Temporal and spatial turnpikes in Ito-diffusion markets under forward performance criteria*

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Abstract

We present turnpike-type asymptotic results for the relative forward risk tolerance under time-monotone forward performance criteria in Ito-diffusion markets. We show that, contrary to the classical turnpike results in lognormal markets, the temporal (large time) and spatial (large wealth) limits do not coincide and, furthermore, depend crucially on the support of the risk preference measure, used to construct the underlying forward utility. Specifically, the spatial limit coincides with the right-end of the support while the temporal limit with the left-end one. Key role plays the spatial inverse of a space-time harmonic function that solves the ill-posed heat equation. We construct and analyze in detail two representative examples, one with discrete and the other with continuous measure support.

Key words: Turnpike portfolios, forward performance criterion, risk preference measure, forward risk tolerance function, time-monotone forward utilities.

1 Introduction

Turnpike portfolios provide an intriguing connection among the behavior of optimal investment policies, long horizons and asymptotic wealth properties of the agent's terminal utility. They have a long history with the early works going back to the classical papers [25] and [20], as well as [36], [14] and [16], for discrete time optimal portfolio models. In continuous time, turnpike questions were first

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examined by [7] (see, also, [17]) and the turnpike property was established using contingent claim methods. Their results were later extended in [15] using an autonomous pde of fast-diffusion type satisfied by the optimal feedback investment strategy and viscosity solutions arguments. Within the same lognormal model, the authors of [4] studied the rate of convergence. Duality methods were used in [8] for complete markets where the assumption of independent returns was relaxed. The semi-martingale incomplete market case was extensively studied in [13] where various turnpike notions (abstract, classic and explicit) were introduced. An important contribution of this work is the study of the interplay between turnpike-type behavior and the so-called extra hedging demand, the component that arises beyond the myopic term in the optimal portfolios.

A closer look at *all* existing turnpike results yields that we are essentially working in a single investment horizon setting, $[0,T]$, $T<\infty$, which is taken to be very long. As a result, in order to properly define the optimization problem, one needs to pre-commit to a market model for this long horizon. This choice cannot be modified later on if time consistency is desired; on the other hand, knowing the model dynamics for a very long horizon is not realistic. Besides the stringent constraints on model (no matter how general) choice, one also pre-commits at initial time to a utility function for very far in the future, which is also a rather restrictive assumption. Finally, we remark that no matter how large T is, the optimal investment problem is not defined beyond this point, for the utility function is chosen only for T and, thus, the underlying problem is well defined on $[0,T]$ only.

Such considerations motivate us to initiate a study of turnpike portfolios within the extended class of utilities, called *forward performance processes* (or forward utilities) which were proposed to precisely accommodate the above limitations in optimal portfolio selection. These forward optimization criteria were introduced by Musiela and the second author in [28] (see, also, [29]) and offer performance measurement flexibility under model adaptation, model ambiguity, alternative market views, rolling horizons, and others. We recall its definition herein and for further details we refer the reader, among others, to [2], [9], [10], [18], [19], [21], [22], [24], [30], [29], [38], [39] and [40]; see, also, the recent review article [27]. In [31], it was shown that forward utilities solve a forward in time (ill-posed) SPDE, with their volatility being a modeling input, see equation (76) herein.

The fundamental difference between the classical and the forward setting is that in the former we commit to a single horizon $[0,T]$ while in the latter we define, from the beginning, the investment problem for all times $t\in[0,\infty)$. Moreover, instead of choosing at the initial time the utility $U_T(x)$ for T, we choose the utility $U(x,0)$ at this initial time. Finally, the model coefficients are not pre-specified at initiation but are being updated in a progressively measurable way forward in time. As a result, for turnpike problems we see that we are now able to define the problem for all future times, and not just up to a fixed T which is taken to be very long, to work with arbitrary Ito-diffusion models without pre-committing to market coefficients for all future times and, lastly, to choose the utility at initial time instead at a remote point in time.

Our study herein focuses on the subclass of *time-monotone* forward utilities (zero volatility) in general Ito-diffusion markets. The motivation is twofold. Firstly, this class has a well-understood structure which permits us to differentiate the effects of market movements from the changes in the agent's individual profile (cf. (8)). This, in turn, allow us to produce rather exact answers to various turnpike questions. Secondly, there are various similarities between the risk tolerance functions within the classical and the forward framework, and this facilitates the comparative study between the two frameworks. The general case of non-zero forward volatility is deferred to future study (see section 6). Time-monotone criteria (see [32]) are given by processes $U(x,t)$, $t \geq 0$, $x \geq 0$, of the form $U(x,t) = u(x, A_t)$, with $u(x,t)$ being a deterministic function solving a non-linear equation (cf. (9)) and the market input process $A_t = \int_0^t \|\lambda_s\|^2 ds$, λ_t being the market price of risk. Furthermore, the optimal investment process π_t^* is given by

$$\pi_t^* = \sigma_t^+ \lambda_t r(X_t^*, A_t) \quad \text{with} \quad r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)},$$

where σ_t^+ is the Moore-Penrose pseudo-inverse of the volatility matrix and X_t^* the optimal wealth generated by π_t^* (cf. (7)). We will refer to $r(x,t)$ as the forward risk tolerance function.

Next, we introduce the turnpike problem in the forward setting and state the main results. Firstly, we recall the classical turnpike property in a lognormal model with marginal utility $U_T'(x)$ (see, for example, [4] and [15]). If $r(x_0, t; T)$ is the associated dynamic rick tolerance function and $\gamma \in (0,1)$, then

$$\lim_{x \uparrow \infty} U_T'(x) = x^{\gamma - 1} \quad \Longrightarrow \quad \lim_{T \uparrow \infty} \frac{r(x_0, t; T)}{x_0} = \frac{1}{1-\gamma}, \tag{1}$$

for each $x_0 > 0$. In other words, if the marginal utility behaves like a power function for large x then, if the horizon T is very long, the associated dynamic risk tolerance function (or, equivalently, the optimal feedback portfolio function $\pi^*(x,t;T)$) approximates the one corresponding to this power utility. Furthermore, it follows easily, that for fixed $T_0 > 0$ and $x_0 > 0$, respectively, we have

$$\lim_{x \uparrow \infty} \frac{r(x,t;T_0)}{x} = \lim_{T \uparrow \infty} \frac{r(x_0,t;T)}{x_0} = \frac{1}{1-\gamma}. \tag{2}$$

We are, thus, motivated to investigate if similar results hold for the forward setting, keeping in mind that asymptotic conditions must be now put in the initial marginal utility $u_x(x,0)$. The first striking observation, as demonstrated in the example of subsection 2.2, is that in general the analogous limits in (2) do *not coincide.* As a result, one needs to investigate *separately* the *spatial* and *temporal* limits of the relative forward risk tolerance function,

$$\lim_{x \uparrow \infty} \frac{r(x,t_0)}{x} \quad \text{and} \quad \lim_{t \uparrow \infty} \frac{r(x_0,t)}{x_0}, \tag{3}$$

for $t_0 \geq 0$, $x_0 > 0$, under respective conditions on the asymptotic behavior of the initial datum for large x. These are the turnpike problems we investigate herein.

Pivotal role for determining these limits is played by a positive finite Borel measure, μ, to be called the *risk preference measure,* which is the defining element in the construction of the time monotone forward processes. Specifically, it was shown in [32] that u in (9) is uniquely related (modulo an immaterial constant) to a space-time harmonic function $h : \mathbb{R}\times [0,\infty) \longrightarrow \mathbb{R}^+$, which itself is (uniquely) characterized by an integral transform. Namely, we have that $u_x\left(h\left(z,t\right),t\right) = e^{-z+\frac{t}{2}}$ and for some a, b with $0 \leq a \leq b \leq \infty$,

$$h\left(z,t\right) = \int_a^b e^{yz-\frac{1}{2}y^2 t}\mu\left(dy\right). \tag{4}$$

It also follows that the forward risk tolerance satisfies $r\left(h(z,t),t\right) = h_z\left(z,t\right)$ and, thus, we have the implicit representation

$$r\left(x,t\right) = \int_a^b y e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu\left(dy\right). \tag{5}$$

As discussed later, the measure μ is an investor specific input and, in particular, is fully specified by the initial (inverse) marginal utility (cf. (14)).

Main results: We establish that, under respective asymptotic assumptions on the initial risk preferences, the *spatial* limit of the relative forward risk tolerance function coincides with the *right-end* point of the support of the risk preference measure while the *temporal* limit coincides with the *left-end one.* In other words, for a and b as in (4), we have for each $t_0 \geq 0$ and $x_0 > 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x} = b \quad \text{and} \quad \lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0} = a. \tag{6}$$

For the spatial turnpike limit, we first show that the analogous assumption to (1), namely,

$$u_x\left(x,0\right) \sim\ x^{\gamma-1}, \quad x \ \text{ large}, \gamma \in (0,1)\,,$$

holds if and only if the support of the measure is finite, with its right-end point being $b = \frac{1}{1-\gamma}$ and, in addition, $\mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1$. We then establish the first turnpike limit in (6) using representation (5), equation (9) and various convexity properties of h and its derivatives.

For the temporal turnpike limit, we first establish that if there exists $\gamma \in (0,1)$ such that for all $\gamma' \in (\gamma,1)$ and all $\gamma'' \in (0,\gamma)\,,$

$$\lim_{x\uparrow\infty}\frac{u_x\left(x,0\right)}{x^{\gamma'-1}} = 0 \qquad \text{and} \qquad \lim_{x\uparrow\infty}\frac{u_x\left(x,0\right)}{x^{\gamma''-1}} = \infty,$$

then, the right-end of the support must satisfy $b = \frac{1}{1-\gamma}$ and vice versa. In turn, we establish the temporal limit in (6), which is the genuine analogue of the classical turnpike results. Obtaining this limit is considerably harder than in the classical case due to the ill-posed nature of the problem. Indeed, the methodology used in [15] is inapplicable due to the lack of comparison results for the ergodic version of the equation satisfied by $r(x,t)$. The approach of [4] does not apply either because of lack of connection between the solutions of the ill-posed heat equation and Feynman-Kac type representation of its solution. Therefore, one needs to work directly with the implicit representation (5) where, however, the spatial inverse $h^{(-1)}$ is involved, which is not explicitly known. The key step in obtaining the temporal limit is to show that, for each $x_0 > 0$, the spatial inverse of h satisfies $\lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t} = \frac{a}{2}$, a result of independent interest. We also provide the rate of convergence for the temporal limit.

We present two representative examples. In the first one, the risk preference measure is a finite sum of Dirac functions while, in the second, it is taken to be the Lebesgue measure. To calculate the limits in (6), we first derive asymptotic expansions for both the auxiliary function $h^{(-1)}$ and the forward risk tolerance function.

The paper is structured as follows. In section 2, we introduce the market model, we recall the forward performance criterion and present a motivating example demonstrating that the temporal and spatial limits do not in general coincide. In section 3 and 4 we present, respectively, the spatial and temporal forward turnpike results. In section 5 we provide two representative examples and conclude in section 6 where we state future research directions.

2 The model and the forward investment criterion

The market environment consists of one riskless and k risky securities. The prices of the risky securities are modelled as Ito-diffusion processes, namely, the price S_t^i, $t \geq 0$, of the i^{th} risky asset follows

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right),$$

with $S_0^i > 0$, for $i = 1, ..., k$. The process $W_t = \left(W_t^1, ..., W_t^d\right)$, $t \geq 0$, is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}$, $t \geq 0$.

The coefficients μ_t^i and $\sigma_t^i = \left(\sigma^{1i}, ..., \sigma_t^{di}\right)$, $t \geq 0$, $i = 1, ..., k$, are $\mathcal{F}_t$-adapted processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote by σ_t the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} asset. We may, then, alternatively, write the above equation as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right).$$

The riskless asset (the savings account) is taken to be the numeraire and has price process B_t, $t \geq 0$, satisfying $dB_t = r_t B_t dt$ with $B_0 = 1$, and for a nonnegative $\mathcal{F}_t-$adapted interest rate process r_t, $t \geq 0$. We, also, denote by μ_t the k-dimensional vector with coordinates μ_t^i and by $\mathbf{1}$ the k-dimensional vector with every component equal to one. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that $\mu_t - r_t\mathbf{1} \in Lin\left(\sigma_t^T\right)$ where $Lin\left(\sigma_t^T\right)$ denotes the linear space generated by the columns of the transpose σ_t^T. This implies that the vector $\lambda_t := \left(\sigma_t^T\right)^+ (\mu_t - r_t\mathbf{1})$, $t \geq 0$, known as the market price of risk, is well defined and solves the equation $\sigma_t^T z = \mu_t - r_t\mathbf{1}$, where $\left(\sigma_t^T\right)^+$ is the Moore-Penrose pseudo-inverse[1] of σ_t^T. It is assumed that there exists a deterministic constant $c > 0$ such that $\|\lambda_t\| \leq c$, $t \geq 0$.

Starting at $t = 0$ with initial endowment $x \geq 0$, the investor invests at all times $t > 0$ in the riskless and risky assets. The (discounted by the numeraire) amounts invested are denoted by the processes π_t^0 and π_t^i, $t \geq 0$, $i = 1, ..., k$, respectively, and are taken to be self-financing. The present value of her total investment is given by the wealth process X_t^π, $t \geq 0$, with $X_t^\pi = \sum_{i=0}^N \pi_t^i$, which solves

$$dX_t^\pi = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t)\,, \quad X_0^\pi = x \geq 0, \tag{7}$$

with the (column) vector $\pi_t = \left(\pi_t^i; i = 1, ..., k\right)$. The wealth process is taken to satisfy the non-negativity constraint $X_t^\pi \geq 0$, $t > 0$. The set of admissible policies is given by

$$\mathcal{A} = \left\{ \pi : \text{self-financing},\ \pi_t \in \mathcal{F}_t,\ E_{\mathbb{P}} \int_0^t \|\sigma_s \pi_s\|^2 \, ds < \infty,\ X_t^\pi \geq 0,\ t > 0 \right\}.$$

The performance of admissible investment strategies is measured via the so-called *forward performance criteria* introduced in [28] (see also, the references mentioned in the Introduction). For the reader's convenience, we review their definition next.

We will be using throughout the domain notations $\mathbb{D}_+ = \mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{D} = \mathbb{R} \times \mathbb{R}_+$.

Definition 1 *An $\mathcal{F}_t$-adapted process $U(x,t)$, $(x,t) \in \mathbb{D}_+$, is a forward performance criterion if,*

i) for each $t \geq 0$, the mapping $x \to U(x,t)$ is strictly increasing and strictly concave,

ii) for each $\pi \in \mathcal{A}$, $U(X_t^\pi, t)$ is a (local) supermartingale,

iii) there exists $\pi^* \in \mathcal{A}$ such that $U(X_t^{\pi^*}, t)$ is a (local) martingale.

Herein we focus on the family of *time-monotone* forward performance processes, which constitute a rich enough class of forward criteria. They were extensively studied in [32] and we refer the reader therein for all technical details. We only review the main results used herein, some of which have been already stated in the Introduction.

[1] For further details see [32]; for general references, see [35] and [23].

2.1 Review of time-monotone forward performance criteria

Time-monotone forward performance criteria are uniquely represented by processes of the form

$$U(x,t) = u(x, A_t), \tag{8}$$

where $u : \mathbb{D}_+ \to \mathbb{R}_+$ expresses the *individual dynamic risk preferences;* it is strictly increasing and strictly concave in x, and satisfies

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}. \tag{9}$$

The *market input* processes A_t and M_t, $t \geq 0$, are defined as

$$M_t = \int_0^t \lambda_s \cdot dW_s \quad \text{and} \quad A_t = \langle M \rangle_t = \int_0^t \|\lambda_s\|^2 \, ds. \tag{10}$$

Central role in the entire construction plays a space-time harmonic function $h : \mathbb{D} \to \mathbb{R}_+$, defined by

$$u_x(h(z,t),t) = e^{-z+\frac{t}{2}}. \tag{11}$$

It solves, as it follows from (9) and (11), the ill-posed heat equation

$$h_t + \frac{1}{2}h_{zz} = 0, \tag{12}$$

and, moreover, for each $t \geq 0$, it is positive and strictly increasing in z. It was shown in [32] (see, also, [33]) that such solutions are uniquely represented in the integral form

$$h(z,t) = \int_a^b \frac{e^{yz-\frac{1}{2}y^2 t} - 1}{y} \nu(dy) + C,$$

where C is a generic constant. The measure $\nu \in \mathcal{B}^+(\mathbb{R})$, where $\mathcal{B}^+(\mathbb{R})$ is the set of positive Borel measures, with the additional properties that, for $z \in \mathbb{R}$,

$$\nu((-\infty,0]) = 0, \quad \int_a^b e^{yz}\nu(dy) < \infty \quad \text{and} \quad \int_a^b \frac{1}{y}\nu(dy) < \infty.$$

To simplify the presentation we choose without loss of generality the constant $C = \int_a^b \frac{1}{y} d\nu(y)$ and introduce the normalized measure $\mu\,(dy) = \frac{1}{y}\nu(dy)$. Then, the function h admits the representation, for $(z,t) \in \mathbb{D}$,

$$h(z,t) = \int_a^b e^{yz-\frac{1}{2}y^2 t}\mu(dy), \text{ with } 0 \leq a \leq b \leq \infty. \tag{13}$$

In turn, from (11) and (13), we obtain that $u(x,t)$ is represented as

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_z\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz.$$

Note that, for $t = 0$, we have that $h(z,0) = \int_a^b e^{yz}\mu(dy)$ and the forward initial datum is given by

$$U(x,0) = u(x,0) = \int_0^x e^{-h^{(-1)}(z,0)}dz,$$

which is, thus, fully specified by the risk preference measure through $h^{(-1)}(x,0)$. In other words, the initial inverse marginal utility *must be* of the form

$$(U'(x,0))^{(-1)} = (u_x(x,0))^{(-1)} = \int_a^b x^{-y}\mu(dy). \tag{14}$$

We stress that (14) expresses an *if and only if* characterization in the sense that only initial utility data with inverse marginals of the above structure are admissible, otherwise equation (9) does not have a well-defined solution for all times in $[0,\infty)$. Inverse marginal utilities of form (14) were recently extensively studied in [26] in the classical setting.

The *forward risk tolerance* function $r : \mathbb{D}_+ \to \mathbb{R}_+$ is defined as

$$r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{15}$$

and (11) yields its implicit representation

$$r(x,t) = h_z\left(h^{(-1)}(x,t),t\right) = \int_a^b y e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy). \tag{16}$$

It, also, satisfies the ill-posed fast-diffusion type equation

$$r_t + \frac{1}{2}r^2 r_{xx} = 0, \quad r(x,0) = \int_a^b y e^{yh^{(-1)}(x,0)}\mu(dy), \tag{17}$$

and, for each $t \geq 0$, $\lim_{x\downarrow 0} r(x,t) = r(0,t) = 0$.

The optimal portfolio process can be written as

$$\pi_t^* = \sigma_t^+ \lambda_t r(X_t^*, A_t).$$

It is easily seen that, for each $t \geq 0$, the function $h\,(.,t)$ is absolutely monotonic, since $\frac{\partial^i h(z,t)}{\partial z^i} > 0$, $i \geq 1$. Such functions satisfy, for each $t \geq 0$, the well-known inequality

$$\frac{\partial^{i+1}h\,(z,t)}{\partial z^{i+1}}\frac{\partial^{i-1}h\,(z,t)}{\partial z^{i-1}} - \left(\frac{\partial^i h\,(z,t)}{\partial z^i}\right)^2 \geq 0.$$

In turn, for each $t \geq 0$, $r\,(.,t)$ is strictly increasing and strictly convex, since

$$r_x(x,t) = \frac{h_{zz}\left(h^{(-1)}(x,t),t\right)}{r(x,t)} = \frac{1}{r(x,t)}\int_a^b y^2 e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy) > 0,$$

and

$$r_{xx}(x,t) = \frac{1}{r^3(x,t)} \left(h_{zzz}(z,t)h_z(z,t) - h_{zz}^2(z,t)\big|_{z=h^{(-1)}(x,t)} \right) > 0.$$

Equation (17) and the above convexity property yields that that $r(x,.)$ is strictly decreasing, $r_t(x,t) < 0$.

We note that throughout we will frequently differentiate under the integral sign in (13) and in similar integrals, which is permitted as explained in [32][2].

As stated in the Introduction, the aim herein is to investigate the spatial and temporal limits of $\frac{r(x,t)}{x}$, with $r(x,t)$ as in (15). We first provide an example which shows that, contrary to classical turnpike results (see [15] and [4]), these two limits do *not* in general coincide in the forward setting.

2.2 A motivating example

Case 1: *Single Dirac function*

The risk preference measure is Dirac, $\mu = \delta_{\frac{1}{1-\gamma}}$, $\gamma \in (0,1)$. From (13) and (11) we have, for $(z,t) \in \mathbb{D}$ and $(x,t) \in \mathbb{D}_+$, respectively,

$$h(z,t) = e^{\frac{1}{1-\gamma}z - \frac{1}{2(1-\gamma)^2}t} \quad \text{and} \quad u_x(x,t) = x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t}.$$

Therefore, the forward risk tolerance is given by $r(x,t) = \frac{1}{1-\gamma}x$ and we, easily, conclude that the spatial and temporal limits are equal, given by (for fixed t_0 and x_0, respectively).

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \frac{1}{1-\gamma} \quad \text{and} \quad \lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = \frac{1}{1-\gamma}.$$

Case 2: *Sum of two Dirac functions*

The risk preference measure is given, for $\theta, \gamma \in (0,1)$, by

$$\mu = \delta_{\frac{1}{1-\theta}} + \delta_{\frac{1}{1-\gamma}} \quad \text{with} \quad \frac{1}{1-\gamma} = 2\frac{1}{1-\theta}. \tag{18}$$

To ease the presentation, we set $\kappa = \frac{1}{1-\theta}$. Then, (13) yields

$$h(z,0) = e^{\kappa z} + e^{2\kappa z},$$

and, from (11),

$$\left(u_x(x,0)\right)^{(-1)} = x^{-\frac{1}{1-\theta}} + x^{-\frac{1}{1-\gamma}}. \tag{19}$$

Therefore, $u_x(x,0) = 2^{1-\theta}\left(\sqrt{1+4x}-1\right)^{\theta-1}$, and, thus,

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{x\uparrow\infty} \frac{2^{2(1-\gamma)}\left(\sqrt{1+4x}-1\right)^{2(\gamma-1)}}{x^{\gamma-1}} = 1. \tag{20}$$

[2]It can be also seen directly since, after differentiation, the relevant integrands are jointly continuous in their respective arguments - see Theorem 24.5 in [1] and the remark following it .

Furthermore, (12) and $h(z,0)$ above yield, for $(z,t)\in\mathbb{D}$,

$$h(z,t)=e^{\kappa z-\frac{1}{2}\kappa^2 t}+e^{2\kappa z-2\kappa^2 t}, \tag{21}$$

and, thus, for $(x,t)\in\mathbb{D}_+$,

$$h^{(-1)}(x,t)=\frac{1}{2}\kappa t+\frac{1}{\kappa}\ln\frac{2x}{1+\sqrt{1+4xe^{-\kappa^2 t}}}. \tag{22}$$

Next, we calculate the forward risk tolerance function using (15). Introducing $f\left(x,t\right):=1+\sqrt{1+4xe^{-\kappa^2 t}}$, rewriting the above as

$$h^{(-1)}(x,t)=\frac{1}{\kappa}\ln\frac{2xe^{\frac{\kappa^2 t}{2}}}{f(x,t)},$$

and observing from (21) that $h_z(z,t)=\kappa e^{\kappa z-\frac{1}{2}\kappa^2 t}+2\kappa e^{2\kappa z-2\kappa^2 t}$, we compute

$$r(x,t)=h_z\left(h^{(-1)}(x,t),t\right)=\kappa\exp\left(\ln\frac{2xe^{\frac{\kappa^2 t}{2}}}{f(x,t)}-\frac{1}{2}\kappa^2 t\right)$$

$$+2\kappa\exp\left(2\ln\frac{2xe^{\frac{\kappa^2 t}{2}}}{f(x,t)}-2\kappa^2 t\right)=2\kappa\frac{x}{f\left(x,t\right)}+8\kappa\frac{x^2}{f^2\left(x,t\right)}e^{-\kappa^2 t}.$$

Note that, for each $x_0>0$, $\lim_{t\uparrow\infty}f\left(x_0,t\right)=2$ but, for each $t_0\geq 0$, $\lim_{x\uparrow\infty}\frac{1}{f(x,t_0)}=0$. Therefore, for each $x_0>0$,

$$\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}=\lim_{t\uparrow\infty}\frac{2\kappa}{f\left(x_0,t\right)}=\kappa.$$

On the other hand, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\lim_{x\uparrow\infty}\frac{8\kappa x}{f^2\left(x,t_0\right)}e^{-\kappa^2 t_0}=\lim_{y\uparrow\infty}\frac{8\kappa xe^{-\kappa^2 t_0}}{\left(1+\sqrt{1+4xe^{-\kappa^2 t_0}}\right)^2}=2\kappa.$$

In summary, reverting to the original notation, we have that, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\frac{2}{1-\theta}=\frac{1}{1-\gamma}, \tag{23}$$

while, for each $x_0>0$,

$$\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}=\frac{1}{1-\theta}. \tag{24}$$

Thus, the spatial and temporal limits of the forward relative risk tolerance do *not* coincide.

Next, we make two important observations. Firstly, from (18) we have for the support of the risk preference measure,

$$supp\left(\mu\right)=\left(\left\{\frac{1}{1-\theta}\right\},\left\{\frac{1}{1-\gamma}\right\}\right),$$

with θ and γ as in (18). Therefore, the *temporal* limit (24) coincides with the *left-end* of the support while the *spatial* limit (23) with the *right-end*.

Secondly, for each $x_0 > 0$, the temporal limit of the ratio $\frac{h^{(-1)}(x_0,t)}{t}$ is equal to *half* of the left-end point of the support; indeed, (22) yields

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}=\frac{1}{2(1-\theta)}.$$

In section 4 we show that both these properties are always valid. In particular, we will see that it is precisely the limit of the ratio $\frac{h^{(-1)}(x,t)}{t}$ that plays the key role in establishing the temporal turnpike property for general risk preference measures.

To juxtapose the above results with the ones in the classical expected utility setting, we compute analogous quantities and associated limits for the cases analyzed in [15] and [4] for lognormal markets (Merton problem) because, as mentioned in the Introduction, their optimal feedback portfolio policies resemble the ones with time-monotone forward criteria. Without loss of generality, we consider a market with a single lognormal stock with mean rate of return μ and volatility σ, and a riskless account of constant interest rate r.

To this end, we fix an investment horizon $T > 0$ and, in analogy to (19), we take the *terminal* inverse marginal utility, $I_T(x) = (U_T')^{(-1)}(x)$, to be of the form

$$I_T(x)=x^{-\frac{1}{1-\theta}}+x^{-\frac{1}{1-\gamma}},$$

for $x > 0$ and θ,γ as in (18). This corresponds to terminal marginal utility $U_T'(x) = 2^{1-\gamma}\left(\sqrt{1+4x}-1\right)^{\gamma-1}$ and, thus, in analogy to (20), we have that

$$\lim_{x\uparrow\infty}\frac{U_T'(x)}{x^{\gamma-1}}=1.$$

We consider the value function, denoted by $u(x,t;T)$, of the associated Merton problem, for $t\in[0,T]$. Letting $\tau = T-t$ be the time to the end of the investment horizon, we deduce, using well-known results, that the function $\tilde{u}(x,\tau)\equiv u(x,T-t;T)$ satisfies, for $(x,\tau)\in\mathbb{R}_+\times[0,T]$ and $\lambda := \frac{\mu-r}{\sigma}$, the Hamilton-Jacobi-Bellman equation

$$\tilde{u}_\tau+\frac{1}{2}\lambda^2\frac{\tilde{u}_x^2}{\tilde{u}_{xx}}=0,$$

with $\tilde{u}(x,0)=U_T(x)$.

In turn, the inverse spatial marginal value function, $\tilde{v}:\mathbb{R}_+\times[0,T)\to\mathbb{R}_+$ solves $\tilde{v}_\tau = \frac{1}{2}\lambda^2x^2\tilde{v}_{xx}+\lambda^2x\tilde{v}_x$, with $\tilde{v}(x,0)=I_T(x)$. We easily deduce that $\tilde{v}(x,\tau)=e^{\alpha\tau}x^{-\alpha}+e^{\beta\tau}x^{-2\alpha}$ with $\alpha=\frac{\gamma}{2(1-\gamma)^2}\lambda^2$ and $\beta=\frac{1+\gamma}{(1-\gamma)^2}\lambda^2$. Note that $\beta>2\alpha$.

Taking the spatial inverse of $\tilde{v}(x,\tau)$ yields

$$\tilde{u}_x(x,\tau)=\left(\frac{e^{\alpha\tau}+\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}}}{2x}\right)^{1-\theta}.$$

Therefore, the related dynamic risk tolerance function $\tilde{r}(x,\tau)$ satisfies (using that $\frac{1}{r(x,\tau)} = -\frac{\partial}{\partial x}\ln \tilde{u}_x(x,\tau)$),

$$\tilde{r}(x,\tau) = \frac{1}{1-\theta}\left(\frac{1}{x} - \frac{2e^{\beta\tau}}{e^{a\tau}\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}} + (e^{2\alpha\tau}+4xe^{\beta\tau})}\right)^{-1}$$

and, thus,

$$\frac{\tilde{r}(x,\tau)}{x} = \frac{1}{1-\theta}\left(1 - x\frac{2e^{\beta\tau}}{e^{a\tau}\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}} + (e^{2\alpha\tau}+4xe^{\beta\tau})}\right)^{-1}.$$

Direct calculations yield that, for each $\tau_0 > 0$ and each $x_0 > 0$, respectively, the spatial and temporal limits are given by

$$\lim_{x\uparrow\infty}\frac{\tilde{r}(x,\tau_0)}{x} = \frac{1}{1-\gamma} \quad \text{and} \quad \lim_{\tau\uparrow\infty}\frac{\tilde{r}(x_0,\tau)}{x_0} = \frac{1}{1-\gamma}.$$

The two limits are equal and, furthermore, coincide with the right-end point $\frac{1}{1-\gamma}$.

Motivated by this example, we are investigating the spatial and temporal asymptotic limits (3) of the forward relative risk tolerance function. To simplify the presentation, the risk aversion coefficient γ is taken to belong to $(0,1)$, as the case $\gamma \leq 0$ can be similarly analyzed.

3 Spatial forward turnpike property

We examine the spatial asymptotic behavior of the relative forward risk tolerance, $\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}$, for each $t_0 \geq 0$, under asymptotic assumptions for large wealth levels of the investor's initial risk preferences. In accordance with a similar assumption in [15] and [4], we impose it on the initial marginal utility $u_x(x,0)$ and not on $u(x,0)$ itself.

Assumption 1: *The initial utility datum $u(x,0)$ is such that, for some $\gamma \in (0,1)$,*

$$\lim_{x\uparrow\infty}\frac{u_x(x,0)}{x^{\gamma-1}} = 1. \tag{25}$$

Given the key role that the risk preference measure μ plays, we first examine what Assumption 1 implies for it. As the next result shows, (25) yields that the right-end of the support must be $\frac{1}{1-\gamma}$ with unit mass therein, $\mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1$, and vice-versa.

Lemma 2 *Assumption 1 holds if and only if the risk preference measure μ in (13) satisfies $b = \frac{1}{1-\gamma}$ and $\mu(\{b\}) = 1$, i.e.,*

$$supp(\mu) \subseteq \left(0, \frac{1}{1-\gamma}\right] \quad \text{and} \quad \mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1. \tag{26}$$

Proof. From (11), (25) and the fact that $h(x,0)$ is strictly increasing and of full range, we have

$$1=\lim_{x\uparrow\infty}\frac{u_x(x,0)}{x^{\gamma-1}}=\lim_{z\uparrow\infty}\frac{u_x(h(z,0),0)}{h^{\gamma-1}(z,0)}=\lim_{z\uparrow\infty}\left(\frac{h(z,0)}{e^{\frac{1}{1-\gamma}z}}\right)^{1-\gamma}.$$

Therefore, representation (13) gives

$$\lim_{z\uparrow\infty}\int_a^b e^{(y-\frac{1}{1-\gamma})z}\mu(dy)=1. \tag{27}$$

If $a=b=\frac{1}{1-\gamma}$, then (26) follows directly. If $a<b$, it must be that $a\leq\frac{1}{1-\gamma}$, otherwise we get a contradiction.

Next, let $\varepsilon>0$. Then,

$$\int_a^b e^{(y-\frac{1}{1-\gamma})z}\mu(dy)\geq\int_{\frac{1}{1-\gamma}+\varepsilon}^b e^{(y-\frac{1}{1-\gamma})z}\mu(dy)\geq e^{\varepsilon z}\mu\left(\left[\frac{1}{1-\gamma}+\varepsilon,b\right]\right).$$

Sending $\varepsilon\downarrow 0$ and using (27) yields that $\mu\left(\left(\frac{1}{1-\gamma},b\right]\right)=0$ and, thus, $supp(\mu)\subseteq\left(a,\frac{1}{1-\gamma}\right]$, $a\geq 0$. On the other hand, we have from (27) that

$$1=\lim_{z\uparrow\infty}\int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{(y-\frac{1}{1-\gamma})z}\mu(dy)+\mu(\{\frac{1}{1-\gamma}\})=\mu(\{\frac{1}{1-\gamma}\}),$$

and we easily conclude. ■

Next, we state the spatial forward turnpike result.

Proposition 3 *Let Assumption 1 be satisfied. Then, for each $t_0\geq 0$, the relative forward risk tolerance converges to the right-end of the support of the risk preference measure,*

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\frac{1}{1-\gamma}. \tag{28}$$

Proof. Let $t_0\geq 0$. From Lemma 2 we have that

$$h(z,t_0)=\int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{yz-\frac{1}{2}y^2t_0}\mu(dy)+e^{\frac{1}{1-\gamma}z-\frac{1}{2}\frac{1}{(1-\gamma)^2}t_0}.$$

Then, by dominated convergence we obtain

$$\lim_{z\uparrow\infty}\frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\frac{1}{(1-\gamma)^2}t_0}}=1, \tag{29}$$

since

$$\lim_{z\uparrow\infty}\frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\frac{1}{(1-\gamma)^2}t_0}}=\lim_{z\uparrow\infty}\int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{(y-\frac{1}{1-\gamma})z-\frac{1}{2}t_0(y^2-\frac{1}{(1-\gamma)^2})}\mu(dy)+1=1.$$

Therefore, from (11) together with the strict monotonicity and the full range of $h(z,t_0)$, we deduce that

$$\lim_{x\uparrow\infty}\frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}=1, \tag{30}$$

since

$$\lim_{x\uparrow\infty}\frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}=\lim_{z\uparrow\infty}\frac{e^{-z+\frac{t_0}{2}}}{(h(z,t_0))^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}$$
$$=\lim_{z\uparrow\infty}\left(\frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2t_0}}\right)^{1-\gamma}=1.$$

Next, we claim that

$$\lim_{x\uparrow\infty}\frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}=\gamma-1. \tag{31}$$

To prove this, it suffices to show that, for each $t_0\geq 0$, $u_x(x,t_0)$ is convex since the above would then follow from the arguments in Lemma 3.1 in [32]. To this end, differentiating (11) yields

$$u_{xxx}\left(h(z,t_0),t_0\right)\left(h_z(z,t_0)\right)^2+u_{xx}(h(z,t_0),t_0)h_{zz}(z,t_0)=e^{-z+\frac{t_0}{2}}.$$

Then, the strict convexity of h and the strict concavity of u give that $u_{xxx}\left(h(z,t_0),t_0\right)>0$, and using the strict monotonicity and the full range of h we conclude. Combining (30) and (31) we then deduce

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\lim_{x\uparrow\infty}\left(-\frac{u_x(x,t_0)}{xu_{xx}(x,t_0)}\right)$$
$$=\lim_{x\uparrow\infty}\left(-\frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}\left(\frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}\right)^{-1}\right)=\frac{1}{1-\gamma}.$$

■

4 Temporal forward turnpike property

We investigate the temporal asymptotic limit of the relative forward risk tolerance, $\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}$, for each $x_0>0$. This is the genuine turnpike analogue of similar results in classical expected utility models and one of the main findings herein. It shows that, for each $x_0>0$, $\frac{r(x_0,t)}{x_0}$ converges to the *left-end* of the support of the risk preference measure μ.

As in the spatial case, we first relate the finiteness of the measure's support to the asymptotic behavior of the initial (marginal) utility datum, but now using a weaker than (25) assumption.

Assumption 2: *There exists* $\gamma\in(0,1)$ *such that, for all* $\gamma'\in(\gamma,1)$,

$$\lim_{x\uparrow\infty}\frac{u_x\left(x,0\right)}{x^{\gamma'-1}}=0, \tag{32}$$

while, for all $\gamma'' \in (0, \gamma)$,

$$\lim_{x \uparrow \infty} \frac{u_x(x,0)}{x^{\gamma''-1}} = \infty. \tag{33}$$

The above assumption is directly related to the "regular variation" condition introduced in [8] and [16] for a discrete and a continuous time setting, respectively.

Lemma 4 *Assumption 2 holds if and only if the initial marginal utility* $u_x(x,0)$ *varies regularly at infinity with exponent* $\gamma - 1$, *i.e. for each* $k > 0$,

$$\lim_{x \uparrow \infty} \frac{u_x(kx,0)}{u_x(x,0)} = k^{\gamma-1}.$$

The proof follows by routine albeit tedious arguments and is omitted.

Assumption 2, weaker than Assumption 1, implies that the risk preference measure has finite support with its right-end point being equal $\frac{1}{1-\gamma}$, but without necessarily having a mass therein. We prove this next.

Lemma 5 *Assumption 2 is equivalent to the risk preference measure* μ *in (13) having finite support with its right boundary at* $\frac{1}{1-\gamma}$, *namely,*

$$\inf \{y > 0 : \mu((y, \infty)) = 0\} = \frac{1}{1-\gamma}. \tag{34}$$

Proof. We first show that Assumption 2 implies (34). We know from the results in [32] that the support of the measure must be of the form $(a, b]$, with $a \geq 0$, and $b \leq \infty$.

Using the strict monotonicity and the full range of $h(z,0)$, we deduce from (32) that, for each $\gamma' \in (\gamma, 1)$,

$$0 = \lim_{x \uparrow \infty} \frac{u_x(x,0)}{x^{\gamma'-1}} = \lim_{z \uparrow \infty} \frac{u_x(h(z,0),0)}{(h(z,0))^{\gamma'-1}} = \lim_{z \uparrow \infty} \left(\frac{h(z,0)}{e^{\frac{z}{1-\gamma'}}} \right)^{1-\gamma'},$$

and, thus,

$$\lim_{z \uparrow \infty} \int_a^b e^{\left(y - \frac{1}{1-\gamma'}\right)z} \mu(dy) = 0. \tag{35}$$

Therefore, it must be that for each $\gamma' \in (\gamma, 1)$, $b \leq \frac{1}{1-\gamma'}$ and, thus, $b \leq \frac{1}{1-\gamma}$, otherwise we get a contradiction.

Next, we assume that there exists $\gamma' \in (\gamma, 1)$ such that $b = \frac{1}{1-\gamma'}$. Then, for each $\tilde{\gamma} \in (\gamma, \gamma')$ we have that $\frac{1}{1-\tilde{\gamma}} < \frac{1}{1-\gamma'}$ and (35) gives that, for ε small enough,

$$\lim_{z \uparrow \infty} \left(\int_a^{\left(\frac{1}{1-\tilde{\gamma}} + \varepsilon\right)^-} e^{z\left(y - \frac{1}{1-\tilde{\gamma}}\right)} \mu(dy) + \int_{\frac{1}{1-\tilde{\gamma}} + \varepsilon}^b e^{z\left(y - \frac{1}{1-\tilde{\gamma}}\right)} \mu(dy) \right) = 0.$$

Therefore, it must be that $\mu\left(\left[\frac{1}{1-\tilde{\gamma}}+\varepsilon, b\right]\right)=0$. Sending $\varepsilon \downarrow 0$ gives $\mu\left(\left(\frac{1}{1-\tilde{\gamma}}, b\right]\right)=0$, which is a contradiction. Therefore, we must have $b \leq \frac{1}{1-\gamma}$. Using (33) and working similarly, we obtain that $b \geq \frac{1}{1-\gamma}$ and, thus, it must be that $b=\frac{1}{1-\gamma}$.

To show the reverse direction, we first observe that (34) and dominated convergence yield that, for any $\varepsilon>0$,

$$\lim_{z \uparrow \infty} \frac{h(z, 0)}{e^{\left(\frac{1}{1-\gamma}+\varepsilon\right) z}}=\lim_{z \uparrow \infty} \int_a^{\frac{1}{1-\gamma}} e^{\left(y-\left(\frac{1}{1-\gamma}+\varepsilon\right)\right) z} \mu(d y)=0 .$$

Then, choosing $\gamma^{\prime}$ such that $\frac{1}{1-\gamma^{\prime}}=\frac{1}{1-\gamma}+\varepsilon$ (i.e. $\gamma^{\prime}=1-\frac{1-\gamma}{1+\varepsilon(1-\gamma)}$), we deduce (32) for all $\gamma^{\prime} \in(\gamma, 1)$. It remains to show (33). Let $\delta>0$. For $z>0$, we have

$$\begin{aligned}
0<\frac{e^{\left(\frac{1}{1-\gamma}-\delta\right) z}}{h(z, 0)} & =\frac{e^{-\frac{\delta}{2} z}}{\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^{-}} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right) z} \mu(d y)+\int_{\frac{1}{1-\gamma}-\frac{\delta}{2}}^{\frac{1}{1-\gamma}} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right) z} \mu(d y)} \\
& \leq \frac{e^{-\frac{\delta}{2} z}}{\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^{-}} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right) z} \mu(d y)+\mu\left(\left[\frac{1}{1-\gamma}-\frac{\delta}{2}, \infty\right)\right)} \\
& \leq \frac{e^{-\frac{\delta}{2} z}}{\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^{-}} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right) z} \mu(d y)+\mu\left(\left(\frac{1}{1-\gamma}-\frac{\delta}{2}, \infty\right)\right)} .
\end{aligned}$$

Using (34) we obtain that $\mu\left(\left(\frac{1}{1-\gamma}-\frac{\delta}{2}, \infty\right)\right)>0$. Passing to the limit above as $z \uparrow \infty$, and using that $\lim_{z \uparrow \infty} \int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^{-}} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right) z} \mu(d y)=0$ and dominated convergence, we deduce that $\lim_{z \uparrow \infty} \frac{e^{\left(\frac{1}{1-\gamma}-\delta\right) z}}{h(z, 0)}=0$. We easily conclude. ■

Next, we turn our attention to the left boundary of the support of the risk preference measure,

$$a=\inf \{y \geq 0: \mu((0, y])>0\} . \tag{36}$$

We will, frequently, use the identity

$$x=\int_a^{\frac{1}{1-\gamma}} e^{y h^{(-1)}(x, t)-\frac{1}{2} y^2 t} \mu(d y), \quad x>0, \tag{37}$$

which follows from (13). We recall that $h^{(-1)}(x, t): \mathbb{D}_{+} \rightarrow \mathbb{R}$ is the spatial inverse of h, which is well-defined since $h(., t)$ is strictly increasing for each $t \geq 0$. We continue with some auxiliary results on $h^{(-1)}$.

Lemma 6 *Let $h^{(-1)}: \mathbb{D}_{+} \rightarrow \mathbb{R}$ be the spatial inverse of h and a as in (36). Then, for each $x_0>0, t \geq 0$,*

$$\frac{a}{2} \leq \frac{\partial}{\partial t} h^{(-1)}\left(x_0, t\right) \leq \frac{1}{2(1-\gamma)} . \tag{38}$$

Furthermore, $\lim_{t\uparrow\infty}\frac{\partial}{\partial t}h^{(-1)}(x_0,t)$ *and* $\lim_{t\uparrow\infty}h^{(-1)}(x_0,t)$ *exist with*

$$\frac{a}{2}\leq\lim_{t\uparrow\infty}\frac{\partial}{\partial t}h^{(-1)}(x_0,t)\leq\frac{1}{2\,(1-\gamma)}\quad \textit{and}\quad \lim_{t\uparrow\infty}h^{(-1)}(x_0,t)=\infty. \tag{39}$$

Proof. Let $x_0>0$. We first show (38). To simplify the presentation, let $f(x,t):=h^{(-1)}\,(x,t)$. Then, differentiating (37) twice yields

$$\int_a^{\frac{1}{1-\gamma}}\left(yf_t(x_0,t)-\frac{1}{2}y^2\right)e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)=0,$$

and

$$\int_a^{\frac{1}{1-\gamma}}yf_{tt}(x_0,t)e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)$$
$$+\int_a^{\frac{1}{1-\gamma}}\left(yf_t(x_0,t)-\frac{1}{2}y^2\right)^2e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)=0.$$

As mentioned earlier, differentiation under the integrals appearing herein is valid given the properties of their respective integrands. From the above, we deduce that

$$f_t(x_0,t)=\frac{1}{2}\frac{\int_a^{\frac{1}{1-\gamma}}y^2e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu\,(dy)}{\int_a^{\frac{1}{1-\gamma}}ye^{yf(x_0,t)-\frac{1}{2}y^2t}\mu\,(dy)},$$

and inequality (38) follows. Furthermore,

$$f_{tt}(x_0,t)=-\frac{\int_a^{\frac{1}{1-\gamma}}\left(yf_t(x_0,t)-\frac{1}{2}y^2\right)^2e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)}{\int_a^{\frac{1}{1-\gamma}}ye^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)}<0. \tag{40}$$

Therefore, $f_t(x_0,t)=\frac{\partial}{\partial t}h^{(-1)}(x_0,t)$ is bounded from below and decreasing in t, and hence its limit as $t\uparrow\infty$ exists.

To show the second limit in (39), we note that for each $x_0>0$, $f_t\,(x_0,t)>0$, and thus $\lim_{t\uparrow\infty}h^{(-1)}(x_0,t)$ exists. On the other hand, it must be that $\lim_{t\uparrow\infty}h^{(-1)}(x_0,t)=\infty$, otherwise sending $t\uparrow\infty$ equation (37) gives a contradiction. ■

We are now ready to present one of the main findings herein, a result interesting on its own right, that yields the temporal asymptotic behavior as $t\uparrow\infty$ of the ratio $\frac{h^{(-1)}(x_0,t)}{t}$, for each $x_0>0$. We show that it converges to half of the left-end of the support of the risk preference measure. We also provide the rate of convergence.

Proposition 7 *Let* $h^{(-1)}:\mathbb{D}_+\to\mathbb{R}$ *be the spatial inverse of* h *and* a *as in (36). Then, for each* $x_0>0$*, the following assertions hold:*

i) The ratio $\frac{h^{(-1)}(x_0,t)}{t}$ *converges to* $\frac{a}{2}$,

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}=\frac{a}{2}. \tag{41}$$

ii) Let

$$\Delta\left(x_0,t\right):=\frac{h^{(-1)}(x_0,t)}{t}-\frac{a}{2}. \tag{42}$$

If $a>0,$ *then*

$$\left|\Delta\left(x_0,t\right)\right|\leq\frac{1}{at}\ln\left(\frac{\mu\left(\left[a,\frac{1}{1-\gamma}\right]\right)}{x_0}\right),\quad \textit{if}\ \ \Delta\left(x_0,t\right)<0, \tag{43}$$

and

$$x_0\geq\mu\left(\left[a,a+\Delta\left(x_0,t\right)\right]\right)e^{\frac{1}{2}ta\Delta(x_0,t)},\quad \textit{if}\ \Delta\left(x_0,t\right)>0. \tag{44}$$

If $a=0^+$, *then* $\Delta\left(x_0,t\right)>0$ *and, moreover, for each* $\theta\in(0,1)$ *,*

$$x_0\geq\mu\left(\left[\Delta\left(x_0,t\right),\left(1+\theta\right)\Delta\left(x_0,t\right)\right]\right)e^{t\Delta^2(x_0,t)\frac{1-\theta^2}{2}}. \tag{45}$$

Proof. *Part (i):*

Let $x_0>0$. First observe that

$$A(x_0):=\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}=\lim_{t\uparrow\infty}\frac{\partial}{\partial t}h^{(-1)}(x_0,t), \tag{46}$$

as it follows from L'Hôpital's rule and Lemma 6. Inequality (38) then gives

$$\frac{a}{2}\leq A(x_0)\leq\frac{1}{2(1-\gamma)}.$$

We look at the following three cases.

a. If $a=\frac{1}{1-\gamma}$, then $a=b$ and $h^{(-1)}(x_0,t)=\ln x_0^{1-\gamma}+\frac{1}{2}\frac{1}{(1-\gamma)}t$, and the result follows directly.

b. Let $0<a<\frac{1}{1-\gamma}$.

We first claim that $A\left(x_0\right)<\frac{1}{2(1-\gamma)}$. We argue by contradiction assuming there exists x_0 such that $A\left(x_0\right)=\frac{1}{2(1-\gamma)}$. Then, for $\varepsilon>0$, there exists $t_0=t_0(x_0,\varepsilon)$ such that, for $t\geq t_0$,

$$-\varepsilon\leq\frac{h^{(-1)}(x_0,t)}{t}-\frac{1}{2(1-\gamma)}\leq\varepsilon.$$

In turn, for $\delta>0$ small enough, the above inequality and (37) give

$$x_0\geq\int_a^{\left(\frac{1}{1-\gamma}-2\varepsilon-\delta\right)^-}e^{ty\left(\frac{1}{2(1-\gamma)}-\varepsilon-\frac{1}{2}y\right)}\mu(dy)+\int_{\frac{1}{1-\gamma}-2\varepsilon-\delta}^{\frac{1}{1-\gamma}}e^{ty\left(\frac{1}{2(1-\gamma)}-\varepsilon-\frac{1}{2}y\right)}\mu(dy),$$

which yields a contradiction as $t \uparrow \infty$, for the first integral would then blow up.

Next, we assume that there exists $x_0 > 0$ such that

$$\frac{a}{2} < A(x_0) < \frac{1}{2(1-\gamma)}.$$

Then, for $\varepsilon, \delta > 0$ small enough, we have

$$a < 2(A(x_0) - \varepsilon) - \delta < 2(A(x_0) - \varepsilon) < \frac{1}{1-\gamma}. \tag{47}$$

From (37), we deduce that, for $t \geq t_0(\varepsilon, x_0)$,

$$x_0 \geq \int_a^{\frac{1}{1-\gamma}} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy).$$

If $\mu(\{a\}) \neq 0$, then $x_0 \geq e^{\frac{ta}{2}(2(A(x_0)-\varepsilon)-a)}\mu(\{a\})$, and sending $t \uparrow \infty$ yields a contradiction because $2(A(x_0) - \varepsilon) - a > \delta > 0$.

If $\mu(\{a\}) = 0$ then,

$$x_0 \geq \int_a^{\frac{1}{1-\gamma}} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy) \geq \int_a^{2(A(x_0)-\varepsilon)-\delta} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy).$$

Next, we consider the quadratic $B(y) := y(A(x_0) - \varepsilon) - \frac{1}{2}y^2$. We have that $B(y_1) = B(y_2) = 0$ for $y_1 = 0$ and $y_2 = 2(A(x_0) - \varepsilon)$, $B(y) > 0$ for $0 < y < 2(A(x_0) - \varepsilon)$, and $B(y)$ achieves a maximum at $y^* = A(x_0) - \varepsilon$.

We, also, look at its minimum $y_* = \min_{a \leq y \leq 2(A(x_0)-\varepsilon)-\delta} B(y)$ and claim that $y_* = 2(A(x_0) - \varepsilon) - \delta$. Indeed, if $0 < a \leq y^*$, choosing $\delta < a$, direct calculations yield that $B(a) > B(y_*)$. If $y^* < a$, then (47) yields that $a < y_* < y_2$ and, thus, the minimum also occurs at y_*. Clearly, because $y_1 < y_* < y_2$, we have that $B(y_*) = \frac{1}{2}\delta(2(A(x_0) - \varepsilon) - \delta) > 0$. Therefore, for $t \geq t_0(x_0, \varepsilon)$,

$$x_0 \geq \int_a^{2(A(x_0)-\varepsilon)-\delta} e^{tB(y_*)}\mu(dy). \tag{48}$$

As $t \uparrow \infty$, the right hand side of (48) converges to ∞, unless it holds that $\mu([a, 2(A(x_0) - \varepsilon) - \delta]) = 0$. Sending $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we obtain that $\mu([a, 2A(x_0)]) = 0$, which, however, contradicts (36). Therefore, it must be that, for each $x_0 > 0$, $A(x_0) \leq \frac{a}{2}$, and we easily conclude.

c. If $a = 0^+$, similar arguments yield that for every $\theta \in (0, A(x_0)]$ it must be that $\mu([\theta, 2A(x_0)]) = 0$. Sending $\theta \downarrow 0$ yields that $\mu((0, A(x_0)]) = 0$, which contradicts (36).

Part ii)

a. Let $a > 0$ and $\Delta(x_0, t)$ as in (42).

If $\Delta(x_0, t) < 0$, equation (37) yields

$$x_0 = \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\Delta(x_0,t)+\frac{1}{2}(a-y)\right)}\mu(dy)$$

$$\leq e^{ta\Delta(x_0,t)} \int_a^{\frac{1}{1-\gamma}} e^{\frac{1}{2}ty(a-y)} \mu\left(dy\right) \leq e^{ta\Delta(x_0,t)} \mu\left(\left[a, \frac{1}{1-\gamma}\right]\right),$$

and (43) follows.

If $\Delta\left(x_0,t\right) > 0$, then (41) gives that, for ε small enough and $t \geq t_0\left(x_0,\varepsilon\right)$ large enough, the inequality $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \varepsilon$ holds. Choosing ε such that $\varepsilon < \frac{1}{2(1-\gamma)} - \frac{a}{2}$ yields

$$0 < \frac{h^{(-1)}\left(x_0,t\right)}{t} - \frac{a}{2} < \frac{1}{2(1-\gamma)} - \frac{a}{2},$$

and, using that $a < \frac{1}{1-\gamma}$, we deduce that

$$\frac{a}{2} + \frac{h^{(-1)}\left(x_0,t\right)}{t} \leq \frac{1}{1-\gamma}.$$

In turn, from (37) we obtain that

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)} \mu\left(dy\right).$$

Next, we observe that the quadratic $H(y) := y\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{y}{2}\right)$ in the above integrand becomes zero at $y_1 = 0$ and $y_3 = 2\frac{h^{(-1)}(x_0,t)}{t} > a$ and, therefore, its minimum occurs at one of the end-points of the integral above, a or $\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t}$. Note that $a < \frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t} < y_3$. If the minimum occurs at a, then $H(a) = a\Delta\left(x_0,t\right)$, while if it occurs at $\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t}$, then

$$H\left(\frac{a}{2} + \frac{h^{(-1)}\left(x_0,t\right)}{t}\right) = \frac{1}{2}\left(\frac{a}{2} + \frac{h^{(-1)}\left(x_0,t\right)}{t}\right)\Delta\left(x_0,t\right) > \frac{1}{2}a\Delta\left(x_0,t\right).$$

Combining the above gives (44) since

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}} e^{\frac{1}{2}ta\Delta(x_0,t)} \mu\left(dy\right) = \mu\left(\left[a, a+\Delta\left(x_0,t\right)\right]\right) e^{\frac{1}{2}ta\Delta(x_0,t)}.$$

b. Let $a = 0^+$.

Then, $\Delta\left(x_0,t\right) = \frac{h^{(-1)}(x_0,t)}{t}$. Recall that $\lim_{t\uparrow\infty} h^{(-1)}(x_0,t) = \infty$ and, thus, $\frac{h^{(-1)}(x_0,t)}{t} > 0$, for t large. For $\varepsilon \in \left(\frac{h^{(-1)}(x_0,t)}{t}, 2\frac{h^{(-1)}(x_0,t)}{t}\right)$ we then have that

$$x_0 \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)} \mu\left(dy\right) \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{t\varepsilon\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{\varepsilon}{2}\right)} \mu\left(dy\right)$$

$$\geq \mu\left(\left[\Delta, (1+\theta)\,\Delta\right]\right).$$

Setting $\varepsilon = (1+\theta)\frac{h^{(-1)}(x_0,t)}{t}$, inequality (45) follows. ■

We are now ready to prove one of the main results herein. It yields the temporal limit of the relative forward risk tolerance and, also, provides the related rate of convergence.

Proposition 8 *Let a be the left-end of the support of the risk preference measure μ and $\Delta(x_0,t)$ as in (42). Then, the following assertions hold:*

i) For each $x_0 > 0$,

$$\lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = a. \tag{49}$$

ii) For each $x_0 > 0$, there exists a function $G(x_0,t)$, $t \geq 0$, given by

$$G(x_0,t) := \begin{cases} \int_a^{\frac{1}{1-\gamma}} (y-a)e^{-ty(\frac{y-a}{2})}\mu(dy), & \text{if } \Delta(x_0,t) < 0 \\ 2\Delta(x_0,t)\,x_0 + \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy), & \text{if } \Delta(x_0,t) > 0, \end{cases}$$

satisfying $\lim_{t\uparrow\infty} G(x_0,t) = 0$ *and, for t large enough,*

$$0 \leq r(x_0,t) - ax_0 \leq G(x_0,t). \tag{50}$$

Proof. *Part i).* Differentiating (11) gives, for $x > 0$, $t \geq 0$,

$$u_{xt}(x,t) = \left(\frac{1}{2} - \frac{\partial}{\partial t}h^{(-1)}(x,t)\right)u_x(x,t).$$

Moreover, (9) and (15) imply that $u_t(x,t) = -\frac{1}{2}u_x(x,t)r(x,t)$ and, in turn,

$$u_{tx}(x,t) = -\frac{1}{2}u_{xx}(x,t)\,r(x,t) - \frac{1}{2}u_x(x,t)\,r_x(x,t).$$

Combining the above we deduce that

$$\frac{1}{2}r_x(x_0,t) = \frac{\partial}{\partial t}h^{(-1)}(x_0,t). \tag{51}$$

In turn, from Proposition 7 and (46) we obtain that, for each $x_0 > 0$,

$$\lim_{t\uparrow\infty} r_x(x_0,t) = \lim_{t\uparrow\infty} 2\frac{\partial}{\partial t}h^{(-1)}(x_0,t) = a.$$

On the other hand,

$$r(x_0,t) = \int_0^{x_0} r_x(x,t)dx, \tag{52}$$

which follows from the facts that $\lim_{c\downarrow 0^+}\int_c^{x_0} r_x(x,t)dx = r(x_0,t) - \lim_{c\downarrow 0^+} r(c,t)$ and,moreover, for all $t \geq 0$, $\lim_{x\downarrow 0} r(x,t) = 0$. We observe from (51) and (40) that $r_{xt}(x,t) = 2\frac{\partial^2}{\partial t^2}h^{(-1)}(x_0,t) < 0$. Therefore, for $x_0 > 0$, $t > 0$ and

$x \in (0, x_0]$, we have that $r_x(x,t) \leq r_x(x_0,0)$. Finally, note that for each $x_0 > 0$, $r_x(x_0,0) < \infty$. This follows directly from (16), (13) and the full range of $h(x,0)$, since

$$r_x\left(h\left(z,0\right),0\right) = \frac{h_{zz}\left(z,0\right)}{h_z\left(z,0\right)} = \frac{\int_a^{\frac{1}{1-\gamma}} y^2 e^{yz - \frac{1}{2}t^2 y} \mu\left(dy\right)}{\int_a^{\frac{1}{1-\gamma}} y e^{yz - \frac{1}{2}t^2 y} \mu\left(dy\right)} \leq \frac{1}{1-\gamma}.$$

Using (52), (51), (41), dominated convergence and passing to the limit as $t \uparrow \infty$ in (51), we deduce (49).

Part ii). We provide the rate of convergence, which also provides an alternative convergence proof. To this end, first note that

$$r(x_0,t) - ax_0 \geq 0. \tag{53}$$

This follows directly from (16) and (13) since

$$r\left(x_0,t\right) = \int_a^{\frac{1}{1-\gamma}} y e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2}y\right)} \mu(dy) \geq a \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2}y\right)} \mu(dy).$$

Furthermore, from (16), (13) and (42), we have

$$r(x_0,t) - ax_0 = \int_a^{\frac{1}{1-\gamma}} (y-a) e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)} \mu(dy) \geq 0. \tag{54}$$

a. If $\Delta\left(x_0,t\right) < 0$ (which occurs only if $a > 0$, as shown in the previous proof), the above equality yields

$$r(x_0,t) - ax_0 \leq \int_a^{\frac{1}{1-\gamma}} (y-a) e^{-ty\left(\frac{y-a}{2}\right)} \mu(dy),$$

and (50) follows directly with $G\left(t\right) := \int_a^{\frac{1}{1-\gamma}} (y-a) e^{-ty\left(\frac{y-a}{2}\right)} \mu(dy)$.

b. Let $\Delta\left(x_0,t\right) > 0$ and $a > 0$ or $a = 0^+$. If $a = \frac{1}{1-\gamma}$, then the result follows trivially.

For $0 \leq a < \frac{1}{1-\gamma}$, observe that for t large enough, $0 < a + 2\Delta\left(x_0,t\right) < \frac{1}{1-\gamma}$ and, thus, representation (54) gives

$$\begin{aligned} r\left(x_0,t\right) - ax_0 = & \int_a^{(a+2\Delta(x_0,t))^-} (y-a) e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)} \mu(dy) \\ & + \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a) e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)} \mu(dy). \end{aligned}$$

Introduce $C_1\left(x_0,t\right) := \int_a^{(a+2\Delta(x_0,t))^-} (y-a) e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)} \mu(dy)$ and observe that

$$C_1\left(x_0,t\right) \leq 2\Delta\left(x_0,t\right) \int_a^{(a+2\Delta(x_0,t))^-} e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)} \mu(dy) \leq 2\Delta\left(x_0,t\right) x_0,$$

where we used (37). Thus,

$$\lim_{t\uparrow\infty} C_1\left(x_0,t\right)=0.$$

Let also $C_2\left(x_0,t\right):=\int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}}(y-a)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy)$ and introduce

$$F\left(y,t,x_0\right):=\left(y-a\right)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)},\quad y\in\left[a+2\Delta\left(x_0,t\right),\frac{1}{1-\gamma}\right].$$

Then, $F\left(a+2\Delta\left(x_0,t\right),t,x_0\right)=2\Delta\left(x_0,t\right)$ and, thus,

$$\lim_{t\uparrow\infty} F\left(a+2\Delta\left(x_0,t\right),t,x_0\right)=0.$$

Furthermore, for each $y\in\left(a+2\Delta\left(x_0,t\right),\frac{1}{1-\gamma}\right]$, we also have $\lim_{t\uparrow\infty}F\left(y,t,x_0\right)=0$. In turn, dominated convergence gives

$$\lim_{t\uparrow\infty} C_2\left(x_0,t\right)=0.$$

Setting $G\left(x_0,t\right):=C_1\left(x_0,t\right)+C_2\left(x_0,t\right)$ we easily conclude. ■

5 Examples

We present two examples in which the risk preference measure is, respectively, a sum of Dirac functions and the Lebesgue measure.

5.1 Finite sum of Dirac functions

We assume that, for some $\gamma\in(0,1)$, the risk preference measure is given by

$$\mu=\sum_{n=1}^{N}\delta_{y_n},\quad 0<y_1<\cdots<y_N=\frac{1}{1-\gamma}.$$

Then, $h(z,0)=\sum_{n=1}^{N}e^{y_nz}$ and, thus, $\lim_{z\uparrow\infty}h(z,0)e^{-y_Nz}=1$. In turn, (11) yields

$$\lim_{x\uparrow\infty}\frac{u_x\left(x,0\right)}{x^{\gamma-1}}=1,$$

which confirms the results of Lemma 2. Furthermore, we easily obtain (cf. (13)) that, for $(z,t)\in\mathbb{D}$,

$$h(z,t)=\sum_{n=1}^{N}\exp\left(y_nz-\frac{1}{2}y_n^2t\right). \tag{55}$$

Therefore, for $x>0$,

$$x=\sum_{n=1}^{N}\exp\left(y_nt\left(\frac{h^{(-1)}(x,t)}{t}-\frac{1}{2}y_n\right)\right). \tag{56}$$

We first provide the temporal and spatial asymptotic behavior of $h^{(-1)}(x,t)$ for large t and large x, respectively.

5.1.1 Temporal asymptotics of $h^{(-1)}$

We claim that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0, t) = \frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0 + o(1). \tag{57}$$

Indeed, the limit in (41) gives

$$\lim_{t \uparrow \infty} \left(\frac{h^{(-1)}(x_0, t)}{t} - \frac{1}{2} y_n \right) \begin{cases} < 0, & 1 < n \leq N \\ = 0, & n = 1 \end{cases}.$$

Therefore, as $t \uparrow \infty$, all terms in (56) vanish except for the first one. In turn,

$$x_0 = \lim_{t \uparrow \infty} \exp \left(y_1 h^{(-1)}(x_0, t) - \frac{1}{2} y_1^2 t \right),$$

and taking logarithm and rearranging terms yields (57). Note, also, that for each $t > 0$,

$$h^{(-1)}(x_0, t) - \frac{1}{2} y_1 t \leq \frac{1}{y_1} \log x_0.$$

5.1.2 Spatial asymptotics of $h^{(-1)}$

We claim that, for each $t_0 \geq 0$, as $x \uparrow \infty$,

$$h^{(-1)}(x, t_0) = (1 - \gamma) \ln x + \frac{1}{2(1 - \gamma)} t + o(1). \tag{58}$$

We first establish that, for each $t_0 \geq 0$,

$$\lim_{x \uparrow \infty} \frac{h^{(-1)}(x, t_0)}{\ln x} = 1 - \gamma, \tag{59}$$

independently of t_0. Indeed, fix $t_0 \geq 0$, let $\delta \in (0, \frac{1}{1-\gamma})$ and assume that

$$\liminf_{x \uparrow \infty} \frac{h^{(-1)}(x, t)}{\ln x} < \frac{1}{\frac{1}{1-\gamma} + \delta}.$$

Then, using (56) and that $h^{(-1)}(x, t_0) > 0$ for large x, we obtain

$$1 = \liminf_{x \uparrow \infty} \frac{1}{x} \sum_{n=1}^{N} \exp \left(y_n \ln x \frac{h^{(-1)}(x, t_0)}{\ln x} - \frac{1}{2} y_n^2 t_0 \right)$$

$$\leq N \liminf_{x \uparrow \infty} x^{\frac{1}{1-\gamma} \left(\frac{h^{(-1)}(x, t_0)}{\ln x} - 1 \right)} < N \liminf_{x \uparrow \infty} x^{-\frac{\delta}{\frac{1}{1-\gamma} + \delta}} = 0,$$

which yields a contradiction. Since δ is arbitrary, we deduce that

$$\liminf_{x \uparrow \infty} \frac{h^{(-1)}(x, t_0)}{\ln x} \geq 1 - \gamma.$$

Similarly, assume that, for $\delta \in \left(0, \frac{1}{1-\gamma}\right)$,

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} > \frac{1}{\frac{1}{1-\gamma} - \delta}.$$

Then, using (56) once more, we get a contradiction since

$$1 \geq \limsup_{x\uparrow\infty} \frac{1}{x} \exp\left(\frac{1}{1-\gamma} \ln x \frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0\right)$$

$$= \limsup_{x\uparrow\infty} x^{\frac{1}{1-\gamma}\frac{h^{(-1)}(x,t_0)}{\ln x} - 1} e^{-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0} = \infty,$$

where we used that $\frac{1}{1-\gamma}\frac{1}{\frac{1}{1-\gamma}-\delta} = \frac{\delta(1-\gamma)}{1-\delta(1-\gamma)} > 0$. Since δ is arbitrary, we obtain that

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} \leq 1-\gamma,$$

and we easily conclude. Next, we rewrite (56) as

$$1 = \sum_{n=1}^{N} \exp\left(y_n h^{(-1)}(x,t_0) - \frac{1}{2}y_n^2 t_0 - \ln x\right)$$

$$= \sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n}\right) - \frac{1}{2}y_n^2 t_0\right). \tag{60}$$

Note that from (59) we have

$$\lim_{x\uparrow\infty} \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n}\right) \begin{cases} < 0, & 1 \leq n < N \\ = 0, & n = N. \end{cases}.$$

Therefore, as $x \uparrow \infty$, the first $N-1$ terms in (60) vanish and we deduce that

$$\lim_{x\uparrow\infty} \exp\left(\frac{1}{1-\gamma} h^{(-1)}(x,t_0) - \ln x - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0\right) = 1,$$

and (58) follows.

5.1.3 Spatial and temporal asymptotics of forward risk tolerance

From (16) and (55), we deduce that

$$r(x,t) = \sum_{n=1}^{N} y_n \exp\left(y_n h^{(-1)}(x,t) - \frac{1}{2}y_n^2 t\right). \tag{61}$$

Let $t_0 \geq 0$. From (61), we get

$$\lim_{x\uparrow\infty} r(x,t_0) = \lim_{x\uparrow\infty} \sum_{n=1}^{N} y_n \exp\left(y_n((1-\gamma)\ln x + \frac{1}{2\,(1-\gamma)} t_0) - \frac{1}{2} y_n^2 t_0 \right),$$

and we easily deduce that, as $x \uparrow \infty$,

$$r(x,t_0) = \sum_{n=1}^{N} y_n \exp\left(\frac{1}{2} y_n t_0 (\frac{1}{1-\gamma} - y_n) \right) x^{(1-\gamma)y_n} + o(1).$$

Next, let $x_0 > 0$. From (61),

$$r(x_0,t) \leq \sum_{n=1}^{N} y_n \exp\left(y_n(\frac{1}{2} y_1 t + \frac{1}{y_1}\ln x_0) - \frac{1}{2} y_n^2 t \right)$$

$$= y_1 x_0 + \sum_{n=2}^{N} y_n \exp\left(\frac{1}{2} y_n (y_1 - y_n) t \right) x_0^{\frac{y_n}{y_1}},$$

and, therefore, as $t \uparrow \infty$,

$$r(x_0,t) = y_1 x_0 + O\left(e^{\frac{1}{2} y_2 (y_1 - y_2) t} \right).$$

In summary, for each $x_0 > 0$ and each $t_0 \geq 0$, respectively,

$$\lim_{x\uparrow\infty} \frac{r\,(x,t_0)}{x} = \frac{1}{1-\gamma} = y_N \quad \text{and} \quad \lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = y_1,$$

and these spatial and temporal limits are consistent with the findings in Proposition 3 and Proposition 7.

5.2 Lebesgue measure

The risk preference measure μ is Lebesgue on $\left[a, \frac{1}{1-\gamma}\right]$, with $a \geq 0$. The analysis that follows is tedious so, to ease the presentation, some intermediate steps are omitted. We first note that this measure satisfies Assumption 2. Indeed, from (13) we have,

$$h(z,0) = \int_a^{\frac{1}{1-\gamma}} e^{yz} dy = \frac{1}{z}\left(e^{\frac{1}{1-\gamma} z} - e^{az} \right), \ z \in \mathbb{R}.$$

Therefore, for any $\tilde{\gamma} \in (0,1)$, we have

$$\lim_{x\uparrow\infty} \frac{u_x\,(x,0)}{x^{\tilde{\gamma}-1}} = \lim_{z\uparrow\infty} \frac{u_x\,(h(z,0),0)}{h(z,0)^{\tilde{\gamma}-1}}$$

$$= \lim_{z\uparrow\infty} \left(e^{-\frac{z}{1-\tilde{\gamma}}} h(z,0) \right)^{1-\tilde{\gamma}} = \lim_{z\uparrow\infty} \left(\frac{e^{z\left(\frac{1}{1-\gamma} - \frac{1}{1-\tilde{\gamma}}\right)} - e^{z\left(a - \frac{1}{1-\tilde{\gamma}}\right)}}{z} \right).$$

Let $L := \lim_{z\uparrow\infty} \frac{e^{z\left(\frac{1}{1-\gamma}-\frac{1}{1-\tilde{\gamma}}\right)}-e^{z\left(a-\frac{1}{1-\tilde{\gamma}}\right)}}{z}$.

If $\tilde{\gamma} \in (\gamma, 1)$,then $\frac{1}{1-\tilde{\gamma}} > \frac{1}{1-\gamma} > a$, and $L = 0$.

If $\tilde{\gamma} \in (0, \gamma)$, then there are two cases: $a < \frac{1}{1-\tilde{\gamma}} < \frac{1}{1-\gamma}$ and $\frac{1}{1-\tilde{\gamma}} < a < \frac{1}{1-\gamma}$. For the former, we trivially deduce that $L = \infty$. For the latter, we have that $L = \lim_{z\uparrow\infty}\left(e^{z\left(a-\frac{1}{1-\tilde{\gamma}}\right)}\frac{e^{z\left(\frac{1}{1-\gamma}-a\right)}-1}{z}\right) = \infty$.

Case $a > 0$.

We introduce the functions $\varphi(z) = e^{-\frac{z^2}{2}}$ and $\Phi(z) = \int_{-\infty}^{z}\varphi(y)dy$, $z \in \mathbb{R}$. Then, it follows easily that

$$h(z,t) = \int_a^{\frac{1}{1-\gamma}} e^{yz-\frac{1}{2}y^2t}dy = \frac{e^{z^2/2t}}{\sqrt{t}}\int_{a\sqrt{t}-z/\sqrt{t}}^{b\sqrt{t}-z/\sqrt{t}}\varphi(y)dy. \tag{62}$$

We also have (cf. (37)), for $x \geq 0$,

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yt\left(\frac{h^{(-1)}(x,t)}{t}-\frac{1}{2}y\right)}dy = \frac{1}{\sqrt{t}}e^{\frac{\left(h^{(-1)}(x,t)\right)^2}{2t}}\int_{a\sqrt{t}-\frac{h^{(-1)}(x,t)}{\sqrt{t}}}^{\frac{1}{1-\gamma}\sqrt{t}-\frac{h^{(-1)}(x,t)}{\sqrt{t}}}\varphi(y)dy. \tag{63}$$

5.2.1 Temporal asymptotics of $h^{(-1)}$

We show that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0,t) = \frac{1}{2}at + \frac{1}{a}\left(\ln t + \ln x_0 + \ln\frac{a}{2}\right) + o(1). \tag{64}$$

For this, we first establish that

$$x_0 = \lim_{t\uparrow\infty}\frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{\frac{1}{2}at}. \tag{65}$$

To this end, using (63) and that $\Phi(z) \leq -\frac{\varphi(z)}{z}$, for $z < 0$, we obtain, for t large enough,

$$x_0 \leq \frac{1}{\sqrt{t}}\exp\left(\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}\right)\Phi\left(-a\sqrt{t}+\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)$$

$$\leq \frac{1}{\sqrt{t}}\frac{1}{a\sqrt{t}-\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}}\exp\left(\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}\right)\varphi\left(-a\sqrt{t}+\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)$$

$$= \frac{1}{at-h^{(-1)}(x_0,t)}e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)} \leq \liminf_{t\uparrow\infty}\frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{\frac{1}{2}at},$$

using Proposition 7. Using similar arguments, we show that

$$x_0 \geq \limsup_{t\uparrow\infty}\frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{\frac{1}{2}at},$$

and (65) follows. Indeed, let $1 < k < \frac{1}{a(1-\gamma)}$. From (63) we have that, for t large enough,

$$
\begin{aligned}
x_0 &= \frac{1}{\sqrt{t}} e^{\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}} \left(\Phi(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) - \Phi(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) \right) \\
&\geq \frac{1}{\sqrt{t}} \frac{1}{ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} e^{\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}} \\
&\quad \times \left(\varphi(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) - \varphi(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) \right) \\
&= \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)}.
\end{aligned}
$$

From Proposition 7 and since $k > 1$, we obtain that

$$
\lim_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} = \lim_{t\uparrow\infty} \frac{e^{ka^2 t(\frac{h^{(-1)}(x_0,t)}{at} - \frac{k}{2})}}{at\left(k - \frac{h^{(-1)}(x_0,t)}{at}\right)} = 0.
$$

Therefore,

$$
\begin{aligned}
x_0 &\geq \limsup_{t\uparrow\infty} \frac{1}{kat - h^{(-1)}(x_0,t)} \left(e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)} \right) \\
&\geq \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{kat - h^{(-1)}(x_0,t)} - \lim_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} \\
&= \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{kat - h^{(-1)}(x_0,t)},
\end{aligned}
$$

and sending $k \downarrow 1$ we conclude.

We now utilize the Lambert W function, $W(x)$, defined as the inverse of $F(x) = xe^x$. Setting

$$
\delta(x_0,t) := h^{(-1)}(x_0,t) - \frac{1}{2}at,
$$

we deduce from (65) that there exists $\varepsilon(t)$ with $\lim_{t\uparrow\infty} \varepsilon(t) = 0$, such that

$$
\frac{e^{a\delta(x_0,t)}}{\frac{1}{2}at - \delta(x_0,t)} = (1 + \varepsilon(t))x_0.
$$

Rewriting yields

$$
a(\frac{1}{2}at - \delta(x_0,t))e^{a(\frac{1}{2}at - \delta(x_0,t))} = \frac{a}{(1+\varepsilon(t))x_0} e^{\frac{1}{2}a^2 t},
$$

and, therefore,

$$\delta(x_0,t)=\frac{1}{2}at-\frac{1}{a}W\left(\frac{a}{(1+\varepsilon(t))x_0}e^{\frac{1}{2}a^2t}\right).$$

It was established in [6] that the asymptotic expansion of $W(x)$, for large x, is given by

$$W(x)=\ln x-\ln\ln x+o(1).$$

Thus,

$$\begin{aligned}\delta(x_0,t)&=\frac{1}{2}at-\frac{1}{a}\ln\left(\frac{a}{(1+\varepsilon(t))x_0}e^{\frac{1}{2}a^2t}\right)\\&+\frac{1}{a}\ln\ln\left(\frac{a}{(1+\varepsilon(t))x_0}e^{\frac{1}{2}a^2t}\right)+o(1)\\&=\frac{1}{a}\left(\ln\frac{x_0}{a}+\ln(1+\varepsilon(t))+\ln\left(\frac{1}{2}a^2t+\ln\frac{a}{(1+\varepsilon(t))x_0}\right)\right)+o(1).\end{aligned}$$

Using that, as $t\uparrow\infty$, $\ln(1+\varepsilon(t))=o(1)$ and that

$$\ln\left(\frac{1}{2}a^2t+\ln\frac{a}{(1+\varepsilon(t))x_0}\right)=\ln(\frac{1}{2}a^2t)+o(1),$$

assertion (64) follows.

5.2.2 Spatial asymptotics of $h^{(-1)}$

We show that, for each $t_0\geq 0$, as $x\uparrow\infty$,

$$h^{(-1)}(x,t_0)=\frac{1}{2(1-\gamma)}t_0+(1-\gamma)\left(\ln x+\ln\ln x-\ln\frac{1}{1-\gamma}\right)+o(1).\tag{66}$$

We first establish that

$$\lim_{x\uparrow\infty}\frac{h^{(-1)}(x,t_0)}{\ln x}=1-\gamma.\tag{67}$$

Indeed, let $f(z,t):=\frac{1}{z}e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2t}$. Then,

$$\begin{aligned}\lim_{z\uparrow\infty}\frac{h(z,t_0)}{f(z,t_0)}&=\lim_{z\uparrow\infty}\int_a^{\frac{1}{1-\gamma}}ze^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\\&=\lim_{z\uparrow\infty}\left(\int_a^{\frac{1}{1-\gamma}}(z-yt_0)e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right.\\&\left.+\int_a^{\frac{1}{1-\gamma}}yt_0e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right)=\lim_{z\uparrow\infty}\left(1-e^{(a-\frac{1}{1-\gamma})x-\frac{1}{2}(a^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}\right.\\&\left.+\int_a^{\frac{1}{1-\gamma}}yte^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right)=1,\end{aligned}$$

where we used that $a < \frac{1}{1-\gamma}$ and monotone convergence. Therefore, for each $t_0 \geq 0$,

$$\lim_{z\uparrow\infty} \frac{h(z,t_0)}{f(z,t_0)} = 1.$$

We now use an auxiliary result on inverses of asymptotic functions from [11] (Theorem 2(i)) to prove (67) by verifying the necessary assumptions. To this end, let $g(z) := (1-\gamma)\ln z$, $z \geq 0$, and notice that, for large z,

$$g(f(z,t_0)) = -(1-\gamma)\ln z + z - \frac{1}{2(1-\gamma)}t_0 \sim z.$$

Thus, $\lim_{z\uparrow\infty} z^{-1}f(z,t_0) = 1$. Since, on the other hand, $\lim_{z\uparrow\infty} f(z,t_0) = \infty$, we deduce that $f^{(-1)}(x,t) \sim g(x)$, as $x \uparrow \infty$. Moreover, $g(x)$ is strictly increasing and the ratio $\frac{g_x(x,t)}{g(x,t)} \sim \frac{1}{x\ln x} = O(\frac{1}{x})$ for sufficiently large x. It, then, follows from the aforementioned result that $\lim_{x\uparrow\infty} \frac{g(x)}{h^{(-1)}(x,t_0)} = 1$ and (67) follows.

Next, we establish that, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x\ln x} = 1-\gamma. \tag{68}$$

Indeed, if $t_0 = 0$, we have from (63) that

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yh^{(-1)}(x,0)}dy = \frac{1}{h^{(-1)}(x,0)}\left(e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)} - e^{ah^{(-1)}(x,0)}\right),$$

and (67) yields

$$\lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}}{x\ln x} = \lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}(1-e^{(a-\frac{1}{1-\gamma})h^{(-1)}(x,0)})}{xh^{(-1)}(x,0)}\frac{h^{(-1)}(x,0)}{\ln x} = 1-\gamma.$$

For $t_0 > 0$, we deduce from (63) that

$$x = \frac{1}{\sqrt{t_0}}e^{\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}}\left(\Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right) - \Phi\left(a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)\right). \tag{69}$$

Then, for large x,

$$\begin{aligned}
1 &\leq \frac{1}{x\sqrt{t_0}}\exp\left(\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}\right)\Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)\\
&\leq \frac{1}{x}\frac{1}{\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} - \frac{1}{1-\gamma}\sqrt{t_0}}\exp\left(\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}\right)\\
&\times\frac{1}{\sqrt{t_0}}\varphi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right) = \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0)}.
\end{aligned}$$

In turn,

$$1 \le \liminf_{x\uparrow\infty} \left(\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0)-\frac{1}{1-\gamma}t_0} \right)$$

$$= \liminf_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \lim_{x\uparrow\infty} \left(\frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0)-\frac{1}{1-\gamma}t_0} \right),$$

and, thus,

$$1 \le \liminf_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}. \tag{70}$$

Next, we use the inequality $\Phi(b)-\Phi(a) \ge \frac{\varphi(a)-\varphi(b)}{b}$, for $a<b<0$, and deduce from (69) that, for large x,

$$1 \ge \frac{1}{x}\exp\left(\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}\right)\frac{1}{\sqrt{t_0}}\frac{1}{a\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}}$$

$$\times\left(\varphi\left(a\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)-\varphi\left(\frac{1}{1-\gamma}\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)\right)$$

$$= \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0)-at_0)} - \frac{e^{a(h^{(-1)}(x,t_0)-\frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0)-at_0)}.$$

Proceeding with analogous convergence arguments we used to establish (70), we obtain that

$$\limsup_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \le 1. \tag{71}$$

Combining (70) and (71), we deduce (68). Taking the logarithm of both sides of (68) gives

$$\lim_{x\uparrow\infty}\left(\frac{1}{1-\gamma}\left(h^{(-1)}(x,t_0)-\frac{1}{2(1-\gamma)}t_0\right)-\ln x-\ln\ln x\right)=\ln\frac{1}{1-\gamma},$$

and (66) follows.

5.2.3 Temporal and spatial asymptotics of forward risk tolerance

We first observe that (16) and (62) yield

$$r(x,t)=\int_a^{\frac{1}{1-\gamma}}\frac{1}{t}(yt-h^{(-1)}(x,t))e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}dy \tag{72}$$

$$+\frac{h^{(-1)}(x,t)}{t}\int_a^{\frac{1}{1-\gamma}}e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}dy$$

$$= \frac{e^{a(h^{(-1)}(x,t)-\frac{1}{2}at)}}{t} - \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t)-\frac{1}{2(1-\gamma)}t)}}{t} + \frac{h^{(-1)}(x,t)}{t}x.$$

We show that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$r(x_0,t) = ax_0 - \left(\frac{1}{2}ax_0\right)^{\frac{1}{a(1-\gamma)}} t^{\frac{1}{a(1-\gamma)}-1} e^{\frac{1}{2(1-\gamma)}(a-\frac{1}{1-\gamma})t}$$

$$+\frac{x_0}{at}(\ln t + \ln x_0 + \ln \frac{a}{2}) + o(1).$$

From (64), we obtain that

$$\lim_{t\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x_0,t)-\frac{1}{2(1-\gamma)}t)}}{t}$$

$$= \lim_{t\uparrow\infty} \exp\left(\frac{1}{1-\gamma}\left(h^{(-1)}(x_0,t) - \frac{1}{2\,(1-\gamma)}t - (1-\gamma)\ln t\right)\right)$$

$$= \lim_{t\uparrow\infty} \exp\left(\frac{1}{1-\gamma}(h^{(-1)}(x_0,t) - \frac{1}{2}at - \frac{1}{a}\ln t) - \frac{1}{2}\frac{1}{1-\gamma}(\frac{1}{1-\gamma} - a)t\right) e^{(\frac{1}{a(1-\gamma)}-1)\ln t}$$

$$= \lim_{t\uparrow\infty} \exp\left(\frac{1}{a\,(1-\gamma)}(\ln x_0 + \ln\frac{a}{2}) + \frac{1}{2\,(1-\gamma)}(a - \frac{1}{1-\gamma})t\right) t^{\frac{1}{a(1-\gamma)}-1}$$

$$= \lim_{t\uparrow\infty} \left(\frac{1}{2}ax_0\right)^{\frac{1}{a(1-\gamma)}} t^{\frac{1}{a(1-\gamma)}-1} e^{\frac{1}{2(1-\gamma)}(a-\frac{1}{1-\gamma})t}.$$

Furthermore,

$$\lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t}x_0 = \frac{1}{2}ax_0 + \lim_{t\uparrow\infty}\left(\frac{x_0}{at}(\ln t + \ln x + \ln\frac{a}{2})\right),$$

and we easily conclude.

Next, we establish the spatial asymptotics of $r\,(x,t)$ for large x.

If $t_0 > 0$, we easily deduce from (72) that, as $x \uparrow \infty$,

$$r(x,t_0) = \frac{1-\gamma}{t_0}x\ln\ln x + \frac{1}{t_0}\left((1-\gamma)\,x\ln x\right)^{a(1-\gamma)} e^{\frac{1}{2}a(\frac{1}{1-\gamma}-a)t_0} \tag{73}$$

$$+\frac{1}{2\,(1-\gamma)}x - \frac{1-\gamma}{t_0}x\ln\frac{1}{1-\gamma} + o(1).$$

If $t_0 = 0$,

$$r(x,0) = \frac{1}{1-\gamma}x - \left(\frac{1}{1-\gamma} - a\right)\frac{e^{ah^{(-1)}(x,0)}}{h^{(-1)}(x,0)},$$

and, thus, for large x,

$$r(x,0) = \frac{1}{1-\gamma}x(1-\frac{1}{\ln x}) + o(1). \tag{74}$$

In summary, we have, for each $x_0 > 0$,

$$\lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = a,$$

which is in accordance with the results of Proposition 7. From (73) and (74), we obtain that for $t_0 > 0$ and $t_0 = 0$, respectively,

$$r(x,t_0) \sim \frac{1-\gamma}{t_0} x \ln\ln x \quad \text{and} \quad r(x,0) \sim \frac{1}{1-\gamma} x.$$

Observe that the spatial turnpike property (28) does not hold.

Case $a = 0$.

We obtain, for $t_0 \geq 0$, the same spatial asymptotics for $h^{(-1)}(x,t_0)$ and $r(x,t_0)$, and the lack of the spatial turnpike limit, as in the case $a > 0$ above.

For the temporal asymptotics of $h^{(-1)}(x,t)$, we claim that, for each $x_0 > 0$, and large t,

$$\frac{h^{(-1)}(x_0,t)}{t} = \frac{\sqrt{\ln t + 2\ln x - \ln 2\pi}}{\sqrt{t}} + o(\frac{1}{\sqrt{t}}).$$

To see this, notice that (63) becomes $x_0 = \int_0^{\frac{1}{1-\gamma}} e^{y(h^{(-1)}(x_0,t) - \frac{1}{2}yt)} dy$. Taking logarithm of both sides yields

$$2\ln x_0 = \left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2 - \ln t \tag{75}$$

$$+2\ln\left(\Phi\left(\sqrt{t}(\frac{1}{1-\gamma} - \frac{h^{(-1)}(x_0,t)}{t})\right) - \Phi\left(-\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right).$$

Next, we claim that $l := \liminf_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{\sqrt{t}} = \infty$. Indeed, if $l < \infty$, then as $t \uparrow \infty$, (75) would give

$$2\ln x_0 = l^2 - \lim_{t\uparrow} \ln t + 2\ln(1 - \Phi(-l)) = -\infty,$$

which is a contradiction. Therefore, it must be that $l = \infty$, which combined with the fact that $\lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t} = 0$, from Proposition 7, implies that the third term on the right hand side of (75) converges to $2\ln\sqrt{2\pi}$. Thus,

$$2\ln x_0 = \lim_{t\uparrow\infty} \left(\left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2 - \ln t + 2\ln\sqrt{2\pi}\right),$$

from which we deduce that $h^{(-1)}(x_0,t) = \sqrt{t(\ln t + 2\ln x - \ln 2\pi)} + o(\sqrt{t})$, and we easily conclude. The rest of the analysis follows easily.

6 Conclusions and extensions

We studied turnpike-type limiting properties of the forward relative risk tolerance function in an Itô-diffusion market and under time monotone forward performance criteria. We showed that, contrary to existing turnpike results in the classical expected utility framework, the asymptotic temporal and spatial limits in the forward setting do *not* in general coincide. Rather, they depend critically on the left and right points of the support of the underlying risk preference measure. In particular, the spatial limit coincides with the right end point of the support while the temporal one with the left end point. Central role in the analysis is played by the asymptotic properties of the spatial inverse of the underlying space-time harmonic function which enters in the construction of the time-monotone preferences.

The work may be extended in the following directions. Firstly, within the time-monotone class, one may study the spatial and temporal asymptotic behavior (large x and large t, respectively) of the optimal wealth and portfolio processes, exploiting their explicit forms $X_t^{*,x} = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right)$ and $\pi_t^{*,x} = \sigma_t^+ \lambda_t h_z\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right)$ and with M_t and A_t as in (10), derived in [32] (see, also, [18] and [24]). Naturally, the long term behavior of the market coefficients σ_t and λ_t will play a role. Preliminary results for the lognormal case (constant coefficients) may be found in [12].

Secondly, one may work with forward utilities which have non-zero forward volatility. As shown in [31], general forward processes are expected to satisfy the ill-posed stochastic PDE,

$$dU(x,t) = \frac{\left(\lambda_t U_x(x,t) + \sigma_t \sigma_t^+ a_x(x,t)\right)^2}{2U_{xx}(x,t)} dt + a(x,t) dW_t, \tag{76}$$

with the forward volatility process $a(x,t)$ being a model input itself. To date, several results exist for finite-dimensional solutions of the above equation; see, for example, [3], [5], [21], [33], [34] and [37].

Structurally, the forward allocations in these models resemble the ones of models with stochastic factors in the classical expected utility setting. Therein, the value function solves a high-dimensional HJB equation with a fully nonlinear and a linear part (corresponding to the generator of the stochastic factor processes). Furthermore, the optimal feedback allocations have two components, the so-called myopic component and the one yielding the extra hedging demand. In analogy, the forward criterion reduces to a multi-dimensional value function process which is associated with an ill-posed high-dimensional Hamilton-Jacobi-Bellman (HJB) equation. The optimal policy processes also have two components with the second (the forward analogue of the extra hedging demand) depending directly on the forward volatility process $a(x,t)$. As in [13], where various notions of turnpike behavior were proposed in general markets, we could formulate alternative forward turnpike notions, which is in our opinion a very interesting question. Two fundamental difficulties are, firstly, the extra hedging demand depends on the forward volatility which is an investor specific input (contrary to the classical case where it is generated by the solution itself) and, secondly, the emerging forward HJB equations are ill-posed which, among others, do not admit comparison results.

References

[1] Aliprantis, D.C. and Burkinshaw, O.: *Principles of Real Analysis*, 3rd Edition, Academic Press, 1998.

[2] Anthropelos, M.: Forward exponential performances: pricing and optimal risk sharing, *SIAM Journal on Financial Mathematics*, 5(1), 626-655, 2014.

[3] Avanesyan, L., Shkolnikov, M. and Sircar, R.: Construction of a class of forward performance processes in stochastic factor models and an extension of Widder's theorem, *Finance and Stochastics,* 24(4), 981-1011, 2020.

[4] Bian, B. and Zheng, H.: Turnpike property and convergence rate for an investment model with general utility functions, *Journal of Economic Dynamics and Control*, 51, 28–49, 2015.

[5] Chong, A., Hu, Y., Liang, G. and Zariphopoulou, T.: An ergodic BSDE approach to forward entropic risk measures: representation and large-maturity dynamics, *Finance and Stochastics,* 23(1), 239-273, 2019.

[6] Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J. and Knuth, D.E.: On the Lambert W function, *Advances in Computational Mathematics* 5(4), 329–359, 1996.

[7] Cox, J. and Huang, C.-F.: A continuous-time portfolio turnpike theorem, *Journal of Economic Dynamics and Control*, 16(3), 491-507, 1992.

[8] Dybvig, P.H., Rogers, L.C.G. and Back, K.: Portfolio Turnpikes, *The Review of Financial Studies*,12(1),165–195, 1999.

[9] El Karoui, N. and Mrad, M.: An exact connection between two solvable SDEs and a nonlinear utility stochastic pde, *SIAM Journal on Financial Mathematics,* 4(1), 697-736, 2014.

[10] El Karoui, N., Hillairet, C. and Mrad, M.: Ramsey rule with forward/backward utility for long-term yield curves modeling, *Decisions in Economics and Finance,* 45, 375-414, 2022.

[11] Entringer, R.C: Functions and inverses of asymptotic functions, *The American Mathematical Monthly*, 74(9), 1095–1097, 1967.

[12] Geng, T. and Zariphopoulou, T.: On the asymptotic properties of the optimal wealth and portfolio processes under time-monotone forward performance criteria in Itô-diffusion markets, preprint, 2024.

[13] Guasoni, P., Kardaras, C., Robertson, S. and Xing, H.: Abstract, classic, and explicit turnpikes, *Finance and Stochastics*, 18(1), 75–114, 2014.

[14] Hakansson, N.H.: Convergence to isoelastic utility and policy in multiperiod portfolio choice, *Journal of Financial Economics,* 1(3), 201-224, 1974.

[15] Huang, C.-F. and Zariphopoulou, T.: Turnpike behavior of long-term investments, *Finance and Stochastics*, 3(1), 15–34, 1999.

[16] Huberman, G. and Ross, S.: Portfolio turnpike theorems, risk aversion and regularly varying utility functions, *Econometrica,* 51(5), 1345-1361, 1983.

[17] Jin, X.: Consumption and portfolio turnpike theorems in a continuous-time finance model, *Journal of Economic Dynamics and Control*, 22(7), 1001-1026, 1998.

[18] Kallblad, S.: Black's inverse investment problem and forward criteria with consumption, *SIAM Journal on Financial Mathematics,* 11(2), 494-525, 2020.

[19] Kallblad, S., Obloj, J. and Zariphopoulou, T.: Dynamically consistent investment under model uncertainty: the robust forward case, *Finance and Stochastics,* 22, 879-918, 2018.

[20] Leland, H.: On turnpike portfolios, *Mathematical models in investment and finance,* K. Shell and G. Szego, eds., North-Holland, Amsterdam, 1972.

[21] Li, J., Li, W. and Liang, G.: A game theoretical approach to homothetic robust forward investment performance processes in stochastic factor models, *SIAM Journal on Financial Mathematics,* 12(2), 867-897, 2021.

[22] Liang, G., Strub, M. and Wang, Y.: Predictable forward performance preferences: infrequent evaluation and applications to human-machine interactions, *Mathematical Finance,* 33(4), 1248-1286, 2023.

[23] Lipster, R.S. and Shirayev, A.N.: *Statistics of Random Processes, II: Applications* (2nd edition), Springer, Berlin, 2000.

[24] Monin, P.: On a dynamic adaptation of the Distribution Builder approach to investment decisions, *Quantitative Finance,* 14(5), 749-760, 2014.

[25] Mossin, J.: Optimal multiperiod portfolio policies, *Journal of Business,* 41(2), 215-229, 1968.

[26] Mostoviy, O., Sirbu, M. and Zariphopoulou, T.: On the analyticity of the value function of optimal investment and stochastically dominant markets, *Pure and Applied Functional Analysis*, 9(3), 825-862, 2024.

[27] Musiela, M.: Preface for "Recent Advances in Forward Performance Processes", *Probability, Uncertainty and Quantitative Risk,* 9(1), 1-12, 2024.

[28] Musiela, M. and Zariphopoulou, T.: Investments and forward utilities, technical report, 2006.

[29] Musiela, M. and Zariphopoulou, T.: Derivative pricing, investment management and the term structure of exponential utilities: The case of binomial model, *Indifference Pricing,* R. Carmona ed. , Princeton University Press, Princeton, 3-41, 2009.

[30] Musiela, M. and Zariphopoulou, T.: Portfolio choice under dynamic investment performance criteria, *Quantitative Finance,* 9(2), 161-170, 2009.

[31] Musiela, M. and T. Zariphopoulou: Stochastic partial differential equations and portfolio choice, *Contemporary Quantitative Finance,* C. Chiarella and A. Novikov eds., Berlin, 195-215, 2010.

[32] Musiela, M. and Zariphopoulou, T.: Portfolio choice under space-time monotone performance criteria, *SIAM Journal on Financial Mathematics*, 1, 326–365, 2010.

[33] Nadtochiy, S. and Tehranchi, M.: Optimal investment for all time horizons and Martin boundary of space-time diffusions, *Mathematical Finance,* 27(2), 438-470, 2015.

[34] Nadtochiy, S. and Zariphopoulou, T.: A class of homothetic forward investment performance process with non-zero volatility, *Inspired by Finance: The Musiela Festschrift,* Y. Kabanov et al. eds., Springer, Berlin, 475-505, 2013.

[35] Penrose, R.: A generalized inverse for matrices, *Proceedings of Cambridge Philosophical Society,* 51, 406-413, 1955.

[36] Ross, S.: Portfolio turnpike theorems for constant policies, *Journal of Financial Economics,* 1(2), 171-198, 1974.

[37] Shkolnikov, M., Sircar, R. and Zariphopoulou, T.: Forward performance processes in incomplete markets and ill-posed HJB equations, *SIAM Journal on Financial Mathematics,* 7, 588-618, 2016.

[38] Strub, M. and Zhou, X.Y.: Evolution of the Arrow-Pratt measure of risk tolerance for predictable forward utility processes, *Finance and Stochastics,* 25, 331-358, 2021.

[39] Waldon, H.: Forward robust portfolio selection: The binomial case, *Probability, Uncertainty and Quantitative Risk,* 9(1), 107-122, 2024.

[40] Zitkovic, G.: A dual characterization of self-generation and exponential forward performances, *The Annals of Applied Probability,* 19(6), 2176-2210, 2009.