^0(\mathcal{F}_t). \tag{3.17}$$

Then,

(i) the control process $\pi^{*,P}$ defined by (3.16) is optimal for the primal problem (3.1), namely, $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ and

$$\begin{aligned} u^P\left(t, \hat{\xi}; T\right) &= \mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_u^{*,P}\left(\theta_u du + dW_u^1\right) + P\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, -\tilde{U}_z(T, D_T)\right) \middle| \mathcal{F}_t\right], \end{aligned}$$

(ii) the bidual relation

$$\tilde{u}^P(t, \eta; T) = \operatorname*{esssup}_{\xi \in L^0(\mathcal{F}_t)} \left(u^P(t, \xi; T) - \xi\eta\right) = u^P\left(t, \hat{\xi}; T\right) - \hat{\xi}\eta$$

holds.

The converse side of Theorem 3.11 is when the initial wealth ξ is arbitrary, but the initial state $\hat{\eta}$ is specified as a particular class of initial states for the density processes, depending on ξ. Its proof closely resembles that of Lemma 3.6 and is therefore omitted.

Corollary 3.12 *Let $T > 0, P \in L^\infty(\mathcal{F}_T)$ and $\xi \in L^0(\mathcal{F}_t)$, $0 \le t \le T$. Let, also, $(D, \tilde{Y}, \tilde{Z})$ be a solution to FBSDE (3.14) on $[t,T]$ with initial-terminal condition $(\hat{\eta}, P)$ satisfying $\tilde{Z}^i \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$ and*

$$\hat{\eta} = U_x\left(t, \xi + \tilde{Y}_t\right) \in L^{0,+}(\mathcal{F}_t).$$

Then,

(i) the control process $\pi^{*,P}$ defined in (3.16) is optimal for the primal problem (3.1), namely, $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ and

$$\begin{aligned} u^P(t, \xi; T) &= \mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_u^{*,P}\left(\theta_u du + dW_u^1\right) + P\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, -\tilde{U}_z(T, D_T)\right) \middle| \mathcal{F}_t\right], \end{aligned}$$

(ii) the bidual relation

$$u^P(t, \xi; T) = \operatorname*{essinf}_{\eta \in L^{0,+}(\mathcal{F}_t)} \left(\tilde{u}^P(t, \eta; T) + \xi\eta\right) = \tilde{u}^P(t, \hat{\eta}; T) + \xi\hat{\eta}$$

holds.

3.3.3 Relations between the primal and dual FBSDEs

The primal FBSDE (3.5) and the dual FBSDE (3.14) form a convex dual pair. Their relationship is a direct analogue to the convex duality between the primal problem (3.1) and the dual problem (3.2).

Proposition 3.13 *Let $T > 0$, $P \in L^{\infty}(\mathcal{F}_T)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \leq t \leq T$. Let, also, $(D, \tilde{Y}, \tilde{Z})$ be a solution to FBSDE (3.14) on $[t, T]$ with initial-terminal condition (η, P), satisfying $\tilde{Z}^i \in \mathbb{L}^2_{BMO}[t, T]$, $i = 1, 2$. Assume that the conditions in Theorem 3.1 hold, with $-\tilde{U}_z(T, D_T)$ in lieu of $X_T^* + P$, namely, i) $\mathbb{E}[|U_x(T, -\tilde{U}_z(T, D_T))|^p] = E[|D_T|^p] < \infty$, for some $p > 1$, and ii) for any $\pi \in \mathcal{A}_{[t,T]}$,*

$$\frac{1}{\varepsilon}\left|U\left(T, -\tilde{U}_z\left(T, D_T\right) + \varepsilon \int_t^T \pi_u\left(\theta_u du + dW_u^1\right)\right) - U\left(T, -\tilde{U}_z\left(T, D_T\right)\right)\right|$$

is uniformly integrable in $\varepsilon \in (0, 1)$. Then, for $\hat{\xi}$ defined in (3.17), the triplet (X, Y, Z) given, for $t \leq s \leq T$, by

$$\begin{aligned}
X_s &:= -\tilde{U}_z\left(s, D_s\right) - \tilde{Y}_s, \\
Y_s &:= \tilde{Y}_s, \\
Z_s &:= \tilde{Z}_s,
\end{aligned}$$

is a solution to FBSDE (3.5) on $[t, T]$ with initial-terminal condition $(\hat{\xi}, P)$.

Proposition 3.14 *Let $T > 0$, $P \in L^{\infty}(\mathcal{F}_T)$ and $\xi \in L^0(\mathcal{F}_t)$, $0 \leq t \leq T$. Let, also, (X, Y, Z) be a solution to FBSDE (3.5) on $[t, T]$ with initial-terminal condition (ξ, P), satisfying $Z^i \in \mathbb{L}^2_{BMO}[t, T]$, $i = 1, 2$. Assume that the conditions in Theorem 3.8 hold, with $U_x(T, X_T + P)$ in lieu of ηZ_T^*, namely, i) $\mathbb{E}[|X_T U_x(T, X_T + P)|^p] < \infty$, for some $p > 1$, and ii) for any $q \in \mathcal{Q}_{[t,T]}$,*

$$\begin{aligned}
\frac{1}{\varepsilon}\Bigg|\tilde{U}\left(T, U_x\left(T, X_T + P\right) \exp\left(-\int_t^T \varepsilon q_u dW_u^2 - \frac{1}{2}\int_t^T \left|\varepsilon q_u\right|^2 du - \int_t^T \varepsilon q_u q_u^{*,P} du\right)\right) \\
-\tilde{U}\left(T, U_x\left(T, X_T + P\right)\right)\Bigg|
\end{aligned}$$

is uniformly integrable in $\varepsilon \in (0, 1)$. Then, for $\hat{\eta}$ defined in (3.9), the triplet $(D, \tilde{Y}, \tilde{Z})$ given, for $t \leq s \leq T$, by

$$\begin{aligned}
D_s &:= U_x\left(s, X_s + Y_s\right), \\
\tilde{Y}_s &:= Y_s, \\
\tilde{Z}_s &:= Z_s,
\end{aligned}$$

is a solution to FBSDE (3.14) on $[t, T]$ with initial-terminal condition $(\hat{\eta}, P)$.

3.4 The case of complete market with random endowment

We discuss the complete market case, where both the primal FBSDE (3.5) and the dual FBSDE (3.14) admit closed form solution. More general cases are deferred, due to the length of their exposition, to the companion work [29]. To this end, assume that the Brownian motion W is one-dimensional and that $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is generated by W. Hence, the controlled state equation (2.1) becomes

$$dX_t^{\pi} = \pi_t\left(\theta_t dt + dW_t\right), \quad t \geq 0,\ X_0^{\pi} = x \in \mathbb{R}.$$

Due to market completeness, there exists a unique state price density process given by

$$\mathcal{E}\left(-\int \theta dW\right)_t = \exp\left(-\frac{1}{2}\int_0^t \left|\theta_u\right|^2 du - \int_0^t \theta_u dW_u\right), \quad t \geq 0,$$

where, we recall, θ is a bounded $\mathbb{F}$-progressively measurable process.

3.4.1 Primal FBSDE

Since $W^2 \equiv 0$, the solution component Z^2 in FBSDE (3.5) vanishes and (3.5) reduces to

$$\begin{cases} X_s = \xi - \displaystyle\int_t^s \left(\frac{U_x\left(u, X_u + Y_u\right)\theta_u + \alpha_x\left(u, X_u + Y_u\right)}{U_{xx}\left(u, X_u + Y_u\right)} + Z_u \right) \left(\theta_u du + dW_u\right), \\ Y_s = P - \displaystyle\int_s^T Z_u \theta_u du - \int_s^T Z_u dW_u. \end{cases} \tag{3.18}$$

The above BSDE admits a unique solution (Y, Z) with

$$Y_s = \mathbb{E}\left[\left. \frac{\mathcal{E}\left(-\int \theta dW\right)_T}{\mathcal{E}\left(-\int \theta dW\right)_s} P \right| \mathcal{F}_s \right] = \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_s\right], \quad t \le s \le T, \tag{3.19}$$

where $\mathbb{Q}^0$ is the unique equivalent martingale measure defined by

$$\left.\frac{d\mathbb{Q}^0}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(-\int \theta dW\right)_T. \tag{3.20}$$

We also have $Z \in \mathbb{L}^2_{BMO}[t, T]$ by the a priori estimates of the BSDE (see, for example, [39, Section 7.2]). Hence, if X is the solution to the SDE in (3.18) with Y in (3.19) and $Z_s = \frac{d\langle Y, W\rangle_s}{ds}$, then we obtain, by Theorem 3.3, that the control process

$$\pi_s^{*,P} := -\frac{U_x\left(s, X_s + Y_s\right)\theta_s + \alpha_x\left(s, X_s + Y_s\right)}{U_{xx}\left(s, X_s + Y_s\right)} - Z_s, \quad t \le s \le T,$$

is optimal for the primal problem (3.1). Furthermore, by Corollary 3.6, we have that $q^{*,P} \equiv 0$ is the optimal control process for the dual problem (3.2).

3.4.2 Dual FBSDE

Analogously, the solution component $\tilde{Z}^2$ in FBSDE (3.14) vanishes, and (3.14) reduces to

$$\begin{cases} D_s = \eta - \displaystyle\int_s^T D_u \theta_u dW_u, \\ \tilde{Y}_s = P - \displaystyle\int_s^T \tilde{Z}_u \theta_u du - \int_s^T \tilde{Z}_u dW_u. \end{cases}$$

It is then straightforward to check that there exists a unique solution given by

$$D_s = \eta \mathcal{E}\left(-\textstyle\int \theta dW\right)_s,$$
$$\tilde{Y}_s = Y_s = \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_s\right], \quad t \le s \le T,$$

and $\tilde{Z} = Z \in \mathbb{L}^2_{BMO}[t, T]$. Therefore, Theorem 3.10 yields that $q^{*,P} \equiv 0$ is the optimal control process for the dual problem (3.2). Furthermore, Corollary 3.12 yields that the control process

$$\pi_s^{*,P} := \tilde{U}_{zz}\left(s, \hat{\eta}\mathcal{E}\left(-\textstyle\int \theta dW\right)_s\right) \hat{\eta}\mathcal{E}\left(-\textstyle\int \theta dW\right)_s \theta_s - \tilde{\alpha}_z\left(s, \hat{\eta}\mathcal{E}\left(-\textstyle\int \theta dW\right)_s\right) - \tilde{Z}_s, \quad t \le s \le T,$$

is optimal for the primal problem (3.1) with $\hat{\eta} := U_x(t, \xi + \mathbb{E}^{\mathbb{Q}^0}[P|\mathcal{F}_t])$.

Finally, we deduce, using Proposition 3.13, that X, given by

$$X_s = -\tilde{U}_z\left(s, U_x\left(t, \xi + \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_t\right]\right)\mathcal{E}\left(-\textstyle\int \theta dW\right)_s\right) - \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_s\right], \quad t \le s \le T,$$

is the solution to the SDE in (3.18) with initial condition $X_t = \xi$.

3.5 Maturity independence of the value functions in forward performance process with random endowment

We now set the ground for the upcoming notion of forward OCE by pointing out the fundamental maturity-independence property of the value functions $u^P(t,\xi;T)$ and $\tilde{u}^P(t,\eta;T)$. Recall the primal problem (3.1), rewritten below for convenience, in a slightly different form and with some abuse of notation, $u^P(t,\xi;T) = u^{P_T}(t,\xi;T)$, where, for $0 < T \leq T'$,

$$u^{P_T}(t,\xi;T') = \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T']}} \mathbb{E}\left[U\left(T', \xi + \int_t^{T'} \pi_u \left(\theta_u du + dW_u^1\right) + P_T\right) \middle| \mathcal{F}_t \right]. \tag{3.21}$$

We essentially parameterize the value function by the time T that the random endowment P_T arrives as well as by T' at which we set the forward performance criterion. Problem (3.21) can be then thought as a classical expected utility maximization problem in $[t,T']$ with terminal random utility $U(T',x)$ and random endowment P_T received at time T. We now claim that $u^{P_T}(t,\xi;T')$ is independent of T' in the sense that the horizon-invariance/maturity-independence property

$$u^{P_T}(t,\xi;T) = u^{P_T}(t,\xi;T')$$

holds. Note that the above property cannot hold for any $T' > T$ in the classical setting due to the pre-chosen, arbitrary but fixed, terminal horizon at which the (static) utility is allocated. We state the general result next.

Proposition 3.15 *Let $T > 0$, $P_T \in L^\infty(\mathcal{F}_T)$ and $t \leq T \leq T'$. Assume that SDE (2.5) admits a strong solution on $[T,T']$ with initial condition $X_T = -\tilde{U}_z(T, \eta Z_T^{t,q})$ for any $q \in \mathcal{Q}_{[t,T]}$. Then,*

$$\tilde{u}^{P_T}(t,\eta;T) = \tilde{u}^{P_T}(t,\eta;T'), \text{ for } \eta \in L^{0,+}(\mathcal{F}_t). \tag{3.22}$$

If, furthermore, FBSDE (3.5) admits a solution (X,Y,Z) on $[t,T]$ with initial-terminal condition (ξ,P) satisfying $Z^i \in \mathbb{L}^2_{BMO}[t,T]$, $i=1,2$, then,

$$u^{P_T}(t,\xi;T) = u^{P_T}(t,\xi;T'), \text{ for } \xi \in L^0(\mathcal{F}_t). \tag{3.23}$$

Reverting to $u^P(t,\xi;T)$, i.e., when $T = T'$, we see that the dependence on T is generated exclusively by the fact that this is the arrival time of the random endowment. This then motives us to introduce the following notion of endowment maturity which will be useful in building a universal framework across all times. Consider random endowments in the general space L defined by

$$L := \cup_{T \geq 0} L^\infty(\mathcal{F}_T).$$

For any random endowment $P \in L$, we define its *maturity* by

$$T_P := \inf\left\{t \geq 0 : P \text{ is } \mathcal{F}_t\text{-measurable}\right\}. \tag{3.24}$$

In turn, the value functions u^P and $\tilde{u}^P$ can be defined as

$$\begin{aligned} u^P(t,\xi) &:= u^P(t,\xi;T_P) \\ &= \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T_P]}} \mathbb{E}\left[U\left(T_P, \xi + \int_t^{T_P} \pi_u \left(\theta_u du + dW_u^1\right) + P\right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T_P \text{ and } \xi \in L^0(\mathcal{F}_t), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\tilde{u}^{P}(t,\eta) &:= \tilde{u}^{P}(t,\eta;T_P)\\ &= \underset{q\in\mathcal{Q}_{[t,T_P]}}{\operatorname{essinf}}\ \mathbb{E}\left[\tilde{U}\left(T_P,\eta Z_{T_P}^{t,q}\right)+\eta Z_{T_P}^{t,q}P\,\middle|\,\mathcal{F}_t\right],\quad 0\le t\le T_P \text{ and } \eta\in L^{0,+}(\mathcal{F}_t),\end{aligned}\tag{3.26}$$

which only depend on the random endowment itself.

The above property essentially allows us to define the value functions $u^P(t,\xi)$ and $\tilde{u}^P(t,\eta)$ no matter what the maturity of the random endowment is. We stress that this does *not* imply that $u^P(t,\xi)$ and $\tilde{u}^P(t,\eta)$ do not depend on when the endowment arrives, an obviously wrong statement. Rather, it expresses how problems (3.25) and (3.26) can be well-defined for *all* times and *all* endowments using the flexible forward performance framework. Once more, note that in the classical framework this cannot be done as the entire optimization problem is tied down to the a priori chosen terminal horizon.

This "maturity-independent" construction was first developed in [41] and, later in [13] for forward entropic risk measures. It will also play a fundamental role in the new notion of forward optimized certainty equivalent that we develop next.

4 Forward optimized certainty equivalent

The concept of optimized certainty equivalent (OCE) was first introduced by Ben-Tal and Teboulle in [5] and yields a valuation criterion rooted in expected utility theory. It is a *static* criterion, defined through the (static) optimization problem,

$$S(P) := \sup_{r\in\mathbb{R}}\left(\mathbb{E}\left[u(P-r)\right]+r\right),\tag{4.1}$$

where u is the utility of the investor and P is a random endowment. Essentially, the OCE represents the optimal split of funds if the investor has the option to choose to spend a portion r of the random endowment, and receive the present value of this r plus the expected utility of $P-r$.

One of the key results established by Ben-Tal and Teboulle in their followup work [6] is a dual representation for OCE, which we recall next. Let $\tilde{u}$ be the convex conjugate of u, namely, $\tilde{u}(z) := \sup_{x\in\mathbb{R}}(u(x)-xz)$. Then, the OCE of P admits the dual representation,

$$S(P) = \inf_{\mathbb{Q}\in\mathcal{Q}}\left(\mathbb{E}^{\mathbb{Q}}[P]+I_{\tilde{u}}(\mathbb{Q},\mathbb{P})\right),\tag{4.2}$$

where $\mathcal{Q}$ is the set of probability measures equivalent to $\mathbb{P}$, and $I_{\tilde{u}}(\mathbb{Q},\mathbb{P})$ is the associated penalty function, defined as

$$I_{\tilde{u}}(\mathbb{Q},\mathbb{P}) := \mathbb{E}\left[\tilde{u}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right].$$

In the context of risk measures, the quantity $\rho(P) := -S(P)$ turns out to be a convex risk measure for P and admits the dual representation

$$\rho(P) = \sup_{\mathbb{Q}\in\mathcal{Q}}\left(\mathbb{E}^{\mathbb{Q}}[-P]-I_{\tilde{u}}(\mathbb{Q},\mathbb{P})\right),$$

also known as a divergence risk measure (see [11]).

A challenging problem, which we attempt to address herein, is how to produce a genuinely dynamic extension of OCE in (4.1). Specifically, we are interested in building a valuation framework that is viable for all claims and all maturities and, furthermore, yields time-consistent OCE.

We face several difficulties here, both conceptual and technical. Firstly, it is not clear how the static valuation rule (4.1) should be modified across times. The obvious choice to merely replace the utility u by a utility, say U, at a given future horizon T, will not work as it will generate, to say the least, similar difficulties with the ones we face in the random endowment and the indifference valuation settings. Among others, such a framework will neither allow for valuation beyond T, nor will allow for adaptive model revision. One may naively propose to work in an infinite horizon to avoid horizon limitations. However, this will introduce very stringent constraints on a pre-chosen market model in $[0,\infty)$.

As mentioned in the Introduction, efforts have been made to build dynamic extensions of OCE. Recently, Backhoff-Veraguas et al. (see [7] and [8]) proposed a dynamic version working with the convex dual and, in addition, introduced an additional variable to guarantee time-consistency. This approach can be compared with the convex dual representation of dynamic risk measures in [15]. However, the related OCE will be bound to the fixed horizon T as we mentioned earlier.

4.1 OCE as the convex conjugate of utility maximization with random endowment

Before we introduce the forward OCE, we provide a key observation which, to the best of our knowledge, has not been employed so far. It interprets the existing static OCE *via the value function of the convex dual of a utility maximization problem with random endowment.*

To this end, note that the OCE of the random endowment $P \in L^{\infty}(\mathcal{F}_T)$ can be rewritten as

$$S\left(P\right)=\sup_{x\in\mathbb{R}}\left(\mathbb{E}\left[u\left(x+P\right)\right]-x\right)\ \text{ with } x=-r.$$

Let us then consider the utility maximization problem with the random endowment P,

$$v^{P}\left(x\right)=\sup_{X_T\in L^0(\mathcal{F}_T)}\mathbb{E}\left[u\left(X_T+P\right)\right],$$

under the constraint $\mathbb{E}^{\mathbb{Q}}\left[X_T\right]\le x$ for all equivalent martingale measures $\mathbb{Q}$, and X_T being the terminal wealth generated by trading strategies with initial wealth $x\in\mathbb{R}$. If we assume that, for any $X_T\in L^0(\mathcal{F}_T)$,

$$X_T \text{ is independent of } P \text{ and } \mathbb{E}[X_T]=x, \text{ i.e., } \mathbb{P} \text{ itself is a martingale measure,} \tag{4.3}$$

then, by Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\left[u\left(X_T+P\right)\right]&=\mathbb{E}\left[\left.\mathbb{E}\left[u\left(X_T+p\right)\right]\right|_{p=P}\right]\\&\le\mathbb{E}\left[\left.u\left(\mathbb{E}\left[X_T+p\right]\right)\right|_{p=P}\right]\\&=\mathbb{E}\left[u\left(x+P\right)\right].\end{aligned}$$

In other words, we obtain that the optimal wealth is given by $X_T^*=x$, and the value function is given by

$$v^{P}\left(x\right)=\mathbb{E}\left[u\left(x+P\right)\right].$$

Therefore, $S(P)$ can be expressed as the convex conjugate of v^P at value 1,

$$S\left(P\right)=\sup_{x\in\mathbb{R}}\left(v^{P}\left(x\right)-x\cdot1\right)=:\left(v^{P}\right)^{*}\left(1\right), \tag{4.4}$$

where $(v^P)^*$ represents the convex conjugate of the value function v^P. In other words, $S(P)$ can be seen as the convex conjugate, evaluated at value 1, of a utility maximization problem involving P as the random endowment in an auxiliary financial market that is orthogonal to this random endowment. In this setting, the optimal policy involves holding only the riskless asset, so that $X_T^*=x$.

4.2 Forward OCE

We are now ready to introduce the novel concept of forward OCE, which, to the best of our knowledge, is new. The definition makes full use of the key observation above and employs the suitable primal and dual forward counterparts of the static utility in (4.1) and (4.2).

Definition 4.1 *Let $T > 0$ be arbitrary and fixed. The forward OCE of $P \in L^{\infty}(\mathcal{F}_T)$, at time $0 \le t \le T$, is defined by*

$$\begin{aligned} F(t,\eta,P;T) &:= \operatorname*{esssup}_{\xi \in L^0(\mathcal{F}_t)} \left(u^P(t,\xi;T) - \xi\eta\right) \\ &= \operatorname*{esssup}_{\xi \in L^0(\mathcal{F}_t)} \left(\operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_u \left(\theta_u du + dW_u^1\right) + P\right) \middle| \mathcal{F}_t \right] - \xi\eta \right), \end{aligned}$$

where $\eta \in L^{0,+}(\mathcal{F}_t)$ and satisfies $\mathbb{E}[\eta] = 1$. In particular, at $t = 0$, we define $F(P;T)$ by

$$\begin{aligned} F(P;T) :=& F(0,1,P;T) \\ =& \sup_{x \in \mathbb{R}} \left(\sup_{\pi \in \mathcal{A}_{[0,T]}} \mathbb{E}\left[U\left(T, x + \int_0^T \pi_u \left(\theta_u du + dW_u^1\right) + P\right)\right] - x \right). \end{aligned}$$

It is worth noting that the dual variable η, which is used to determine the forward OCE of P, can be naturally interpreted as a deflator, serving to convert nominal values of current wealth ξ into real values. A similar concept has been explored in [7], albeit without the context of utility maximization. The forward OCE represents the result of an optimal dynamic allocation of funds. An investor faces the choice of saving a portion of their current wealth ξ at time t in addition to the random endowment P at time T. The consequence of this decision is a sacrifice in the real value due to reduced spending, which amounts to $\xi\eta$. Thus, the forward OCE at time t mirrors the optimal balance between maximizing the forward performance process from the saving amount of ξ at time t alongside the random endowment P at time T and reducing spending due to this saving choice. In the newly defined framework for forward OCE, we introduce an associated auxiliary financial market where the underlying asset can be utilized to partially hedge the risk associated with P. In contrast, in the classical OCE framework, this auxiliary market is assumed to be orthogonal to P, rendering its hedging efforts futile.

Similar to the classical OCE, we also obtain the following dual representation for the forward OCE, which follows from Theorems 3.10 and 3.11.

Theorem 4.2 *Let $T > 0$, $P \in L^{\infty}(\mathcal{F}_T)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$ with $\mathbb{E}[\eta] = 1$. Let, also, $(D, \tilde{Y}, \tilde{Z})$ be a solution to the dual FBSDE (3.14) on $[t,T]$ with initial-terminal condition (η, P) satisfying $\tilde{Z}^i \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$. Then, the forward OCE of P at time t admits the dual representation*

$$F(t,\eta,P;T) = \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T, \eta Z_T^{t,q}\right) + \eta Z_T^{t,q} P \middle| \mathcal{F}_t \right], \quad 0 \le t \le T,$$

and the corresponding optimal control $q^{*,P}$ is given by

$$q_s^{*,P} := \frac{\tilde{Z}_s^2 + \tilde{\alpha}_z^2(s, D_s)}{D_s \tilde{U}_{zz}(s, D_s)}, \quad t \le s \le T.$$

In light of Proposition 3.15, we obtain directly the *maturity-independence* property of the forward OCE.

Proposition 4.3 *Let $0 < T \le T'$, $P \in L^{\infty}(\mathcal{F}_T)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$ with $\mathbb{E}[\eta] = 1$, $0 \le t \le T$. Assume that SDE (2.5) admits a strong solution on $[T, T']$ with initial condition $X_T = -\tilde{U}_z(T, \eta Z_T^{t,q})$ for any $q \in \mathcal{Q}_{[t,T]}$. Then,*

$$F(t,\eta,P;T) = F(t,\eta,P;T').$$

4.2.1 Connection with the classical (static) OCE

We discuss how the forward OCE relates to classical OCE. Firstly, note that the two assumptions in (4.3) play a crucial role in transitioning from the perspective of utility maximization to the classical OCE definition. These assumptions involve the orthogonality between the market and the random endowment, as well as the exclusive use of the riskless asset as the optimal policy.

In the context of the forward performance framework, we need to consider the counterparts of (4.3), which ultimately lead to the recovery of classical OCE. Specifically, we assume the following conditions in the forward performance framework:

(i) the market price of risk $\theta \equiv 0$;

(ii) the random endowment $P \in L^{\infty}(\mathcal{F}_T^2)$, where $\mathbb{F}^2 = (\mathcal{F}_t^2)_{t\geq 0}$ is the natural filtration of W^2;

(iii) the volatility component $\alpha^1 \equiv 0$ in the forward performance SPDE (2.4).

Under the above assumptions, the dynamics of the forward performance process U are

$$dU\left(t,x\right) = \alpha^2\left(t,x\right)dW_t^2. \tag{4.5}$$

Additionally, the value function in (3.1) takes the form:

$$u^P\left(t,\xi;T\right) = \operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T,\xi+\int_t^T \pi_u dW_u^1 + P\right)\middle|\mathcal{F}_t\right], \quad 0\leq t\leq T \text{ and } \xi\in L^0\left(\mathcal{F}_t\right).$$

From Theorem 3.3, we deduce that if (Y,Z^2) is a solution to the BSDE

$$Y_s = P + \int_s^T \left(\frac{1}{2}\frac{U_{xxx}\left(u,\xi+Y_u\right)}{U_{xx}\left(u,\xi+Y_u\right)}\left|Z_u^2\right|^2 + \frac{\alpha_{xx}^2\left(u,\xi+Y_u\right)}{U_{xx}\left(u,\xi+Y_u\right)}Z_u^2\right)du - \int_s^T Z_u^2 dW_u^2,$$

namely, $(\xi, Y, (0, Z^2))$ solves FBSDE (3.5) under assumptions (i)-(iii), then the control process $\pi^{*,P} = 0$ is optimal for $u^P(t,\xi;T)$. Thus,

$$u^P\left(t,\xi;T\right) = \mathbb{E}\left[U\left(T,\xi+P\right)\middle|\mathcal{F}_t\right],$$

and the forward OCE of P at time t is given by

$$F\left(t,\eta,P;T\right) = \operatorname*{esssup}_{\xi\in L^0(\mathcal{F}_t)}\left(\mathbb{E}\left[U\left(T,\xi+P\right)\middle|\mathcal{F}_t\right] - \xi\eta\right).$$

In particular, at $t=0$, we have

$$\begin{aligned} F\left(P;T\right) &= \sup_{x\in\mathbb{R}}\left(\mathbb{E}\left[U\left(T,x+P\right)\right]-x\right) \\ &= \sup_{r\in\mathbb{R}}\left(\mathbb{E}\left[U\left(T,P-r\right)\right]+r\right). \end{aligned}$$

Thus, if, furthermore, the volatility component $\alpha^2 \equiv 0$, then $U(t,x) \equiv u(x)$ and the forward OCE coincides with the classical OCE, $F\left(P;T\right) = S\left(P\right)$.

Under assumptions (i)-(iii), we have a more explicit form of the dual representation,

$$F\left(t,\eta,P;T\right) = \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T,\eta\frac{\mathcal{E}\left(-\int qdW^2\right)_T}{\mathcal{E}\left(-\int qdW^2\right)_t}\right) + \eta\frac{\mathcal{E}\left(-\int qdW^2\right)_T}{\mathcal{E}\left(-\int qdW^2\right)_t}P\middle|\mathcal{F}_t\right],$$

where

$$\mathcal{E}\left(-\int qdW^2\right)_s := \exp\left(-\frac{1}{2}\int_0^s\left|q_u\right|^2du - \int_0^s q_u dW_u^2\right).$$

In particular, at $t=0$, we have

$$F\left(P;T\right) = \inf_{q\in\mathcal{Q}_{[0,T]}} \mathbb{E}\left[\tilde{U}\left(T,\mathcal{E}\left(-\textstyle\int qdW^2\right)_T\right) + \mathcal{E}\left(-\textstyle\int qdW^2\right)_T P\right].$$

4.2.2 Properties of the forward OCE

We conclude this section by presenting several key properties of the forward OCE. The proofs easily follow from the properties of the forward performance process. We first introduce a modification of the forward OCE through normalization. The *normalized* forward OCE, denoted by $\tilde{F}(t, \eta, P; T)$, is defined as follows:

$$\tilde{F}(t, \eta, P; T) := F(t, \eta, P; T) - F(t, \eta, 0; T). \tag{4.6}$$

Proposition 4.4 *Let $T > 0$ be arbitrary and fixed, and $\eta \in L^{0,+}(\mathcal{F}_t)$ with $\mathbb{E}[\eta] = 1$, $0 \le t \le T$. Then the (normalized) forward OCE has the following properties:*

(i) Monotonicity: for $P^i \in L^\infty(\mathcal{F}_T)$, $i = 1, 2$ and $P^1 \ge P^2$,

$$F(t, \eta, P^1; T) \ge F(t, \eta, P^2; T).$$

(ii) Cash invariance: for $P \in L^\infty(\mathcal{F}_T)$ and $c \in L^\infty(\mathcal{F}_t)$,

$$F(t, \eta, P + c; T) = F(t, \eta, P; T) + \eta c.$$

(iii) Concavity: for $P^i \in L^\infty(\mathcal{F}_T)$, $i = 1, 2$, and $\lambda \in (0, 1)$,

$$\lambda F\left(t, \eta, P^1; T\right) + (1 - \lambda) F(t, \eta, P; T) \le F\left(t, \eta, P^\lambda; T\right),$$

where $P^\lambda := \lambda P^1 + (1 - \lambda) P^2$.

(iv) Replication invariance: for $P \in L^\infty(\mathcal{F}_T)$ and for any $\pi \in \mathcal{A}_{[t,T]}$,

$$F\left(t, \eta, P + \int_t^T \pi_u \left(\theta_u du + dW_u^1\right); T\right) = F(t, \eta, P; T).$$

(v) Positivity: for nonnegative $P \in L^\infty(\mathcal{F}_T)$, $\tilde{F}(t, \eta, P; T) \ge 0$.

(vi) Constancy: for $c \in L^\infty(\mathcal{F}_t)$,

$$\tilde{F}(t, \eta, c; T) = \eta c.$$

Remark 4.5 *The normalized forward OCE $\tilde{F}$ defined in (4.6) satisfies Proposition 4.3 and Proposition 4.4 (i)-(iv). Furthermore, for the exponential forward performance process, this normalized forward OCE aligns with the negative of the forward entropic risk measure, which will be discussed in Section 5.4.*

5 Exponential forward performance process in a single stochastic factor model with random endowment

We analyze exponential forward performance processes. In contrast to the complete market case presented in Section 3.4, where explicit solutions are available, the case of exponential forward performance processes leads to an incomplete market. To characterize the exponential forward performance process, an ergodic BSDE can be utilized, as described in [31]. Following this work, Chong et al. [13] studied the corresponding exponential forward utility maximization with random endowment and its application to forward entropic risk measures.

In this section, we demonstrate that the general results derived in Section 3 cover the exponential results in [31] and [13] as a special case. Additionally, the newly proposed forward OCE aligns with the negative of the forward entropic risk measure.

5.1 Single stochastic factor model

The single stochastic factor model, taken from [31] and [13], assumes that the stock price process follows

$$\frac{dS_t}{S_t} = \mu\left(V_t\right)dt + \sigma\left(V_t\right)dW_t^1, \quad S_0 = S > 0,$$

where μ and $\sigma > 0$ are deterministic functions. The stochastic factor solves

$$dV_t = \eta\left(V_t\right)dt + \left(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right), \quad V_0 = v \in \mathbb{R}, \quad \rho \in [0,1].$$

The controlled state equation (2.1) then becomes

$$dX_t^\pi = \pi_t\left(\theta\left(V_t\right)ds + dW_t^1\right),$$

with $\theta\left(v\right) := \frac{\mu(v)}{\sigma(v)}$. As in [13] and [31], we introduce the following assumptions:

(i) there exists a large enough $C > 0$ such that

$$\left(\eta\left(v\right) - \eta\left(v'\right)\right)\left(v - v'\right) \leq -C\left|v - v'\right|^2;$$

(ii) θ is uniformly bounded and Lipschitz continuous.

5.2 Ergodic BSDE representation for exponential forward performance process

Due to the homothetic property of the exponential forward performance process and the single stochastic factor Markovian setup, SPDE (2.4) actually simplifies to an ergodic BSDE, derived in [31]. Indeed, consider the ergodic BSDE

$$dY_t^e = \left(\theta\left(V_t\right)Z_t^{e,1} + \frac{1}{2}\left|\theta\left(V_t\right)\right|^2 - \frac{1}{2}\left|Z_t^{e,2}\right|^2 + \lambda t\right)dt + \left(Z_t^e\right)^\top dW_t. \tag{5.1}$$

We easily deduce that it admits a unique Markovian solution $(Y_t^e, Z_t^e, \lambda) = (y^e(V_t), z^e(V_t), \lambda)$, $t \geq 0$, where $y^e : \mathbb{R} \to \mathbb{R}$ has at most linear growth and $z^e : \mathbb{R} \to \mathbb{R}^2$ is bounded. Then, the process given by

$$U\left(t,x\right) = -e^{-\gamma x + Y_t^e - \lambda t}, \quad \gamma > 0, \tag{5.2}$$

is an exponential forward performance process. It provides a solution to SPDE (2.4) which takes the form

$$dU\left(t,x\right) = \beta\left(t,x\right)dt + \alpha^\top\left(t,x\right)dW_t$$

with drift

$$\beta\left(t,x\right) = \frac{1}{2}U\left(t,x\right)\left|\theta\left(V_t\right) + Z_t^{e,1}\right|^2$$

and volatility

$$\alpha^i\left(t,x\right) = U\left(t,x\right)Z_t^{e,i}, \ i = 1,2. \tag{5.3}$$

Moreover, the optimal control process π^* in (2.3) is given by

$$\pi_t^* = \frac{\theta\left(V_t\right) + Z_t^{e,1}}{\gamma}.$$

It follows that the process U in (5.2) satisfies all the conditions in Assumption 2.3.

The ergodic BSDE (5.1) can be also used to characterize the convex conjugate of U, which takes the form

$$\tilde{U}(t,z) = -\frac{z}{\gamma} + \frac{z}{\gamma}\ln\frac{z}{\gamma} - \frac{z}{\gamma}\left(Y_t^e - \lambda t\right). \tag{5.4}$$

It is then straightforward to verify that $\tilde{U}$ satisfies the dual SPDE

$$d\tilde{U}(t,z) = \tilde{\beta}(t,z)\,dt + \tilde{\alpha}^\top(t,z)\,dW_t$$

with drift

$$\tilde{\beta}(t,z) = -\frac{z}{2\gamma}\left|\theta(V_t) + Z_t^{e,1}\right|^2 + \frac{z}{2\gamma}\left|Z_t^e\right|^2$$

and volatility

$$\tilde{\alpha}(t,z) = \alpha\left(t, -\tilde{U}_z(t,z)\right) = -\frac{z}{\gamma}Z_t^e. \tag{5.5}$$

In turn, the primal problem (3.1) and the dual problem (3.2) simplify, respectively, to

$$\begin{aligned} u^P(t,\xi;T) = &\operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}\left[-e^{-\gamma\left(\xi+\int_t^T \pi_u\left(\theta(V_u)du + dW_u^1\right)+P\right)+Y_T^e-\lambda T}\middle|\,\mathcal{F}_t\right] \\ =& e^{-\gamma\xi}u^P(t,0;T), \end{aligned} \tag{5.6}$$

and

$$\begin{aligned} \tilde{u}^P(t,\eta;T) = &\operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[-\frac{\eta Z_T^{t,q}}{\gamma} + \frac{\eta Z_T^{t,q}}{\gamma}\ln\frac{\eta Z_T^{t,q}}{\gamma} - \frac{\eta Z_T^{t,q}}{\gamma}\left(Y_T^e - \lambda T\right) + \eta Z_T^{t,q}P \middle|\,\mathcal{F}_t\right] \\ =&\eta\tilde{u}^P(t,1;T) + \frac{1}{\gamma}\eta\ln\eta. \end{aligned} \tag{5.7}$$

5.3 FBSDEs for the primal and dual problems

For this class of exponential forward performance processes, both the primal FBSDE (3.5) and the dual FBSDE (3.14) decouple. Specifically, (3.5) reduces to

$$\begin{cases} X_s = \xi + \displaystyle\int_t^s \left(\frac{\theta(V_u) + Z_u^{e,1}}{\gamma} - Z_u^1\right)\left(\theta(V_u)\,du + dW_u^1\right), \\ Y_s = P - \displaystyle\int_s^T \left(Z_u^1\theta(V_u) + \frac{\gamma}{2}\left|Z_u^2\right|^2 - Z_u^{e,2}Z_u^2\right)du - \int_s^T Z_u^\top dW_u. \end{cases} \tag{5.8}$$

Recalling Theorem 3.2 in [13], we know that there exists a unique solution (Y,Z) with Y being bounded and $Z^i \in \mathbb{L}^2_{BMO}[t,T],\ i=1,2$. Then, by Theorem 3.3, the optimal control process for $u^P(t,\xi;T)$ is given by

$$\pi_s^{*,P} = \frac{\theta(V_s) + Z_s^{e,1}}{\gamma} - Z_s^1, \quad t\le s\le T. \tag{5.9}$$

Moreover, applying Corollary 3.6, we also derive the optimal control process for $\tilde{u}^P(t,\eta;T)$, which is given by

$$q_s^{*,P} = \gamma Z_s^2 - Z_s^{e,2}, \quad t\le s\le T. \tag{5.10}$$

On the other hand, the dual FBSDE (3.14) reduces to

$$\begin{cases} D_s = \eta - \displaystyle\int_s^T D_u\left(\theta(V_u)\,dW_u^1 + \left(\gamma\tilde{Z}_u^2 - Z_u^{e,2}\right)dW_u^2\right), \\ \tilde{Y}_s = P - \displaystyle\int_s^T \left(\tilde{Z}_u^1\theta(V_u) + \frac{\gamma}{2}\left|\tilde{Z}_u^2\right|^2 - Z_u^{e,2}\tilde{Z}_u^2\right)du - \int_s^T \tilde{Z}_u^\top dW_u. \end{cases}$$

We note that above BSDE coincides with the one in (5.8), namely, $(\tilde{Y}, \tilde{Z}) = (Y, Z)$. Then, from Theorem 3.8, we have that the optimal control process $q^{*,P}$ for $\tilde{u}^P(t,\eta;T)$ is

$$q_s^{*,P} = \gamma \tilde{Z}_s^2 - Z_s^{e,2}, \quad t \leq s \leq T,$$

which aligns with (5.10). Moreover, by Corollary 3.12, the optimal control process $\pi^{*,P}$ for $u^P(t,\xi;T)$ is

$$\pi_s^{*,P} = \frac{\theta\left(V_s\right) + Z_s^{e,1}}{\gamma} - \tilde{Z}_s^1, \quad t \leq s \leq T,$$

which aligns with (5.9).

Finally, we derive the explicit forms of the value functions $u^P\left(t,\xi;T\right)$ and $\tilde{u}^P\left(t,\eta;T\right)$ using the explicit representations of U in (5.2) and $\tilde{U}$ in (5.4). First, observe that

$$U\left(t,x\right) = -\frac{1}{\gamma} U_x\left(t,x\right).$$

In turn,

$$u^P\left(t,\xi;T\right) = \mathbb{E}\left[\left. U\left(T, X_T + P\right) \right| \mathcal{F}_t\right] = -\frac{1}{\gamma}\mathbb{E}\left[\left. U_x\left(T, X_T + P\right) \right| \mathcal{F}_t\right].$$

It follows by Remark 3.2 that $U_x(s, X_s + Y_s)$ is a true martingale, which implies that

$$u^P\left(t,\xi;T\right) = -\frac{1}{\gamma} U_x\left(t, \xi + Y_t\right).$$

Next, note that

$$\tilde{U}\left(t,z\right) = -\frac{z}{\gamma} + z\tilde{U}_z\left(t,z\right),$$

and, as a result,

$$\begin{aligned}
\tilde{u}^P\left(t,\eta;T\right) &= u^P\left(t, -\tilde{U}_z\left(t,\eta\right) - Y_t; T\right) - \left(-\tilde{U}_z\left(t,\eta\right) - Y_t\right)\eta \\
&= -\frac{1}{\gamma} U_x\left(t, -\tilde{U}_z\left(t,\eta\right)\right) - \left(-\tilde{U}_z\left(t,\eta\right) - Y_t\right)\eta \\
&= -\frac{\eta}{\gamma} + \left(\tilde{U}_z\left(t,\eta\right) + Y_t\right)\eta \\
&= \tilde{U}\left(t,\eta\right) + \eta Y_t. \qquad (5.11)
\end{aligned}$$

5.4 Forward OCE under exponential forward performance process

We discuss the associated *normalized* forward OCE, defined in (4.6), and demonstrate its direct connection with the negative of the forward entropic risk measure introduced in [13].

For $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and stochastic deflator $\eta \in L^{0,+}(\mathcal{F}_t)$ with $\mathbb{E}[\eta] = 1$, $0 \leq t \leq T$, the definition of the normalized forward OCE in (4.6) and Theorem 4.2 yield that

$$\begin{aligned}
\tilde{F}\left(t,\eta,P;T\right) &= \tilde{u}^P\left(t,\eta;T\right) - \tilde{u}^0\left(t,\eta;T\right) \\
&= \tilde{u}^P\left(t,\eta;T\right) - \tilde{U}\left(t,\eta\right),
\end{aligned}$$

where we used that $\tilde{u}^0(t,\eta;T) = \tilde{U}(t,\eta)$, as it follows from Remark 3.7.

On one hand, as shown in (5.11), the expression of the normalized forward OCE in terms of the solution component Y of FBSDE (5.8) is given by

$$\tilde{F}\left(t,\eta,P;T\right) = \eta Y_t. \qquad (5.12)$$

This result is one of the major findings established in [13, Theorem 3.2]. Indeed, by choosing a deterministic deflator $\eta = 1$, we observe that the primal representation (5.12) for the normalized exponential forward OCE is actually the negative of the forward entropic risk measure,

$$\tilde{F}(t, 1, P; T) = -\rho(t, P; T),$$

where, according to [13, Definition 3.1], ρ is defined as the utility indifference price of P, namely,

$$U(t, x) = u^P(t, x + \rho(t, P; T)), \quad 0 \le t \le T.$$

On the other hand, using the dual representation in (5.7) yields

$$\begin{aligned}\tilde{F}(t, \eta, P; T) &= \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}\left[\eta Z_T^{t,q}\left(\frac{\ln Z_T^{t,q}}{\gamma} - \frac{Y_T^e - \lambda T}{\gamma} + P\right)\middle| \mathcal{F}_t\right] + \frac{\eta}{\gamma}(Y_t^e - \lambda t) \\ &= \eta \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}^{\mathbb{Q}^q}\left[\frac{\ln Z_T^{t,q}}{\gamma} - \frac{Y_T^e - \lambda T}{\gamma} + P\middle| \mathcal{F}_t\right] + \frac{\eta}{\gamma}(Y_t^e - \lambda t), \end{aligned} \tag{5.13}$$

where

$$\left.\frac{d\mathbb{Q}^q}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z_T^{t,q}.$$

Note that

$$\ln Z_T^{t,q} = \frac{1}{2}\int_t^T \left(|\theta(V_u)|^2 + |q_u|^2\right) du - \int_t^T \theta(V_u)\, dW_u^{1,\theta} - \int_t^T q_u dW_u^{2,q},$$

with

$$(dW_u^{1,\theta}, dW_u^{2,q}) := (dW_u^1 + \theta(V_u)\, du, dW_u^2 + q_u du),$$

and, in accordance with the ergodic BSDE (5.1), we have

$$\begin{aligned} Y_T^e - \lambda T = Y_t^e - \lambda t &+ \int_t^T \left(\frac{1}{2}|\theta(V_u)|^2 - \frac{1}{2}\left|Z_u^{e,2}\right|^2 - Z_u^{e,2} q_u\right) du \\ &+ \int_t^T Z_u^{e,1} dW_u^{1,\theta} + \int_t^T Z_u^{e,2} dW_u^{2,q}. \end{aligned}$$

As a result, the dual representation (5.13) can be expressed more explicitly as

$$\tilde{F}(t, \eta, P; T) = \eta \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}^{\mathbb{Q}^q}\left[P + \frac{1}{2\gamma}\int_t^T \left|Z_u^{e,2} + q_u\right|^2 du\middle| \mathcal{F}_t\right], \tag{5.14}$$

which is the negative of the dual representation for the forward entropic risk measure established in [13, Theorem 3.5] for $\eta = 1$.

6 Backward SPDEs for forward performance processes with random endowment

To conclude the paper, we briefly discuss how to tackle the primal problem (3.1) and the dual problem (3.2) by directly characterizing their value functions in terms of the solutions of backward SPDEs. We also explore how to use these backward SPDE solutions to construct the solutions of the primal FBSDE (3.5) and dual FBSDE (3.14). Since the solvability of the corresponding backward SPDEs is far from clear, the discussion that follows is formal.

6.1 Formal derivation of backward SPDEs

For given $T > 0$ and random endowment $P \in L^{\infty}(\mathcal{F}_T)$, recall the value function for the primal problem (3.1),

$$u^P(t,x;T) = \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, X_T^{\pi} + P\right) \middle| \mathcal{F}_t, X_t^{\pi} = x \right], \quad 0 \le t \le T \text{ and } x \in \mathbb{R}. \tag{6.1}$$

Assume that u^P is strictly increasing and strictly concave, admits the Itô-decomposition

$$du^P(t,x;T) = b^P(t,x;T)\,dt + \left(a^P(t,x;T)\right)^{\top} dW_t,$$

and is regular enough for the upcoming calculations to hold. Then, applying the Itô-Ventzell formula yields

$$\begin{aligned} du^P\left(t, X_t^{\pi}; T\right) = &\left(b^P\left(t, X_t^{\pi}; T\right) + \left(u_x^P\left(t, X_t^{\pi}; T\right)\theta_t + a_x^{P,1}\left(t, X_t^{\pi}; T\right)\right)\pi_t + \frac{1}{2}u_{xx}^P\left(t, X_t^{\pi}; T\right)|\pi_t|^2 \right) dt \\ &+ \left(a^{P,1}\left(t, X_t^{\pi}; T\right) + u_x^P\left(t, X_t^{\pi}; T\right)\pi_t\right) dW_t^1 + a^{P,2}\left(t, X_t^{\pi}; T\right) dW_t^2. \end{aligned}$$

Following the standard martingale optimality condition (as described, for example, in [31]), we select the drift $b^P(t,x;T)$ to ensure that $u^P(t, X_t^{\pi}; T)$ is a supermartingale for any $\pi \in \mathcal{A}_{[t,T]}$ and a martingale for an optimal control process. This yields

$$\begin{aligned} b^P(t,x;T) &= -\sup_{\pi \in \mathbb{R}^n} \left(\frac{1}{2}u_{xx}^P(t,x;T)|\pi|^2 + \left(u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)\right)\pi \right) \\ &= \frac{1}{2}\frac{\left|u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)\right|^2}{u_{xx}^P(t,x;T)}, \end{aligned}$$

with the above optimum given by

$$\pi^{*,P}(t,x;T) = -\frac{u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)}{u_{xx}^P(t,x;T)}. \tag{6.2}$$

Thus, u^P is expected to satisfy the backward SPDE

$$du^P(t,x;T) = \frac{1}{2}\frac{\left|u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)\right|^2}{u_{xx}^P(t,x;T)}dt + \left(a^P(t,x;T)\right)^{\top} dW_t \tag{6.3}$$

with terminal condition

$$u^P(T,x;T) = U(T, x+P). \tag{6.4}$$

Note that, different from the SPDE characterization (2.4) of the forward performance U where the volatility is a model input, (6.3) is a *backward* SPDE with the volatility a^P being part of the solution.

Differentiating (6.3) in x yields

$$du_x^P(t,x;T) = b_x^P(t,x;T)\,dt + \left(a_x^P(t,x;T)\right)^{\top} dW_t, \quad 0 \le t \le T \text{ and } x \in \mathbb{R}, \tag{6.5}$$

with terminal condition

$$u_x^P(T,x;T) = U_x(T, x+P), \tag{6.6}$$

where

$$\begin{aligned} b_x^P(t,x;T) = &\left(u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)\right)\theta_t + \frac{\left(u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)\right)a_{xx}^{P,1}(t,x;T)}{u_{xx}^P(t,x;T)} \\ &- \frac{1}{2}\frac{\left|u_x^P(t,x;T)\theta_t + a_x^{P,1}(t,x;T)\right|^2 u_{xxx}^P(t,x;T)}{\left|u_{xx}^P(t,x;T)\right|^2}. \end{aligned} \tag{6.7}$$

Next, we define the convex conjugate of u^P,

$$\tilde{u}^P(t,z;T) := \sup_{x\in\mathbb{R}} \left(u^P(t,x;T) - xz\right).$$

We know that u^P and $\tilde{u}^P$ satisfy the same relations (2.10)-(2.17) as U and $\tilde{U}$ do. As in Section 2.2, we have the dual relation

$$u_x^P\left(t, -\tilde{u}_z^P(t,z;T); T\right) = z, \tag{6.8}$$

where $\tilde{u}_z^P$ satisfies the backward SPDE

$$d\tilde{u}_z^P(t,z;T) = \tilde{b}_z^P(t,z;T)\,dt + \left(\tilde{a}_z^P(t,z;T)\right)^\top dW_t \tag{6.9}$$

with

$$\tilde{u}_z^P(T,z;T) = \tilde{U}_z(T,z) + P, \tag{6.10}$$

and drift given by

$$\begin{aligned}\tilde{b}_z^P(t,z;T) =& -\tilde{u}_{zz}^P(t,z;T)\,z\,|\theta_t|^2 + \tilde{a}_{zz}^{P,1}(t,z;T)\,z\theta_t + \tilde{a}_z^{P,1}(t,z;T)\,\theta_t - \frac{1}{2}\tilde{u}_{zzz}^P(t,z;T)\,|z\theta_t|^2 \\ & - \frac{1}{2}\frac{\tilde{u}_{zzz}^P(t,z;T)}{\left|\tilde{u}_{zz}^P(t,z;T)\right|^2}\left|\tilde{a}_z^{P,2}(t,z;T)\right|^2 + \frac{\tilde{a}_z^{P,2}(t,z;T)\,\tilde{a}_{zz}^{P,2}(t,z;T)}{\tilde{u}_{zz}^P(t,z;T)}.\end{aligned} \tag{6.11}$$

6.2 Primal backward SPDE and FBSDE

We will use backward SPDE (6.5) for u_x^P to construct a solution of the primal FBSDE (3.5) and an optimal control for the primal problem (3.1) as well as the corresponding quantities for the dual problem (3.2).

Proposition 6.1 *Let w be a solution to the backward SPDE (6.5) satisfied by u_x^P, namely,*

$$\begin{cases} dw(t,x) = h(t,x)\,dt + d^\top(t,x)\,dW_t, \quad 0\le t\le T \text{ and } x\in\mathbb{R}, \\ w(T,x) = U_x(T,x+P), \end{cases} \tag{6.12}$$

where

$$h(t,x) = \left(w(t,x)\,\theta_t + d^1(t,x)\right)\left(\theta_t + \frac{d_x^1(t,x)}{w_x(t,x)}\right) - \frac{1}{2}\frac{\left|w(t,x)\,\theta_t + d^1(t,x)\right|^2 w_{xx}(t,x)}{|w_x(t,x)|^2},$$

and assume that it is regular enough to apply the Itô-Ventzell formula. Then, for $\xi \in L^0(\mathcal{F}_t)$, the triplet (X,Y,Z) given by

$$X_s := \xi - \int_t^s \frac{w(u,X_u)\,\theta_u + d^1(u,X_u)}{w_x(u,X_u)}\left(\theta_u du + dW_u^1\right), \tag{6.13}$$

$$Y_s := -\tilde{U}_z(s, w(s,X_s)) - X_s,$$

and

$$\begin{aligned} Z_s^1 &:= \frac{w(s,X_s)\,\theta_s + d^1(s,X_s)}{w_x(s,X_s)} - \tilde{\alpha}_z^1(s,w(s,X_s)) + \tilde{U}_{zz}(s,w(s,X_s))\,w(s,X_s)\,\theta_s, \\ Z_s^2 &:= -\tilde{\alpha}_z^2(s,w(s,X_s)) - d^2(s,X_s)\,\tilde{U}_{zz}(s,w(s,X_s)), \end{aligned} \tag{6.14}$$

is a solution to FBSDE (3.5) on $[t,T]$ with initial-terminal condition (ξ,P).

To show this, we apply the Itô-Ventzell formula to $w(s, X_s)$ and deduce that

$$dw\left(s, X_s\right) = -w\left(s, X_s\right)\theta_s dW_s^1 + d^2\left(s, X_s\right) dW_s^2,$$

and

$$\begin{aligned}
dY_s =& -d\tilde{U}_z\left(s, w\left(s, X_s\right)\right) - X_s \\
=& \Bigg(\left(\tilde{U}_{zz}\left(s, w\left(s, X_s\right)\right) w\left(s, X_s\right) - \tilde{\alpha}_z^1\left(s, w\left(s, X_s\right)\right) + \frac{d^1\left(s, X_s\right) + w\left(s, X_s\right)\theta_s}{w_x\left(s, X_s\right)}\right)\theta_s \\
&+ \frac{1}{2}\tilde{U}_{zzz}\left(s, w\left(s, X_s\right)\right)\left(\left|\frac{\tilde{\alpha}_z^2\left(s, w\left(s, X_s\right)\right)}{\tilde{U}_{zz}\left(s, w\left(s, X_s\right)\right)}\right|^2 - \left|d^2\left(s, X_s\right)\right|^2\right) \\
&- \frac{\tilde{\alpha}_z^2\left(s, w\left(s, X_s\right)\right)\tilde{\alpha}_{zz}^2\left(s, w\left(s, X_s\right)\right)}{\tilde{U}_{zz}\left(s, w\left(s, X_s\right)\right)} - \tilde{\alpha}_z^2\left(s, w\left(s, X_s\right)\right) d^2\left(s, X_s\right)\Bigg) ds \\
&+ Z_s^\top dW_s.
\end{aligned}$$

Note, however, that by the definition of Y, $U_x\left(s, X_s + Y_s\right) = w\left(s, X_s\right)$, which, together with the definition of Z in (6.14) as well as (2.13), (2.14), (2.21), and (2.22), implies that (X, Y, Z) satisfies FBSDE (3.5).

Corollary 6.2 *In addition to the assumptions in Proposition 6.1, we further assume that the processes*

$$\frac{w\left(s, X_s\right)\theta_s + d^1\left(s, X_s\right)}{w_x\left(s, X_s\right)} \in \mathbb{L}^2_{BMO}[t, T], \text{ and } d^2\left(s, X_s\right)\tilde{U}_{zz}\left(s, w\left(s, X_s\right)\right) \in \mathbb{L}^2_{BMO}[t, T].$$

Then, $Z^i \in \mathbb{L}^2_{BMO}[t, T]$*,* $i = 1, 2$*, and* $\pi^{*,P}$ *given by*

$$\pi_s^{*,P} := -\frac{w\left(s, X_s\right)\theta_s + d^1\left(s, X_s\right)}{w_x\left(s, X_s\right)}$$

is optimal for $u^P(t, \xi; T)$*. Respectively,* $q^{*,P}$ *given by*

$$q_s^{*,P} := -\frac{d^2\left(s, X_s\right)}{w\left(s, X_s\right)} \tag{6.15}$$

is optimal for $\tilde{u}^P\left(t, \hat{\eta}; T\right)$ *with* $\hat{\eta} := w(t, \xi)$*.*

Furthermore, for any $\eta \in L^{0,+}(\mathcal{F}_t)$ *and* $\hat{\xi} := w^{-1}(t, \eta)$*,* $q^{*,P}$ *defined by (6.15) with* X *satisfying* $X_t = \hat{\xi}$ *is optimal for* $\tilde{u}^P\left(t, \eta; T\right)$*.*

To verify the optimality of the above processes, we first verify that $Z^i \in \mathbb{L}^2_{BMO}[t, T]$, $i = 1, 2$, using (6.14), (2.13), (2.21) and Assumption 2.3 (iv). In turn, we use Theorems 3.3, 3.5 and Corollary 3.6 to conclude.

6.3 Dual backward SPDE and FBSDE

Similarly to Proposition 6.1 and Corollary 6.2, we obtain the following results using backward SPDE (6.9).

Proposition 6.3 *Let* $\tilde{w}$ *be a solution to the backward SPDE (6.9) satisfied by* $\tilde{u}_z^P$*, namely,*

$$\begin{cases} d\tilde{w}\left(t, z\right) = \tilde{h}\left(t, z\right) dt + \tilde{d}^\top\left(t, z\right) dW_t, \quad 0 \le t \le T \text{ and } z > 0, \\ \tilde{w}\left(T, z\right) = \tilde{U}_z\left(T, z\right) + P, \end{cases} \tag{6.16}$$

with

$$\tilde{h}(t,x) = -\tilde{w}_z(t,z)\,z\,|\theta_t|^2 + \tilde{d}^1_z(t,z)\,z\theta_t + \tilde{d}^1(t,z)\,\theta_t - \frac{1}{2}\tilde{w}_{zz}(t,z)\,|z\theta_t|^2 - \frac{1}{2}\frac{\tilde{w}_{zz}(t,z)}{|\tilde{w}_z(t,z)|^2}\left|\tilde{d}^2(t,z)\right|^2 + \frac{\tilde{d}^2(t,z)\,\tilde{d}^2_z(t,z)}{\tilde{w}_z(t,z)},$$

and assume that $\tilde{w}$ is regular enough to apply the Itô-Ventzell formula. Then, for $\eta \in L^{0,+}(\mathcal{F}_t)$, the triplet $(D, \tilde{Y}, \tilde{Z})$ given by

$$D_s := \eta - \int_t^s \left(D_u\theta_u dW^1_u + \frac{\tilde{d}^2(u,D_u)}{\tilde{w}_z(u,D_u)}dW^2_u\right),$$

$$\tilde{Y}_s := \tilde{w}(s,D_s) - \tilde{U}_z(s,D_s),$$

and

$$\tilde{Z}^1_s := \left(\tilde{w}_z(s,D_s) - \tilde{U}_{zz}(s,D_s)\right)D_s\theta_s - \tilde{\alpha}^1_z(s,D_s),$$
$$\tilde{Z}^2_s := \left(\tilde{w}_z(s,D_s) - \tilde{U}_{zz}(s,D_s)\right)\frac{\tilde{d}^2(s,D_s)}{\tilde{w}_z(s,D_s)} - \tilde{\alpha}^2_z(s,D_s),$$

solves FBSDE (3.14) on $[t,T]$ with initial-terminal condition (η, P).

Corollary 6.4 *In addition to the assumptions of Proposition 6.3, we further assume that*

$$\tilde{w}_z(s,D_s)D_s \in \mathbb{L}^2_{BMO}[t,T], \text{ and } \frac{\left(\tilde{w}_z(s,D_s) - \tilde{U}_{zz}(s,D_s)\right)\tilde{d}^2(s,D_s)}{\tilde{w}_z(s,D_s)} \in \mathbb{L}^2_{BMO}[t,T].$$

Then, $\tilde{Z}^i \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$, and $q^{*,P}$ defined by

$$q^{*,P}_s := \frac{\tilde{d}^2(s,D_s)}{\tilde{w}_z(s,D_s)\,D_s}$$

is optimal for $\tilde{u}^P(t,\eta;T)$. Respectively, $\pi^{*,P}$ defined by

$$\pi^{*,P}_s := -\tilde{w}_z(s,D_s)\,D_s\theta_s \tag{6.17}$$

is optimal for $u^P(t,\hat{\xi};T)$ with $\hat{\xi} := -\tilde{w}(t,\eta)$.

Furthermore, for any $\xi \in L^0(\mathcal{F}_t)$ and $\hat{\eta} := \tilde{w}^{-1}(t,-\xi)$, $\pi^{*,P}$ defined by (6.17) with D satisfying $D_t = \hat{\eta}$ is optimal for $u^P(t,\xi;T)$.

Remark 6.5 *Let w and $\tilde{w}$ be the solutions to backward SPDEs (6.12) and (6.16), respectively. Applying the Itô-Ventzell formula to these two equations, one can verify that*

$$w(t,-\tilde{w}(t,z)) = z \text{ and } -\tilde{w}(t,w(t,x)) = x. \tag{6.18}$$

which gives the relation between the primal and dual backward SPDEs.

7 Conclusions

In this paper, we extended the notion of forward performance criteria to settings with random endowment in incomplete markets and studied the related stochastic optimization problems. For this, we developed a new methodology by directly studying the candidate optimal control processes for both the primal and dual problems. We constructed two new system of FBSDEs and established necessary and sufficient conditions for optimality, and various equivalences between the two problems. This new approach is general and complements the existing one based on backward SPDEs for the related value functions.

Building on these results, we introduced and developed the novel concept of forward optimized certainty equivalent, which offers a genuinely dynamic valuation mechanism that accommodates progressively adaptive market model updates, stochastic risk preferences, and incoming claims with arbitrary maturities. We, also, considered representative examples for both forward performance criteria with random endowment and forward OCE; for the case of exponential forward performance criteria, we investigated the connection of forward OCE with the forward entropic risk measure.

The existence and uniqueness of FBSDE solutions are currently examined by the authors in the companion paper [29] using the decoupling field approach, and are not included in this paper.

Appendix. Proofs

A.1 Proof of Lemma 2.1

Let $X^{t,\pi} := \int_t^\cdot \pi_u \left(\theta_u du + dW_u^1\right)$. Then, Itô's formula yields that $X^{t,\pi} Z^{t,q}$ is a local martingale, given by

$$d\left(X_s^{t,\pi} Z_s^{t,q}\right) = \left(-X_s^{t,\pi} Z_s^{t,q}\theta_s + Z_s^{t,q}\pi_s\right) dW_s^1 - X_s^{t,\pi} Z_s^{t,q} q_s dW_s^2.$$

We observe that for $n > 2$, since $\pi \in \mathbb{L}^2_{BMO}[0,T]$, the energy inequality implies that

$$\mathbb{E}\left[\left(\int_0^T |\pi_u|^2 du\right)^{\frac{n}{2}}\right] \leq (\frac{n}{2})! || \int_0^\cdot \pi_u dW_u^1 ||_{BMO}^n < \infty,$$

where $||\cdot||_{BMO}$ denotes the BMO norm. Then, using Burkholder-Davis-Gundy inequality and the uniform boundedness of θ, we deduce that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq s\leq T}\left|X_s^{t,\pi}\right|^n\right] &\leq C\mathbb{E}\left[\sup_{t\leq s\leq T}\left(\left|\int_t^s \pi_u\theta_u du\right|^n + \left|\int_t^s \pi_u dW_u^1\right|^n\right)\right] \\ &\leq C\mathbb{E}\left[\left(\int_t^T |\pi_u\theta_u| du\right)^n\right] + C\mathbb{E}\left[\left(\int_t^T |\pi_u|^2 du\right)^{\frac{n}{2}}\right] < \infty. \end{aligned} \tag{A.1}$$

Next, we show that $\int_t^\cdot X_u^{t,\pi} Z_u^{t,q} q_u dW_u^2$ is a true martingale. Note that the stochastic exponential $Z^{t,q}$ is a uniformly integrable martingale due to the boundedness of θ and the fact that $q \in \mathbb{L}^2_{BMO}[t,T]$. Using Doob's inequality and the reverse Hölder's inequality yield

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\left|Z_s^{t,q}\right|^{p_0}\right] \leq \left(\frac{p_0}{p_0-1}\right)^{p_0} \mathbb{E}\left[|Z_T^{t,q}|^{p_0}\right] < \infty,$$

for some $p_0 > 1$. Using again Burkholder-Davis-Gundy inequality, we obtain that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le s\le T}\left|\int_t^s X_u^{t,\pi} Z_u^{t,q} q_u dW_u^2\right|\right] &\le C\mathbb{E}\left[\left(\int_t^T \left|X_u^{t,\pi} Z_u^{t,q} q_u\right|^2 du\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[\sup_{t\le s\le T}\left|Z_s^{t,q}\right| \left(\int_t^T \left|X_u^{t,\pi} q_u\right|^2 du\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[\sup_{t\le s\le T}\left|Z_s^{t,q}\right|^{p_0}\right] + C\mathbb{E}\left[\left(\int_t^T \left|X_u^{t,\pi} q_u\right|^2 du\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right].
\end{aligned}
$$

Moreover, inequality (A.1) and property $q \in \mathbb{L}^2_{BMO}[t,T]$ yield that

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_t^T \left|X_u^{t,\pi} q_u\right|^2 du\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right] &\le \mathbb{E}\left[\sup_{t\le s\le T}\left|X_s^{t,\pi}\right|^{\frac{p_0}{p_0-1}} \left(\int_t^T \left|q_u\right|^2 du\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right] \\
&\le \mathbb{E}\left[\sup_{t\le s\le T}\left|X_s^{t,\pi}\right|^{\frac{2p_0}{p_0-1}}\right] + \mathbb{E}\left[\left(\int_t^T \left|q_u\right|^2 du\right)^{\frac{p_0}{p_0-1}}\right] < \infty,
\end{aligned}
$$

and, thus, $\int_t^\cdot X_u^{t,\pi} Z_u^{t,q} q_u dW_u^2$ is a true martingale. Similarly, we deduce that

$$
\int_t^\cdot \left(-X_u^{t,\pi} Z_u^{t,q}\theta_u + Z_u^{t,q}\pi_u\right) dW_s^1
$$

is also a martingale.

A.2 Proof of Theorem 3.1

Step 1. Since $\pi^{*,P}$ is optimal, for any $\pi \in \mathcal{A}_{[t,T]}$ and $\varepsilon \in (0,1)$ we must have

$$
\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[U\left(T, X_T^{\varepsilon,\pi} + P\right) - U\left(T, X_T^* + P\right)\right|\mathcal{F}_t] \le 0, \tag{A.2}
$$

where

$$
X_s^{\varepsilon,\pi} := \xi + \int_t^s \left(\pi_u^{*,P} + \varepsilon\pi_u\right)\left(\theta_u du + dW_u^1\right), \quad t \le s \le T.
$$

We can easily check that

$$
\frac{1}{\varepsilon}\left[U\left(T, X_T^{\varepsilon,\pi} + P\right) - U\left(T, X_T^* + P\right)\right] = \int_0^1 U_x\left(T, X_T^{\delta\varepsilon,\pi} + P\right) d\delta \int_t^T \pi_u\left(\theta_u du + dW_u^1\right),
$$

and that there exists $\delta_0 \in (0,1)$ such that

$$
\begin{aligned}
\int_0^1 U_x\left(T, X_T^{\delta\varepsilon,\pi} + P\right) d\delta &= U_x\left(T, X_T^{\delta_0\varepsilon,\pi} + P\right) \\
&= U_x\left(T, \xi + \int_t^T \left(\pi_u^{*,P} + \delta_0\varepsilon\pi_u\right)\left(\theta_u du + dW_u^1\right) + P\right).
\end{aligned}
$$

Thus, by (A.2), the uniformly integrability assumption and the dominated convergence theorem, we obtain

$$
\begin{aligned}
0 &\geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[U\left(T, X_T^{\varepsilon,\pi} + P\right) - U\left(T, X_T^* + P\right) \middle| \mathcal{F}_t \right] \\
&= \lim_{\varepsilon \to 0} \mathbb{E}\left[U_x\left(T, \xi + \int_t^s \left(\pi_u^{*,P} + \delta_0 \varepsilon \pi_u\right)\left(\theta_u du + dW_u^1\right) + P\right) \int_t^T \pi_u \left(\theta_u du + dW_u^1\right) \middle| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \int_t^T \pi_u \left(\theta_u du + dW_u^1\right) \middle| \mathcal{F}_t \right]. \qquad (A.3)
\end{aligned}
$$

Replacing π by $-\pi$ implies that

$$
0 \leq \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \int_t^T \pi_u \left(\theta_u du + dW_u^1\right) \middle| \mathcal{F}_t \right].
$$

It follows by (A.3) that

$$
0 = \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \int_t^T \pi_u \left(\theta_u du + dW_u^1\right) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[N_T A_T^\pi \middle| \mathcal{F}_t \right], \qquad (A.4)
$$

where, for $t \leq s \leq T$,

$$
N_s := \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \middle| \mathcal{F}_s \right] \qquad (A.5)
$$

and

$$
A_s^\pi := \int_t^s \pi_u \left(\theta_u du + dW_u^1\right). \qquad (A.6)
$$

Since N is a martingale on $[t, T]$, there exists a density process, say $\varphi = (\varphi^1, \varphi^2)$ on $[t, T]$, such that

$$
dN_s = \varphi_s^\top dW_s, \quad t \leq s \leq T; \qquad (A.7)
$$

see, for example, [26, Problem 3.4.16].

Step 2. We define an $\mathbb{F}$-progressively measurable process Y on $[t, T]$ such that

$$
U_x\left(s, X_s^* + Y_s\right) = N_s = \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \middle| \mathcal{F}_s \right], \quad t \leq s \leq T. \qquad (A.8)
$$

In other words, $Y_s := U_x^{-1}(s, N_s) - X_s^*$ for $t \leq s \leq T$ and $Y_T = P$. By the dual relation (2.10), we know that

$$
Y_s = -\tilde{U}_z(s, N_s) - X_s^*, \quad t \leq s \leq T. \qquad (A.9)
$$

Applying the Itô-Ventzell formula to the dynamics of $\tilde{U}_z$ in (2.18) and N in (A.7) yields that

$$
\begin{aligned}
dY_s &= - d\tilde{U}_z(s, N_s) - \pi_s^{*,P}\left(\theta_s ds + dW_s^1\right) \\
&= \left(-\tilde{\beta}_z(s, N_s) - \frac{1}{2}\tilde{U}_{zzz}(s, N_s)\left|\varphi_s\right|^2 - \tilde{\alpha}_{zz}^\top(s, N_s)\varphi_s - \pi_s^{*,P}\theta_s \right) ds \\
&\quad + \left(-\tilde{\alpha}_z^1(s, N_s) - \tilde{U}_{zz}(s, N_s)\varphi_s^1 - \pi_s^{*,P} \right) dW_s^1 + \left(-\tilde{\alpha}_z^2(s, N_s) - \tilde{U}_{zz}(s, N_s)\varphi_s^2 \right) dW_s^2. \qquad (A.10)
\end{aligned}
$$

By (2.10), (2.13) and (2.14), we have

$$
\tilde{U}_{zz}(s, N_s) = -\frac{1}{U_{xx}\left(s, X_s^* + Y_s\right)}, \qquad (A.11)
$$

$$
\tilde{U}_{zzz}(s, N_s) = \frac{U_{xxx}\left(s, X_s^* + Y_s\right)}{\left(U_{xx}\left(s, X_s^* + Y_s\right)\right)^3}. \qquad (A.12)
$$

It follows by (2.21) that

$$\tilde{\alpha}_z(s, N_s) = \frac{\alpha_x(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)}, \tag{A.13}$$

$$\tilde{\alpha}_{zz}(s, N_s) = -\frac{U_{xxx}(s, X_s^* + Y_s)\,\alpha_x(s, X_s^* + Y_s)}{(U_{xx}(s, X_s^* + Y_s))^3} + \frac{\alpha_{xx}(s, X_s^* + Y_s)}{|U_{xx}(s, X_s^* + Y_s)|^2}, \tag{A.14}$$

and by (2.19) that

$$\begin{aligned}\tilde{\beta}_z(s, N_s) =& \frac{\beta_x(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)} + \frac{1}{2}\frac{U_{xxx}(s, X_s^* + Y_s)\,|\alpha_x(s, X_s^* + Y_s)|^2}{(U_{xx}(s, X_s^* + Y_s))^3} \\ & - \frac{\alpha_x^\top(s, X_s^* + Y_s)\,\alpha_{xx}(s, X_s^* + Y_s)}{|U_{xx}(s, X_s^* + Y_s)|^2},\end{aligned} \tag{A.15}$$

where

$$\begin{aligned}\beta_x(s, X_s^* + Y_s) =& \left(U_x(s, X_s^* + Y_s)\,\theta_s + \alpha_x^1(s, X_s^* + Y_s)\right)\left(\theta_s + \frac{\alpha_{xx}^1(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)}\right) \\ & - \frac{1}{2}\frac{\left|U_x(s, X_s^* + Y_s)\,\theta_s + \alpha_x^1(s, X_s^* + Y_s)\right|^2 U_{xxx}(s, X_s^* + Y_s)}{|U_{xx}(s, X_s^* + Y_s)|^2}.\end{aligned} \tag{A.16}$$

Combining (A.9)-(A.16) yields that

$$\begin{aligned}dY_s =& -\left(\frac{\beta_x(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)} + \frac{1}{2}\frac{U_{xxx}(s, X_s^* + Y_s)\,|\varphi_s - \alpha_x(s, X_s^* + Y_s)|^2}{(U_{xx}(s, X_s^* + Y_s))^3}\right. \\ & \left. + \frac{\alpha_{xx}^\top(s, X_s^* + Y_s)\,(\varphi_s - \alpha_x(s, X_s^* + Y_s))}{|U_{xx}(s, X_s^* + Y_s)|^2} + \pi_s^{*,P}\theta_s\right) ds \\ & + \left(\frac{\varphi_s^1 - \alpha_x^1(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)} - \pi_s^{*,P}\right) dW_s^1 + \frac{\varphi_s^2 - \alpha_x^2(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)} dW_s^2.\end{aligned}$$

For $t \le s \le T$, let

$$Z_s^1 := \frac{\varphi_s^1 - \alpha_x^1(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)} - \pi_s^{*,P} \text{ and } \quad Z_s^2 := \frac{\varphi_s^2 - \alpha_x^2(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)}. \tag{A.17}$$

It then follows that

$$\varphi_s^1 = \alpha_x^1(s, X_s^* + Y_s) + U_{xx}(s, X_s^* + Y_s)\,\pi_s^{*,P} + U_{xx}(s, X_s^* + Y_s)\,Z_s^1 \tag{A.18}$$

and

$$\varphi_s^2 = \alpha_x^2(s, X_s^* + Y_s) + U_{xx}(s, X_s^* + Y_s)\,Z_s^2. \tag{A.19}$$

Therefore,

$$\begin{aligned}dY_s =& -\left(\frac{\beta_x(s, X_s^* + Y_s)}{U_{xx}(s, X_s^* + Y_s)} + \frac{1}{2}\frac{U_{xxx}(s, X_s^* + Y_s)\left(\left|Z_s^1 + \pi_s^{*,P}\right|^2 + \left|Z_s^2\right|^2\right)}{U_{xx}(s, X_s^* + Y_s)}\right. \\ & \left. + \frac{\alpha_{xx}^1(s, X_s^* + Y_s)\left(Z_s^1 + \pi_s^{*,P}\right) + \alpha_{xx}^2(s, X_s^* + Y_s)\,Z_s^2}{U_{xx}(s, X_s^* + Y_s)} + \pi_s^{*,P}\theta_s\right) ds \\ & + Z_s^\top dW_s \\ =& - f(s, X_s^*, Y_s, Z_s)\,ds + Z_s^\top dW_s,\end{aligned} \tag{A.20}$$

where

$$
\begin{aligned}
f(s,x,y,z) := {} & \frac{\beta_x(s,x+y)}{U_{xx}(s,x+y)} + \frac{1}{2}\frac{U_{xxx}(s,x+y)\left(\left|z^1+\pi_s^{*,P}\right|^2+\left|z^2\right|^2\right)}{U_{xx}(s,x+y)} \\
& + \frac{\alpha_{xx}^1(s,x+y)\left(z^1+\pi_s^{*,P}\right)+\alpha_{xx}^2(s,x+y)z^2}{U_{xx}(s,x+y)} + \pi_s^{*,P}\theta_s. \qquad \text{(A.21)}
\end{aligned}
$$

Step 3. Applying Itô's formula to N in (A.7) with φ in (A.18) and (A.19) and A^π in (A.6), we obtain that

$$
\begin{aligned}
dN_sA_s^\pi = {} & \pi_s\left(N_s\theta_s+\alpha_x^1(s,X_s^*+Y_s)+U_{xx}(s,X_s^*+Y_s)\pi_s^{*,P}+U_{xx}(s,X_s^*+Y_s)Z_s^1\right)ds \\
& +\left(A_s^\pi\varphi_s^1+N_s\pi_s\right)dW_s^1+A_s^\pi\varphi_s^2dW_s^2.
\end{aligned}
$$

By Lemma A.1, we know that $\int_t^\cdot \left(A_s^\pi\varphi_s^1+N_s\pi_s\right)dW_s^1+\int_t^\cdot A_s^\pi\varphi_s^2dW_s^2$ is a true martingale. Then, integrating both sides from t to T and taking conditional expectation, by (A.4), we have

$$
0=\mathbb{E}\left[\int_t^T \pi_s\left(N_s\theta_s+\alpha_x^1(s,X_s^*+Y_s)+U_{xx}(s,X_s^*+Y_s)\pi_s^{*,P}+U_{xx}(s,X_s^*+Y_s)Z_s^1\right)ds\,\middle|\,\mathcal{F}_t\right].
$$

Since π is arbitrary, we deduce that

$$
\begin{aligned}
\pi_s^{*,P} &= -\frac{N_s\theta_s+\alpha_x^1(s,X_s^*+Y_s)}{U_{xx}(s,X_s^*+Y_s)}-Z_s^1 \\
&= -\frac{U_x(s,X_s^*+Y_s)\theta_s+\alpha_x^1(s,X_s^*+Y_s)}{U_{xx}(s,X_s^*+Y_s)}-Z_s^1, \quad t\le s\le T. \qquad \text{(A.22)}
\end{aligned}
$$

Step 4. Combining (A.21), (2.8) and (A.22) yields

$$
f(s,x,y,z)=-z^1\theta_s+\frac{\frac{1}{2}U_{xxx}(s,x+y)\left|z^2\right|^2+\alpha_{xx}^2(s,x+y)z^2}{U_{xx}(s,x+y)}. \qquad \text{(A.23)}
$$

In turn, we easily deduce the BSDE with generator f in (A.23),

$$
dY_s=-f(s,X_s^*,Y_s,Z_s)\,ds+Z_s^\top dW_s, \quad Y_T=P,
$$

with $X^*=X$ in (3.3) and $\pi^{*,P}$ in (A.22), and the forward part of FBSDE (3.5) is obtained.

Lemma A.1 *For any $\pi\in\mathcal{A}_{[t,T]}$, the process*

$$
\int_t^\cdot \left(A_u^\pi\varphi_u^1+N_u\pi_u\right)dW_u^1+\int_t^\cdot A_u^\pi\varphi_u^2dW_u^2
$$

is a true martingale on $[t,T]$.

Proof. By (A.1), we know that $\mathbb{E}[\sup_{t\le s\le T}|A_s^\pi|^2]<\infty$. Burkholder-Davis-Gundy inequality yields

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le s\le T}\left|\int_t^s A_u^\pi\varphi_u^i dW_u^i\right|\right] &\le C\mathbb{E}\left[\left(\int_t^T\left|A_u^\pi\varphi_u^i\right|^2du\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[\sup_{t\le s\le T}|A_s^\pi|^2\right]+C\mathbb{E}\left[\int_t^T\left|\varphi_u^i\right|^2du\right]<\infty.
\end{aligned}
$$

Moreover, by the assumption of L^p-integrability of $U_x(T, X_T^*+P)$ and Doob's L^p-inequality, we deduce that

$$\mathbb{E}\left[\sup_{t\le s\le T}|N_s|^p\right]\le C\mathbb{E}\left[U_x\left(T, X_T^*+P\right)^p\right]<\infty.$$

Using similar arguments as in Lemma 2.1, we obtain that $\mathbb{E}[\sup_{t\le s\le T}|\int_t^s N_u\pi_u dW_u^1|]<\infty$. ■

A.3 Proof of Theorem 3.3

By definition of $\pi^{*,P}$, we have

$$X_s=\xi+\int_t^s \pi_u^{*,P}\left(\theta_u du+dW_u^1\right)=X_s^*,\quad t\le s\le T.$$

For any $\pi\in\mathcal{A}_{[t,T]}$, let

$$X_s^\pi:=\xi+\int_t^s \pi_u\left(\theta_u du+dW_u^1\right),\quad t\le s\le T.$$

We show that, for any $\pi\in\mathcal{A}_{[t,T]}$,

$$\mathbb{E}\left[\left.U\left(T, X_T^\pi+P\right)\right|\mathcal{F}_t\right]\le\mathbb{E}\left[\left.U\left(T, X_T+P\right)\right|\mathcal{F}_t\right]. \tag{A.24}$$

From Remark 3.2, we introduce a probability measure using the optimal state price density process $Z^{t,q^{*,P}}$, namely,

$$\left.\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right|_{\mathcal{F}_T}=Z_T^{t,q^{*,P}}=\frac{U_x\left(T,X_T+P\right)}{U_x\left(t,\xi+Y_t\right)}.$$

Then, Girsanov's theorem yields that

$$\left(dW_s^{*,1},dW_s^{*,2}\right):=\left(dW_s^1+\theta_s ds, dW_s^2+q_s^{*,P}ds\right),$$

is a Brownian motion under $\mathbb{Q}^*$.

To show (A.24) we work as follows. Using the spatial concavity of U, we have

$$\begin{aligned}
&\mathbb{E}\left[\left.U\left(T, X_T^\pi+P\right)\right|\mathcal{F}_t\right]-\mathbb{E}\left[\left.U\left(T, X_T+P\right)\right|\mathcal{F}_t\right] \\
&\le\mathbb{E}\left[\left.U_x\left(T, X_T+P\right)\left(X_T^\pi-X_T\right)\right|\mathcal{F}_t\right] \\
&=\mathbb{E}\left[\left.\frac{U_x\left(T,X_T+P\right)}{U_x\left(t,\xi+Y_t\right)}\left(X_T^\pi-X_T\right)\right|\mathcal{F}_t\right]U_x\left(t,\xi+Y_t\right) \\
&=\mathbb{E}^{\mathbb{Q}^*}\left[\left.\int_t^T\left(\pi_u-\pi_u^{*,P}\right)dW_u^{*,1}\right|\mathcal{F}_t\right]U_x\left(t,\xi+Y_t\right).
\end{aligned} \tag{A.25}$$

Applying the reverse Hölder's inequality to $Z^{t,q^{*,P}}$, we establish the existence of $p_0>1$ such that $\mathbb{E}[|Z_T^{t,q^{*,P}}|^{p_0}]<\infty$. Moreover, observing that $\pi-\pi^{*,P}\in\mathbb{L}^2_{BMO}[t,T]$ and employing the energy inequality, we deduce that

$$\mathbb{E}\left[\left(\int_t^T\left|\pi_u-\pi_u^{*,P}\right|^2du\right)^{\frac{n}{2}}\right]<\infty,\qquad \text{for } n>2.$$

In turn, Burkholder-Davis-Gundy inequality yields that

$$\mathbb{E}^{\mathbb{Q}^*}\left[\sup_{t\le s\le T}\left|\int_t^s\left(\pi_u-\pi_u^{*,P}\right)dW_u^{*,1}\right|\right]\le C\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_t^T\left|\pi_u-\pi_u^{*,P}\right|^2du\right)^{\frac{1}{2}}\right]$$
$$\le C\mathbb{E}\left[\left|Z_T^{t,q^{*,P}}\right|^{p_0}\right]+C\mathbb{E}\left[\left(\int_t^T\left|\pi_u-\pi_u^{*,P}\right|^2du\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right]<\infty,$$

which further implies that the right hand side of (A.25) is zero and (A.24) follows.

A.4 Proof of Theorem 3.5

(i). By Remark 3.2, we know that $q^{*,P}\in\mathcal{Q}_{[t,T]}$. We want to show that, for any $q\in\mathcal{Q}_{[t,T]}$,

$$\mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}Z_T^{t,q^{*,P}}\right)+\hat{\eta}Z_T^{t,q^{*,P}}P\middle|\mathcal{F}_t\right]\le\mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}Z_T^{t,q}\right)+\hat{\eta}Z_T^{t,q}P\middle|\mathcal{F}_t\right].\tag{A.26}$$

Note that by Remark 3.2 and the definition of $\hat{\eta}$,

$$\hat{\eta}Z_T^{t,q^{*,P}}=U_x\left(T,X_T+P\right).\tag{A.27}$$

By Theorem 3.3,

$$X_T=\xi+\int_t^T\pi_u^{*,P}\left(\theta_udu+dW_u^1\right),$$

with $\pi^{*,P}\in\mathcal{A}_{[t,T]}$ in (3.7). It follows from (2.12), (A.27) and Lemma 2.1 that

$$\begin{aligned}&\mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}Z_T^{t,q^{*,P}}\right)+\hat{\eta}Z_T^{t,q^{*,P}}P\middle|\mathcal{F}_t\right]\\&=\mathbb{E}\left[\tilde{U}\left(T,U_x\left(T,X_T+P\right)\right)+U_x\left(T,X_T+P\right)P\middle|\mathcal{F}_t\right]\\&=\mathbb{E}\left[U\left(T,X_T+P\right)-U_x\left(T,X_T+P\right)X_T\middle|\mathcal{F}_t\right]\\&=\mathbb{E}\left[U\left(T,X_T+P\right)\middle|\mathcal{F}_t\right]-\xi\hat{\eta}.\end{aligned}\tag{A.28}$$

On the other hand, for any $q\in\mathcal{Q}_{[t,T]}$, by (2.9) and Lemma 2.1, we easily deduce that

$$\begin{aligned}&\mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}Z_T^{t,q}\right)+\hat{\eta}Z_T^{t,q}P\middle|\mathcal{F}_t\right]\\&\ge\mathbb{E}\left[U\left(T,X_T+P\right)-\hat{\eta}Z_T^{t,q}X_T\middle|\mathcal{F}_t\right]\\&=\mathbb{E}\left[U\left(T,X_T+P\right)\middle|\mathcal{F}_t\right]-\xi\hat{\eta},\end{aligned}$$

which together with (A.28) proves (A.26).

(ii). By (2.11) and Lemma 2.1, we have, for any $\pi\in\mathcal{A}_{[t,T]}$, $q\in\mathcal{Q}_{[t,T]}$, $\xi\in L^0(\mathcal{F}_t)$ and $\eta\in L^{0,+}(\mathcal{F}_t)$, that

$$\begin{aligned}&\mathbb{E}\left[U\left(T,\xi+\int_t^T\pi_u\left(\theta_udu+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right]\\&\le\mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q}\right)+\eta Z_T^{t,q}\left(\xi+\int_t^T\pi_u\left(\theta_udu+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right]\\&=\mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q}\right)+\eta Z_T^{t,q}P\middle|\mathcal{F}_t\right]+\xi\eta.\end{aligned}\tag{A.29}$$

Thus, due to the arbitrariness of π, q and η, we obtain

$$u^P(t,\xi;T) \le \operatorname*{essinf}_{\eta\in L^{0,+}(\mathcal{F}_t)} \left(\tilde{u}^P(t,\eta;T)+\xi\eta\right). \tag{A.30}$$

On the other hand, for $q^{*,P}$ defined in (3.6), using Theorem 3.3, equality (A.28) and part (i) of this theorem gives

$$\begin{aligned} u^P(t,\xi;T) &= \mathbb{E}\left[U(T,X_T+P)\right|\mathcal{F}_t] \\ &= \mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}Z_T^{t,q^{*,P}}\right)+\hat{\eta}Z_T^{t,q^{*,P}}P\middle|\mathcal{F}_t\right]+\xi\hat{\eta} \\ &= \tilde{u}^P(t,\hat{\eta};T)+\xi\hat{\eta}, \end{aligned} \tag{A.31}$$

which together with (A.30) proves the desired result.

A.5 Proof of Corollary 3.6

For any given $\eta\in L^{0,+}(\mathcal{F}_t)$, applying Theorem 3.5 to $\hat{\xi}$ in (3.10) yields (i). By (A.29) and the arbitrariness of $\xi\in L^0(\mathcal{F}_t), \pi\in\mathcal{A}_{[t,T]}$ and $q\in\mathcal{Q}_{[t,T]}$, we have that, for any $\eta\in L^{0,+}(\mathcal{F}_t)$,

$$\tilde{u}^P(t,\eta;T) \ge \operatorname*{esssup}_{\xi\in L^0(\mathcal{F}_t)}\left(u^P(t,\xi;T)-\xi\eta\right).$$

Replacing ξ by $\hat{\xi}$ in (A.31) yields that $\tilde{u}^P(t,\eta;T)=u^P(t,\hat{\xi};T)-\hat{\xi}\eta$, which proves (ii).

A.6 Proof of Theorem 3.8

Step 1. Since $q^{*,P}$ is optimal, for any $q\in\mathcal{Q}_{[t,T]}$ and $\varepsilon\in(0,1)$, we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\left(\tilde{U}(T,Z_T^{\varepsilon,q})+Z_T^{\varepsilon,q}P\right)-\left(\tilde{U}(T,Z_T^*)+Z_T^*P\right)\middle|\mathcal{F}_t\right]\ge 0, \tag{A.32}$$

where $Z^{\varepsilon,q}:=\eta Z^{t,q^*+\varepsilon q}$. Note that

$$\frac{1}{\varepsilon}\left(\left(\tilde{U}(T,Z_T^{\varepsilon,q})+Z_T^{\varepsilon,q}P\right)-\left(\tilde{U}(T,Z_T^*)+Z_T^*P\right)\right)=\frac{1}{\varepsilon}\int_0^1\left(\tilde{U}_z\left(T,Z_T^{\delta\varepsilon,q}\right)+P\right)\partial_\delta Z_T^{\delta\varepsilon,q}d\delta,$$

and

$$\begin{aligned}\partial_\delta Z_T^{\delta\varepsilon,q} &= \eta\partial_\delta\exp\left(-\int_t^T\left(\theta_u dW_u^1+(q_u^*+\delta\varepsilon q_u)dW_u^2\right)-\frac{1}{2}\int_t^T\left(|\theta_u|^2+|q_u^*+\delta\varepsilon q_u|^2\right)du\right)\\ &= Z_T^{\delta\varepsilon,q}\left(-\int_t^T\varepsilon q_u dW_u^2-\int_t^T(q_u^*+\delta\varepsilon q_u)\varepsilon q_u du\right).\end{aligned}$$

Therefore, there exists $\delta_0\in(0,1)$ such that

$$\begin{aligned}&\frac{1}{\varepsilon}\left(\left(\tilde{U}(T,Z_T^{\varepsilon,q})+Z_T^{\varepsilon,q}P\right)-\left(\tilde{U}(T,Z_T^*)+Z_T^*P\right)\right)\\ =&\frac{1}{\varepsilon}\int_0^1\left(\tilde{U}_z\left(T,Z_T^{\delta\varepsilon,q}\right)+P\right)Z_T^{\delta\varepsilon,q}\left(-\int_t^T\varepsilon q_u dW_u^2-\int_t^T(q_u^*+\delta\varepsilon q_u)\varepsilon q_u du\right)d\delta\\ =&\left(\tilde{U}_z\left(T,Z_T^{\delta_0\varepsilon,q}\right)+P\right)Z_T^{\delta_0\varepsilon,q}\left(-\int_t^T q_u dW_u^2-\int_t^T(q_u^*+\delta_0\varepsilon q_u)q_u du\right).\end{aligned}$$

By the uniformly integrability assumption and the dominated convergence theorem, (A.32) implies that

$$
\begin{aligned}
0 &\leq \mathbb{E}\left[\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\left(\tilde{U}\left(T, Z_T^{\varepsilon,q}\right)+Z_T^{\varepsilon,q}P\right)-\left(\tilde{U}\left(T, Z_T^{*}\right)+Z_T^{*}P\right)\right]\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\lim_{\varepsilon\to 0}\left(\tilde{U}_z\left(T, Z_T^{\delta_0\varepsilon,q}\right)+P\right)Z_T^{\delta_0\varepsilon,q}\left(-\int_t^T q_u dW_u^2-\int_t^T\left(q_u^*+\delta_0\varepsilon q_u\right)q_u du\right)\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{E}\left[\left(\tilde{U}_z\left(T, Z_T^{*}\right)+P\right)Z_T^{*}\left(-\int_t^T q_u dW_u^2-\int_t^T q_u^* q_u du\right)\middle|\,\mathcal{F}_t\right].
\end{aligned}
$$

Replacing q by $-q$, we obtain that

$$
0=\mathbb{E}\left[\left(\tilde{U}_z\left(T, Z_T^{*}\right)+P\right)Z_T^{*}\left(\int_t^T q_u dW_u^2+\int_t^T q_u^* q_u du\right)\middle|\,\mathcal{F}_t\right]=\mathbb{E}\left[\,M_T H_T^q\middle|\,\mathcal{F}_t\right], \tag{A.33}
$$

where, for $t \leq s \leq T$, M and H^q are defined as

$$
M_s := \mathbb{E}\left[\left(\tilde{U}_z\left(T, Z_T^{*}\right)+P\right)Z_T^{*}\middle|\,\mathcal{F}_s\right] \tag{A.34}
$$

and

$$
H_s^q := \int_t^s q_u dW_u^2+\int_t^s q_u^* q_u du. \tag{A.35}
$$

Since M is a martingale on $[t,T]$, there exists a density process $\psi=(\psi^1,\psi^2)$ on $[t,T]$ such that

$$
dM_s=\psi_s^{\top}dW_s,\quad t\leq s\leq T; \tag{A.36}
$$

see, for example, [26, Problem 3.4.16].

Step 2. We define an $\mathbb{F}$-progressively measurable process $\tilde{Y}$ on $[t,T]$ such that

$$
\left(\tilde{U}_z\left(s, Z_s^{*}\right)+\tilde{Y}_s\right)Z_s^{*}=M_s=\mathbb{E}\left[\left(\tilde{U}_z\left(T, Z_T^{*}\right)+P\right)Z_T^{*}\middle|\,\mathcal{F}_s\right],\quad t\leq s\leq T,
$$

In other words,

$$
\tilde{Y}_s := \frac{M_s}{Z_s^*}-\tilde{U}_z\left(s, Z_s^{*}\right),\quad \text{for } t\leq s\leq T \text{ and } \tilde{Y}_T=P. \tag{A.37}
$$

Itô's formula yields

$$
d\left(\frac{1}{Z_s^*}\right)=\frac{\left|\theta_s\right|^2+\left|q_s^{*,P}\right|^2}{Z_s^*}ds+\frac{1}{Z_s^*}\left(\theta_s dW_s^1+q_s^{*,P}dW_s^2\right),
$$

and, in turn,

$$
\begin{aligned}
d\left(\frac{M_s}{Z_s^*}\right)=&\left(\frac{M_s}{Z_s^*}\left(\left|\theta_s\right|^2+\left|q_s^{*,P}\right|^2\right)+\frac{\psi_s^1\theta_s+\psi_s^2 q_s^{*,P}}{Z_s^*}\right)ds \\
&+\frac{1}{Z_s^*}\left(\left(M_s\theta_s+\psi_s^1\right)dW_s^1+\left(M_s q_s^{*,P}+\psi_s^2\right)dW_s^2\right).
\end{aligned} \tag{A.38}
$$

Therefore,

$$
\begin{aligned}
&d\tilde{U}_z\left(s, Z_s^{*}\right) \\
&=\left(\tilde{\beta}_z\left(s, Z_s^{*}\right)+\frac{1}{2}\tilde{U}_{zzz}\left(s, Z_s^{*}\right)\left|Z_s^*\right|^2\left(\left|\theta_s\right|^2+\left|q_s^{*,P}\right|^2\right)-\tilde{\alpha}_{zz}^1\left(s, Z_s^{*}\right)Z_s^*\theta_s-\tilde{\alpha}_{zz}^2\left(s, Z_s^{*}\right)Z_s^* q_s^{*,P}\right)ds \\
&\quad+\left(\tilde{\alpha}_z^1\left(s, Z_s^{*}\right)-\tilde{U}_{zz}\left(s, Z_s^{*}\right)Z_s^*\theta_s\right)dW_s^1+\left(\tilde{\alpha}_z^2\left(s, Z_s^{*}\right)-\tilde{U}_{zz}\left(s, Z_s^{*}\right)Z_s^* q_s^{*,P}\right)dW_s^2,
\end{aligned}
$$

which, combined with (2.25) and (A.38), gives

$$
\begin{aligned}
d\tilde{Y}_s =& d\left(\frac{M_s}{Z_s^*}\right) - d\tilde{U}_z\left(s, Z_s^*\right) \\
=& \Bigg(\frac{M_s}{Z_s^*}\left(\left|\theta_s\right|^2 + \left|q_s^{*,P}\right|^2\right) + \frac{\psi_s^1\theta_s + \psi_s^2 q_s^{*,P}}{Z_s^*} \\
& + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* \left|\theta_s\right|^2 - \tilde{\alpha}_z^1\left(s, Z_s^*\right)\theta_s - \frac{\tilde{\alpha}_z^2\left(s, Z_s^*\right)\tilde{\alpha}_{zz}^2\left(s, Z_s^*\right)}{\tilde{U}_{zz}\left(s, Z_s^*\right)} + \frac{1}{2}\tilde{U}_{zzz}\left(s, Z_s^*\right)\frac{\left|\tilde{\alpha}_z^2\left(s, Z_s^*\right)\right|^2}{\left|\tilde{U}_{zz}\left(s, Z_s^*\right)\right|^2} \\
& - \frac{1}{2}\tilde{U}_{zzz}\left(s, Z_s^*\right)\left|Z_s^* q_s^{*,P}\right|^2 + \tilde{\alpha}_{zz}^2\left(s, Z_s^*\right) Z_s^* q_s^{*,P}\Bigg) ds \\
& + \left(\frac{M_s\theta_s + \psi_s^1}{Z_s^*} - \tilde{\alpha}_z^1\left(s, Z_s^*\right) + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^*\theta_s\right) dW_s^1 \\
& + \left(\frac{M_s q_s^{*,P} + \psi_s^2}{Z_s^*} - \tilde{\alpha}_z^2\left(s, Z_s^*\right) + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P}\right) dW_s^2.
\end{aligned} \tag{A.39}
$$

Let, for $t \leq s \leq T$,

$$
\begin{aligned}
\tilde{Z}_s^1 &:= \frac{M_s\theta_s + \psi_s^1}{Z_s^*} - \tilde{\alpha}_z^1\left(s, Z_s^*\right) + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^*\theta_s, \\
\tilde{Z}_s^2 &:= \frac{M_s q_s^{*,P} + \psi_s^2}{Z_s^*} - \tilde{\alpha}_z^2\left(s, Z_s^*\right) + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P}.
\end{aligned} \tag{A.40}
$$

It then follows that

$$
\begin{aligned}
\psi_s^1 &= \left(\tilde{Z}_s^1 + \tilde{\alpha}_z^1\left(s, Z_s^*\right) - \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^*\theta_s\right) Z_s^* - M_s\theta_s, \\
\psi_s^2 &= \left(\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, Z_s^*\right) - \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P}\right) Z_s^* - M_s q_s^{*,P}.
\end{aligned} \tag{A.41}
$$

Substituting (A.41) into (A.39) yields

$$
\begin{aligned}
d\tilde{Y}_s =& \Bigg(\tilde{Z}_s^1\theta_s + \tilde{Z}_s^2 q_s^{*,P} + \tilde{\alpha}_z^2\left(s, Z_s^*\right) q_s^{*,P} - \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^*\left|q_s^{*,P}\right|^2 \\
& - \frac{\tilde{\alpha}_z^2\left(s, Z_s^*\right)\tilde{\alpha}_{zz}^2\left(s, Z_s^*\right)}{\tilde{U}_{zz}\left(s, Z_s^*\right)} + \frac{1}{2}\tilde{U}_{zzz}\left(s, Z_s^*\right)\frac{\left|\tilde{\alpha}_z^2\left(s, Z_s^*\right)\right|^2}{\left|\tilde{U}_{zz}\left(s, Z_s^*\right)\right|^2} \\
& - \frac{1}{2}\tilde{U}_{zzz}\left(s, Z_s^*\right)\left|Z_s^* q_s^{*,P}\right|^2 + \tilde{\alpha}_{zz}^2\left(s, Z_s^*\right) Z_s^* q_s^{*,P}\Bigg) ds + \tilde{Z}_s^\top dW_s \\
=& -\tilde{f}\left(s, Z_s^*, \tilde{Y}_s, \tilde{Z}_s\right) ds + \tilde{Z}_s^\top dW_s,
\end{aligned} \tag{A.42}
$$

where

$$
\begin{aligned}
\tilde{f}\left(s, d, y, z\right) :=& - z^1\theta_s - z^2 q_s^{*,P} - \tilde{\alpha}_z^2\left(s, d\right) q_s^{*,P} + \tilde{U}_{zz}\left(s, d\right) d\left|q_s^{*,P}\right|^2 \\
& + \frac{\tilde{\alpha}_z^2\left(s, d\right)\tilde{\alpha}_{zz}^2\left(s, d\right)}{\tilde{U}_{zz}\left(s, d\right)} - \frac{1}{2}\tilde{U}_{zzz}\left(s, d\right)\frac{\left|\tilde{\alpha}_z^2\left(s, d\right)\right|^2}{\left|\tilde{U}_{zz}\left(s, d\right)\right|^2} \\
& + \frac{1}{2}\tilde{U}_{zzz}\left(s, d\right)\left|d q_s^{*,P}\right|^2 - \tilde{\alpha}_{zz}^2\left(s, d\right) d q_s^{*,P}.
\end{aligned} \tag{A.43}
$$

Step 3. Applying Itô's formula to M in (A.36) with ψ as in (A.41) and H^q in (A.35), we obtain that, for any $q \in \mathcal{Q}_{[t,T]}$,

$$
\begin{aligned}
d\left(M_s H_s^q\right) = q_s &\left(\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, Z_s^*\right) - \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P}\right) Z_s^* ds \\
&+ H_s^q \psi_s^1 dW_s^1 + \left(M_s q_s + H_s^q \psi_s^2\right) dW_s^2.
\end{aligned}
$$

From the definition of H^q and Burkholder-Davis-Gundy inequality we have that $\mathbb{E}[\sup_{t\leq s\leq T}|H_s^q|^2] < \infty$, and, thus, applying similar arguments used in Lemma A.1 we deduce that $\int_t^{\cdot} H_u^q \psi_u^1 dW_u^1 + \int_t^{\cdot}\left(M_u q_u + H_u^q \psi_u^2\right) dW_u^2$ is a true martingale. Thus, (A.33) gives

$$
\mathbb{E}\left[M_T H_T^q \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T q_s \left(\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, Z_s^*\right) - \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P}\right) Z_s^* ds \middle| \mathcal{F}_t\right] = 0,
$$

which, together with the arbitrariness of q, gives

$$
\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, Z_s^*\right) - \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P} = 0,
$$

and, thus,

$$
q_s^{*,P} = \frac{\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, Z_s^*\right)}{\tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^*}. \tag{A.44}
$$

Step 4. Combining (A.44) and (A.43) gives

$$
\tilde{f}\left(s, d, y, z\right) = -z^1\theta_s + \frac{1}{2}\tilde{U}_{zzz}\left(s, d\right)\frac{\left|z^2\right|^2}{\left|\tilde{U}_{zz}\left(s, d\right)\right|^2} + \tilde{U}_{zzz}\left(s, d\right)\frac{\tilde{\alpha}_z^2\left(s, d\right) z^2}{\left|\tilde{U}_{zz}\left(s, d\right)\right|^2} - \frac{\tilde{\alpha}_{zz}^2\left(s, d\right) z^2}{\tilde{U}_{zz}\left(s, d\right)}.
$$

and we easily conclude.

A.7 Proof of Theorem 3.10

From (2.10), (2.16) and (2.20), we have

$$
q_s^{*,P} = \left(\frac{\alpha_x^2\left(s, -\tilde{U}_z\left(s, D_s\right)\right)}{U_{xx}\left(s, -\tilde{U}_z\left(s, D_s\right)\right)} + \tilde{Z}_s^2\right)\left(\frac{-U_{xx}\left(s, -\tilde{U}_z\left(s, D_s\right)\right)}{U_x\left(s, -\tilde{U}_z\left(s, D_s\right)\right)}\right),
$$

which, together with Assumption 2.3 (iv) yields that $q^{*,P} \in \mathcal{Q}_{[t,T]}$.

Next we show that, for any $q \in \mathcal{Q}_{[t,T]}$,

$$
\mathbb{E}\left[\tilde{U}\left(T, D_T\right) + D_T P \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[\tilde{U}\left(T, \eta Z_T^{t,q}\right) + \eta Z_T^{t,q} P \middle| \mathcal{F}_t\right]. \tag{A.45}
$$

Indeed, from the definition of $q^{*,P}$, we have

$$
\eta Z^{t,q^{*,P}} = D,
$$

which is a true martingale on $[t, T]$. From Remark 3.9 and Lemma 2.1, we obtain that $(\tilde{U}_z(s, D_s) + \tilde{Y}_s)D_s$ is a true martingale, and in turn (2.15) yields

$$
\begin{aligned}
\mathbb{E}\left[\tilde{U}\left(T, D_T\right) + D_T P \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) + D_T\left(\tilde{U}_z\left(T, D_T\right) + P\right) \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) \middle| \mathcal{F}_t\right] + \eta\left(\tilde{U}_z\left(t, \eta\right) + \tilde{Y}_t\right).
\end{aligned} \tag{A.46}
$$

On the other hand, from (2.9) and (3.15), we have

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{U}\left(T,\eta Z_T^{t,q}\right)+\eta Z_T^{t,q}P\middle|\mathcal{F}_t\right]\\
&\geq \mathbb{E}\left[U\left(T,-\tilde{U}_z\left(T,D_T\right)\right)+\eta Z_T^{t,q}\tilde{U}_z\left(T,D_T\right)+\eta Z_T^{t,q}P\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}\left[U\left(T,-\tilde{U}_z\left(T,D_T\right)\right)+\eta Z_T^{t,q}\left(\tilde{U}_z\left(t,\eta\right)+\tilde{Y}_t-\int_t^T\pi_u^{*,P}\left(\theta_u du+dW_u^1\right)\right)\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}\left[U\left(T,-\tilde{U}_z\left(T,D_T\right)\right)\middle|\mathcal{F}_t\right]+\eta\left(\tilde{U}_z\left(t,\eta\right)+\tilde{Y}_t\right),
\end{aligned}
\tag{A.47}
$$

where in the last equality we used Lemma 2.1 for $\pi^{*,P}$ and $q\in\mathbb{L}^2_{BMO}[t,T]$. Combining (A.46) and (A.47) gives (A.45).

A.8 Proof of Theorem 3.11

(i). From Remark 3.9 we have that $\pi^{*,P}\in\mathcal{A}_{[t,T]}$ and

$$
\mathbb{E}\left[U\left(T,\hat{\xi}+\int_t^T\pi_u^{*,P}\left(\theta_u du+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right]=\mathbb{E}\left[U\left(T,-\tilde{U}_z\left(T,D_T\right)\right)\middle|\mathcal{F}_t\right].
\tag{A.48}
$$

Next, we verify that, for any $\pi\in\mathcal{A}_{[t,T]}$,

$$
\mathbb{E}\left[U\left(T,\hat{\xi}+\int_t^T\pi_u\left(\theta_u du+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right]\leq\mathbb{E}\left[U\left(T,\hat{\xi}+\int_t^T\pi_u^{*,P}\left(\theta_u du+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right].
\tag{A.49}
$$

By the concavity of U, (3.15) and (2.10), we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[U\left(T,\hat{\xi}+\int_t^T\pi_u\left(\theta_u du+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right]-\mathbb{E}\left[U\left(T,\hat{\xi}+\int_t^T\pi_u^{*,P}\left(\theta_u du+dW_u^1\right)+P\right)\middle|\mathcal{F}_t\right]\\
&\leq\mathbb{E}\left[U_x\left(T,\hat{\xi}+\int_t^T\pi_u^{*,P}\left(\theta_u du+dW_u^1\right)+P\right)\int_t^T\left(\pi_u-\pi_u^{*,P}\right)\left(\theta_u du+dW_u^1\right)\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}\left[U_x\left(T,-\tilde{U}_z\left(T,D_T\right)\right)\int_t^T\left(\pi_u-\pi_u^{*,P}\right)\left(\theta_u du+dW_u^1\right)\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}\left[D_T\int_t^T\left(\pi_u-\pi_u^{*,P}\right)\left(\theta_u du+dW_u^1\right)\middle|\mathcal{F}_t\right].
\end{aligned}
\tag{A.50}
$$

Note that $D=\eta Z^{t,q^{*,P}}$ with $q^{*,P}\in\mathcal{Q}_{[t,T]}$ by Theorem 3.10, and $\pi-\pi^{*,P}\in\mathcal{A}_{[t,T]}$. Thus, Lemma 2.1 gives

$$
\mathbb{E}\left[D_T\int_t^T\left(\pi_u-\pi_u^{*,P}\right)\left(\theta_u du+dW_u^1\right)\middle|\mathcal{F}_t\right]=0,
$$

which together with (A.50) proves (A.49).

(ii). Working as in the proof of Corollary 3.6, we obtain

$$
\tilde{u}^P\left(t,\eta;T\right)\geq\underset{\xi\in L^0(\mathcal{F}_t)}{\operatorname{esssup}}\left(u^P\left(t,\xi;T\right)-\xi\eta\right).
$$

On the other hand, by (i) and (2.15), we have that

$$\begin{aligned}u^P\left(t,\hat{\xi};T\right)&=\mathbb{E}\left[U\left(T,-\tilde{U}_z\left(T,D_T\right)\right)\Big|\,\mathcal{F}_t\right]\\&=\mathbb{E}\left[\tilde{U}\left(T,D_T\right)-D_T\tilde{U}_z\left(T,D_T\right)\Big|\,\mathcal{F}_t\right]\\&=\mathbb{E}\left[\tilde{U}\left(T,D_T\right)+D_TP\Big|\,\mathcal{F}_t\right]-\mathbb{E}\left[D_T\left(\tilde{U}_z\left(T,D_T\right)+P\right)\Big|\,\mathcal{F}_t\right],\end{aligned}$$

which, combined with Theorem 3.10, (3.15) and Lemma 2.1, yields

$$u^P\left(t,\hat{\xi};T\right)=\tilde{u}^P\left(t,\eta;T\right)+\hat{\xi}\eta.$$

A.9 Proof of Proposition 3.13

By Theorem 3.11, $\pi^{*,P}$, defined in (3.16), is optimal for $u^P(t,\hat{\xi};T)$, and, according to Remark 3.9,

$$\pi_s^{*,P}\left(\theta_s ds+dW_s^1\right)=-d\left(\tilde{U}_z\left(s,D_s\right)+\tilde{Y}_s\right).$$

Denote

$$X_s^*:=\hat{\xi}+\int_t^s\pi_u^{*,P}\left(\theta_u du+dW_u^1\right)=-\tilde{U}_z\left(s,D_s\right)-\tilde{Y}_s,\quad t\le s\le T. \tag{A.51}$$

Thus, $X_T^*=-\tilde{U}_z\left(T,D_T\right)-P$.

Applying Theorem 3.1 gives that $X_s=X_s^*=-\tilde{U}_z\left(s,D_s\right)-\tilde{Y}_s$. By (A.9) in the proof of Theorem 3.1, we have that, for $t\le s\le T$,

$$Y_s=-\tilde{U}_z\left(s,N_s\right)-X_s^*,$$

with

$$N_s=\mathbb{E}\left[U_x\left(T,X_T^*+P\right)\middle|\,\mathcal{F}_s\right]=\mathbb{E}\left[U_x\left(T,-\tilde{U}_z\left(T,D_T\right)\right)\Big|\,\mathcal{F}_s\right]=\mathbb{E}\left[D_T\middle|\,\mathcal{F}_s\right].$$

On the other hand, since $\tilde{Z}^2\in\mathbb{L}^2_{BMO}[t,T]$, D is a true martingale, which together with (A.51) implies that

$$Y_s=-\tilde{U}_z\left(s,D_s\right)-X_s=\tilde{Y}_s,\quad t\le s\le T. \tag{A.52}$$

By (A.17), we have, for $t\le s\le T$,

$$Z_s^1=\frac{\varphi_s^1-\alpha_x^1\left(s,X_s^*+Y_s\right)}{U_{xx}\left(s,X_s^*+Y_s\right)}-\pi_s^{*,P},\ \text{and}\ Z_s^2=\frac{\varphi_s^2-\alpha_x^2\left(s,X_s^*+Y_s\right)}{U_{xx}\left(s,X_s^*+Y_s\right)},$$

where φ^i, $i=1,2$, are given by (A.7). Since $N_s=D_s$, we obtain that

$$\varphi_s^1=-D_s\theta_s,\ \text{and}\ \varphi_s^2=-\frac{\tilde{\alpha}_z^2\left(s,D_s\right)+\tilde{Z}_s^2}{\tilde{U}_{zz}\left(s,D_s\right)}.$$

Combining (3.16), (A.52), (2.16) and (2.21), we easily deduce that $Z=\tilde{Z}$.

A.10 Proof of Proposition 3.14

From Theorem 3.5, $q^{*,P}$ defined in (3.6) is optimal for the dual problem (3.2), and, by Remark 3.2,

$$dU_x\left(s,X_s+Y_s\right)=-U_x\left(s,X_s+Y_s\right)\left(\theta_s dW_s^1+q_s^{*,P}dW_s^2\right).$$

Let $Z_s^* := \hat{\eta} Z_s^{t,q^{*,P}} = U_x(s, X_s + Y_s)$. Applying Theorem 3.8, we have

$$D_s = Z_s^* = U_x\left(s, X_s + Y_s\right), \quad t \leq s \leq T. \tag{A.53}$$

By (A.37) in the proof of Theorem 3.8, we know that

$$\tilde{Y}_s := \frac{M_s}{Z_s^*} - \tilde{U}_z\left(s, Z_s^*\right), \quad t \leq s \leq T,$$

where

$$\begin{aligned}
M_s &= \mathbb{E}\left[\left(\tilde{U}_z\left(T, Z_T^*\right) + P\right) Z_T^* \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\left(\tilde{U}_z\left(T, U_x\left(T, X_T + P\right)\right) + P\right) D_T \middle| \mathcal{F}_s\right] \\
&= -\mathbb{E}\left[X_T D_T \middle| \mathcal{F}_s\right].
\end{aligned}$$

Since $Z^i \in \mathbb{L}^2_{BMO}[t,T]$, $q^{*,P} \in \mathcal{Q}_{[t,T]}$ and

$$X_s = \xi + \int_t^s \pi_u^{*,P}\left(\theta_u du + dW_u^1\right),$$

where $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ by Theorem 3.3. Then, Lemma 2.1 yields that $M_s = -X_s D_s$ and

$$dM_s = \left(X_s D_s \theta_s - D_s \pi_s^{*,P}\right) dW_s^1 + X_s D_s q_s^{*,P} dW_s^2.$$

Thus,

$$\tilde{Y}_s = -\frac{X_s D_s}{D_s} - \tilde{U}_z\left(s, U_x\left(s, X_s + Y_s\right)\right) = Y_s, \quad t \leq s \leq T.$$

Finally, from (A.40), we have that, for $t \leq s \leq T$,

$$\begin{aligned}
\tilde{Z}_s^1 &:= \frac{M_s \theta_s + \psi_s^1}{Z_s^*} - \tilde{\alpha}_z^1\left(s, Z_s^*\right) + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* \theta_s, \\
\tilde{Z}_s^2 &:= \frac{M_s q_s^{*,P} + \psi_s^2}{Z_s^*} - \tilde{\alpha}_z^2\left(s, Z_s^*\right) + \tilde{U}_{zz}\left(s, Z_s^*\right) Z_s^* q_s^{*,P},
\end{aligned}$$

which, together with (3.6), (A.53), (2.13) and (2.20), gives that $\tilde{Z} = Z$.

A.11 Proof of Proposition 3.15

The proof of the maturity independence of the value function $\tilde{u}^P$ relies on the self-generation property of $\tilde{U}$. For $t \leq T \leq T'$, note that, for any $q \in \mathcal{Q}_{[t,T']}$, we have that

$$Z_{T'}^{t,q} = Z_T^{t,q} Z_{T'}^{T,q} \text{ and } \mathbb{E}[Z_{T'}^{T,q} | \mathcal{F}_T] = 1,$$

since $q \in \mathbb{L}^2_{BMO}[t,T']$. Then, by (3.2) and the tower property of conditional expectation, we obtain

$$\begin{aligned}
\tilde{u}^P\left(t, \eta; T'\right) &= \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T']}} \mathbb{E}\left[\tilde{U}\left(T', \eta Z_{T'}^{t,q}\right) + \eta Z_{T'}^{t,q} P \middle| \mathcal{F}_t\right] \\
&= \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T']}} \mathbb{E}\left[\mathbb{E}\left[\tilde{U}\left(T', \eta Z_T^{t,q} Z_{T'}^{T,q}\right) + \eta Z_T^{t,q} Z_{T'}^{T,q} P \middle| \mathcal{F}_T\right] \middle| \mathcal{F}_t\right] \\
&= \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T']}} \mathbb{E}\left[\mathbb{E}\left[\tilde{U}\left(T', \eta Z_T^{t,q} Z_{T'}^{T,q}\right) \middle| \mathcal{F}_T\right] + \eta Z_T^{t,q} P \middle| \mathcal{F}_t\right].
\end{aligned} \tag{A.54}$$

From Remark 3.7, we have that

$$\mathbb{E}\left[\tilde{U}\left(T', \eta Z_T^{t,q} Z_{T'}^{T,q}\right)\Big|\,\mathcal{F}_T\right] \geq \tilde{U}\left(T, \eta Z_T^{t,q}\right), \quad \text{for any } q \in \mathcal{Q}_{[T,T'],}$$

and, in addition, there exists $q^* \in \mathcal{Q}[T,T']$ such that

$$\tilde{U}\left(t, \eta Z_T^{t,q}\right) = \mathbb{E}\left[\tilde{U}\left(T', \eta Z_T^{t,q} Z_{T'}^{T,q^*}\right)\Big|\,\mathcal{F}_T\right].$$

It then follows that

$$\tilde{u}^P\left(t,\eta;T'\right) \geq \underset{q\in\mathcal{Q}_{[t,T]}}{\operatorname{essinf}}\, \mathbb{E}\left[\tilde{U}\left(T, \eta Z_T^{t,q}\right) + \eta Z_T^{t,q} P\Big|\,\mathcal{F}_t\right],$$

and

$$\begin{aligned}
\tilde{u}^P\left(t,\eta;T'\right) &\leq \underset{q\in\mathcal{Q}_{[t,T]}}{\operatorname{essinf}}\, \mathbb{E}\left[\mathbb{E}\left[\tilde{U}\left(T', \eta Z_T^{t,q} Z_{T'}^{T,q^*}\right)\Big|\,\mathcal{F}_T\right] + \eta Z_T^{t,q} P\Big|\,\mathcal{F}_t\right] \\
&= \underset{q\in\mathcal{Q}_{[t,T]}}{\operatorname{essinf}}\, \mathbb{E}\left[\tilde{U}\left(t, \eta Z_T^{t,q}\right) + \eta Z_T^{t,q} P\Big|\,\mathcal{F}_t\right].
\end{aligned}$$

Thus,

$$\tilde{u}^P\left(t,\eta;T'\right) = \underset{q\in\mathcal{Q}_{[t,T]}}{\operatorname{essinf}}\, \mathbb{E}\left[\tilde{U}\left(t, \eta Z_T^{t,q}\right) + \eta Z_T^{t,q} P\Big|\,\mathcal{F}_t\right] = \tilde{u}^P\left(t,\eta;T\right).$$

To show the maturity independence of the value function u^P, we observe that FBSDE (3.5) on $[t,T]$ with initial-terminal condition (ξ, P) can be extended to $[t,T']$. Indeed, if (X,Y,Z) is a solution to FBSDE (3.5) on $[t,T]$ with initial-terminal condition (ξ,P), then $(\bar{X},\bar{Y},\bar{Z})$ defined by

$$\begin{aligned}
\bar{X}_s &:= X_s \mathbf{1}_{[t,T)}(s) + X_s^{T',P}\mathbf{1}_{[T,T']}(s), \\
\bar{Y}_s &:= Y_s \mathbf{1}_{[t,T)}(s) + P\mathbf{1}_{[T,T']}(s), \\
\bar{Z}_s &:= Z_s \mathbf{1}_{[t,T]}(s) + 0\cdot\mathbf{1}_{(T,T']}(s)
\end{aligned}$$

is a solution to FBSDE (3.5) on $[t,T']$ with initial-terminal condition (ξ,P), where $X^{T',P}$ satisfies

$$dX_s^{T',P} = -\frac{U_x\left(s, X_s^{T',P}\right)\theta_s + \alpha_x^1\left(s, X_s^{T',P}\right)}{U_{xx}\left(s, X_s^{T',P}\right)}\left(\theta_s ds + dW_s^1\right),$$

with initial condition $X_T^{T',P} = -\tilde{U}_z(T, \hat{\eta} Z_T^{t,q^{*,P}})$, and $\hat{\eta}$ and $q^{*,P}$ are respectively defined by (3.9) and (3.6). Thus, according to the first part of the Proposition and Theorem 3.5 (ii), we obtain that

$$\begin{aligned}
u^P\left(t,\xi;T'\right) &= \underset{\eta\in L^{0,+}(\mathcal{F}_t)}{\operatorname{essinf}}\left(\tilde{u}^P\left(t,\eta;T'\right) + \xi\eta\right) \\
&= \underset{\eta\in L^{0,+}(\mathcal{F}_t)}{\operatorname{essinf}}\left(\tilde{u}^P\left(t,\eta;T\right) + \xi\eta\right) \\
&= u^P\left(t,\xi;T\right),
\end{aligned}$$

and we easily conclude.

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