Temporal and spatial turnpikes in Ito-diffusion markets under forward performance criteria*

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Abstract

We present turnpike-type asymptotic results for the relative risk tolerance in Ito-diffusion markets and under time monotone forward performance criteria. We show that, contrary to the classical case, the temporal (large time) and spatial (large wealth) limits do not coincide and, furthermore, depend crucially on the support of the risk preference measure, used to construct the underlying forward criterion. Specifically, the spatial limit coincides with the right end of the support while the temporal limit with the left end one. Key role plays the spatial inverse of a space-time harmonic function that solves the ill-posed heat equation. We construct two representative examples, one with discrete and the other with continuous measure support, and analyze the asymptotic behavior of the dynamic relative risk tolerance for each case.

1 Introduction

Turnpike results in maximal expected utility models yield the behavior of optimal portfolio functions when the investment horizon is long and under asymptotic assumptions on the investor's risk preferences.

The essence of the turnpike result (stated, for simplicity, for a single log-normal stock with coefficients μ and σ) is the following: consider a (pre-chosen) investment horizon $[0,T]$ and assume that the investor's terminal utility U_T behaves like a power function for large wealth levels, i.e., for some $\gamma \in (0,1)$,

$$U_T(x) \sim \frac{1}{\gamma} x^{\gamma}, \quad x \text{ large.} \tag{1}$$

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Then, if this specific horizon T is very long, the associated optimal portfolio function $\pi^*(x,t;T)$ approximates the one corresponding to this power utility, i.e., for *each* $x>0$, $t\in[0,T]$,

$$\frac{\pi^*(x,t;T)}{x}\sim\frac{\mu}{\sigma^2}\frac{1}{1-\gamma},\quad T\ \text{ large.} \tag{2}$$

In other words, the asymptotic *spatial* behavior of the terminal datum dictates the long-horizon *temporal* behavior of the portfolio function for *every* wealth level. The function $\pi^*(x,t;T)$ is the one the determines the optimal wealth process in feedback form, in that the optimal wealth process X_t^*, $t\in[0,T]$, is generated by the investment strategy $\pi_t^*=\pi^*(X_t^*,t;T)$.

Turnpike results can be found in [5] (see, also, [14]) where a continuous time model was first considered and the turnpike properties were established using contingent claim methods. Their results were later extended in [12] using an autonomous pde of fast-diffusion type satisfied by $\pi^*(x,t;T)$ and viscosity solutions arguments. Duality methods were used in [6] for complete markets and the incomplete market case was studied in [11]. The authors of [3] established the rate of convergence in a log-normal model, showing that there exist a positive constant c and a function $D(x)$, such that, for each $x>0$, $\left|\pi^*(x,t;T)-\frac{\mu}{\sigma^2}\frac{1}{1-\gamma}x\right|\leq D(x)\,e^{-c(T-t)}$.

A closer look at all existing turnpike results yields that we are essentially working in a single investment horizon setting, $[0,T]$, which is taken to be very long. As a result, in order to properly define the optimization problem, one needs to pre-commit to a market model for this long horizon. This choice cannot be modified later on if time consistency is desired but, on the other hand, knowing the model dynamics for a very long horizon might not realistic. Besides the stringent constraints on model choice, one also pre-commits at initial time to a utility function for very far in the future, which is also a rather restrictive assumption. Finally, we remark that no matter how big T is, the optimal investment problem is not defined beyond this point, for the utility function is chosen only for T and, thus, the underlying problem is well defined on $[0,T]$ only.

Herein, we take an alternative point of view and consider a new asymptotic investment problem. Instead of committing to a single long horizon $[0,T]$ with T large, we define from the beginning an investment problem for *all* times $t\in[0,\infty)$. Moreover, instead of choosing at the initial time the utility U_T for the remote horizon T, we choose the utility at this initial time. We, also, depart from the log-normal setting and work with a general Ito-diffusion multi-security market model. However, we do not pre-specify the model coefficients but instead we update them going forward.

We measure the performance of investment strategies via the so called *forward performance criterion.* This alternative class of stochastic utilities was introduced by Musiela and the second author in [21] (see, also, [22]) and offers flexibility for performance measurement under model adaptation, model ambiguity, alternative market views, rolling horizons, and others. We recall its definition and refer the reader, among others, to [2], [7], [8], [15], [16], [17], [23], [24], [29], [30] and [31]; see, also, the recent review article [20].

Herein, we develop forward turnpike type results working with the class of *time monotone* forward utilities, developed and studied in [25]; we briefly review them in the next section. These forward criteria are given by a time-decreasing and adapted to the market information process, $U(x,t)$, $t \geq 0$, $x \geq 0$, of the form

$$U(x,t) = u(x, A_t),$$

where $u(x,t)$ is a deterministic function (cf. (13)) and $A_t = \int_0^t |\lambda_s|^2 \, ds$, with the process λ_t, $t \geq 0$, being the market price of risk. In other words, $U(x,t)$ is a compilation of a *deterministic investor-specific input*, $u(x,t)$, and a *stochastic market-specific input*, A_t. Furthermore, the optimal investment process π_t^*, $t \geq 0$, is given by

$$\pi_t^* = \sigma_t^+ \lambda_t r(X_t^*, A_t) \qquad \text{with} \qquad r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{3}$$

where σ_t^+ is the pseudo-inverse of the volatility matrix, and X_t^*, $t \geq 0$, the optimal wealth generated by this investment strategy π_t^* (cf. (11)). The function $r(x,t)$ is the dynamic risk tolerance and will be the main object of study herein.

Contrary to the classical case, in which a terminal datum is pre-assigned for T and the solution is then constructed for $t \in [0,T)$, in the forward setting, the forward criterion is defined for all times, starting with an initial (and not terminal) datum $U(x,0)$.

In analogy to the classical setting, we are thus motivated to study the following turnpike-type question in the forward framework: if the initial condition $u(x,0)$ is such that, for some $\gamma \in (0,1)$,

$$u(x,0) \sim \frac{1}{\gamma} x^\gamma, \quad x \ \text{ large}, \tag{4}$$

does this imply that, for each $x > 0$,

$$\frac{r(x,t)}{x} \sim \frac{1}{1-\gamma}, \quad t \ \text{ large ?}$$

There are fundamental differences between the classical and the forward settings as one is not a mere variation of the other by a time reversal. Rather, the classical problem is well-posed while the forward is an inverse and, in general, ill-posed problem. As a result, various properties used for the classical turnpike results fail, with the most important being the lack of comparison principle for the various PDEs (cf. (13) and (24)) at hand.

The first striking difference between the two settings is the *distinct* nature of the *temporal* and *spatial* limits. Indeed, in the traditional turnpike results in [12] and [3], the temporal limit in (2) coincides with the spatial one, in that for fixed time T_0 and wealth level x_0, respectively,

$$\lim_{x\uparrow\infty}\frac{\pi\left(x,t;T_0\right)}{x}=\lim_{T\uparrow\infty}\frac{\pi\left(x_0,t;T\right)}{x_0}.$$

However, this is *not* the case in the forward setting. Indeed, the temporal and spatial limits of the function $\frac{r(x,t)}{x}$ do *not* coincide. This can be seen, for instance, in the motivational example in subection 2.1.

The aim herein then becomes the study of the *spatial* and *temporal* limits of the dynamic relative risk tolerance function,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x} \qquad \text{and} \qquad \lim_{t\uparrow\infty}\frac{r(x_0,t)}{x}, \tag{5}$$

for fixed $t_0\geq 0, x_0>0$, respectively, under appropriate conditions on the asymptotic behavior of the initial datum $U\left(x,0\right)$, for large x.

Pivotal role for determining these limits is played by a positive finite Borel measure, μ, which is the defining element in the construction of the time monotone forward processes. Specifically, it was shown in [25] that the above function u is uniquely (up to an additive constant) related to a space-time harmonic function $h:\mathbb{R}\times\left[0,\infty\right)\longrightarrow\mathbb{R}^+$ which is (uniquely) characterized by an integral transform, namely,

$$u_x\left(h\left(z,t\right),t\right)=e^{-z+\frac{t}{2}} \qquad \text{with} \qquad h\left(z,t\right)=\int_a^b e^{zy-\frac{1}{2}y^2t}\mu\left(dy\right), \tag{6}$$

for some $0\leq a\leq b\leq\infty$. An immediate consequence of this general solution is that the initial datum, in particular the initial inverse marginal utility, is directly constructed through this measure μ, in that $\left(u_x(x,0)\right)^{(-1)}$ needs to be of the integral form

$$\left(u_x(x,0)\right)^{(-1)}=\int_a^b x^{-y}\mu\left(dy\right).$$

As a result, it is natural to expect that the asymptotic properties of $u\left(x,0\right)$, which enter crucially in the turnpike assumptions, are also directly linked to the form and properties of μ. Furthermore, this measure also appears in the specification of the dynamic risk tolerance function. Indeed, we deduce from (3) and (6) that $r\left(x,t\right)$ is represented as

$$r\left(x,t\right)=h_z\left(h^{(-1)}\left(x,t\right),t\right), \tag{7}$$

with both h_z and $h^{(-1)}$ depending on μ. We will be calling μ the *risk preference measure.*

The main results herein are that, under certain asymptotic assumptions for large x on the initial risk preferences, the *spatial* limit of the dynamic relative risk tolerance function coincides with the *right end* point of the support of the risk preference measure while its *temporal* limit coincides with the *left end.* In other words, for a and b as in (6), we have, for each $t_0 \geq 0$ and $x_0 > 0$, respectively,

$$\lim_{t\uparrow\infty} \frac{r(x_0,t)}{x} = a \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = b. \tag{8}$$

The first step in obtaining the above limits is to establish equivalences between the asymptotic, for large x, behavior of the initial marginal utility and the structure of the measure, and in particular, the finiteness of its support and the existence of masses at its end points. We summarize the main findings next.

Spatial turnpike limit: To establish the spatial limit in (8), we first show that the asymptotic assumption (4), stated in terms of the marginal,

$$u_x\,(x,0) \sim \; x^{\gamma-1}, \quad x \;\text{ large}, \tag{9}$$

for some $\gamma \in (0,1)$, holds if and only if the right end of the measure's support satisfies both $b = \frac{1}{1-\gamma}$ and $\mu\left(\{b\}\right) = 1$. In other words, condition (9) implies that the measure must have finite support with its right boundary equal to $\frac{1}{1-\gamma}$ and, furthermore, with a (unit) mass at this point. Conversely, for the measure to have these properties, condition (9) must hold. We, in turn, establish the spatial limit in (8) using representation (6), equation (13) and various convexity properties of h and its derivatives. We stress that the requirement that $\mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) \neq 0$ cannot be relaxed. Indeed, we show in subsection 5.2, where the measure is Lebesgue, that the spatial turnpike property *fails.*

Temporal turnpike limit: To establish the temporal limit in (8), we first relate the finiteness of the measure's support with a weaker version of (9). Specifically, we show that if there exists $\gamma \in (0,1)$ such that for all $\gamma' \in (\gamma,1)$ and all $\gamma'' \in (0,\gamma)$,

$$\lim_{x\uparrow\infty} \frac{u_x\,(x,0)}{x^{\gamma'-1}} = 0 \qquad \text{and} \qquad \lim_{x\uparrow\infty} \frac{u_x\,(x,0)}{x^{\gamma''-1}} = \infty, \tag{10}$$

then the right boundary of the support must satisfy $b = \frac{1}{1-\gamma}$ and vice versa. This "regular variation" assumption is weaker than (9), required for the spatial limit and, naturally, yields a weaker result. Indeed, while the support has to be finite with right boundary equal to $\frac{1}{1-\gamma}$, it does not need to have a mass at $\frac{1}{1-\gamma}$. In turn, we establish the temporal limit in (8), which is the genuine analogue of the classical turnpike results. Obtaining this limit is considerably more challenging than in the classical case due to the ill-posed nature of the problem. Indeed, the methodology used in [12] is inapplicable due to the lack of comparison results for the ergodic version of the equation satisfied by $r\,(x,t)$. The approach of [3] does not apply either because of the lack of connection between the solutions of the ill-posed heat equation and Feynman-Kac type stochastic representation of its solution. Therefore, one needs to work directly with the function $r\,(x,t)$, which, from (6) and (7), is given in the implicit form

$$r\,(x,t) = \int_a^{\frac{1}{1-\gamma}} y e^{y h^{(-1)}(x,t) - \frac{1}{2}y^2 t} \mu\,(dy)\,,$$

where, however, the spatial inverse $h^{(-1)}$ is involved, which is not explicitly known. The key step in obtaining the temporal limit is to show that, for each $x > 0$,

$$\lim_{t\uparrow\infty} \frac{h^{(-1)}(x,t)}{t} = \frac{a}{2}.$$

In turn, we establish the temporal limit in (8) as well as the rate of convergence using the implicit representation

$$r(x,t) - ax = \int_a^{\frac{1}{1-\gamma}} (y-a)\, e^{ty\left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2}y\right)} \mu(dy).$$

In addition to the general spatial and temporal convergence results in (8), we present two representative examples. In the first, the measure is a finite sum of Dirac functions while, in the second, it is taken to be the Lebesgue measure. To calculate the limits in (8) we first derive asymptotic expansions for both the auxiliary function $h^{(-1)}(x,t)$ and the dynamic risk tolerance function.

The paper is structured as follows. In section 2, we present the market model, the forward performance criterion and a motivating example demonstrating that the temporal and spatial limits do not in general coincide. In section 3 and 4 we analyze, respectively, the spatial and temporal asymptotic behavior of the dynamic relative risk tolerance. In section 5 we present the two representative examples and conclude in section 6 providing future research directions.

2 The model and the forward investment criterion

The market environment consists of one riskless and k risky securities. The prices of the risky securities are modelled as Itô-diffusion processes, namely, the price S_t^i, $t \geq 0$, of the i^{th} risky asset follows

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right),$$

with $S_0^i > 0$, for $i = 1, ..., k$. The process $W_t = \left(W_t^1, ..., W_t^d\right)$, $t \geq 0$, is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with natural filtration $\{\mathcal{F}_t\}$, $t \geq 0$.

The coefficients μ_t^i and $\sigma_t^i = \left(\sigma^{1i}, ..., \sigma_t^{di}\right)$, $t \geq 0$, $i = 1, ..., k$, are $\mathcal{F}_t$-adapted processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote by σ_t the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} asset. We may, then, alternatively, write the above equation as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right).$$

The riskless asset (the savings account) is taken to be the numeraire and has price process B_t, $t \geq 0$, satisfying $dB_t = r_t B_t dt$ with $B_0 = 1$, and for a nonnegative $\mathcal{F}_t-$adapted interest rate process r_t, $t \geq 0$. We, also, denote by μ_t the k-dimensional vector with coordinates μ_t^i and by $\mathbf{1}$ the k-dimensional vector with every component equal to one. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that $\mu_t - r_t \mathbf{1} \in Lin\left(\sigma_t^T\right)$, where $Lin\left(\sigma_t^T\right)$ denotes the linear space generated by the columns of σ_t^T. Therefore, the equation $\sigma_t^T z = \mu_t - r_t \mathbf{1}$ has a solution, known as the market price of risk, $\lambda_t = \left(\sigma_t^T\right)^+ \left(\mu_t - r_t \mathbf{1}\right)$. It is assumed that there exists a deterministic constant $c > 0$, such that $|\lambda_t| \leq c$, $t \geq 0$.

Starting at $t = 0$ with an initial endowment $x \geq 0$, the investor invests at any time $t > 0$ in the riskless and risky assets. The present value of the amounts invested are denoted by the processes π_t^0 and π_t^i, $t \geq 0, i = 1, ..., k$, respectively, and are taken to be self-financing. The present value of her investment is given by the (discounted) wealth process X_t^π, $t \geq 0$, with $X_t^\pi = \sum_{i=1}^N \pi_t^i$, which solves

$$dX_t^\pi = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t), \quad X_0^\pi = x \geq 0, \tag{11}$$

with the (column) vector $\pi_t = \left(\pi_t^i; i = 1, ..., k\right)$. It is taken to satisfy the nonnegativity constraint $X_t^\pi \geq 0$, $t > 0$.

The set of admissible policies is given by

$$\mathcal{A} = \left\{ \pi : \text{self-financing}, \ \pi_t \in \mathcal{F}_t, \ E_{\mathbb{P}} \int_0^t |\sigma_s \pi_s|^2 \, ds < \infty, \ X_t^\pi \geq 0, \ t > 0 \right\}.$$

The performance of admissible investment strategies is evaluated via the so-called forward performance criteria introduced in [21] (see also, the references mentioned in the Introduction). We review their definition next.

We introduce the domain notation $\mathbb{D}_+ = \mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{D} = \mathbb{R} \times \mathbb{R}_+$.

Definition 1 *An $\mathcal{F}_t$-adapted process $U(x,t)$, $(x,t) \in \mathbb{D}_+$, is a forward performance criterion if,*

i) for each $t \geq 0$, the mapping $x \to U(x,t)$ is strictly increasing and strictly concave,

ii) for each $\pi \in \mathcal{A}$, $U(X_t^\pi, t)$ is a (local) supermartingale,

iii) there exists $\pi^* \in \mathcal{A}$ such that $U(X_t^{\pi^*}, t)$ is a (local) martingale.

Herein we focus on the class of *time monotone* forward performance processes, which constitute a rich enough class of forward criteria. They were extensively studied in [25] and we refer the reader therein for all technical details. We only review the main results that we will use, some of which have been already stated in the Introduction.

Time monotone forward performance criteria are uniquely represented by processes of the form

$$U(x,t) = u(x, A_t), \tag{12}$$

where $u : \mathbb{D}_+ \to \mathbb{R}_+$, and for each $t \geq 0$, it is strictly increasing and strictly concave in x, and satisfies

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}. \tag{13}$$

The *market input* processes A_t and M_t, $t \geq 0$, are defined as

$$M_t = \int_0^t \lambda_s \cdot dW_s \quad \text{and} \quad A_t = \langle M \rangle_t = \int_0^t |\lambda_s|^2 \, ds. \tag{14}$$

Central role in the entire construction is played by the space-time harmonic function $h : \mathbb{D} \to \mathbb{R}_+$, defined by

$$u_x(h(z,t),t) = e^{-z+\frac{t}{2}}. \tag{15}$$

It solves, as it follows from (13) and (15), the ill-posed heat equation

$$h_t + \frac{1}{2} h_{zz} = 0, \tag{16}$$

and, moreover, it is positive and strictly increasing in z, for each $t \geq 0$. It was shown in [25] that such solutions are *uniquely* represented in the integral form

$$h(z,t) = \int_a^b \frac{e^{yz-\frac{1}{2}y^2 t} - 1}{y} \nu(dy) + C,$$

where the measure $\nu \in \mathcal{B}^+(\mathbb{R})$, the set of positive Borel measures, with the additional properties that, for $z \in \mathbb{R}$,

$$\nu((-\infty, 0]) = 0, \quad \int_a^b e^{yz} \nu(dy) < \infty \quad \text{and} \quad \int_a^b \frac{1}{y} \nu(dy) < \infty.$$

To simplify the presentation we choose without loss of generality the constant $C = \int_a^b \frac{1}{y} d\nu(y)$ and introduce the normalized measure $\mu(dy) = \frac{1}{y}\nu(dy)$. Then, the function h admits the (unique) representation, for $(z,t) \in \mathbb{D}$,

$$h(z,t) = \int_a^b e^{yz-\frac{1}{2}y^2 t} \mu(dy), \text{ with } 0 \leq a \leq b \leq \infty. \tag{17}$$

From (15) and (17), we obtain that $u(x,t)$ is represented as

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_z \left(h^{(-1)}(x,s), s \right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz. \tag{18}$$

Note that, for $t = 0$, the initial datum of the forward process is given by

$$U(x,0) = u(x,0) = \int_0^x e^{-h^{(-1)}(z,0)} dz, \tag{19}$$

which is fully specified by the risk preference measure through $h^{(-1)}(x,0)$. Furthermore, the initial inverse marginal utility, $(u_x(x,0))^{(-1)}$, must be of the form

$$(U'(x,0))^{-1} = (u_x(x,0))^{(-1)} = \int_a^b x^{-y} \mu(dy). \tag{20}$$

We stress that (19) and (20) are *if and only if* characterizations in the sense that only initial utility data with inverse marginals of the above structure are admissible, otherwise equation (13) does not have a well defined solution for all times in $[0,\infty)$. For modeling purposes, the risk preference measure is extracted from the choice of the initial utility or its marginal. Inverse marginal utilities of form (20) were extensively studied in [19] in the classical setting.

It was shown in [25] that h, together with the market input processes A_t and M_t, yield the optimal allocation process π_t^* and the associated optimal wealth X_t^*, $t\geq 0$. Specifically, if the measure satisfies the additional assumption, $\int_a^b ye^{yz+\frac{1}{2}y^2t}\mu(dy)<\infty$, $z\in\mathbb{R}$, then the optimal processes are given, respectively, by

$$X_t^*=h\left(h^{(-1)}(x,0)+A_t+M_t,A_t\right)\ \text{and}\ \pi_t^*=\sigma_t^+\lambda_t h_z\left(h^{(-1)}(X_t^*,A_t),A_t\right). \tag{21}$$

The *dynamic risk tolerance* function $r:\mathbb{D}_+\to\mathbb{R}_+$, defined as

$$r(x,t):=-\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{22}$$

can be represented as

$$r(x,t)=h_z\left(h^{(-1)}(x,t),t\right)=\int_a^b e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}\mu(dy). \tag{23}$$

It, also, satisfies the ill-posed fast-diffusion type equation

$$r_t+\frac{1}{2}r^2r_{xx}=0,\quad r(x,0)=\int_a^b e^{yh^{(-1)}(x,0)}\mu(dy), \tag{24}$$

and, for each $t\geq 0$, $\lim_{x\downarrow 0}r(x,t)=r(0,t)=0$.

The optimal portoflio process can be written as

$$\pi_t^*=\sigma_t^+\lambda_t r(X_t^*,A_t). \tag{25}$$

It is easily seen that, for each $t\geq 0$, the function $h\left(.,t\right)$ is absolutely monotonic, since $\frac{\partial^i h(z,t)}{\partial z^i}>0$, $i=1,....$ Such functions satisfy, for each $t\geq 0$, the well known inequality

$$\frac{\partial^{i+1}h\left(z,t\right)}{\partial z^{i+1}}\frac{\partial^{i-1}h\left(z,t\right)}{\partial z^{i-1}}-\left(\frac{\partial^{i}h\left(z,t\right)}{\partial z^{i}}\right)^2\geq 0. \tag{26}$$

In turn, for each $t\geq 0$, $r\left(.,t\right)$ is strictly increasing and strictly convex, since

$$r_x(x,t)=\frac{h_{zz}\left(h^{(-1)}(x,t),t\right)}{r(x,t)}=\frac{1}{r(x,t)}\int_a^b y^2e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}\mu(dy)>0,$$

and

$$r_{xx}(x,t)=\frac{1}{r^3(x,t)}\left(h_{zzz}(z,t)h_z(z,t)-h_{zz}^2(z,t)\big|_{z=h^{(-1)}(x,t)}\right)>0,$$

where we used (26).

We note that throughout we will frequently differentiate under the integral sign in (17) and in similar integrals, which is permitted as explained in [25][1].

As stated in the Introduction, the aim herein is to investigate the spatial and temporal limits of $\frac{r(x,t)}{x}$, with $r(x,t)$ as in (22). We first provide an example which shows that, contrary to classical turnpike results ([12], [3] and others), these two limits do *not* in general coincide in the forward setting.

2.1 A motivating example

Case 1: *Single Dirac function*

The risk preference measure is Dirac, $\mu = \delta_{\frac{1}{1-\gamma}}$, $\gamma \in (0,1)$. From (17) and (15) we have, for $(z,t) \in \mathbb{D}$ and $(x,t) \in \mathbb{D}_+$, respectively,

$$h(z,t) = e^{\frac{1}{1-\gamma}z - \frac{1}{2(1-\gamma)^2}t} \quad \text{and} \quad u_x(x,t) = x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t}.$$

Therefore, the dynamic risk tolerance function is given by $r(x,t) = \frac{1}{1-\gamma}x$ and we easily conclude that the spatial and temporal limits are equal, given by (for fixed t_0 and x_0, respectively).

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \frac{1}{1-\gamma} \quad \text{and} \quad \lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = \frac{1}{1-\gamma}.$$

Case 2: *Sum of two Dirac functions*

The risk preference measure is given, for $\theta, \gamma \in (0,1)$, by

$$\mu = \delta_{\frac{1}{1-\theta}} + \delta_{\frac{1}{1-\gamma}} \quad \text{with} \quad \frac{1}{1-\gamma} = 2\frac{1}{1-\theta}. \tag{27}$$

To ease the presentation, we set $\kappa = \frac{1}{1-\theta}$. Then, (17) yields

$$h(z,0) = e^{\kappa z} + e^{2\kappa z}, \tag{28}$$

and, from (15),

$$\left(u_x(x,0)\right)^{(-1)} = x^{-\frac{1}{1-\theta}} + x^{-\frac{1}{1-\gamma}} . \tag{29}$$

Therefore, $u_x(x,0) = 2^{1-\theta}\left(\sqrt{1+4x}-1\right)^{\theta-1}$, and, thus,

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{x\uparrow\infty} \frac{2^{2(1-\gamma)}\left(\sqrt{1+4x}-1\right)^{2(\gamma-1)}}{x^{\gamma-1}} = 1. \tag{30}$$

Furthermore, (16) and (28) yield, for $(z,t) \in \mathbb{D}$,

$$h(z,t) = e^{\kappa z - \frac{1}{2}\kappa^2 t} + e^{2\kappa z - 2\kappa^2 t}, \tag{31}$$

[1] It can be also seen directly since, after differentiation, the relevant integrands are jointly continuous in their respective arguments - see Theorem 24.5 in [1] and the remark following it .

and, therefore, for $(x,t)\in\mathbb{D}_+$,

$$h^{(-1)}(x,t)=\frac{1}{2}\kappa t+\frac{1}{\kappa}\ln\frac{2x}{1+\sqrt{1+4xe^{-\kappa^2 t}}}. \tag{32}$$

Next, we calculate the risk tolerance using (22). Introducing $f(x,t):=1+\sqrt{1+4xe^{-\kappa^2 t}}$, rewriting (32) as

$$h^{(-1)}(x,t)=\frac{1}{\kappa}\ln\frac{2xe^{\frac{\kappa^2 t}{2}}}{f(x,t)},$$

and observing from (31) that $h_z(z,t)=\kappa e^{\kappa z-\frac{1}{2}\kappa^2 t}+2\kappa e^{2\kappa z-2\kappa^2 t}$, we compute

$$r(x,t)=h_z\left(h^{(-1)}(x,t),t\right)=\kappa\exp\left(\ln\frac{2xe^{\frac{\kappa^2 t}{2}}}{f(x,t)}-\frac{1}{2}\kappa^2 t\right)$$

$$+2\kappa\exp\left(2\ln\frac{2xe^{\frac{\kappa^2 t}{2}}}{f(x,t)}-2\kappa^2 t\right)=2\kappa\frac{x}{f(x,t)}+8\kappa\frac{x^2}{f^2(x,t)}e^{-\kappa^2 t}.$$

Note that, for each $x_0>0$, $\lim_{t\uparrow\infty}f(x_0,t)=2$ while, for each $t_0\geq 0$, $\lim_{x\uparrow\infty}\frac{1}{f(x,t_0)}=0$.

Therefore, $\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}=\lim_{t\uparrow\infty}\frac{2\kappa}{f(x,t)}=\kappa$. On the other hand, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\lim_{x\uparrow\infty}\frac{8\kappa x}{f^2(x,t)}e^{-\kappa^2 t}=\lim_{y\uparrow\infty}\frac{8\kappa y}{\left(1+\sqrt{1+4y}\right)^2}=2,\ y=xt_0.$$

In summary, reverting to the original notation, we have that, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x}=\frac{2}{1-\theta}=\frac{1}{1-\gamma}, \tag{33}$$

while, for each $x_0>0$,

$$\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}=\frac{1}{1-\theta}. \tag{34}$$

Thus, the spatial and temporal limits of the relative risk tolerance do *not* coincide.

Next, we make the following two important observations. Firstly, we note that (27) yields that the support of the measure is

$$supp\,(\mu)=\left[\frac{1}{1-\theta},\frac{1}{1-\gamma}\right].$$

Therefore, the *temporal* limit (34) coincides with the *left end* of the support while the *spatial* limit (33) with the *right end*. Secondly, for each $x_0>0$, the temporal limit of the ratio $\frac{h^{(-1)}(x_0,t)}{t}$ is equal to *half* of the left end point of the support, since (32) yields

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x,t)}{t}=\frac{1}{2(1-\theta)}.$$

In section 4 we show that both these properties are always valid. In particular, we will see that it is precisely the limit of the ratio $\frac{h^{(-1)}(x,t)}{t}$ that plays the key role in establishing the temporal turnpike property for general risk preference measures.

To juxtapose the above results with the ones in the classical expected utility setting, we compute analogous quantities and associated limits for the cases analyzed in [12] and [3] for log-normal markets (Merton problem) because their optimal feedback portfolio functions resemble the ones with time monotone forward criteria. Without loss of generality, we consider a market with a single log-normal stock with mean rate of return μ and volatility σ, and a riskless account of constant interest rate r.

To this end, we fix an investment horizon $T>0$ and, in analogy to (29), we take the *terminal* inverse marginal utility, $I_T(x)=(U_T')^{(-1)}(x)$, to be of the form

$$I_T(x)=x^{-\frac{1}{1-\theta}}+x^{-\frac{1}{1-\gamma}},$$

for $x>0$ and θ,γ as in (27). This corresponds to terminal marginal utility $U_T'(x)=2^{1-\gamma}\left(\sqrt{1+4x}-1\right)^{\gamma-1}$ and, thus, in analogy to (30), we have that

$$\lim_{x\uparrow\infty}\frac{U_T'(x)}{x^{\gamma-1}}=1.$$

We consider the value function, denoted by $u(x,t;T)$, of the associated Merton problem, for $t\in[0,T]$. Letting $\tau=T-t$ be the time to the end of the investment horizon, we deduce, using well known results, that the function $\tilde{u}(x,\tau)\equiv u(x,T-t;T)$ satisfies, for $(x,\tau)\in\mathbb{R}_+\times[0,T]$ and $\lambda=\frac{\mu-r}{\sigma}$, the Hamilton-Jacobi-Bellman equation

$$\tilde{u}_\tau+\frac{1}{2}\lambda^2\frac{\tilde{u}_x^2}{\tilde{u}_{xx}}=0,$$

with $\tilde{u}(x,0)=U_T(x)$.

In turn, the inverse spatial marginal value function, $\tilde{v}:\mathbb{R}_+\times[0,T)\to\mathbb{R}_+$ solves $\tilde{v}_\tau=\frac{1}{2}\lambda^2x^2\tilde{v}_{xx}+\lambda^2x\tilde{v}_x$, with $\tilde{v}(x,0)=I_T(x)$. We easily deduce that $\tilde{v}(x,\tau)=e^{\alpha\tau}x^{-\alpha}+e^{\beta\tau}x^{-2\alpha}$ with $\alpha=\frac{\gamma}{2(1-\gamma)^2}\lambda^2$ and $\beta=\frac{1+\gamma}{(1-\gamma)^2}\lambda^2$. Note that $\beta>2\alpha$. Taking the spatial inverse of $\tilde{v}(x,\tau)$ yields

$$\tilde{u}_x(x,\tau)=\left(\frac{e^{\alpha\tau}+\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}}}{2x}\right)^{1-\theta}.$$

Therefore, the related dynamic risk tolerance function $\tilde{r}(x,\tau)$ satisfies (using that $\frac{1}{r(x,\tau)} = -\frac{\partial}{\partial x}\ln \tilde{u}_x(x,\tau)$),

$$\tilde{r}(x,\tau) = \frac{1}{1-\theta}\left(\frac{1}{x} - \frac{2e^{\beta\tau}}{e^{a\tau}\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}} + \left(e^{2\alpha\tau}+4xe^{\beta\tau}\right)}\right)^{-1}$$

and, thus,

$$\frac{\tilde{r}(x,\tau)}{x} = \frac{1}{1-\theta}\left(1 - x\frac{2e^{\beta\tau}}{e^{a\tau}\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}} + \left(e^{2\alpha\tau}+4xe^{\beta\tau}\right)}\right)^{-1}.$$

Direct calculations yield that, for each $\tau_0 > 0$ and each $x_0 > 0$, respectively, the spatial and temporal limits are given by

$$\lim_{x\uparrow\infty}\frac{\tilde{r}(x,\tau_0)}{x} = \frac{1}{1-\gamma} \quad \text{and} \quad \lim_{\tau\uparrow\infty}\frac{\tilde{r}(x_0,\tau)}{x_0} = \frac{1}{1-\gamma}.$$

The two limits are equal and, furthermore, they coincide with the right end point $\frac{1}{1-\gamma}$.

Motivated by this example, we are investigating the spatial and temporal asymptotic limits of the dynamic relative risk tolerance function. The coefficient γ is taken to belong to $(0,1)$ to simplify the presentation as the case $\gamma \leq 0$ can be similarly analyzed.

3 Spatial asymptotic behavior of relative risk tolerance

We examine the spatial asymptotic behavior of the local risk tolerance function under asymptotic assumptions for large wealth levels of the investor's initial risk preferences. In accordance with a similar assumption in [12] and [3], we impose it on the initial marginal utility $u_x(x,0)$ and not on $u(x,0)$ itself.

Assumption 1: The initial datum $u(x,0)$ in (19) is such that, for some $\gamma \in (0,1)$,

$$\lim_{x\uparrow\infty}\frac{u_x(x,0)}{x^{\gamma-1}} = 1. \tag{35}$$

We stress that (35) is necessary for the spatial limit (39) to hold in general. In the next section where we look at the temporal limit, the above property will be relaxed.

Given the key role that the risk preference measure μ plays, we first examine what Assumption 1 implies for it. As the next result shows, (35) yields that its support must be finite with its right boundary equal to $\frac{1}{1-\gamma}$ and, furthermore, $\mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1$.

Lemma 2 *Assumption (35) holds if and only if the risk preference measure* μ *in (17) satisfies* $b = \frac{1}{1-\gamma}$ *and* $\mu(\{b\}) = 1$, *i.e.*

$$supp\,(\mu) \subseteq \left(0, \frac{1}{1-\gamma}\right] \quad and \quad \mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1. \tag{36}$$

Proof. From (15), (35) and the fact that $h(x,0)$ is strictly increasing and of full range, we have

$$1 = \lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{z\uparrow\infty} \frac{u_x(h(z,0),0)}{h^{\gamma-1}(z,0)} = \lim_{z\uparrow\infty} \left(\frac{h(z,0)}{e^{\frac{1}{1-\gamma}z}}\right)^{1-\gamma}.$$

Therefore, representation (17) gives

$$\lim_{z\uparrow\infty} \int_a^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) = 1. \tag{37}$$

If $a = b = \frac{1}{1-\gamma}$, then (36) follows directly. If $a < b$, then it must be that $a \le \frac{1}{1-\gamma}$, otherwise we get a contradiction.

Next, let $\varepsilon > 0$. Then,

$$\int_a^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) \ge \int_{\frac{1}{1-\gamma}+\varepsilon}^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) \ge e^{\varepsilon z}\mu\left(\left[\frac{1}{1-\gamma}+\varepsilon, b\right]\right). \tag{38}$$

Sending $\varepsilon \downarrow 0$ and using (37) yields that $\mu\left(\left(\frac{1}{1-\gamma}, b\right]\right) = 0$, and thus $supp(\mu) \subseteq \left(a, \frac{1}{1-\gamma}\right]$. On the other hand, we have from (37) that

$$1 = \lim_{z\uparrow\infty} \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{z(y-\frac{1}{1-\gamma})}\mu(dy) + \mu(\{\frac{1}{1-\gamma}\}) = \mu(\{\frac{1}{1-\gamma}\}),$$

and we easily conclude. ■

Next, we state the main spatial asymptotic result.

Proposition 3 *Suppose that the initial utility datum is such that limit (35) is satisfied. Then, for each* $t_0 \ge 0$, *the relative risk tolerance converges to the right end of the support of the risk preference measure,*

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \frac{1}{1-\gamma}. \tag{39}$$

Proof. Let $t_0 \ge 0$. From Lemma 2 we have that

$$h(z,t_0) = \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{yz-\frac{1}{2}y^2t_0}\mu(dy) + e^{\frac{1}{1-\gamma}z-\frac{1}{2}\frac{1}{(1-\gamma)^2}y^2t_0}.$$

In turn, by dominated convergence we obtain

$$\lim_{z\uparrow\infty} \frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\frac{1}{(1-\gamma)^2}t_0}} = 1, \tag{40}$$

since

$$\lim_{z\uparrow\infty} \frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\frac{1}{(1-\gamma)^2}t_0}} = \lim_{z\uparrow\infty} \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}t_0(y^2-\frac{1}{(1-\gamma)^2})}\mu(dy) + 1 = 1.$$

Therefore, from (15) together with the strict monotonicity and the full range of $h(z,t_0)$, we deduce that

$$\lim_{x\uparrow\infty} \frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} = 1, \tag{41}$$

since

$$\lim_{x\uparrow\infty} \frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} = \lim_{z\uparrow\infty} \frac{e^{-z+\frac{t_0}{2}}}{(h(z,t_0))^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}$$

$$= \lim_{z\uparrow\infty} \left(\frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}} \right)^{1-\gamma} = 1.$$

Next, we claim that

$$\lim_{x\uparrow\infty} \frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} = \frac{1}{\gamma-1}. \tag{42}$$

To prove this, it suffices to show that, for each $t_0 \geq 0$, $u_x(x,t_0)$ is convex since the above would then follow from the arguments in Lemma 3.1 in [25]. To this end, differentiating (15) yields

$$u_{xxx}\left(h(z,t_0),t_0\right)\left(h_z(z,t_0)\right)^2 + u_{xx}(h(z,t_0),t_0)h_{zz}(z,t_0) = e^{-z+\frac{t_0}{2}}. \tag{43}$$

Then, the strict convexity of h and the strict concavity of u give that $u_{xxx}\left(h(z,t_0),t_0\right) > 0$, and using the strict monotonicity and the full range of h we conclude. Combining (41) and (42) we deduce

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \lim_{x\uparrow\infty} \left(-\frac{u_x(x,t_0)}{xu_{xx}(x,t_0)} \right)$$

$$= \lim_{x\uparrow\infty} \left(-\frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} \left(\frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} \right)^{-1} \right) = \frac{1}{1-\gamma}.$$

■

Assumption (35), or equivalently (36), *cannot* be relaxed. Indeed, as we will see in Example 5.2 where we take the measure to be Lebesgue on $\left[a, \frac{1}{1-\gamma}\right]$ and, thus, there is no mass at $\frac{1}{1-\gamma}$, the spatial turnpike property does not hold.

4 Temporal asymptotic behavior of relative risk tolerance

We investigate the temporal limit of the relative risk tolerance, $\frac{r(x_0,t)}{x_0}$, as $t \uparrow \infty$ and for each $x_0 > 0$. This is the genuine turnpike analogue of similar results in classical expected utility models and one of the main findings herein. It shows that, for each space argument, the relative risk tolerance converges to the *left end* of the support of the risk preference measure μ. As in the spatial case, we first relate the properties of the measure to the asymptotic behavior of the initial (marginal) utility datum, but now using a weaker than (35) assumption.

Assumption 2: There exists $\gamma \in (0,1)$ *such that, for all* $\gamma' \in (\gamma, 1)$,

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma'-1}} = 0, \tag{44}$$

while, for all $\gamma'' \in (0,\gamma)$,

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma"-1}} = \infty. \tag{45}$$

The above assumption is directly related to a condition (regular variation) introduced in [6] and [13], for a discrete and a continuous time setting, respectively.

Lemma 4 *Assumption 2 is equivalent to the initial marginal utility* $u_x(x,0)$ *to be varying regularly at infinity with exponent* $\gamma - 1$, *i.e. for each* $k > 0$,

$$\lim_{x\uparrow\infty} \frac{u_x(kx,0)}{u_x(x,0)} = k^{\gamma-1}.$$

The proof follows by routine albeit tedious arguments and is omitted.

Assumption 2, weaker than Assumption 1, implies that the measure has finite support with its right end point being equal $\frac{1}{1-\gamma}$, but without necessarily having a mass therein. We prove this next.

Lemma 5 *Assumption 2 is equivalent to the risk preference measure* μ *in (17) having finite support with its right boundary at* $\frac{1}{1-\gamma}$, *namely,*

$$\inf\{y > 0 : \mu((y,\infty)) = 0\} = \frac{1}{1-\gamma}. \tag{46}$$

Proof. We first show that Assumption 2 implies (46). We know by the results in [25] that the support of the measure must be of the form $(a,b]$, with $a \geq 0$, and $b \leq \infty$. Using the strict monotonicity and the full range of $h(z,0)$, we deduce from (44) that, for each $\gamma' \in (\gamma,1)$,

$$0 = \lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma'-1}} = \lim_{z\uparrow\infty} \frac{u_x(h(z,0),0)}{(h(z,0))^{\gamma'-1}} = \lim_{z\uparrow\infty} \left(\frac{h(z,0)}{e^{\frac{z}{1-\gamma'}}}\right)^{1-\gamma'},$$

and, thus,

$$\lim_{z\uparrow\infty}\int_a^b e^{z\left(y-\frac{1}{1-\gamma'}\right)}\mu(dy)=0. \tag{47}$$

Therefore, if $b\geq 1$, the above gives a contradiction and, thus, it must be that $b<1$.

Next, we assume that there exists $\gamma'\in(\gamma,1)$ such that $b=\frac{1}{1-\gamma'}$. Then, for each $\tilde{\gamma}\in(\gamma,\gamma')$ we have that $\frac{1}{1-\tilde{\gamma}}<\frac{1}{1-\gamma'}$ and (47) gives that, for ε small enough,

$$\lim_{z\uparrow\infty}\left(\int_a^{\left(\frac{1}{1-\tilde{\gamma}}+\varepsilon\right)^-} e^{z\left(y-\frac{1}{1-\tilde{\gamma}}\right)}\mu\left(dy\right)+\int_{\frac{1}{1-\tilde{\gamma}}+\varepsilon}^{b} e^{z\left(y-\frac{1}{1-\tilde{\gamma}}\right)}\mu\left(dy\right)\right)=0.$$

Therefore, it must be that $\mu\left(\left[\frac{1}{1-\tilde{\gamma}}+\varepsilon,b\right]\right)=0$.

Sending $\varepsilon\downarrow 0$ gives $\mu\left(\left(\frac{1}{1-\tilde{\gamma}},b\right]\right)=0$, which is a contradiction. Therefore, we must have $b\leq\frac{1}{1-\gamma}$. Using (45) and working similarly, we obtain that $b\geq\frac{1}{1-\gamma}$ and, thus, it must be that $b=\frac{1}{1-\gamma}$.

To show the reverse direction, we first observe that (46) and dominated convergence yield that, for any $\varepsilon>0$,

$$\lim_{z\uparrow\infty}\frac{h(z,0)}{e^{\left(\frac{1}{1-\gamma}+\varepsilon\right)z}}=\lim_{z\uparrow\infty}\int_a^{\frac{1}{1-\gamma}} e^{z\left(y-\left(\frac{1}{1-\gamma}+\varepsilon\right)\right)}\mu(dy)=0.$$

Then, choosing γ' such that $\frac{1}{1-\gamma'}=\frac{1}{1-\gamma}+\varepsilon$ (i.e. $\gamma'=1-\frac{1-\gamma}{1+\varepsilon(1-\gamma)}$), we deduce (44) for all $\gamma'\in(\gamma,1)$. It remains to show (45). Let $\delta>0$. For $z>0$, we have

$$\begin{aligned}
0<\frac{e^{\left(\frac{1}{1-\gamma}-\delta\right)z}}{h(z,0)}&=\frac{e^{-\frac{\varepsilon}{2}z}}{\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^-} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right)z}\mu(dy)+\int_{\frac{1}{1-\gamma}-\frac{\delta}{2}}^{\frac{1}{1-\gamma}} e^{\left(y-\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)\right)z}\mu(dy)}\\
&\leq\frac{e^{-\frac{\delta}{2}z}}{\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^-} e^{\left(y-\frac{1}{1-\gamma}+\frac{\delta}{2}\right)z}\mu(dy)+\mu\left(\left[\frac{1}{1-\gamma}-\frac{\delta}{2},\infty\right)\right)}\\
&\leq\frac{e^{-\frac{\delta}{2}z}}{\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^-} e^{\left(y-\frac{1}{1-\gamma}+\frac{\delta}{2}\right)z}\mu(dy)+\mu\left(\left(\frac{1}{1-\gamma}-\frac{\delta}{2},\infty\right)\right)}.
\end{aligned}$$

Using (46) for $\varepsilon=\frac{\delta}{2}$, we obtain that $\mu\left(\left(\frac{1}{1-\gamma}-\frac{\delta}{2},\infty\right)\right)>0$. Passing to the limit above as $z\uparrow\infty$, and using that $\lim_{z\uparrow\infty}\int_a^{\left(\frac{1}{1-\gamma}-\frac{\delta}{2}\right)^-} e^{\left(y-\frac{1}{1-\gamma}+\frac{\delta}{2}\right)z}\mu(dy)=0$ and dominated convergence, we deduce that $\lim_{z\uparrow\infty}\frac{e^{\left(\frac{1}{1-\gamma}-\delta\right)z}}{h(z,0)}=0$. We easily conclude. ■

Next, we turn our attention to the left boundary of the support of the risk preference measure,

$$a:=\inf\{y\geq 0:\mu\left((0,y]\right)>0\}. \tag{48}$$

We will frequently use the identity

$$x=\int_a^{\frac{1}{1-\gamma}} e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy), \tag{49}$$

for $x>0$, which follows from (17). Herein, $h^{(-1)}(x,t):\mathbb{D}_+\to\mathbb{R}$ is the spatial inverse of h, which is well defined since h is strictly increasing for each $t\geq 0$.

Lemma 6 *Let $h^{(-1)}:\mathbb{D}_+\to\mathbb{R}$ be the spatial inverse of h and a as in (48). Then, for each $x_0>0$, $\lim_{t\uparrow\infty}\frac{\partial}{\partial t}h^{(-1)}(x_0,t)$ exists and, moreover, for each $t\geq 0$,*

$$\frac{a}{2}\leq\frac{\partial}{\partial t}h^{(-1)}(x_0,t)\leq\frac{1}{2(1-\gamma)}. \tag{50}$$

Proof. Let $x_0>0$. To simplify the presentation, let $f(x,t):=h^{(-1)}(x,t)$. Then, equation (16) gives

$$f_t(x_0,t)=\frac{1}{2}\frac{h_{zz}\left(f\left(x_0,t\right),t\right)}{h_z\left(f\left(x_0,t\right),t\right)}=\frac{1}{2}\frac{\int_a^{\frac{1}{1-\gamma}}y^2 e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu\left(dy\right)}{\int_a^{\frac{1}{1-\gamma}}ye^{yf(x_0,t)-\frac{1}{2}y^2t}\mu\left(dy\right)},$$

and inequality (50) follows, since $a\leq y\leq\frac{1}{1-\gamma}$. As mentioned earlier, differentiation under the integrals appearing herein is valid given the properties of their integrands. Differentiating once more gives

$$f_{tt}(x_0,t)=-\frac{\int_a^{\frac{1}{1-\gamma}}\left(yf_t(x_0,t)-\frac{1}{2}y^2\right)^2 e^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)}{\int_a^{\frac{1}{1-\gamma}}ye^{yf(x_0,t)-\frac{1}{2}y^2t}\mu(dy)}<0. \tag{51}$$

Therefore, we conclude that $f_t(x_0,t)$ is bounded from below and decreasing in t, and hence its limit as $t\uparrow\infty$ exists. ■

We are now ready to present one of the main findings herein, a result interesting on its own right, that yields the temporal asymptotic behavior as $t\uparrow\infty$ of the ratio $\frac{h^{(-1)}(x_0,t)}{t}$, for each $x_0>0$. We show that it converges to half of the left end of the support of the risk preference measure and, also, provide the rate of convergence.

Proposition 7 *Let $h^{(-1)}:\mathbb{D}_+\to\mathbb{R}$ be the spatial inverse of h and a as in (48). Then, for each $x_0>0$, the following assertions hold:*

i) The ratio $\frac{h^{(-1)}(x_0,t)}{t}$ converges to $\frac{a}{2}$,

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}=\frac{a}{2}. \tag{52}$$

ii) Let

$$\Delta\left(x_0,t\right):=\frac{h^{(-1)}(x_0,t)}{t}-\frac{a}{2}. \tag{53}$$

If $a > 0$, then

$$|\Delta(x_0, t)| \le \frac{1}{at} \ln\left(\frac{\mu\left(\left[a, \frac{1}{1-\gamma}\right]\right)}{x_0}\right), \quad \Delta(x_0, t) < 0, \tag{54}$$

and

$$x_0 \ge \mu\left([a, a + \Delta(x_0, t)]\right) e^{\frac{1}{2} ta\Delta(x_0,t)}, \quad \Delta(x_0, t) > 0. \tag{55}$$

If $a = 0^+$, then $\Delta(x_0, t) > 0$ and, moreover, for each $\theta \in (0, 1)$,

$$x_0 \ge \mu\left([\Delta(x_0, t), (1 + \theta)\Delta(x_0, t)]\right) e^{t\Delta(x_0,t)\frac{1-\theta^2}{2}}. \tag{56}$$

Proof. *Part (i):*

Let $x_0 > 0$. Recall that, from Lemma 6, $\lim_{t\uparrow\infty} h^{(-1)}(x_0, t)$ exists. Moreover, rewriting (49) as

$$x_0 = \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2}y\right)} \mu(dy), \tag{57}$$

we see that $\lim_{t\uparrow\infty} h^{(-1)}(x_0, t) = \infty$, otherwise, sending $t \uparrow \infty$ we get a contradiction.

Next, let

$$A(x_0) := \lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0, t)}{t} = \lim_{t\uparrow\infty} \frac{\partial}{\partial t} h^{(-1)}(x_0, t), \tag{58}$$

where the last inequality follows from L'Hôpital's rule and the full range of h. Inequality (50) then gives

$$\frac{a}{2} \le A(x_0) \le \frac{1}{2(1-\gamma)}. \tag{59}$$

We claim that, for each x_0, $A(x_0) < \frac{1}{2(1-\gamma)}$. We look at the following three cases.

a. If $a = \frac{1}{1-\gamma}$, then $a = b$ and $h^{(-1)}(x_0, t) = \ln x_0^{1-\gamma} + \frac{1}{2}\frac{1}{(1-\gamma)}t$, and the result follows directly.

b. If $0 < a < \frac{1}{1-\gamma}$, we argue by contradiction assuming that there exists x_0 such that $A(x_0) = \frac{1}{2(1-\gamma)}$. Then, for $\varepsilon > 0$, there exists $t_0(x_0, \varepsilon))$ such that, for $t \ge t_0$,

$$-\varepsilon \le \frac{h^{(-1)}(x_0, t)}{t} - \frac{1}{2(1-\gamma)} \le \varepsilon.$$

In turn, for $\delta > 0$ small enough, the above inequality and (49) give

$$x_0 \ge \int_a^{\left(\frac{1}{1-\gamma} - 2\varepsilon - \delta\right)^-} e^{ty\left(\frac{1}{2(1-\gamma)} - \varepsilon - \frac{1}{2}y\right)} \mu(dy) + \int_{\frac{1}{1-\gamma} - 2\varepsilon - \delta}^{\frac{1}{1-\gamma}} e^{ty\left(\frac{1}{2(1-\gamma)} - \varepsilon - \frac{1}{2}y\right)} \mu(dy),$$

which yields a contradiction as $t \uparrow \infty$. Next, assume that there exists $x_0 > 0$ such that

$$\frac{a}{2} < A(x_0) < \frac{1}{2(1-\gamma)}.$$

Then, for $\varepsilon, \delta > 0$ small enough, we have

$$a < 2(A(x_0) - \varepsilon) - \delta < 2(A(x_0) - \varepsilon) < \frac{1}{1-\gamma}. \tag{60}$$

From (49), we deduce that, for $t \geq t_0(\varepsilon, x_0)$,

$$x_0 \geq \int_a^{\frac{1}{1-\gamma}} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)} \mu(dy). \tag{61}$$

If $\mu(\{a\}) \neq 0$, then $x_0 \geq e^{\frac{ta}{2}(2(A(x_0)-\varepsilon)-a)} \mu(\{a\})$, and sending $t \uparrow \infty$ yields a contradiction. If $\mu(\{a\}) = 0$, then

$$x_0 \geq \int_a^{\frac{1}{1-\gamma}} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)} \mu(dy) \geq \int_a^{2(A(x_0)-\varepsilon)-\delta} e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)} \mu(dy).$$

Next, we consider the quadratic $B(y) := y(A(x_0) - \varepsilon) - \frac{1}{2}y^2$.

We have that $B(y_1) = B(y_2) = 0$, for $y_1 = 0$ and $y_2 = 2(A(x_0) - \varepsilon)$, $B(y) > 0$, for $0 < y < 2(A(x_0) - \varepsilon)$, and $B(y)$ achieves a maximum at $y^* = A(x_0) - \varepsilon$.

We also look at its minimum $y_* = \min_{a \leq y \leq 2(A(x_0)-\varepsilon)-\delta} B(y)$ and claim that $y_* = 2(A(x_0) - \varepsilon) - \delta$. Indeed, if $0 < a \leq y^*$, choosing $\delta < a$, direct calculations yield that $B(a) > B(y_*)$. If $y^* < a$, then (60) yields that $a < y_* < y_2$ and, thus, the minimum also occurs at y_*. Clearly, because $y_1 < y_* < y_2$, we have that $B(y_*) = \frac{1}{2}\delta\,(2(A(x_0) - \varepsilon) - \delta) > 0$. Therefore, for $t \geq t_0(x_0, \varepsilon)$,

$$x_0 \geq \int_a^{2(A(x_0)-\varepsilon)-\delta} e^{tB(y_*)} \mu(dy). \tag{62}$$

As $t \uparrow \infty$, the right hand side of (62) converges to ∞, unless it holds that $\mu([a, 2(A(x_0) - \varepsilon) - \delta]) = 0$. Sending $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we obtain that $\mu([a, 2A(x_0)]) = 0$, which, however, contradicts (48). Therefore, it must be that, for each $x_0 > 0$, $A(x_0) \leq \frac{a}{2}$, and we easily conclude.

c. If $a = 0^+$, similar arguments yield that for every $\theta \in (0, A(x_0)]$ it must be that $\mu([\theta, 2A(x_0)]) = 0$. Sending $\theta \downarrow 0$ yields $\mu((0, A(x_0)]) = 0$, which contradicts (48).

Part ii).

We first assume that $a > 0$ and look at the following cases for $\Delta(x_0, t)$, defined in (53).

If $\Delta(x_0, t) < 0$, (49) yields

$$x_0 = \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\Delta(x_0,t)+\frac{1}{2}(a-y)\right)} \mu(dy)$$

$$\leq e^{ta\Delta(x_0,t)} \int_a^{\frac{1}{1-\gamma}} e^{\frac{1}{2}ty(a-y)} \mu(dy) \leq e^{ta\Delta(x_0,t)} \mu\left(\left[a, \frac{1}{1-\gamma}\right]\right),$$

and (54) follows.

If $\Delta(x_0,t) > 0$, then (52) yields that, for ε small enough and $t \geq t_0(x_0,\varepsilon)$ large enough, the inequality $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \varepsilon$ holds. Choosing ε such that $\varepsilon < \frac{1}{2(1-\gamma)} - \frac{a}{2}$ yields $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \frac{1}{2(1-\gamma)} - \frac{a}{2}$, and using that $a < \frac{1}{1-\gamma}$ gives

$$\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t} \leq \frac{1}{1-\gamma}.$$

In turn, from (49) we deduce that

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{y}{2}\right)} \mu(dy).$$

The quadratic $H(y) := y\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{y}{2}\right)$ in the above integrand becomes zero at $y_1 = 0$ and $y_3 = 2\frac{h^{(-1)}(x_0,t)}{t} > a$ and, therefore, its minimum occurs at one of the end points, a or $\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t}$. Note that $a < \frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t} < y_3$. If the minimum occurs at a, then $H(a) = a\Delta(x_0,t)$, while if it occurs at $\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t}$, then $H\left(\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t}\right) = \frac{1}{2}\left(\frac{a}{2} + \frac{h^{(-1)}(x_0,t)}{t}\right)\Delta(x_0,t) > \frac{1}{2}a\Delta(x_0,t)$.

Combining the above gives

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}} e^{\frac{1}{2}ta\Delta(x_0,t)} \mu(dy) = \mu([a, a+\Delta(x_0,t)])\, e^{\frac{1}{2}ta\Delta(x_0,t)}.$$

Finally, let $a = 0^+$. Then, $\Delta(x_0,t) = \frac{h^{(-1)}(x_0,t)}{t}$.

Recall that $\lim_{t\uparrow\infty} h^{(-1)}(x_0,t) = \infty$ and, thus, $\frac{h^{(-1)}(x_0,t)}{t} > 0$, for t large. For $\varepsilon \in \left(\frac{h^{(-1)}(x_0,t)}{t}, 2\frac{h^{(-1)}(x_0,t)}{t}\right)$, we then have

$$x_0 \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{y}{2}\right)} \mu(dy) \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{t\varepsilon\left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{\varepsilon}{2}\right)} \mu(dy).$$

Setting $\varepsilon = (1+\theta)\frac{h^{(-1)}(x_0,t)}{t}$, inequality (56) follows. ■

We are now ready to prove one of the main results herein. It yields the temporal limit of the relative dynamic risk tolerance and, also, provides the related rate of convergence.

Theorem 8 *Let a be the left end of the support of the risk preference measure μ and $\Delta(x_0,t)$ as in (53). Then, for each $x_0 > 0$,*

$$\lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = a. \tag{63}$$

Furthermore, there exists a function $G(x_0,t)$*, given by*

$$G(x_0,t) := \begin{cases} \int_a^{\frac{1}{1-\gamma}} (y-a)e^{-ty(\frac{y-a}{2})}\mu(dy), & \textit{if } \Delta(x_0,t) < 0 \\ 2\Delta(x_0,t)\, x_0 + \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy), & \textit{if } \Delta(x_0,t) > 0, \end{cases}$$

satisfying $\lim_{t\uparrow\infty} G(x_0,t) = 0$ *and, for* t *large enough,*

$$0 \le r(x_0,t) - ax_0 \le G(x_0,t). \tag{64}$$

Proof. Differentiating (15) gives

$$u_{xt}(x_0,t) = \left(\frac{1}{2} - \frac{\partial}{\partial t} h^{(-1)}(x_0,t)\right) u_x(x_0,t).$$

Moreover, (13) and (22) imply that $u_t(x_0,t) = -\frac{1}{2}u_x(x_0,t)r(x_0,t)$ and, in turn,

$$u_{tx}(x_0,t) = -\frac{1}{2}u_{xx}(x_0,t)\, r(x_0,t) - \frac{1}{2}u_x(x_0,t)\, r_x(x_0,t).$$

Combining the above we deduce

$$\frac{1}{2} r_x(x_0,t) = \frac{\partial}{\partial t} h^{(-1)}(x_0,t), \tag{65}$$

and from Proposition 7 and (58) we obtain that

$$\lim_{t\uparrow\infty} r_x(x_0,t) = \lim_{t\uparrow\infty} 2\frac{\partial}{\partial t} h^{(-1)}(x_0,t) = a.$$

On the other hand,

$$\lim_{c\downarrow 0^+} \int_c^{x_0} r_x(x,t)dx = r(x_0,t) - \lim_{c\downarrow 0^+} r(c,t).$$

Using the fact that, for all $t \ge 0$, $\lim_{x\downarrow 0} r(x,t) = 0$, we get that, for $x_0 > 0$, $r(x_0,t) = \int_a^{x_0} r_x(x,t)dx$. Finally, we deduce from (65) and (51) that $r_{xt}(x_0,t) < 0$, and thus, for $x_0 > 0$, we have, for $y \in (0, x_0]$, that $r_x(y,t) \le r_x(x_0,0)$. However, for each $x_0 > 0$, $r_x(x_0,0) < \infty$. This follows directly from (23), (17) and the full range of $h(x,0)$, since

$$r_x(h(z,0),0) = \frac{h_{zz}(z,0)}{h_z(z,0)} = \frac{\int_a^{\frac{1}{1-\gamma}} y^2 e^{yz-\frac{1}{2}t^2 y}\mu(dy)}{\int_a^{\frac{1}{1-\gamma}} y e^{yz-\frac{1}{2}t^2 y}\mu(dy)} \le \frac{1}{1-\gamma}.$$

Using dominated convergence and passing to the limit as $t \uparrow \infty$ in (65), we deduce (63).

Next, we give an alternative convergence proof which also yields the rate of convergence. First note that

$$r(x_0,t) - ax_0 \ge 0. \tag{66}$$

This follows directly from (23) and (17) since

$$r\left(x_0,t\right)=\int_a^{\frac{1}{1-\gamma}} y e^{t\left(y\frac{h^{(-1)}(x_0,t)}{t}-\frac{1}{2}y^2\right)}\mu(dy)\geq a\int_a^{\frac{1}{1-\gamma}} e^{t\left(y\frac{h^{(-1)}(x_0,t)}{t}-\frac{1}{2}y^2\right)}\mu(dy).$$

Furthermore, from (23), (17) and (53), we have

$$r(x_0,t)-ax_0=\int_a^{\frac{1}{1-\gamma}}(y-a)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy)\geq 0. \tag{67}$$

If $\Delta\left(x_0,t\right)<0$ (which occurs only if $a>0$, as shown in the previous proof), the above equality yields

$$r(x_0,t)-ax_0\leq\int_a^{\frac{1}{1-\gamma}}(y-a)e^{-ty\left(\frac{y-a}{2}\right)}\mu(dy),$$

and (64) follows directly with $G\left(t\right):=\int_a^{\frac{1}{1-\gamma}}(y-a)e^{-ty\left(\frac{y-a}{2}\right)}\mu(dy)$.

Let $\Delta\left(x_0,t\right)>0$ and $a>0$ or $a=0^+$. If $a=\frac{1}{1-\gamma}$, then the result follows trivially.

For $0\leq a<\frac{1}{1-\gamma}$, observe that for t large enough, $0<a+2\Delta\left(x_0,t\right)<\frac{1}{1-\gamma}$ and, thus, representation (67) gives

$$\begin{aligned} r\left(x_0,t\right)-ax_0 &= \int_a^{(a+2\Delta(x_0,t))^-}(y-a)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy) \\ &\quad+\int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}}(y-a)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy). \end{aligned}$$

Introduce $C_1\left(x_0,t\right):=\int_a^{(a+2\Delta(x_0,t))^-}(y-a)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy)$ and observe that

$$C_1\left(x_0,t\right)\leq 2\Delta\left(x_0,t\right)\int_a^{(a+2\Delta(x_0,t))^-}e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy)\leq 2\Delta\left(x_0,t\right)x_0,$$

where we used (49). Thus,

$$\lim_{t\uparrow\infty}C_1\left(x_0,t\right)=0. \tag{68}$$

Let also $C_2\left(x_0,t\right):=\int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}}(y-a)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)}\mu(dy)$ and

$$F\left(y,t,x_0\right):=\left(y-a\right)e^{ty\left(\frac{2\Delta(x_0,t)+a-y}{2}\right)},\quad y\in\left[a+2\Delta\left(x_0,t\right),\frac{1}{1-\gamma}\right].$$

Then, $F\left(a+2\Delta\left(x_0,t\right),t,x_0\right)=2\Delta\left(x_0,t\right)$ and, thus,

$$\lim_{t\uparrow\infty}F\left(a+2\Delta\left(x_0,t\right),t,x_0\right)=0.$$

Furthermore, for each $y \in \left(a + 2\Delta(x_0, t), \frac{1}{1-\gamma}\right]$, we also have $\lim_{t\uparrow\infty} F(y, t, x_0) = 0$. In turn, dominated convergence gives

$$\lim_{t\uparrow\infty} C_2(x_0, t) = 0. \tag{69}$$

Setting $G(x_0, t) := C_1(x_0, t) + C_2(x_0, t)$, and using (68) and (69), we obtain (64). ■

5 Examples

We present two representative examples in which the risk preference measure is, respectively, a sum of Dirac functions and the Lebesgue measure. The first one generalizes the results in subsection 2.1 while the second demonstrates that the spatial turnpike property fails if there is no mass at the right end of the support of the risk preference measure.

5.1 Finite sum of Dirac functions

We assume that, for some $\gamma \in (0, 1)$, the risk preference measure is given by

$$\mu = \sum_{n=1}^{N} \delta_{y_n}, \quad 0 < y_1 < \cdots < y_N = \frac{1}{1-\gamma}.$$

Then, $h(z, 0) = \sum_{n=1}^{N} e^{y_n z}$ and, thus, $\lim_{z\uparrow\infty} h(z, 0)e^{-y_N z} = 1$. In turn, (15) yields

$$\lim_{x\uparrow\infty} \frac{u_x(x, 0)}{x^{\gamma-1}} = 1,$$

which confirms the results of Lemma 2. Furthermore, we easily obtain (cf. (17)) that, for $(z, t) \in \mathbb{D}$,

$$h(z, t) = \sum_{n=1}^{N} \exp\left(y_n z - \frac{1}{2} y_n^2 t\right). \tag{70}$$

Therefore, for $x > 0$,

$$x = \sum_{n=1}^{N} \exp\left(y_n t \left(\frac{h^{(-1)}(x, t)}{t} - \frac{1}{2} y_n\right)\right). \tag{71}$$

We first provide the temporal and spatial asymptotic behavior of $h^{(-1)}(x, t)$ for large t and large x, respectively.

5.1.1 Temporal asymptotics of $h^{(-1)}$

We claim that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0, t) = \frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0 + o(1). \tag{72}$$

Indeed, the limit in (52) gives

$$\lim_{t \uparrow \infty} \left(\frac{h^{(-1)}(x_0, t)}{t} - \frac{1}{2} y_n \right) \begin{cases} < 0, & 1 < n \le N \\ = 0, & n = 1 \end{cases}.$$

Therefore, as $t \uparrow \infty$, all terms in (71) vanish except for the first one. In turn,

$$x_0 = \lim_{t \uparrow \infty} \exp \left(y_1 h^{(-1)}(x_0, t) - \frac{1}{2} y_1^2 t \right), \tag{73}$$

and taking logarithm and rearranging terms yields (72). Note, also, that for each $t > 0$,

$$h^{(-1)}(x_0, t) - \frac{1}{2} y_1 t \le \frac{1}{y_1} \log x_0. \tag{74}$$

5.1.2 Spatial asymptotics of $h^{(-1)}$

We claim that, for each $t_0 \ge 0$, as $x \uparrow \infty$,

$$h^{(-1)}(x, t_0) = (1 - \gamma) \ln x + \frac{1}{2(1 - \gamma)} t + o(1). \tag{75}$$

We first establish that, for each $t_0 \ge 0$,

$$\lim_{x \uparrow \infty} \frac{h^{(-1)}(x, t_0)}{\ln x} = 1 - \gamma, \tag{76}$$

independently of t_0. Indeed, fix $t_0 \ge 0$, let $\delta \in (0, \frac{1}{1-\gamma})$ and assume that

$$\liminf_{x \uparrow \infty} \frac{h^{(-1)}(x, t)}{\ln x} < \frac{1}{\frac{1}{1-\gamma} + \delta}.$$

Then, using (71) and that $h^{(-1)}(x, t_0) > 0$ for large x, we obtain

$$\begin{aligned} 1 &= \liminf_{x \uparrow \infty} \frac{1}{x} \sum_{n=1}^{N} \exp \left(y_n \ln x \frac{h^{(-1)}(x, t_0)}{\ln x} - \frac{1}{2} y_n^2 t_0 \right) \\ &\le N \liminf_{x \uparrow \infty} x^{\frac{1}{1-\gamma} \left(\frac{h^{(-1)}(x, t_0)}{\ln x} - 1 \right)} < N \liminf_{x \uparrow \infty} x^{-\frac{\delta}{\frac{1}{1-\gamma} + \delta}} = 0, \end{aligned}$$

which yields a contradiction. Since δ is arbitrary, we deduce that

$$\liminf_{x \uparrow \infty} \frac{h^{(-1)}(x, t_0)}{\ln x} \ge (1 - \gamma). \tag{77}$$

Similarly, assume that, for $\delta \in \left(0, \frac{1}{1-\gamma}\right)$,

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} > \frac{1}{\frac{1}{1-\gamma} - \delta}.$$

Then, using (71) once more, we get a contradiction since

$$1 \geq \limsup_{x\uparrow\infty} \frac{1}{x} \exp\left(\frac{1}{1-\gamma} \ln x \frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0\right)$$

$$= \limsup_{x\uparrow\infty} x^{\frac{1}{1-\gamma}\frac{h^{(-1)}(x,t_0)}{\ln x} - 1} e^{-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0} = \infty,$$

where we used that $\frac{1}{1-\gamma}\frac{1}{\frac{1}{1-\gamma}-\delta} = \frac{\delta(1-\gamma)}{1-\delta(1-\gamma)} > 0$. Since δ is arbitrary, we deduce that

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} \leq 1-\gamma, \tag{78}$$

and we easily conclude.

Next, we rewrite (71) as

$$1 = \sum_{n=1}^{N} \exp\left(y_n h^{(-1)}(x,t_0) - \frac{1}{2} y_n^2 t_0 - \ln x\right)$$

$$= \sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n}\right) - \frac{1}{2} y_n^2 t_0\right). \tag{79}$$

Note that from (76), we have

$$\lim_{x\uparrow\infty}\left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n}\right) \begin{cases} < 0, & 1 \leq n < N \\ = 0, & n = N. \end{cases}.$$

Therefore, as $x \uparrow \infty$, the first $N-1$ terms in (79) vanish, and we deduce that

$$1 = \lim_{x\uparrow\infty} \exp\left(\frac{1}{1-\gamma} h^{(-1)}(x,t_0) - \ln x - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0\right),$$

and (75) follows.

5.1.3 Spatial and temporal asymptotics of risk tolerance

From (23) and (70), we deduce that the dynamic risk tolerance function is given by

$$r(x,t) = \sum_{n=1}^{N} y_n \exp\left(y_n h^{(-1)}(x,t) - \frac{1}{2} y_n^2 t\right). \tag{80}$$

Let $t_0 \geq 0$. From (80), we get

$$\lim_{x\uparrow\infty} r(x,t_0) = \lim_{x\uparrow\infty} \sum_{n=1}^{N} y_n \exp\left(y_n((1-\gamma)\ln x + \frac{1}{2(1-\gamma)} t_0) - \frac{1}{2} y_n^2 t_0 \right),$$

and we easily deduce that, as $x \uparrow \infty$,

$$r(x,t_0) = \sum_{n=1}^{N} y_n \exp\left(\frac{1}{2} y_n t_0 (\frac{1}{1-\gamma} - y_n) \right) x^{(1-\gamma)y_n} + o(1).$$

Next, let $x_0 > 0$. From (80),

$$r(x_0,t) \leq \sum_{n=1}^{N} y_n \exp\left(y_n(\frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0) - \frac{1}{2} y_n^2 t \right)$$

$$= y_1 x_0 + \sum_{n=2}^{N} y_n \exp\left(\frac{1}{2} y_n (y_1 - y_n) t \right) x_0^{\frac{y_n}{y_1}},$$

and, therefore, as $t \uparrow \infty$,

$$r(x_0,t) = y_1 x_0 + O\left(e^{\frac{1}{2} y_2 (y_1 - y_2) t} \right).$$

In summary, for each $x_0 > 0$ and each $t_0 \geq 0$, respectively,

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \frac{1}{1-\gamma} = y_N \quad \text{and} \quad \lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = y_1, \tag{81}$$

and these spatial and temporal limits are consistent with the findings in Proposition 3 and Theorem 8.

5.2 Lebesgue measure

We assume that the risk preference measure μ is Lebesgue on $\left[a, \frac{1}{1-\gamma}\right]$, with $a = 0^+$ or $a > 0$ and, thus, it has continuous support without a mass at its right end for the spatial turnpike limit to hold. The analysis that follows is tedious so, to ease the presentation, some intermediate steps are omitted.

Case 1: $a > 0$.

From (17), $h(z,0) = \int_a^{\frac{1}{1-\gamma}} e^{yz} dy$ and, thus, (6) yields

$$\lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = 1.$$

We introduce the functions $\varphi(z) := e^{-\frac{z^2}{2}}$ and $\Phi(z) := \int_{-\infty}^{z} \varphi(y) dy$, $z \in \mathbb{R}$. Then,

$$h(z,t) = \int_a^{\frac{1}{1-\gamma}} e^{yz - \frac{1}{2} y^2 t} dy = \frac{e^{z^2/2t}}{\sqrt{t}} \int_{a\sqrt{t} - z/\sqrt{t}}^{b\sqrt{t} - z/\sqrt{t}} \varphi(y) dy. \tag{82}$$

We also have, for $x \geq 0$,

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yt\left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2}y\right)} dy = \frac{1}{\sqrt{t}} e^{\frac{h^{(-1)}(x,t)^2}{2t}} \int_{a\sqrt{t} - \frac{h^{(-1)}(x,t)}{\sqrt{t}}}^{\frac{1}{1-\gamma}\sqrt{t} - \frac{h^{(-1)}(x,t)}{\sqrt{t}}} \varphi(y) dy. \tag{83}$$

5.2.1 Temporal asymptotics of $h^{(-1)}$

We show that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0, t) = \frac{1}{2}at + \frac{1}{a}\left(\ln t + \ln x_0 + \ln \frac{a}{2}\right) + o(1). \tag{84}$$

For this, we first establish that

$$x_0 = \lim_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{\frac{1}{2}at}. \tag{85}$$

To this end, using (83) and that, for $z < 0$,

$$\Phi(z) \leq -\frac{\varphi(z)}{z}, \tag{86}$$

we obtain, for t large enough,

$$x_0 \leq \frac{1}{\sqrt{t}} \exp\left(\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}\right) \Phi\left(-a\sqrt{t} + \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)$$

$$\leq \frac{1}{\sqrt{t}} \frac{1}{a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} \exp\left(\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}\right) \varphi\left(-a\sqrt{t} + \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)$$

$$= \frac{1}{at - h^{(-1)}(x_0,t)} e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}.$$

Proposition 7 yields

$$x_0 \leq \liminf_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{\frac{1}{2}at}. \tag{87}$$

Next we show that

$$x_0 \geq \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{\frac{1}{2}at},$$

which together with (87) will establish (85). To this end, we use that, for any $\beta > \alpha > 0$, the inequality

$$\Phi(\beta) - \Phi(\alpha) \geq \frac{1}{\beta}\varphi(\alpha) - \varphi(\beta) \tag{88}$$

holds. Let $1 < k < \frac{1}{a(1-\gamma)}$. From (83) and the above inequality, we have that, for t large enough,

$$
\begin{aligned}
x_0 = \frac{1}{\sqrt{t}} e^{\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}} & \left(\Phi(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) - \Phi(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) \right) \\
& \geq \frac{1}{\sqrt{t}} \frac{1}{ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} e^{\frac{\left(h^{(-1)}(x_0,t)\right)^2}{2t}} \\
& \times \left(\varphi(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) - \varphi(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}) \right) \\
& = \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)}.
\end{aligned}
$$

From Proposition 7 and since $k > 1$, we obtain that

$$
\lim_{t \uparrow \infty} \frac{e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} = \lim_{t \uparrow \infty} \frac{e^{ka^2 t(\frac{h^{(-1)}(x_0,t)}{at} - \frac{k}{2})}}{at\left(k - \frac{h^{(-1)}(x_0,t)}{at}\right)} = 0.
$$

Therefore,

$$
\begin{aligned}
x_0 \geq \limsup_{t \uparrow \infty} & \frac{1}{kat - h^{(-1)}(x_0,t)} \left(e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)} \right) \\
& \geq \limsup_{t \uparrow \infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{kat - h^{(-1)}(x_0,t)} - \lim_{t \uparrow \infty} \frac{e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} \\
& = \limsup_{t \uparrow \infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{kat - h^{(-1)}(x_0,t)},
\end{aligned}
$$

and sending $k \downarrow 1$ we conclude.

Next, we utilize the Lambert W function, $W(x)$, defined as the inverse of $F(x) = xe^x$. Setting

$$
\delta(x_0,t) := h^{(-1)}(x_0,t) - \frac{1}{2}at,
$$

we deduce from (85) that there exists $\varepsilon(t)$ with $\lim_{t \uparrow \infty} \varepsilon(t) = 0$, such that

$$
\frac{e^{a\delta(x_0,t)}}{\frac{1}{2}at - \delta(x_0,t)} = (1 + \varepsilon(t))x_0.
$$

Rewriting yields

$$
a(\frac{1}{2}at - \delta(x_0,t))e^{a(\frac{1}{2}at - \delta(x_0,t))} = \frac{a}{(1 + \varepsilon(t))x_0} e^{\frac{1}{2}a^2 t},
$$

and, therefore,

$$\delta(x_0,t)=\frac{1}{2}at-\frac{1}{a}W\left(\frac{a}{(1+\varepsilon\,(t))x_0}e^{\frac{1}{2}a^2t}\right).$$

It was established in [4] that the asymptotic expansion of $W(x)$, for large x, is given by

$$W(x)=\ln x-\ln\ln x+o(1).$$

Thus,

$$\begin{aligned}\delta(x_0,t)&=\frac{1}{2}at-\frac{1}{a}\ln\left(\frac{a}{(1+\varepsilon\,(t))x_0}e^{\frac{1}{2}a^2t}\right)\\&\quad+\frac{1}{a}\ln\ln\left(\frac{a}{(1+\varepsilon\,(t))x_0}e^{\frac{1}{2}a^2t}\right)+o(1)\\&=\frac{1}{a}\left(\ln\frac{x_0}{a}+\ln(1+\varepsilon\,(t))+\ln\left(\frac{1}{2}a^2t+\ln\frac{a}{(1+\varepsilon\,(t))x_0}\right)\right)+o(1).\end{aligned}$$

Using that, as $t\uparrow\infty$, $\ln(1+\varepsilon(t))=o(1)$ and that

$$\ln\left(\frac{1}{2}a^2t+\ln\frac{a}{(1+\varepsilon\,(t))x_0}\right)=\ln(\frac{1}{2}a^2t)+o(1),$$

assertion (84) follows.

5.2.2 Spatial asymptotics of $h^{(-1)}$

We show that, for each $t_0\geq 0$, as $x\uparrow\infty$,

$$h^{(-1)}(x,t_0)=\frac{1}{2(1-\gamma)}t_0+(1-\gamma)\left(\ln x+\ln\ln x-\ln\frac{1}{1-\gamma}\right)+o(1). \tag{89}$$

We first establish that

$$\lim_{x\uparrow\infty}\frac{h^{(-1)}(x,t_0)}{\ln x}=(1-\gamma)\,. \tag{90}$$

Indeed, let $f(z,t):=\frac{1}{z}e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2t}$. Then,

$$\begin{aligned}\lim_{z\uparrow\infty}\frac{h(z,t_0)}{f(z,t_0)}&=\lim_{z\uparrow\infty}\int_a^{\frac{1}{1-\gamma}}ze^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\\&=\lim_{z\uparrow\infty}\left(\int_a^{\frac{1}{1-\gamma}}(z-yt_0)e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right.\\&\quad\left.+\int_a^{\frac{1}{1-\gamma}}yt_0e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right)\\&=\lim_{z\uparrow\infty}\left(1-e^{(a-\frac{1}{1-\gamma})x-\frac{1}{2}(a^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}\right.\end{aligned}$$

$$+\int_a^{\frac{1}{1-\gamma}} yte^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-(\frac{1}{1-\gamma})^2)t_0}dy\Bigg) = 1,$$

where we used that $a < \frac{1}{1-\gamma}$ and monotone convergence. Therefore, for each $t_0 \geq 0$,

$$\lim_{z\uparrow\infty} \frac{h(z,t_0)}{f(z,t_0)} = 1. \tag{91}$$

We now use an auxiliary result on inverses of asymptotic functions from [9] (Theorem 2(i)) to prove (90) by verifying the necessary assumptions. To this end, let $g(z) := (1-\gamma)\ln z$, $z \geq 0$, and notice that, for large z,

$$g(f(z,t_0)) = -(1-\gamma)\ln z + z - \frac{1}{2(1-\gamma)}t_0 \sim z.$$

Thus, $\lim_{z\uparrow\infty} z^{-1}f(z,t_0) = 1$. Since, on the other hand, $\lim_{z\uparrow\infty} f(z,t_0) = \infty$, we deduce that $f^{(-1)}(x,t) \sim g(x)$, as $x \uparrow \infty$. Moreover, $g(x)$ is strictly increasing and the ratio $\frac{g_x(x,t)}{g(x,t)} \sim \frac{1}{x\ln x} = O(\frac{1}{x})$ for sufficiently large x. It, then, follows from the aforementioned result that $\lim_{x\uparrow\infty} \frac{g(x)}{h^{(-1)}(x,t_0)} = 1$ and (90) follows.

Next, we establish that, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x\ln x} = 1-\gamma. \tag{92}$$

Indeed, if $t_0 = 0$, we have from (83) that

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yh^{(-1)}(x,0)}dy = \frac{1}{h^{(-1)}(x,0)}\left(e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)} - e^{ah^{(-1)}(x,0)}\right), \tag{93}$$

and (90) yields

$$\lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}}{x\ln x} = \lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}(1-e^{(a-\frac{1}{1-\gamma})h^{(-1)}(x,0)})}{xh^{(-1)}(x,0)} \frac{h^{(-1)}(x,0)}{\ln x} = 1-\gamma.$$

For $t_0 > 0$, we deduce from (83) that

$$x = \frac{1}{\sqrt{t_0}}e^{\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}}\left(\Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right) - \Phi\left(a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)\right). \tag{94}$$

Then, for large x,

$$\begin{aligned} 1 &\leq \frac{1}{x\sqrt{t_0}}\exp\left(\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}\right)\Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right) \\ &\leq \frac{1}{x}\frac{1}{\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} - \frac{1}{1-\gamma}\sqrt{t_0}}\exp\left(\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}\right) \end{aligned}$$

$$\times \frac{1}{\sqrt{t_0}}\varphi\left(\frac{1}{1-\gamma}\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)=\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0)-\frac{1}{1-\gamma}t_0)}.$$

In turn,

$$1\le \liminf_{x\uparrow\infty}\left(\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}\frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0)-\frac{1}{1-\gamma}t_0}\right)$$

$$=\liminf_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}\lim_{x\uparrow\infty}\left(\frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0)-\frac{1}{1-\gamma}t_0}\right),$$

and, thus,

$$1\le \liminf_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}. \tag{95}$$

Next, we use the inequality

$$\Phi(b)-\Phi(a)\ge\frac{\varphi(a)-\varphi(b)}{b},\quad \text{for}\quad a<b<0, \tag{96}$$

and deduce from (94) that, for large x,

$$1\ge\frac{1}{x}\exp\left(\frac{\left(h^{(-1)}(x,t_0)\right)^2}{2t_0}\right)\frac{1}{\sqrt{t_0}}\frac{1}{a\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}}$$

$$\times\left(\varphi\left(a\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)-\varphi\left(\frac{1}{1-\gamma}\sqrt{t_0}-\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}\right)\right)$$

$$=\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0)-at_0)}-\frac{e^{a(h^{(-1)}(x,t_0)-\frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0)-at_0)}.$$

Proceeding with analogous convergence arguments we used to establish (95), we obtain that

$$1\ge \limsup_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}. \tag{97}$$

Combining (95) and (97), we obtain (92). Taking the logarithm of both sides of (92) gives

$$\lim_{x\uparrow\infty}\left(\frac{1}{1-\gamma}\left(h^{(-1)}(x,t_0)-\frac{1}{2\,(1-\gamma)}t_0\right)-\ln x-\ln\ln x\right)=\ln\frac{1}{1-\gamma},$$

and (89) follows.

5.2.3 Temporal and spatial asymptotics of risk tolerance

We first observe that (23) and (82) yield

$$r(x,t)=\int_a^{\frac{1}{1-\gamma}}\frac{1}{t}(yt-h^{(-1)}(x,t))e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}dy \tag{98}$$

$$+\frac{h^{(-1)}(x,t)}{t}\int_a^{\frac{1}{1-\gamma}}e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}dy$$

$$=\frac{e^{a(h^{(-1)}(x,t)-\frac{1}{2}at)}}{t}-\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t)-\frac{1}{2(1-\gamma)}t)}}{t}+\frac{h^{(-1)}(x,t)}{t}x.$$

We show that, for each $x_0>0$, as $t\uparrow\infty$,

$$r(x_0,t)=ax_0-\left(\frac{1}{2}ax_0\right)^{\frac{1}{a(1-\gamma)}}t^{\frac{1}{a(1-\gamma)}-1}e^{\frac{1}{2(1-\gamma)}(a-\frac{1}{1-\gamma})t} \tag{99}$$

$$+\frac{x_0}{at}(\ln t+\ln x_0+\ln\frac{a}{2})+o(1).$$

From (84), we obtain that

$$\lim_{t\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x_0,t)-\frac{1}{2(1-\gamma)}t)}}{t}$$

$$=\lim_{t\uparrow\infty}\exp\left(\frac{1}{1-\gamma}(h^{(-1)}(x_0,t)-\frac{1}{2\,(1-\gamma)}t-(1-\gamma)\ln t)\right)$$

$$=\lim_{t\uparrow\infty}\exp\left(\frac{1}{1-\gamma}(h^{(-1)}(x_0,t)-\frac{1}{2}at-\frac{1}{a}\ln t)-\frac{1}{2}\frac{1}{1-\gamma}(\frac{1}{1-\gamma}-a)t\right)e^{(\frac{1}{a(1-\gamma)}-1)\ln t}$$

$$=\lim_{t\uparrow\infty}\exp\left(\frac{1}{a\,(1-\gamma)}(\ln x_0+\ln\frac{a}{2})+\frac{1}{2\,(1-\gamma)}(a-\frac{1}{1-\gamma})t\right)t^{\frac{1}{a(1-\gamma)}-1}$$

$$=\lim_{t\uparrow\infty}\left(\frac{1}{2}ax_0\right)^{\frac{1}{a(1-\gamma)}}t^{\frac{1}{a(1-\gamma)}-1}e^{\frac{1}{2(1-\gamma)}(a-\frac{1}{1-\gamma})t}.$$

Furthermore,

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}x_0=\frac{1}{2}ax_0+\lim_{t\uparrow\infty}\left(\frac{x_0}{at}(\ln t+\ln x+\ln\frac{a}{2})\right),$$

and we easily conclude.

Next, we establish the spatial asymptotics of $r\,(x,t)$ for large x.

If $t_0>0$, we easily deduce from (98) that, as $x\uparrow\infty$,

$$r(x,t_0)=\frac{1-\gamma}{t_0}x\ln\ln x+\frac{1}{t_0}\left((1-\gamma)\,x\ln x\right)^{a(1-\gamma)}e^{\frac{1}{2}a(\frac{1}{1-\gamma}-a)t_0} \tag{100}$$

$$+\frac{1}{2\left(1-\gamma\right)}x-\frac{1-\gamma}{t_0}x\ln\frac{1}{1-\gamma}+o(1).$$

If $t_0=0$,

$$r(x,0)=\frac{1}{1-\gamma}x-\left(\frac{1}{1-\gamma}-a\right)\frac{e^{ah^{(-1)}(x,0)}}{h^{(-1)}(x,0)},$$

and, thus, for large x,

$$r(x,0)=\frac{1}{1-\gamma}x(1-\frac{1}{\ln x})+o(1). \tag{101}$$

In summary, we have, for each $x_0>0$,

$$\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0}=a,$$

which is in accordance with the results of Theorem 8. From (100) and (101), we obtain that for $t_0>0$ and $t_0=0$, respectively,

$$r(x,t_0)\sim\frac{1-\gamma}{t_0}x\ln\ln x\quad\text{and}\quad r(x,0)\sim\frac{1}{1-\gamma}x.$$

Therefore, the spatial turnpike property (39) *fails,* for the risk preference measure lacks a Dirac mass on the right end point for Proposition 3 to hold.

Case 2: $a=0$

If the risk preference measure is Lebesgue on $(0,\frac{1}{1-\gamma}]$, we obtain, for $t_0\geq 0$, the same spatial asymptotics for $h^{(-1)}$ and r, and the lack of the spatial turnpike limit, as in the case $a>0$.

For the temporal asymptotics of $h^{(-1)}$, we claim that, for each $x_0>0$,

$$\frac{h^{(-1)}(x_0,t)}{t}=\frac{\sqrt{\ln t+2\ln x-\ln 2\pi}}{\sqrt{t}}+o(\frac{1}{\sqrt{t}}). \tag{102}$$

To see this, notice that (83) becomes $x_0=\int_0^{\frac{1}{1-\gamma}}e^{y(h^{(-1)}(x_0,t)-\frac{1}{2}yt)}dy$. Taking logarithm of both sides yields

$$2\ln x_0=\left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2-\ln t \tag{103}$$

$$+2\ln\left(\Phi\left(\sqrt{t}(\frac{1}{1-\gamma}-\frac{h^{(-1)}(x_0,t)}{t})\right)-\Phi\left(-\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right).$$

Next, we claim that $l:=\liminf_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}=\infty$. Indeed, if $l<\infty$, then as $t\uparrow\infty$, (103) would give

$$2\ln x_0=l^2-\lim_{t\uparrow}\ln t+2\ln(1-\Phi(-l))=-\infty,$$

which is a contradiction. Therefore, it must be that $l = \infty$, which combined with the fact that $\lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t} = 0$, implies that the third term on the right hand side of (103) converges to $2\ln\sqrt{2\pi}$. Thus,

$$2\ln x_0 = \lim_{t\uparrow\infty}\left(\left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2 - \ln t + 2\ln\sqrt{2\pi}\right)$$

from which we deduce that $h^{(-1)}(x_0,t) = \sqrt{t(\ln t + 2\ln x - \ln 2\pi)} + o(\sqrt{t})$, and we easily conclude. The rest of the analysis follows easily.

6 Conclusions and extensions

We studied turnpike-type limiting properties of the dynamic relative risk tolerance function in an Ito-diffusion market and under time monotone forward performance criteria. We showed that, contrary to turnpike results in the classical expected utility framework, the asymptotic temporal and spatial limits in the forward setting do not in general coincide. Rather, they depend critically on the left and right points of the support of the underlying risk preference measure. The spatial limit coincides with the right end point of the support while the temporal one with the left end point. Central role in the analysis is played by the asymptotic properties of the spatial inverse of the underlying space-time harmonic function.

There are various extensions of both the results and the setting we investigated herein. Firstly, one may study the asymptotic properties of the optimal wealth and optimal portfolio policy processes provided in (21). Naturally, their asymptotic study will involve the long term behavior of both the market and the investor-specific inputs.

In a related direction, an interesting problem is how to construct investment policies that yield a *targeted* long-term wealth distribution. In a static model, elicitation of risk preferences from desired distributions was studied in [28] and in a dynamic setting in [18], where markets were assumed to be log-normal and the analysis was done in both the classical and the forward setting. However, in the latter work, there is a strong model commitment which is not a realistic assumption for long-term portfolio management. In the Ito-diffusion market we consider herein, the model is adaptively updated. Preliminary results on the above questions can be found in [10].

Another line of research would be to study spatial and temporal turnpike properties under forward performance criteria with non-zero volatility. In this case, the closed form solutions we used herein do not hold and analogous expressions are not known to date. In Markovian models, one may reduce the forward SPDE derived in [24] to a finite dimensional ill-posed HJB equation (see, among others, [24][26] and [27]) which can be, in turn, used to analyze the related feedback functions. On the other hand, these ill-posed equations do not admit comparison results which makes the analysis quite challenging.

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