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Abstract

We introduce forward recursive aggregator systems in Itô-diffusion markets, bridging the theories of recursive utilities and of forward performance criteria. We derive the associated HJB SPDE and sufficient conditions for consistency and admissibility. We also develop a convex duality theory based on state price densities and identify the density generated by the optimal wealth process. For homothetic aggregators, we characterize a class of forward Epstein–Zin recursive utility processes, derive the optimal investment and consumption policies, and obtain explicit solutions in the Black and Scholes and Heston models.

Key words: recursive utilities, Epstein–Zin utilities, forward utilities, forward recursive aggregators, forward performance processes, stochastic partial differential equations, convex duality, state price densities, optimal investment and consumption, Heston model.

1 Introduction

The paper contributes to the literature of recursive utilities and to optimal portfolio and consumption models under such stochastic preferences. We introduce and develop a novel class of recursive utility processes, the *forward recursive utilities* (or forward recursive performance criteria), which incorporate progressively measurable aggregators, arbitrary or rolling horizons, and progressively measurable adaptation of market model dynamics. Thus, our work bridges the areas of recursive preferences and forward performance processes, and proposes an extended class of utility processes which contain the existing ones as special cases. In parallel, the characterization, construction and specification of forward recursive utilities give rise to new stochastic optimization problems which are interesting in their own right.

Recursive utilities occupy a central place in asset pricing and in optimal consumption and portfolio choice. In deterministic settings, they first appeared in the seminal works of Koopmans [11] and Lucas and Stokey [16] in discrete-time models, while their continuous-time analogue was developed by Epstein in [6]. In stochastic environments, recursive utilities of separable kind were proposed by Epstein [5] for discrete time and by Uzawa [23] in continuous time. The general class of non-separable recursive utilities were subsequently developed by Epstein and Zin [24] in discrete-time and Duffie and Epstein [2] in continuous models. Since then, recursive utilities have been developed and incorporated in many areas, including optimal investment and consumption choice, asset-price equilibrium, optimal growth, insurance, and pensions. Providing a comprehensive list of references lies outside the scope of this paper.

To motivate the forward recursive construction, we first briefly review some key elements and structural characteristics of their existing classical counterparts. Since we will be working in Itô-diffusion markets, we only recall results for such processes (we refer the reader to [2] for further details). To this end, for a given horizon $[0,T]$, $T \leq \infty$, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$, a recursive utility is an $\mathcal{F}_t-$adapted process, denoted by $V_{t,T}$, $t \in [0,T]$, that measures the performance of admissible consumption processes c_t. It is constructed via an *aggregator pair* (f, m), with m resembling a certainty equivalent rule and f acting upon the upcoming consumption stream and its future utility value. Intertemporal consistency requirements yield the relation

$$\frac{d}{ds}\, m(\mathrm{Law}(V_{t+s,T} \,|\, \mathcal{F}_t))\bigg|_{s=0} = -f\,(c_t, V_{t,T})\,, \quad t \in [0,T]\,. \tag{1}$$

Then, such a process $V_{t,T}$ admits the Itô decomposition (taking, for exposition purposes, the Brownian motion W to be one-dimensional),

$$dV_{t,T} = \mu_t dt + \sigma_t dW_t, \quad t \in [0,T]\,, \qquad V_{T,T} = 0,$$

for suitable drift and volatility processes μ and σ satisfying the compatibility condition

$$\mu_t = -f\left(c_t, V_{t,T}\right) - \frac{1}{2} A^{\mathrm{DE}}\left(V_{t,T}\right) \sigma_t^2, \quad t \in [0,T]\,, \tag{2}$$

with $A^{\mathrm{DE}}(\cdot)$, being non-positive and known as the *variance multiplier* of Duffie and Epstein. If such a process $V_{t,T}$ exists, it can be represented as

$$V_{t,T} = \mathbb{E}\left[\int_t^T \left(f\left(c_s, V_{s,T}\right) + \frac{1}{2} A^{\mathrm{DE}}\left(V_{s,T}\right) \frac{d}{ds}\left[V\right]_{s,T}\right) ds \middle| \mathcal{F}_t\right], \; t \in [0,T]\,. \tag{3}$$

The above may be simplified, working with the so-called *normalized* version. Specifically, let (f, m) and $\left(\bar{f}, \bar{m}\right)$ be two aggregator pairs such that, for a given smooth function φ,

$$f\left(c, u\right) = \frac{\bar{f}\left[c, \varphi\left(u\right)\right]}{\varphi'\left(u\right)} \qquad \text{and} \qquad m\left(\mathrm{Law}(\xi)\right) = \varphi^{-1}(\bar{m}\left[\mathrm{Law}(\varphi(\xi))\right])\,,$$

where ξ is a random variable. Then, the normalized (but ordinally equivalent) version of $V_{t,T}$, denoted by $\bar{V}_{t,T}$, is given by

$$\bar{V}_{t,T} = \mathbb{E}\left[\int_t^T \bar{f}\left(c_s, \bar{V}_{s,T}\right) ds \middle| \mathcal{F}_t\right], \quad t \in [0,T]\,. \tag{4}$$

Formulations (3) and (4) (original or normalized) have generated a wide range of interesting problems about the existence, uniqueness, smoothness, time-consistency and further characterization (concavity, monotonicity and others) of recursive utility processes. Furthermore, it turned out that there exists a deep connection between recursive utilities and backward stochastic differential equations (BSDE), which prompted the considerable development of many elegant works in this area of stochastic analysis.

The normalized formulation has important tractability and representation advantages. For the remainder of this introductory review, we work with the normalized pair $\left(\bar{f}, \bar{m}\right)$ but, to ease the presentation, omit the bars, writing (f, m). In section 2, we first introduce the general forward construction and then the corresponding normalized analogue.

Recursive utilities were subsequently incorporated in stochastic optimization problems of controlled diffusion processes,

$$dX_s = b(X_s, c_s, s)ds + \sigma(X_s, c_s, s)dW_s, \quad s \in [t,T]\,, \quad X_t = x > 0,$$

with the related value function process defined as

$$U_{t,T}(x) = \sup_c \mathbb{E}\left[\int_t^T f\left(c_s, U_{s,T}\right) ds + \xi_T \middle| \mathcal{F}_t, X_t = x\right], \quad t \in [0,T]\,, \; x > 0,$$

for a pre-specified aggregator pair (f, m) and a terminal condition ξ_T.

In Markovian environments, we have (with a slight abuse of notation) that $U_{t,T}(x) = u(t,x)$ with $u(t,x)$ solving the Hamilton-Jacobi-Bellman (HJB) equation

$$u_t + \max_c \left(\frac{1}{2}\sigma^2(t,x,c)u_{xx} + b(t,x,c)u_x + f(c,u)\right) = 0, \quad t \in [0,T]\,, \; x > 0, \quad u(T,x) = 0. \tag{5}$$

It is assumed for simplicity that $\xi_T = 0$, but the case of non-zero terminal condition can be incorporated in the same way. Such equations were first studied by Duffie and Lions in [3] and later by others (see, also, [22]). Similar optimization criteria were subsequently developed in non-Markovian market environments and the related problems were studied using BSDE and FBSDE.

More broadly, stochastic recursive utilities have been extensively incorporated and widely studied in models that allow for both intertemporal investment and consumption, with an underlying controlled process X satisfying the well known SDE

$$dX_s^{\pi;c} = b_s \pi_s ds - c_s ds + \sigma_s \pi_s dW_s, \quad 0 \le t \le s \le T, \quad X_t = x > 0, \tag{6}$$

with control processes (π, c) being self-financing and satisfying certain integrability properties. The coefficients

(b,σ) represent the drift and the volatility of the traded stock (under the simplified assumption of zero interest rate). The value function process is then defined as

$$U_{t,T}(x) = \sup_{(\pi,c)} \mathbb{E}\left[\int_t^T f\left(c_s, U_{s,T}\right) ds + \xi_T \middle| \mathcal{F}_t, X_t = x \right], \quad t \in [0,T]\,,\ x > 0. \tag{7}$$

In the Markovian case, a HJB equation similar to (5) arises. For example, if the stock price process is log-normal, with coefficients (b,σ), the related HJB takes the form

$$u_t + \max_{(\pi,c)} \left(\frac{1}{2}\sigma^2\pi^2 u_{xx} + b\pi u_x - cu_x + f(c,u) \right) = 0, \quad t \in [0,T]\,,\ x > 0, \quad u(T,x) = \xi\left(x\right). \tag{8}$$

In general, equations of type (5) and (8) are non-tractable, are fully degenerate, may not have smooth solutions and are typically analyzed in the weak (viscosity) sense. Closed-form solutions are obtained only for various homothetic classes, which we revisit in detail later on.

While recursive utilities play a central role in asset pricing and their study has and, still, continues to give rise to very interesting mathematical problems at the intersection of several areas, there are some modeling elements that may impede their universal applicability. We discuss these modeling issues next and propose an extended framework to accommodate some of the related limitations. We choose to do this by working directly in the context of optimal portfolio and consumption problems in Itô-diffusion markets, in (6) and (7) above, and leave the original foundational construction, as in (1), (2) and (7), for future work.

Inspecting the recursive stochastic optimization model (7), we see that there are three modeling inputs, chosen at initiation time $t = 0$. The first is the aggregator pair (f, m), the foundational building block of the recursive utilities. Indeed, this pair is chosen once and for all at initial time, without any flexibility to change it afterwards. As a result, any evolution of preferences depends heavily on these initially specified elements. However, it is often the case that preferences do change over time, and frequently, after, for example, big market movements, realized losses and relative performance outcomes.

The second modeling element is the pre-determined investment horizon $[0,T]$, which is strongly embedded throughout in the recursive construction. On the other hand, it is also often the case that investment horizons may not be a priori known and may be adapted to incoming information. This is particularly crucial if random endowments are incorporated and their arrival times are not a priori known. The third modeling element is the choice of the market model, denoted by $\mathcal{M}_{[0,T]}$, since the choice of the processes (b,σ) in (6) is made at initial time as well. However, such initial choice may turn out to be erroneous as more information is acquired in upcoming times. We add that one may try to remedy the limitation of pre-specified horizon by working in an infinite horizon setting. This choice, however, makes the pre-choice of the market model even more restrictive.

In summary, one may think of the modeling triplet $\mathcal{T} = \left((f,m), [0,T], \mathcal{M}_{[0,T]}\right)$ as a static choice at $t = 0$, which results in various restrictions on the flexibility of the model to incorporate upcoming preferences, horizon and market model changes. Such issues motivated the last author and Musiela to introduce in the early 2000s the concept of *forward performance criteria* (or forward utilities). These are processes defined for all times and constructed via supermartingale and martingale requirements along admissible and optimal control processes, respectively. In many ways, they resemble the classical value function process which inherently obeys the dynamic programming principle (DPP). However, DPP yields optimality backwards in time and, thus, the pre-specification of triplet $\mathcal{T}$ is unavoidable. In contrast, forward utilities are defined forward in time, which allows the elements of $\mathcal{T}$ to be updated as information arrives.

We continue with some key structural and theoretical properties of forward utility processes, stating them informally to ease the presentation. In Itô-diffusion environments, a forward utility process, $U(t,x)$, is defined such that along an arbitrary admissible policy, say α, generating process X^α, $U(t, X_t^\alpha)$ is a (local) supermartingale while, along an optimal one, $\alpha^\star$, $U(t, X_t^{\alpha^\star})$ becomes a (local) martingale. Assuming the decomposition

$$dU(t,x) = b^U(t,x)dt + \delta(t,x)^\top dW_t,\ t \geq 0,\ x > 0,$$

for suitable drift and volatility processes $b^U(t,x)$ and $\delta(t,x)$, these two properties impose a certain structure on $b^U(t,x)$ for a given forward volatility process $\delta(t,x)$. We stress that, contrary to the classical case where the volatility of the value function process is part of the solution, in the forward setting it is a modeling input. In other words, embedded in the definition of the forward utility process is the choice of $\delta(t,x)$. The conditions resulting on $b^U(t,x)$ essentially lead to an ill-posed SPDE which plays the role of the classical HJB equation (see, e.g., [18]). To date, general existence, uniqueness and regularity results for such SPDE are lacking.

There are, however, three classes of forward utilities that have been extensively studied. The first class is the one of time-monotone forward criteria ($\delta(t,x) \equiv 0$), studied in [19] (see, also, preprint [20]). The second class is in Markovian, stochastic factor models in which $U(t,x)$ has a finite-dimensional representation, $U(t,x) = u(t,x,Y_{1,t},...,Y_{N,t})$, for some "factor" processes $Y_{i,t}$, $i=1,...,N$, with function u solving an ill-posed, but finite-dimensional, HJB equation. Within the homothetic family of forward utilities, further reduction leads to tractable problems that have been analyzed using, among others, the Martin boundary, ergodic BSDE, systems of FBSDE and others. Thirdly, predictable forward utilities have been analyzed with functional and non-local equations. Providing a complete bibliography of forward utilities and related applications, among others in mean field games, entropic risk measures, optimized certainty equivalent, indifference valuation, risk sharing, pension fund management, equilibrium models and many others, lies outside the scope of this paper. We refer the reader to the review article [26] and to the recent volume collections [14] and [15].

For the definition of *forward recursive utility processes*, we focus for now on Itô-diffusion market environments with controlled processes $X^{\pi,c}$ as in (6). We combine elements from the definition of forward utilities and structural characteristics of the classical recursive utilities, and seek a *pair* of random fields, denoted by $U(t,x)$ and $F(t,c,u,z)$, together with a volatility process $\delta(t,x)$, such that the process

$$U(t,X_t^{\pi,c}) + \int_0^t F(s,c_s,U(s,X_s^{\pi,c}),Z_s^{\pi,c})ds,\ t\geq 0,$$

is a (local) supermartingale along an admissible policy (π,c), where process $Z^{\pi,c}$ is defined as

$$Z_t^{\pi,c} = \delta\left(t,X_t^{\pi,c}\right) + U_x(t,X_t^{\pi,c})X_t^{\pi,c}\sigma_t^{\top}\pi_t,\ t\geq 0,$$

and along an optimal policy $(\pi^\star,c^\star)$, the process

$$U(t,X_t^{\pi^\star,c^\star}) + \int_0^t F(s,c_s^\star,U(s,X_s^{\pi^\star,c^\star}),Z_s^{\pi^\star,c^\star})ds,\ t\geq 0,$$

is a (local) martingale. We will refer to the pair of processes (U,F) (or $(U,F;\delta(t,x))$, when specific reference to $\delta(t,x)$ is needed) as a consistent *forward aggregator system,* with $U(t,x)$ called its *forward recursive utility process* and $F(t,c,u,z)$ the related *forward aggregator.* The definition of $Z^{\pi,c}$ displays explicitly how the model input $\delta(t,x)$ enters the forward recursive setting.

Working, in alignment with the normalized classical recursive framework, with forward aggregators of the form

$$F(t,c,u,z) = f(t,c,u) + \frac{1}{2}A(u)|z|^2,$$

for a suitable pair (f,A) with $A\leq 0$ continuous and f independent of z, leads to the (normalized) forward recursive SPDE

$$dU(t,x) = \left(\frac{\left|U_x(t,x)\theta_t + \sigma_t^+\sigma_t\delta_x(t,x)\right|^2}{2U_{xx}(t,x)} - \widetilde{F}(t,U_x(t,x),U(t,x))\right)dt + \delta(t,x)^{\top}dW_t,\quad t\geq 0,\ x>0, \tag{9}$$

where $\widetilde{F}$ is the Fenchel transform of F with respect to the consumption and process θ is the market price of risk.

Here $\sigma_t^+\sigma_t$ denotes the orthogonal projection of $\mathbb{R}^d$ onto the traded subspace $\operatorname{Im}\sigma_t^{\top}$. It enters only through the recursive volatility, which may not lie in this subspace. The market price of risk is left unchanged, $\theta_t = \sigma_t^+\sigma_t\,\theta_t$, $t\geq 0$.

If SPDE (9) admits a sufficiently regular solution, the optimal feedback maps are identified in Theorem 2.10 and derived in its proof.

We also provide a detailed study in the associated dual domain. We associate with the market its family of state price densities and show that the convex conjugate random field $\widetilde{U}$ satisfies the Legendre counterpart of the primal Bellman SPDE. This conjugate equation yields a submartingale inequality for every state price density together with a dual control for which the dual criterion becomes a local martingale. Contrary to the time-additive forward utilities studied in [25] the marginal utility along the optimal wealth is not itself a state price density, in the recursive setting. A state price density is obtained after multiplication by a specific discount factor $\kappa^\star$, namely $Y^\star = \kappa^\star U_x(\cdot,X^\star)$. When F is convex in its continuation-utility argument, a variational transform in the spirit of Matoussi and Xing in [17] yields a global Fenchel bound, which bound we show is attained at the primal optimum and at this same density.

Finally, we address the passage from candidate optimal feedback policies to admissible ones. El Karoui and Mrad [4] connect the utility SPDE with two solvable SDEs, and El Karoui, Hillairet and Mrad [25] develop the corresponding investment and consumption framework. Following this route, we show that explicit bounds on the model inputs render the deflated dual SDE globally well-posed, and that the optimal wealth is then constructed from the inverse marginal field, the two solution families being linked by the identity $U_x(t, X_t^\star(x)) = D_t^\star(U_x(0,x))$. These results are presented in Theorem 3.14, where consistency, admissibility and the primal–dual relations are also established under suitable conditions on the model inputs.

Having defined forward recursive utilities and aggregators and examined the primal and dual problems, we turn to representative cases. Given the popularity of Epstein-Zin recursive utilities in the classical framework, we focus on their forward analogue. In particular, we examine forward recursive aggregator systems of the separable and multiplicative form

$$U(t,x) = \Phi_t u(x) \quad \text{and} \quad F(t,c,v) = \Psi_t f(c,v), \quad t \geq 0,\ x > 0,\ c > 0,$$

where

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0,$$

and

$$f(c,v) = \frac{1}{1-\frac{1}{\eta}} \left(c^{1-\frac{1}{\eta}} \left((1-\gamma)v\right)^{1-\frac{1}{\lambda}} - (1-\gamma)v \right), \quad c > 0,\ (1-\gamma)v > 0,$$

parametrized by constants γ, η and $\lambda = (1-\gamma)/(1-1/\eta)$, and suitable processes Φ and Ψ. We provide various characterization results. In particular, we specify Φ as the solution of a scalar Bernoulli-type SDE driven by Ψ by looking at an auxiliary linear SDE solved by the process $\Xi_t = \Phi_t^{\eta/\lambda}$. We also derive the optimal feedback control processes and the optimal state price density in closed form. We further consider the Heston stochastic volatility model and analyze it in detail.

Organization of the paper. In section 2, we introduce the Itô-diffusion market and the new notion forward recursive aggregator systems. We derive the unnormalized and normalized versions and the related forward recursive SPDEs, and state the analogous verification theorems. In section 3, we develop the duality theory for forward recursive utilities. We study the state price densities and the wealth-conjugate SPDE. Furthermore, we provide a theorem that derives, from explicit bounds on the model inputs, the existence and uniqueness of the dual and primal SDEs (optimal wealth process and state price density), the admissibility of the feedback control pair, and the corresponding primal and dual consistency properties. In section 4, we develop the forward analogue of the popular homothetic Epstein–Zin class. For tractability, we mainly consider the separable class and solve it explicitly, including constant-coefficient and Heston stochastic-volatility models. To ease the presentation, we provide the majority of the proofs in the three Appendices.

2 Forward recursive aggregator systems and forward recursive utilities

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ supporting a standard d-dimensional Brownian motion W, and we assume that the filtration coincides with the one generated by W.

The market consists of a riskless bond, taken to be the numeraire, and n risky assets whose discounted prices $S = (S^1, \ldots, S^n)$ are specified by

$$dS_t = \operatorname{diag}(S_t)(\mu_t\, dt + \sigma_t\, dW_t)\,,\ t \geq 0, \qquad S_0 \in (\mathbb{R}_+^*)^n. \tag{10}$$

The coefficients $\mu_t \in \mathbb{R}^n$ and $\sigma_t \in \mathbb{R}^{n\times d}$ are $\mathcal{F}_t$-predictable processes and σ_t has full row rank $n \leq d$, for $dt \otimes d\mathbb{P}$-a.e. (t,ω). The market price of risk is the $\mathbb{R}^d$-valued process

$$\theta_t = \sigma_t^+ \mu_t,\ t \geq 0,$$

where $\sigma_t^+ = \sigma_t^\top (\sigma_t \sigma_t^\top)^{-1}$ denotes the Moore–Penrose pseudo-inverse of σ_t. In particular, it holds that $\sigma_t\theta_t = \mu_t$, and $\theta_t \in \operatorname{Im} \sigma_t^\top$. We assume that

$$\int_0^T |\theta_t|^2\, dt < \infty \ \ \mathbb{P}\text{-a.s. for each } T > 0. \tag{11}$$

The subspace $\operatorname{Im}\sigma_t^\top \subset \mathbb{R}^d$ contains the directions spanned by the tradable assets. When it is a strict subspace of $\mathbb{R}^d$, the market is incomplete. In what follows, the matrix $\sigma_t^+\sigma_t$ is the orthogonal projection of $\mathbb{R}^d$ onto $\operatorname{Im}\sigma_t^\top$, and $I_d - \sigma_t^+\sigma_t$ the projection onto its orthogonal complement $(\operatorname{Im}\sigma_t^\top)^\perp$. Since $\theta_t \in \operatorname{Im}\sigma_t^\top$, the two projections act on the market price of risk as

$$\sigma_t^+\sigma_t\,\theta_t = \theta_t \quad \text{and} \quad \left(I_d - \sigma_t^+\sigma_t\right)\theta_t = 0, \quad t \geq 0. \tag{12}$$

The wealth process $X^{\pi,c}$ solves

$$dX_t^{\pi,c} = X_t^{\pi,c}\,\pi_t^\top\sigma_t(dW_t + \theta_t\,dt) - c_t\,dt,\ t \geq 0,\ X_0^{\pi,c} = x > 0, \tag{13}$$

where the portfolio process π, valued in $\mathbb{R}^n$, denotes the *proportions* of wealth allocated to the risky assets, so that the amount invested is $X_t^{\pi,c}\pi_t$. The consumption stream c takes non-negative values. We also use the *relative consumption* process $\check{c} = c/X^{\pi,c}$. All processes are expressed in units discounted by the numeraire.

Definition 2.1. A control pair (π, c) is *admissible* if π and c are $\mathcal{F}_t$-progressively measurable processes satisfying $c_t > 0$, for $dt \otimes d\mathbb{P}$-a.e. (t,ω),

$$\int_0^T \left(c_t + |\pi_t^\top\sigma_t|^2\right)\,dt < \infty \quad \mathbb{P}\text{-a.s. for each } T > 0,$$

and the associated SDE (13) has a unique strong solution $X^{\pi,c}$ which is strictly positive. We denote by $\mathcal{A}$ the set of admissible controls.

2.1 Forward recursive aggregator systems

We introduce the new concepts of forward recursive preferences and their aggregators. The forward utility is represented by an Itô random field whose evolution is linked to a progressively measurable aggregator through a martingale optimality principle. The regularity classes for the random fields used below are recalled in Appendix A.

Definition 2.2. Let $\mathcal{U} \subset \mathbb{R}$ be a non-empty open interval, and $\mathbb{D}_U = [0,\infty) \times \Omega \times (0,\infty)$ and $\mathbb{D}_F^{\mathcal{U}} = [0,\infty) \times \Omega \times (0,\infty) \times \mathcal{U} \times \mathbb{R}^d$. A pair (U, F) of random fields

$$U : \mathbb{D}_U \longrightarrow \mathcal{U} \quad \text{and} \quad F : \mathbb{D}_F^{\mathcal{U}} \longrightarrow \mathbb{R},$$

is called a *forward aggregator system on* $\mathcal{U}$ if:

i) for each $x > 0$, the process $U(\cdot,\cdot,x)$ is $\mathcal{F}_t$-progressively measurable, and for each $(c,u,z) \in (0,\infty) \times \mathcal{U} \times \mathbb{R}^d$, the process $F(\cdot,\cdot,c,u,z)$ is $\mathcal{F}_t$-progressively measurable,

ii) $\mathbb{P}$-almost surely, for every $t \geq 0$, the map $x \mapsto U(t,\omega,x)$ is strictly increasing and strictly concave,

iii) for $dt \otimes d\mathbb{P}$-a.e. (t,ω), the map $c \mapsto F(t,\omega,c,u,z)$ is strictly increasing and strictly concave for every $(u,z) \in \mathcal{U} \times \mathbb{R}^d$, and the map $(u,z) \mapsto F(t,\omega,c,u,z)$ is continuous on $\mathcal{U} \times \mathbb{R}^d$ for each $c > 0$.

The properties of U are stated in ii) for a single version of the field on one $\mathbb{P}$-full set, while those of F are stated in (iii) for $dt \otimes d\mathbb{P}$-a.e. (t,ω), since F is used only inside time integrals. Progressive measurability in (t,ω) together with continuity in the spatial variables gives, by the Carathéodory measurability theorem, that F is jointly measurable in (t,ω,c,u,z). We suppress the ω-argument and write $U(t,x)$ and $F(t,c,u,z)$, except where the dependence on ω has to be displayed.

Definition 2.3. Let (U, F) be a forward aggregator system on $\mathcal{U}$ and let U be a $\mathcal{K}_{\text{loc}}^{2,\varepsilon}$-semimartingale random field, for some $\varepsilon \in (0,1]$, with local characteristics (b,δ) in the sense that

$$dU(t,x) = b(t,x)\,dt + \delta(t,x)^\top dW_t,\ t \geq 0,\ x > 0, \tag{14}$$

for a suitable initial condition

$$U(0,x) = u_0(x), \quad x > 0. \tag{15}$$

For every admissible pair $(\pi, c) \in \mathcal{A}$, define

$$Z_t^{\pi,c} = \delta(t, X_t^{\pi,c}) + U_x(t, X_t^{\pi,c})\,X_t^{\pi,c}\,\sigma_t^\top\pi_t, \quad t \geq 0. \tag{16}$$

We stress that the process $Z^{\pi,c}$ is the diffusion coefficient obtained by composing the random field with the controlled wealth, and not a state process with an independently chosen initial value. Its value at zero is determined by the data, $Z_0^{\pi,c} = \delta(0,x) + U_x(0,x)\,x\,\sigma_0^\top \pi_0$, and no separate condition on $Z_0^{\pi,c}$ is imposed.
Whenever

$$\int_0^T |F(s, c_s, U(s, X_s^{\pi,c}), Z_s^{\pi,c})|\, ds < \infty \quad \mathbb{P}\text{-a.s., for every } T > 0, \tag{17}$$

the associated value process is defined by

$$V_t^{\pi,c} = U(t, X_t^{\pi,c}) + \int_0^t F(s, c_s, U(s, X_s^{\pi,c}), Z_s^{\pi,c})\, ds, \qquad t \geq 0. \tag{18}$$

The forward aggregator system (U, F) is said to be *consistent* if the following two conditions hold:
i) for every $(\pi, c) \in \mathcal{A}$ satisfying (17), the process $V^{\pi,c}$ is a local $\mathcal{F}_t$-supermartingale, and
ii) there exists $(\pi^\star, c^\star) \in \mathcal{A}$ satisfying (17) such that the process $V^{\pi^\star, c^\star}$ is a local $\mathcal{F}_t$-martingale.

If (U, F) is a consistent forward aggregator system, then U is called a *forward recursive utility*, F a *forward aggregator*, and any pair $(\pi^\star, c^\star)$ satisfying ii) is called *optimal*.

Remark 2.4. As in the existing forward utility literature, the volatility process δ in (14) is a modeling input, in contrast to its analogue in the classical (backward) recursive utility literature where it is part of the solution process.

Furthermore, the process $Z^{\pi,c}$ in (16) is the diffusion coefficient of $U(t, X_t^{\pi,c})$, produced by the Itô–Ventzel formula. It combines the modeling input δ and the term $U_x(t,x) X \sigma^\top \pi$ that comes from the dynamics of the underlying wealth controlled process. Its role as the z-argument in F is the (forward) analogue of the Z-process in the classical BSDE formulation of recursive utility.

Remark 2.5. In the classical recursive utility literature, one often imposes additional Lipschitz assumptions on F with respect to (u, z), which however fail for the popular Epstein–Zin recursive utilities. Since later on we focus on the forward analogue of the latter, we do not impose any such conditions. In their absence, however, the integrability requirement (17) may restrict the admissible control pairs to which the criterion applies. This restriction depends on the pair (U, F) itself, in analogy with the restricted class of strategies in [10, Remark 3.14] . The role of (17) is the following:
i) The value process (18) is defined only for the admissible control pairs that satisfy (17).
ii) Integrability of the positive part is a consequence of the Bellman inequality (23) below. Indeed, writing $v_t = X_t^{\pi,c} \sigma_t^\top \pi_t$, this inequality gives, for $dt \otimes d\mathbb{P}$-a.e. (t, ω),

$$F(t, c_t, U(t, X_t^{\pi,c}), Z_t^{\pi,c}) \leq -b(t, X_t^{\pi,c}) - U_x(t, X_t^{\pi,c})\big(v_t^\top \theta_t - c_t\big) - \tfrac{1}{2} U_{xx}(t, X_t^{\pi,c}) |v_t|^2 - v_t^\top \delta_x(t, X_t^{\pi,c}). \tag{19}$$

Each term on the right-hand side is $\mathbb{P}$-a.s. integrable on $[0, T]$ for each $T > 0$, by the continuity in x of b, U_x, U_{xx} and δ_x along the continuous process $X^{\pi,c}$, together with the integrability requirements of Definition 2.1 and $\int_0^T |\theta_t|^2 dt < \infty$. Hence,

$$\int_0^T F^+(s, c_s, U(s, X_s^{\pi,c}), Z_s^{\pi,c})\, ds < \infty \qquad \mathbb{P}\text{-a.s. for every } T > 0.$$

iii) Integrability of the negative part is therefore the only part of (17) that is not guaranteed by the consistency requirement, and it remains an admissibility condition on the control pair. Along the optimal pair of Theorem 2.6 it holds automatically, as shown in Appendix A, since it is equal to locally integrable terms.

Consistency is understood locally, in line with the local forward criteria of [10, Definition 2.1]. Whenever expectations at a deterministic horizon are used, we additionally assume that the relevant stopped family is uniformly integrable, equivalently, of class (D) in the setting at hand. These are additional assumptions, under which, the local properties become true (super)martingale properties and the criterion considers the corresponding admissible control pairs in the expected-utility sense (cf. [10, Remark 3.15]). They are not imposed in the general results below, which are therefore, considered only in the local sense.

2.2 The forward recursive HJB SPDE and verification results

The martingale optimality principle leads to the following *forward recursive HJB SPDE*,

$$dU(t,x) = b(t,x)\,dt + \delta(t,x)^\top dW_t, \quad t \ge 0,\ x > 0, \tag{20}$$

where the drift b satisfies, for $dt \otimes d\mathbb{P}$-a.e. (t,ω) and each $x > 0$, the optimality requirement

$$b(t,x) + \sup_{(\pi,c)} \Big(U_x(t,x)\big(x\,\pi^\top \sigma_t \theta_t - c\big) + \tfrac{1}{2} U_{xx}(t,x)\,\big|x\,\pi^\top \sigma_t\big|^2 + x\,\pi^\top \sigma_t\,\delta_x(t,x)$$
$$+ F\big(t, c, U(t,x), \delta(t,x) + U_x(t,x)\,x\,\sigma_t^\top \pi\big)\Big) = 0,$$

together with the initial condition (15). The Itô–Ventzel formula shows that this drift condition is precisely the local supermartingale condition. Theorem 2.6 gives the corresponding verification result.

We introduce the following notation. For $t \ge 0$, $\omega \in \Omega$, $x > 0$, and $\pi \in \mathbb{R}^n$, we define

$$Z(t,\omega,x,\pi) = \delta(t,\omega,x) + U_x(t,\omega,x)\,x\,\sigma_t(\omega)^\top \pi,$$

and

$$\begin{aligned} H(t,\omega,x,\pi,c) = {} & b(t,\omega,x) + U_x(t,\omega,x)\left(x\pi^\top \sigma_t(\omega)\theta_t(\omega) - c\right) \\ & + \tfrac{1}{2} U_{xx}(t,\omega,x)\,|x\pi^\top \sigma_t(\omega)|^2 + x\pi^\top \sigma_t(\omega)\,\delta_x(t,\omega,x) \\ & + F(t,\omega,c,U(t,\omega,x),Z(t,\omega,x,\pi))\,. \end{aligned}$$

These are definitions and not additional requirements. In this notation, the drift condition above reads for $dt\otimes d\mathbb{P}$-a.e. (t,ω) and every $x > 0$,

$$\sup_{(\pi,c)} H(t,x,\pi,c) = 0.$$

In the unnormalized setting, the supremum with respect to π is coupled with the supremum with respect to c. Indeed, the z-argument of F depends on π through $Z(t,x,\pi)$, and the first-order condition in π of the supremum defining the drift condition reads

$$U_x(t,x)\,\sigma_t\theta_t + U_{xx}(t,x)\,x\,\sigma_t\sigma_t^\top \pi + \sigma_t\,\delta_x(t,x) + U_x(t,x)\,\sigma_t\,F_z(t,c,U(t,x),Z(t,x,\pi)) = 0. \tag{21}$$

When $F_z \ne 0$, the last term depends on c through F_z, so that the optimal π depends on c and the optimization in π cannot be separated from the optimization in c. Without additional assumptions, no further simplification is possible.

Theorem 2.6 (Verification). Let (U,F) be a forward aggregator system, where U is a $\mathcal{K}^{2,\varepsilon}_{\text{loc}}$-semimartingale random field for some $\varepsilon \in (0,1]$, with local characteristics (b,δ) as in (14). Moreover, assume that:

i) there exist *feedback controls*, that is, maps $\pi^\star\colon [0,+\infty) \times \Omega \times (0,+\infty) \to \mathbb{R}^n$ and $c^\star\colon [0,+\infty) \times \Omega \times (0,+\infty) \to (0,+\infty)$ that are $\mathcal{F}_t$-progressively measurable in (t,ω) for each $x > 0$ and Borel measurable in x, such that, for $dt \otimes d\mathbb{P}$-a.e. (t,ω) and every $x > 0$,

$$H(t,\omega,x,\pi^\star(t,\omega,x),c^\star(t,\omega,x)) = 0, \tag{22}$$

and

$$H(t,\omega,x,\pi,c) \le 0 \qquad \forall\,(\pi,c) \in \mathbb{R}^n \times (0,+\infty), \tag{23}$$

and

ii) for each $x > 0$, the wealth equation controlled by the feedback controls $(\pi^\star(t,X_t^\star), c^\star(t,X_t^\star))$, namely

$$dX_t^\star = X_t^\star\,\pi^\star(t,X_t^\star)^\top \sigma_t(dW_t + \theta_t\,dt) - c^\star(t,X_t^\star)\,dt, \quad t \ge 0, \quad X_0^\star = x,$$

admits a unique strong solution $X^\star$ which is strictly positive, with $\big(\pi^\star(t,X_t^\star), c^\star(t,X_t^\star)\big)_{t\ge0} \in \mathcal{A}$.

Then, (U,F) is a consistent forward aggregator system, and the feedback control pair $\big(\pi^\star(t,X_t^\star), c^\star(t,X_t^\star)\big)_{t\ge0}$ is optimal.

Proof. The proof is given in Appendix A. □

Following Duffie and Epstein [2], we introduce a normalization under which optimization with respect to investment and consumption decouples.

2.3 Normalization and related results

We work with forward aggregators of the form

$$F(t,c,u,z) = f(t,c,u) + \tfrac{1}{2}A(u)|z|^2, \quad t \geq 0,\ c > 0,\ u \in \mathcal{U},\ z \in \mathbb{R}^d, \tag{24}$$

where $A : \mathcal{U} \to (-\infty, 0]$ is continuous and f does not depend on z. Here A is introduced directly as the coefficient of the quadratic volatility term of the forward aggregator. It represents the forward analogue of the *variance multiplier* introduced in Duffie and Epstein [2].

We collect here the assumptions on the normalization map.

Assumption 2.7. Fix $u_* \in \mathcal{U}$ and let φ be twice continuously differentiable on $\mathcal{U}$, with

$$\varphi' > 0, \qquad \varphi''(u) = A(u)\varphi'(u), \qquad \varphi(u_*) = 0, \qquad \varphi'(u_*) = 1. \tag{25}$$

The differential equation in (25) is precisely what removes the volatility term under normalization. Since A is non-positive and φ' is positive, φ is strictly increasing and concave, and $\bar{\mathcal{U}} = \varphi(\mathcal{U})$ is an open interval. The map φ is determined by the differential equation up to a positive affine transformation $\varphi \mapsto a\varphi + b$, and the last two conditions in (25) fully determine a representative of the class of solutions. Indeed, affine transformations do not affect the induced performance criterion.

Proposition 2.8. Let φ satisfy Assumption 2.7 and set

$$\bar{U} = \varphi \circ U, \quad \bar{\mathcal{U}} = \varphi(\mathcal{U}),$$

and

$$\bar{F}(t,c,\bar{u}) = \varphi'\big(\varphi^{-1}(\bar{u})\big)\, f\big(t,c,\varphi^{-1}(\bar{u})\big)\,, \quad t \geq 0,\ c > 0,\ \bar{u} \in \bar{\mathcal{U}}.$$

Let (U,F) be a consistent forward aggregator system on $\mathcal{U}$ and let $(\pi^\star, c^\star) \in \mathcal{A}$ be an optimal control pair. Then, the following hold:

i) for every $(\pi, c) \in \mathcal{A}$ satisfying (17), the associated value process $\bar{V}^{\pi,c}$ of $(\bar{U}, \bar{F})$ is a local $\mathcal{F}_t$-supermartingale, and

ii) the value process $\bar{V}^{\pi^\star,c^\star}$ is a local $\mathcal{F}_t$-martingale.

Hence, $(\bar{U}, \bar{F})$ is a consistent forward aggregator system on $\bar{\mathcal{U}}$.

Conversely, if (U,F) is a forward aggregator system of form (24) and $(\bar{U}, \bar{F})$ is consistent on $\bar{\mathcal{U}}$, then (U,F) is consistent on $\mathcal{U}$, and the two systems have the same optimal control pairs.

Proof. The proof is given in Appendix A. □

A forward aggregator system is called *normalized* if its aggregator is independent of z. Proposition 2.8 allows us to work with the normalized system when the dependence on z is purely quadratic, and we do so from here on. We, henceforth, omit the bars from the normalized objects.

To display the resulting separation, write $v = x\,\sigma_t^\top \pi$ for the wealth diffusion coefficient of the portfolio generated by π. For $dt \otimes d\mathbb{P}$-a.e. (t,ω) and every $x > 0$, the optimality requirement becomes

$$b(t,x) + \sup_{v \in \operatorname{Im}\sigma_t^\top} \Big(\tfrac{1}{2}\, U_{xx}(t,x)\,|v|^2 + v^\top \big(U_x(t,x)\theta_t + \delta_x(t,x)\big)\Big) + \sup_{c>0}\Big(F(t,c,U(t,x)) - U_x(t,x)\,c\Big) = 0, \quad t \geq 0,\ x > 0,$$

in which the first supremum involves only the wealth diffusion v and the second only the consumption rate c. The reason is precisely the one identified in (21). Specifically, for a normalized aggregator, $F_z \equiv 0$, so the term $U_x\,\sigma_t F_z$ disappears from the first-order condition in π, and the consumption rate no longer enters the portfolio choice. Conversely, the portfolio no longer enters the consumption choice because π appears in the criterion only through v and through the z-argument of F, and the latter has been removed. The maximizations can then be performed one at a time, in either order. Since v is constrained to the subspace $\operatorname{Im}\sigma_t^\top$, only the projection $\sigma_t^+\sigma_t\delta_x$ contributes to the first supremum. The second supremum is the Fenchel transform of F in its consumption argument,

$$\widetilde{F}(t,d,u) = \sup_{c>0}\big(F(t,c,u) - d\,c\big)\,,\ d > 0,\ u \in \mathcal{U}. \tag{26}$$

Thus, the forward recursive HJB SPDE (20) becomes

$$dU(t,x)=\Big(\frac{|U_x(t,x)\theta_t+\sigma_t^+\sigma_t\delta_x(t,x)|^2}{2\,U_{xx}(t,x)}-\widetilde{F}\big(t,U_x(t,x),U(t,x)\big)\Big)dt+\delta(t,x)^\top dW_t,\quad t\ge 0,\ x>0,\tag{27}$$

with initial condition

$$U(0,x)=u_0(x),\quad x>0.$$

Nondegeneracy. The strict concavity in Definition 2.2 and the regularity of U do not by themselves yield a strict sign for the second derivative at every point. We therefore impose, as a standing assumption for the classical form of the forward recursive HJB SPDE and for the duality results of Section 3,

$$U_{xx}(t,x)<0,\qquad t\ge 0,\ x>0,\quad \mathbb{P}\text{-a.s.}\tag{28}$$

The strict inequality is what allows us to divide by U_{xx}, which is done in the drift of (27), in the portfolio feedback (30) below, and in the inverse marginal computations of Section 3. Under regularity conditions on the convex conjugate $\widetilde{U}$ of U, (28) is equivalent to

$$\widetilde{U}_{yy}(t,y)=-\frac{1}{U_{xx}(t,I(t,y))}>0,\qquad y>0,$$

in the notation of (36). We note that condition (28) is not part of the martingale definition of consistency in Definition 2.3, and it is not used in Theorem 2.6.

We introduce the following regularity assumptions for the normalized forward aggregator.
Assumption 2.9. For $dt\otimes d\mathbb{P}$-a.e. (t,ω), the map $(c,u)\mapsto F(t,\omega,c,u)$ is continuously differentiable on $(0,\infty)\times\mathcal{U}$, its derivatives F_c and F_u are jointly continuous in (c,u) on $(0,\infty)\times\mathcal{U}$, and the Inada conditions in c hold, namely,

$$\lim_{c\downarrow 0}F_c(t,\omega,c,u)=\infty\quad\text{ and }\quad\lim_{c\uparrow\infty}F_c(t,\omega,c,u)=0,\quad\text{for every } u\in\mathcal{U}.$$

Recall from Definition 2.2 that map $c\mapsto F(t,c,u)$ is strictly increasing and strictly concave. Together with Assumption 2.9, this gives, for $dt\otimes d\mathbb{P}$-a.e. (t,ω) and each $(d,u)\in(0,\infty)\times\mathcal{U}$, a unique interior maximizer of $c\mapsto F(t,c,u)-d\,c$, at which $F_c=d$, namely the optimizer $F_c^{-1}(t,d,u)$. The Fenchel transform $\widetilde{F}$ in (26) is therefore finite and the maximum is attained at $c=F_c^{-1}(t,d,u)$, and for $dt\otimes d\mathbb{P}$-a.e. (t,ω) and each $(d,u)\in(0,\infty)\times\mathcal{U}$, we have

$$\widetilde{F}(t,d,u)=F\big(t,F_c^{-1}(t,d,u),u\big)-d\,F_c^{-1}(t,d,u).$$

Moreover, $\widetilde{F}$ is continuously differentiable on $(0,\infty)\times\mathcal{U}$ and satisfies the envelope identities

$$\widetilde{F}_d(t,d,u)=-F_c^{-1}(t,d,u)\quad\text{ and }\quad\widetilde{F}_u(t,d,u)=F_u\big(t,F_c^{-1}(t,d,u),u\big),\tag{29}$$

for all $(d,u)\in(0,\infty)\times\mathcal{U}$ and $dt\otimes d\mathbb{P}$-a.e. (t,ω). The first identity in (29) is the differentiability of a convex conjugate at a point with a unique maximizer. The second is the envelope theorem. The joint continuity of F_c and F_u assumed above, together with the strict monotonicity of F_c in c, makes F_c^{-1} continuous in (d,u) and $\widetilde{F}_d$, $\widetilde{F}_u$ continuous as well.

Theorem 2.10 (Verification). Let (U,F) be a normalized forward aggregator system on $\mathcal{U}$ satisfying Assumption 2.9, where U is a $\mathcal{K}^{2,\varepsilon}_{\text{loc}}$-semimartingale random field for some $\varepsilon\in(0,1]$, with local characteristics (b,δ) as in (14) and satisfying the nondegeneracy condition (28). Assume that the random field U satisfies the normalized forward recursive HJB SPDE (27).
Let $\pi^\star:[0,\infty)\times\Omega\times(0,\infty)\mapsto\mathbb{R}^n$, and $c^\star:[0,\infty)\times\Omega\times(0,\infty)\mapsto(0,\infty)$ be defined as

$$\pi^\star(t,x)=-\frac{U_x(t,x)}{x\,U_{xx}(t,x)}\,(\sigma_t^\top)^+\left(\theta_t+\frac{\delta_x(t,x)}{U_x(t,x)}\right),\qquad t\ge 0,\ x>0,\tag{30}$$

and

$$c^\star(t,x)=F_c^{-1}(t,U_x(t,x),U(t,x))\,,\qquad t\ge 0,\ x>0,\tag{31}$$

and assume that the wealth equation controlled by the above feedback controls admits a unique strong solution $X^\star$ which is strictly positive, and that $\big(\pi^\star(t,X_t^\star),c^\star(t,X_t^\star)\big)_{t\geq 0}\in\mathcal{A}$. Then, (U,F) is a consistent forward aggregator system and the control pair $\big(\pi^\star(t,X_t^\star),c^\star(t,X_t^\star)\big)_{t\geq 0}$ is optimal.

Proof. The proof is given in Appendix A. □

Explicit conditions on the model inputs guaranteeing the admissibility of the feedback control pair are provided in Section 3 (Assumption 3.13 and Theorem 3.14). They resemble those in Assumption 2.3 in [13], where the risk tolerance $-U_x/U_{xx}$ is required to be bounded above and below away from zero, and the ratio $|\delta_x/U_{xx}|$ to be bounded.

Remark 2.11. We note that in the term $\widetilde{F}(t,U_x(t,x),U(t,x))$ both the forward recursive utility $U(t,x)$ and its marginal $U_x(t,x)$ appear. This nonlinear coupling makes the problem in general non-tractable. However, for specific but rich enough cases, which we study in Section 4, we are able to produce closed form solutions and draw analogies with their classical counterparts.

3 Duality results for normalized forward recursive aggregator systems

In this section, we develop the convex duality theory for forward recursive aggregator systems. We first introduce the family of state price densities associated with the market with stock prices as in (10) together with the associated deflated process. We then show that the convex conjugate of a forward recursive utility solves a *dual forward recursive HJB SPDE*, which is the Legendre counterpart of the primal one. We establish the corresponding dual consistency on the family of state price densities and, in turn identify the state price density selected by the primal optimum, which turns out to be the marginal utility along the optimal wealth multiplied by the factor that removes its recursive drift. Finally, we provide a theorem that derives, using explicit bounds on the model inputs, the existence and uniqueness of the dual and primal solution families, the admissibility of the feedback control pair, and the resulting primal and dual consistency properties. This approach extends the results of [4] to the recursive setting. These results provide the tools used in Section 4 where we introduce and solve in detail the forward analogue of the classical Epstein-Zin recursive utility maximization problem.

3.1 State price densities

For an admissible control pair $(\pi,c)\in\mathcal{A}$ and a positive Itô process Y, the deflated process $H^{\pi,c}(Y)$ is defined as

$$H_t^{\pi,c}(Y)=X_t^{\pi,c}\,Y_t+\int_0^t Y_s\,c_s\,ds,\quad t\geq 0. \tag{32}$$

Definition 3.1. A *state price density* is a positive Itô process Y such that for every $(\pi,c)\in\mathcal{A}$, $H^{\pi,c}(Y)$ is a local martingale. We denote by $\mathcal{Y}$ the family of state price densities.

Using the wealth dynamics (13), $H^{\pi,c}(Y)$ is a local martingale for every $(\pi,c)\in\mathcal{A}$ if and only if Y satisfies, for some initial value $y>0$ and some predictable process ν with $\nu_t\in(\operatorname{Im}\sigma_t^\top)^\perp$, for $dt\otimes d\mathbb{P}$-a.e. (t,ω),

$$dY_t^{y,\nu}=Y_t^{y,\nu}\big(-\theta_t+\nu_t\big)^\top dW_t,\qquad Y_0^{y,\nu}=y,\qquad t\geq 0. \tag{33}$$

A dual control ν is *admissible* if it is a $\mathcal{F}_t$-predictable process satisfying $\nu_t\in(\operatorname{Im}\sigma_t^\top)^\perp$, for $dt\otimes d\mathbb{P}$-a.e. (t,ω), and

$$\int_0^T|\nu_t|^2\,dt<\infty\ \ \mathbb{P}\text{-a.s., for each } T>0.$$

We denote the set of admissible dual controls by $\mathcal{A}^{\mathrm{dual}}$ and we set

$$\mathcal{Y}(y)=\big\{Y^{y,\nu}:\nu\in\mathcal{A}^{\mathrm{dual}}\big\},\quad y>0,\quad\text{and}\quad\mathcal{Y}=\bigcup_{y>0}\mathcal{Y}(y). \tag{34}$$

This formulation distinguishes the dual control ν, the equation (33) that it parametrizes, and the resulting density $Y^{y,\nu}$. When the initial value plays no role, we write Y^ν for an element of $\mathcal{Y}$. The set $\mathcal{Y}$ depends only on the market processes (μ, σ) through the market price of risk θ and the projection $\sigma_t^+\sigma_t$. Clearly, if the market is complete, $\sigma_t^+\sigma_t \equiv I_d$, $\nu \equiv 0$, and $\mathcal{Y}$ consists of a single state price density, up to a positive scaling.

3.2 Conjugate forward recursive utility and dual forward recursive SPDE

We analyse the conjugate random field $\widetilde{U}$ and derive the equation that it satisfies. As in the literature of forward utilities (see, e.g., [4, 13]), we impose additional regularity conditions on the random field U for the dual forward recursive HJB SPDE to be well defined.

Assumption 3.2. For some $\varepsilon \in (0, 1]$,

i) U is a $\mathcal{K}^{2,\varepsilon}_{\mathrm{loc}}$-semimartingale random field and, $\mathbb{P}$-almost surely, the map $x \mapsto U(t, x)$ is three times continuously differentiable for every $t \geq 0$, with third derivative locally ε-Hölder on compact subsets of $(0, \infty)$,

ii) $\mathbb{P}$-almost surely, for every $t \geq 0$, the marginal $U_x(t, \cdot)$ satisfies the Inada conditions

$$\lim_{x\downarrow 0} U_x(t, x) = \infty \quad \text{and} \quad \lim_{x\uparrow\infty} U_x(t, x) = 0,$$

iii) the inverse marginal field

$$I(t, \omega, y) = \big(x \mapsto U_x(t, \omega, x)\big)^{-1}(y), \qquad t \geq 0,\ \omega \in \Omega,\ y > 0,$$

is a $\mathcal{K}^{2,\varepsilon}_{\mathrm{loc}}$-semimartingale random field.

Strict concavity and the Inada conditions in ii) make $U_x(t, \cdot)$ a bijection from $(0, \infty)$ onto $(0, \infty)$, so that $I(t, y)$ is well defined for all $y > 0$. The regularity of this specific field is assumed in iii), and is not derived from the dual equation. The Itô–Ventzel calculations of Appendix B use (i) for the field U and its derivative field U_x, together with their local characteristics (b, δ) and (b_x, δ_x), and (iii) for the field I along which the conjugate is computed. No Itô decomposition of δ or of U_{xxx} is required. This resembles Assumption 2.3 in [13], and the inverse field condition is standard in the Itô random field setting, see [4].

Next, we define the convex dual

$$\widetilde{U}(t, y) = \sup_{x>0} \big(U(t, x) - x\, y\big) = U(t, I(t, y)) - y\, I(t, y), \quad t \geq 0,\ y > 0. \tag{35}$$

The technical conjugacy calculation is given in Lemma 3.3 in Appendix B. In particular, $\widetilde{U}$ is a semimartingale random field with local characteristics $(\widetilde{b}, \widetilde{\delta})$. The conjugacy identities hold $\mathbb{P}$-a.s. for every $t \geq 0$ and $y > 0$,

$$I(t, y) = -\widetilde{U}_y(t, y), \qquad U_{xx}(t, I(t, y)) = -\frac{1}{\widetilde{U}_{yy}(t, y)}. \tag{36}$$

The differential characteristics satisfy, for $dt \otimes d\mathbb{P}$-a.e. (t, ω) and every $y > 0$,

$$\widetilde{\delta}(t, y) = \delta(t, I(t, y)) \quad \text{and} \quad \widetilde{b}(t, y) = b(t, I(t, y)) - \frac{1}{2}\frac{|\delta_x(t, I(t, y))|^2}{U_{xx}(t, I(t, y))}, \tag{37}$$

and, as a consequence,

$$\delta_x(t, I(t, y)) = -\frac{\widetilde{\delta}_y(t, y)}{\widetilde{U}_{yy}(t, y)}. \tag{38}$$

To ease the presentation, we write

$$\hat{u}(t, y) = \widetilde{U}(t, y) - y\widetilde{U}_y(t, y) = U(t, I(t, y)), \qquad t \geq 0,\ y > 0. \tag{39}$$

In other words, $\hat{u}(t, y)$ is the primal utility level expressed at the dual state y.

When (U, F) solves the primal SPDE (27), the drift characteristic of $\widetilde{U}$ is given, for $dt \otimes d\mathbb{P}$-a.e. (t, ω) and every $y > 0$, by

$$\widetilde{b}(t, y) = -\widetilde{F}\big(t, y, \hat{u}(t, y)\big) - \tfrac{1}{2}\, y^2\, \widetilde{U}_{yy}(t, y)\, |\theta_t|^2 + y\, \theta_t^\top \widetilde{\delta}_y(t, y) + \frac{|(I_d - \sigma_t^+\sigma_t)\, \widetilde{\delta}_y(t, y)|^2}{2\, \widetilde{U}_{yy}(t, y)}. \tag{40}$$

Proposition 3.4 (Primal recursive SPDE $\implies$ dual recursive SPDE). Let (U,F) satisfy the normalized primal forward recursive HJB SPDE (27), Assumption 2.9 and Assumption 3.2. Then, the convex dual $\widetilde{U}$ solves the *dual normalized forward recursive HJB SPDE*

$$
\begin{aligned}
d\widetilde{U}(t,y) = \Big(&-\widetilde{F}\big(t,y,\hat{u}(t,y)\big) - \tfrac{1}{2}\, y^2\, \widetilde{U}_{yy}(t,y)\, |\theta_t|^2 \\
&+ y\, \theta_t^\top \widetilde{\delta}_y(t,y) + \frac{|(I_d - \sigma_t^+\sigma_t)\, \widetilde{\delta}_y(t,y)|^2}{2\, \widetilde{U}_{yy}(t,y)}\Big)\, dt + \widetilde{\delta}(t,y)^\top dW_t, \quad t \geq 0,\ y > 0,
\end{aligned}
\tag{41}
$$

with initial condition

$$\widetilde{U}(0,y) = \tilde{u}_0(y), \quad y > 0.$$

Proof. The proof is given in Appendix B. □

3.3 Dual consistency on $\mathcal{Y}$

The dual value process introduced below is defined only along those dual controls for which the running dual aggregator term is integrable. This is not a property derived from the dual equation but a requirement imposed on the control, and we state it as such: for $\nu \in \mathcal{A}^{\text{dual}}$ satisfying

$$\int_0^T \Big|\widetilde{F}\big(s, Y_s^\nu, \hat{u}(s, Y_s^\nu)\big)\Big|\, ds < \infty \quad \mathbb{P}\text{-a.s., for every } T > 0, \tag{42}$$

the associated *dual value process* is well defined and given by

$$\mathcal{Z}_t^\nu = \widetilde{U}(t, Y_t^\nu) + \int_0^t \widetilde{F}\big(s, Y_s^\nu, \hat{u}(s, Y_s^\nu)\big)\, ds, \quad t \geq 0. \tag{43}$$

Definition 3.5. The conjugate pair $(\widetilde{U}, \widetilde{F})$ is *dual consistent on* $\mathcal{Y}$ if:

(i) for every $\nu \in \mathcal{A}^{\text{dual}}$ satisfying (42), the process $\mathcal{Z}^\nu$ is a local $\mathcal{F}_t$-submartingale, and

(ii) there exists a dual control $\nu^{\text{dual}} \in \mathcal{A}^{\text{dual}}$ satisfying (42) for which $\mathcal{Z}^{\nu^{\text{dual}}}$ is a local $\mathcal{F}_t$-martingale.

If the conjugate pair $(\widetilde{U}, \widetilde{F})$ is *dual consistent on* $\mathcal{Y}$, we call $\widetilde{U}$ a *dual forward recursive utility*, $\widetilde{F}$ its *dual forward aggregator*, and ν^{dual} the *optimal dual control*.

Remark 3.6 (One-sided integrability). If $\widetilde{U}$ satisfies (41), then

$$\int_0^T \widetilde{F}^-\big(s, Y_s^\nu, \hat{u}(s, Y_s^\nu)\big) ds < \infty \qquad \mathbb{P}\text{-a.s. for every } T > 0,$$

as shown in Appendix B. Consequently, (42) may fail only through divergence of the positive part. We stress that this is a consequence of the dual equation and not part of Definition 3.5.

Theorem 3.7 (Dual verification). Let Assumptions 2.9 and 3.2 hold, and suppose that $\widetilde{U}$ satisfies the dual forward recursive HJB SPDE (41). Define the dual feedback map

$$\nu_t^{\text{dual}}(y) = -\frac{(I_d - \sigma_t^+\sigma_t)\widetilde{\delta}_y(t,y)}{y\widetilde{U}_{yy}(t,y)}, \qquad t \geq 0,\ y > 0. \tag{44}$$

Assume that the state price density equation (33), controlled by $\nu_t = \nu_t^{\text{dual}}(Y_t^{\nu^{\text{dual}}})$, has a unique, strictly positive, strong solution and that the resulting control belongs to $\mathcal{A}^{\text{dual}}$. Then, the pair $(\widetilde{U}, \widetilde{F})$ is dual consistent on $\mathcal{Y}$, with dual optimal control ν^{dual} and state price density $Y^{\nu^{\text{dual}}}$.

Proof. The proof is given in Appendix B. □

The above dual consistency uses only the wealth conjugate $\widetilde{U}$ and the Fenchel transform $\widetilde{F}$ in the consumption argument, and it does not require the convexity of F in u. The density $Y^{\nu^{\text{dual}}}$ is the one for which the dual value process (43) is a local $\mathcal{F}_t$-martingale. It should be distinguished from the density $Y^\star$ generated by the primal optimizer in Theorem 3.12, and we study their relation there.

Next, we introduce the variational representation of Matoussi and Xing [17]. It requires the convexity of the aggregator F in its utility argument u, and it involves an auxiliary process α, which plays the role of a stochastic discount rate. Its role is the following. In the recursive setting, the marginal utility along the optimal wealth is not itself a state price density, because the dependence of the aggregator on the utility level adds a finite variation term to its dynamics. Discounting by α removes precisely this term and restores the time additive structure, in which the discounted marginal utility does become a state price density. Any admissible discount process α, combined with any state price density, produces in this way a Fenchel type upper bound for the primal criterion. We show in Proposition 3.10 and Theorem 3.12 that this bound is attained, and we identify where: equality holds at the discount process $\alpha^\star = -F_u$, evaluated along the optimal consumption and utility levels, and at the state price density $Y^\star$ generated by the primal optimizer.

3.4 Convex duality using a variational transform

We establish a link between the convex duality developed above and a family of time-additive discounted dual forward utilities $(\widetilde{U}, G)$ for investment and consumption, indexed by an auxiliary process α. For an ordinary time-additive forward utility, the marginal utility evaluated along the optimal wealth is a state price density. In the present recursive setting, the marginal utility includes an additional finite variation term and, as we show, it must be multiplied by the recursive discount factor before a state price density is obtained. This is why the auxiliary process α is needed.

Assumption 3.8. In addition to Assumption 2.9, we assume that, for $dt \otimes d\mathbb{P}$-a.e. (t,ω) and each $c > 0$, the map $u \mapsto F(t,c,u)$ is convex on $\mathcal{U}$.

Motivated by the variational representation used by Matoussi and Xing [17] for the classical recursive utility problem, we define, under Assumption 3.8,

$$\widehat{F}(t,\omega,c,\alpha) = \inf_{u\in\mathcal{U}} \left(F(t,\omega,c,u) + \alpha\, u\right), \qquad t \geq 0,\ \omega \in \Omega,\ c > 0,\ \alpha \in \mathbb{R}. \tag{45}$$

We suppress the ω-argument from now on, for notational convenience. For $dt \otimes d\mathbb{P}$-a.e. (t,ω) and every $(c,u) \in (0,\infty) \times \mathcal{U}$, the definition of the infimum gives $\widehat{F}(t,c,\alpha) - \alpha u \leq F(t,c,u)$. Since $u \mapsto F(t,c,u)$ is differentiable and convex, equality at u holds for

$$\alpha^\star(t,c,u) = -F_u(t,c,u), \qquad t \geq 0,\ c > 0,\ u \in \mathcal{U}, \tag{46}$$

in which case the infimum in (45) is attained at u itself. Equivalently,

$$F(t,c,u) = \sup_{\alpha\in\mathbb{R}} \big\{\widehat{F}(t,c,\alpha) - \alpha\, u\big\}, \qquad t \geq 0,\ c > 0,\ u \in \mathcal{U}, \tag{47}$$

and the maximizer is $\alpha^\star(t,c,u)$.

Concave case. When $u \mapsto F(t,c,u)$ is concave, the pointwise variational identity has a formal analogue obtained by replacing the infimum in (45) by a supremum and the supremum in (47) by an infimum, as in [17, Remark 2.10]. However, the joint saddle relation with the consumption transform G, defined below in (50), does not follow from the concavity in u alone. A concave analogue of the results below requires separate joint-concavity, finiteness, and attainment assumptions. We therefore state the results of this section only under the convexity condition in Assumption 3.8.

To this end, we first introduce the class of discount processes. A progressively measurable real valued process α belongs to $\mathcal{A}^{\text{disc}}$ if, for every $T > 0$,

$$\int_0^T |\alpha_t|\, dt < \infty \quad \mathbb{P}\text{-a.s.}, \qquad \text{and} \qquad \widehat{F}(t,c,\alpha_t) > -\infty,$$

for $dt \otimes d\mathbb{P}$-a.e. (t,ω) and every $c > 0$. For $\alpha \in \mathcal{A}^{\text{disc}}$, set

$$\kappa^\alpha_{s,t} = \exp\Big(-\int_s^t \alpha_r\, dr\Big), \qquad 0 \leq s \leq t. \tag{48}$$

For a control pair $(\pi, c) \in \mathcal{A}$, define the discounted additive value process associated with the discount α,

$$S_t^{\alpha,\pi,c} = \kappa_{0,t}^{\alpha}\, U(t, X_t^{\pi,c}) + \int_0^t \kappa_{0,s}^{\alpha}\, \widehat{F}(s, c_s, \alpha_s)\, ds, \qquad t \geq 0. \tag{49}$$

We also define the Fenchel transform of $\widehat{F}$ with respect to the consumption argument and the deflated density,

$$G(t, d, \alpha) = \sup_{c>0} \big(\widehat{F}(t, c, \alpha) - d\, c \big), \qquad t \geq 0,\ d > 0,\ \alpha \in \mathbb{R}, \tag{50}$$

and

$$D_t^{\alpha,\nu} = \frac{Y_t^{\nu}}{\kappa_{0,t}^{\alpha}}, \qquad t \geq 0. \tag{51}$$

The corresponding dual process is

$$\mathcal{Z}_t^{\alpha,\nu} = \kappa_{0,t}^{\alpha} \widetilde{U}(t, D_t^{\alpha,\nu}) + \int_0^t \kappa_{0,s}^{\alpha} G(s, D_s^{\alpha,\nu}, \alpha_s)\, ds, \qquad t \geq 0. \tag{52}$$

Integrability issue. The fact that α already belongs to $\mathcal{A}^{\text{disc}}$ makes the integrands in (49) and (52) finite. Indeed, $\widehat{F}(t, c, \alpha_t) > -\infty$ by the definition of $\mathcal{A}^{\text{disc}}$, while $\widehat{F}(t, c, \alpha_t) \leq F(t, c, u) + \alpha_t u < \infty$, for every $u \in \mathcal{U}$. Subtracting $d\,c$ from the latter bound and taking the supremum over $c > 0$ gives $G(t, d, \alpha_t) \leq \widetilde{F}(t, d, u) + \alpha_t u$, which is finite because $\widetilde{F}$ is, under Assumption 2.9, while $G(t, d, \alpha_t) \geq \widehat{F}(t, 1, \alpha_t) - d > -\infty$. Under Assumption 3.2, the conjugate $\widetilde{U}(t, \cdot)$ is in turn real valued on $(0, \infty)$, the supremum defining the conjugate $\widetilde{U}(t, \cdot)$ being attained at $I(t, \cdot)$.

However, the absolute convergence of the corresponding time integrals is not, in general, guaranteed. It has to be imposed separately, for the particular discount process, control pair and density used. We say that a triple $(\alpha, (\pi, c), Y^{\nu}) \in \mathcal{A}^{\text{disc}} \times \mathcal{A} \times \mathcal{Y}$ is *integrable* if, for every $T > 0$,

$$\int_0^T \kappa_{0,s}^{\alpha} \Big(\big|\widehat{F}(s, c_s, \alpha_s)\big| + \big|G(s, D_s^{\alpha,\nu}, \alpha_s)\big| \Big)\, ds < \infty \qquad \mathbb{P}\text{-a.s.} \tag{53}$$

For an integrable triple, the discounted value process (49) and the dual process (52) are real valued, and so is the budget process (32), since $\int_0^T c_s\, ds < \infty$ by Definition 2.1 and Y^{ν} has continuous paths. All three sides of the bound below are then finite, and the two Fenchel equality conditions can be stated as a characterization.

Proposition 3.9 (Convex duality bound). Let Assumptions 2.9 and 3.2 hold and let $\big(\alpha, (\pi, c), Y^{\nu}\big) \in \mathcal{A}^{\text{disc}} \times \mathcal{A} \times \mathcal{Y}$ be an integrable triple. Then, for every $t \geq 0$,

$$S_t^{\alpha,\pi,c} - H_t^{\pi,c}(Y^{\nu}) \leq \mathcal{Z}_t^{\alpha,\nu}, \tag{54}$$

with equality $\mathbb{P}$-a.s. at time t if and only if

$$D_t^{\alpha,\nu} = U_x\big(t, X_t^{\pi,c}\big), \tag{55}$$

and, for $ds \otimes d\mathbb{P}$-a.e. (s, ω) with $s \leq t$,

$$\widehat{F}(s, c_s, \alpha_s) - D_s^{\alpha,\nu} c_s = G(s, D_s^{\alpha,\nu}, \alpha_s). \tag{56}$$

Proof. Fix an integrable triple $\big(\alpha, (\pi, c), Y^{\nu}\big) \in \mathcal{A}^{\text{disc}} \times \mathcal{A} \times \mathcal{Y}$ and $t \geq 0$. Since $\kappa_{0,s}^{\alpha} D_s^{\alpha,\nu} = Y_s^{\nu}$ for every $s \geq 0$, the two Fenchel inequalities (26) and (45) yield

$$\kappa_{0,t}^{\alpha} \Big(U\big(t, X_t^{\pi,c}\big) - X_t^{\pi,c} D_t^{\alpha,\nu} \Big) \leq \kappa_{0,t}^{\alpha} \widetilde{U}\big(t, D_t^{\alpha,\nu}\big), \tag{57}$$

which is the definition of the wealth conjugate $\widetilde{U}$ at $D_t^{\alpha,\nu}$, and

$$\kappa_{0,s}^{\alpha} \Big(\widehat{F}(s, c_s, \alpha_s) - D_s^{\alpha,\nu} c_s \Big) \leq \kappa_{0,s}^{\alpha} G(s, D_s^{\alpha,\nu}, \alpha_s), \qquad s \in [0, t], \tag{58}$$

which is the definition of G at $D^{\alpha,\nu}_s$. Integrating (58) over $[0,t]$ and adding (57) gives

$$\kappa^\alpha_{0,t} U\big(t, X^{\pi,c}_t\big) + \int_0^t \kappa^\alpha_{0,s} \widehat{F}(s, c_s, \alpha_s)\, ds - X^{\pi,c}_t Y^\nu_t - \int_0^t Y^\nu_s c_s\, ds \le \kappa^\alpha_{0,t} \widetilde{U}\big(t, D^{\alpha,\nu}_t\big) + \int_0^t \kappa^\alpha_{0,s} G(s, D^{\alpha,\nu}_s, \alpha_s)\, ds,$$

which yields (54), by (32), (49) and (52).

By (53), every term in (57) and in the integral of (58) over $[0,t]$ is finite. Equality in (54) therefore holds if and only if it holds in (57) and, for $ds \otimes d\mathbb{P}$-a.e. (s,ω) with $s \le t$, in (58).

Equality in (57) holds if and only if $D^{\alpha,\nu}_t = U_x(t, X^{\pi,c}_t)$, because $x \mapsto U(t,x)$ is strictly concave and differentiable. Equality in (58) for $ds \otimes d\mathbb{P}$-a.e. (s,ω) with $s \le t$ holds if and only if c_s attains the supremum defining $G(s, D^{\alpha,\nu}_s, \alpha_s)$, that is, if and only if $\widehat{F}(s, c_s, \alpha_s) - D^{\alpha,\nu}_s c_s = G(s, D^{\alpha,\nu}_s, \alpha_s)$. This concludes the proof. $\square$

Inequality (54) is an upper bound for an arbitrary discount process α in $\mathcal{A}^{\mathrm{disc}}$. The optimum occurs at $\alpha^\star = -F_u(\cdot, c^\star, U^\star)$ where the Fenchel inequality in u holds, with equality along the optimal control pair, by (46). Consequently, whenever the first integral in (53) is finite for every $T > 0$, the discounted process $S^{\alpha^\star,\pi,c}$ is a local supermartingale for every $(\pi, c) \in \mathcal{A}$ that also satisfies (17), and it is a local martingale at the optimum. Proposition 3.10 and Theorem 3.12 make these equalities precise.

A diagram illustrating the relationship between the various functions introduced below is presented in Figure 1, starting from the primal problem in the upper left corner and ending at the dual problem in the lower left corner. In contrast with [17], the two recursive corners are related directly, by the Legendre transformation in the wealth variable.

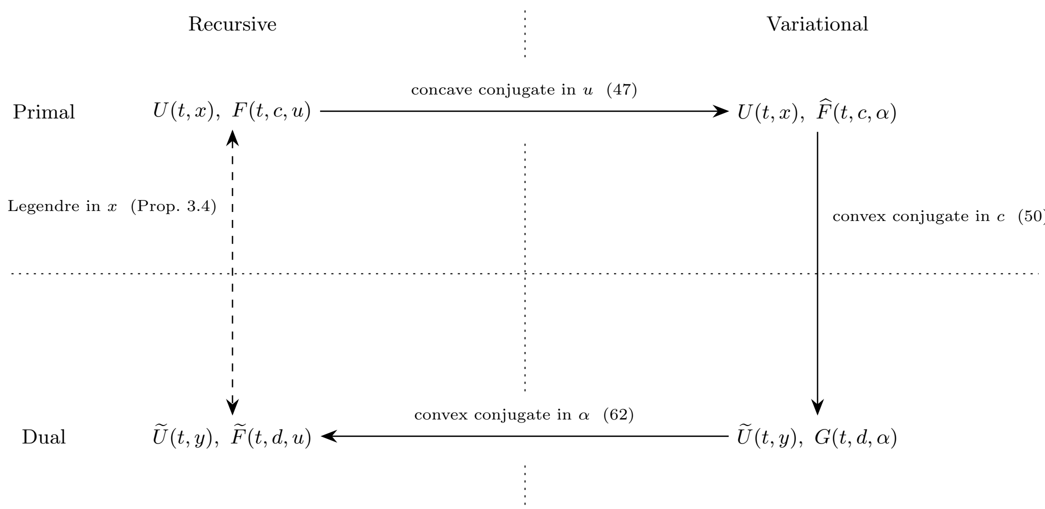

Figure 1: Double Fenchel–Legendre transformation.

3.5 The state price density generated by the primal optimum

We first establish the two equalities in the variational dual bound and then identify the state price density $Y^\star \in \mathcal{Y}$ generated by the primal optimum.

For $D > 0$, define the feedback discounting map

$$\alpha^\star(t, D) = -F_u\big(t, F_c^{-1}(t, D, \hat{u}(t,D)), \hat{u}(t,D)\big)\,, \qquad t \ge 0. \tag{59}$$

A state price density $Y^\nu \in \mathcal{Y}$ is said to *admit a feedback discount* if there exists a progressively measurable process $\alpha^\star \in \mathcal{A}^{\mathrm{disc}}$ such that

$$D_t = \frac{Y^\nu_t}{\kappa^{\alpha^\star}_{0,t}}, \qquad \alpha^\star_t = \alpha^\star(t, D_t), \qquad t \ge 0, \tag{60}$$

the second identity holding for $dt \otimes d\mathbb{P}$-a.e. (t,ω), and such that, for every $T > 0$,

$$\int_0^T \kappa^{\alpha^\star}_{0,s} \big|G(s, D_s, \alpha^\star_s)\big|\, ds < \infty, \qquad \mathbb{P}\text{-a.s.} \tag{61}$$

The first identity in (60) is (51) evaluated at $\alpha^\star$, while the second one evaluates the map (59) along the deflated state D. Taken together, the two identities constitute a pathwise system that couples $\alpha^\star$ and D. Note that $D_0 = Y_0^\nu = y$, the initial value of the density, since $\kappa_{0,0}^{\alpha^\star} = 1$. Condition (61) is precisely the requirement imposed on the term G in (53). It makes the finite variation term of (52) absolutely convergent, so that $\mathcal{Z}^{\alpha^\star,\nu}$ is a well defined continuous process. Along the primal optimum, (61) is automatically satisfied, as shown in Step 2 in the proof of Theorem 3.12.

Proposition 3.10 (Equality in the variational dual bound). Let (U, F) satisfy Assumptions 3.2 and 3.7, and suppose that $\widetilde{U}$ satisfies the dual forward recursive HJB SPDE (41). Then, the following assertions hold.

(i) Pointwise Fenchel equality. For $dt \otimes d\mathbb{P}$-a.e. (t, ω) and every $D > 0$, the functions $\hat{u}$ and $\alpha^\star$ defined in (39) and (59) satisfy

$$G\big(t, D, \alpha^\star(t, D)\big) = \widetilde{F}\big(t, D, \hat{u}(t, D)\big) + \alpha^\star(t, D)\, \hat{u}(t, D). \tag{62}$$

(ii) Dynamics of the process $\mathcal{Z}^{\alpha^\star,\nu}$. Let $Y^\nu \in \mathcal{Y}$ admit a feedback discount $\alpha^\star$. Then,

$$d\mathcal{Z}_t^{\alpha^\star,\nu} = dM_t^\nu + \kappa_{0,t}^{\alpha^\star} \frac{\big|(I_d - \sigma_t^+ \sigma_t)\, \widetilde{\delta}_y(t, D_t) + D_t\, \widetilde{U}_{yy}(t, D_t)\, \nu_t\big|^2}{2\, \widetilde{U}_{yy}(t, D_t)}\, dt, \quad t \geq 0, \tag{63}$$

where M^ν is a local $\mathcal{F}_t$-martingale. In particular, $\mathcal{Z}^{\alpha^\star,\nu}$ is a local $\mathcal{F}_t$-submartingale, and it is a local $\mathcal{F}_t$-martingale if and only if

$$\nu_t = -\frac{(I_d - \sigma_t^+ \sigma_t)\, \widetilde{\delta}_y(t, D_t)}{D_t\, \widetilde{U}_{yy}(t, D_t)}, \qquad \text{for } dt \otimes d\mathbb{P}\text{-a.e. } (t, \omega). \tag{64}$$

Proof. The proof is given in Appendix B. $\square$

3.5.1 Identification of $Y^\star$ along the primal optimum

We now identify the state price density that the primal optimum generates, together with the dynamics of the marginal utility along the optimal wealth.

Theorem 3.12 (The state price density generated by the primal optimum). Let Assumption 3.2 and the hypotheses of Theorem 2.10 hold. Let $X^\star$ be the corresponding wealth process, and let $U^\star$ and $c^\star$ be the forward utility and the consumption processes along it,

$$U_t^\star = U(t, X_t^\star), \qquad \text{and} \qquad c_t^\star = c^\star(t, X_t^\star), \quad t \geq 0.$$

Introduce the process

$$\nu_t^\star = \frac{(I_d - \sigma_t^+ \sigma_t)\, \delta_x(t, X_t^\star)}{U_x(t, X_t^\star)}, \quad t \geq 0, \tag{65}$$

as well as the processes

$$\kappa_t^\star = \exp\!\left(\int_0^t F_u(s, c_s^\star, U_s^\star)\, ds\right) \qquad \text{and} \qquad Y_t^\star = \kappa_t^\star\, U_x(t, X_t^\star), \qquad t \geq 0. \tag{66}$$

Then, the following assertions hold.

(i) Marginal utility dynamics and state price density. The process $\nu^\star$ takes values in $(\operatorname{Im} \sigma_t^\top)^\perp$, and

$$dU_x(t, X_t^\star) = -F_u(t, c_t^\star, U_t^\star)\, U_x(t, X_t^\star)\, dt + U_x(t, X_t^\star)\, (-\theta_t + \nu_t^\star)^\top dW_t, \quad t \geq 0. \tag{67}$$

Furthermore, for every $T > 0$,

$$\int_0^T \big(\big|F_u(t, c_t^\star, U_t^\star)\big| + |\nu_t^\star|^2\big)\, dt < \infty, \qquad \mathbb{P}\text{-a.s.} \tag{68}$$

Process $Y^\star$, defined in (66), is in $\mathcal{Y}$ and satisfies

$$dY_t^\star = Y_t^\star\, (-\theta_t + \nu_t^\star)^\top dW_t, \qquad Y_0^\star = U_x(0, x), \quad t \geq 0. \tag{69}$$

(ii) Equality in the variational dual bound. Suppose that Assumption 3.8 also holds and that the discount process

$$\alpha_t^\star = -F_u(t, c_t^\star, U_t^\star), \qquad t \geq 0,$$

is in $\mathcal{A}^{\mathrm{disc}}$. Let $D_t^\star = Y_t^\star/\kappa_t^\star = U_x(t, X_t^\star)$. Then, the pair $(D^\star, \alpha^\star)$ satisfies the feedback relation

$$\alpha_t^\star = \alpha^\star(t, D_t^\star), \qquad \text{for } dt \otimes d\mathbb{P}\text{-a.e. } (t, \omega),$$

and, consequently, (60) is satisfied along the optimum. This is a conclusion of the theorem and not an additional requirement. Finally, the process $\nu^\star$ satisfies (64), and the corresponding process $\mathcal{Z}^{\alpha^\star,\nu^\star}$ is a local $\mathcal{F}_t$-martingale.

Proof. The proof is given in Appendix B. □

In [4], where there is no aggregator, the marginal utility along the optimal wealth is itself a state price density. Recursive aggregation adds the finite variation term $-F_u U_x$ in (67). As a consequence, the following three processes are distinct and must be distinguished from each other.

- $U_x(t, X_t^\star)$ is the marginal utility along the optimal wealth.
- $\kappa^\star$ is the factor that removes the recursive drift of $U_x(\cdot, X^\star)$.
- $Y_t^\star = \kappa_t^\star U_x(t, X_t^\star)$ is the resulting state price density, with *dual control* $\nu_t^\star$.

The process $D^\star$ is a deflated dual state and should *not* be confused with the state price density.

There are two optimization problems here and they should *not* be conflated. On one hand, the dual forward recursive HJB SPDE selects, for each dual state, the feedback control ν^{dual} in (44), for which the dual value process is a local $\mathcal{F}_t$-martingale. On the other hand, the primal optimum leads to the state price density $Y^\star = \kappa^\star U_x(\cdot, X^\star)$. The two controls do not need to agree in a general nonhomothetic system, since ν^{dual} is evaluated along its own state process. They do agree when the feedback (44) is independent of its state argument, as in the homothetic Epstein–Zin class of Section 4, where the identification follows easily.

3.6 Admissibility and bounds on the model inputs

The admissibility of the optimal feedback controls was assumed in the verification Theorems 2.10 and 3.6. We now investigate its connection with explicit bounds on the model inputs. We recall that the starting point is El Karoui and Mrad [4, Theorem 4.2], where bounds on the orthogonal volatility characteristics connect the forward utility SPDE with two solvable SDEs. It was developed for investment and consumption by El Karoui, Hillairet and Mrad [25]. The orthogonal components of (A1) below reproduce the bounds of [4, Theorem 4.2]. Our first bound is imposed on the full vector δ_x and is, therefore, stronger than its counterpart therein. Its projected component is needed below to control the admissibility of the candidate portfolio process.

We now make the following important observation. In the absence of an aggregator, the marginal utility along the optimal wealth process is itself a state price density. Here Lemma 3.11 shows that $U_x(\cdot, X^\star)$ has an additional drift $-F_u U_x$. The two flows are therefore connected through the deflated process $D^\star$, while $Y^\star = \kappa^\star U_x(\cdot, X^\star)$ is recovered only after applying the recursive discount. This explains the two additional sets of bounds. Condition (A2) controls the admissibility of the consumption and portfolio, whereas (A3) controls the drift generated by the recursive discount. Neither condition is present in [4].

We distinguish the *portfolio volatility* $\sigma_t^{\pi,\star}(x) = \sigma_t^\top \pi^\star(t, x)$ from the *wealth volatility*

$$\sigma^\star(t, x) = x\, \sigma_t^\top \pi^\star(t, x) = -\frac{1}{U_{xx}(t, x)}\, \sigma_t^+ \sigma_t \big(U_x(t, x)\theta_t + \delta_x(t, x)\big), \qquad t \geq 0,\ x > 0. \tag{70}$$

The variational discount in feedback form on wealth is

$$\alpha^\star(t, x) = -F_u\big(t, c^\star(t, x), U(t, x)\big), \qquad t \geq 0,\ x > 0,$$

which is the map (59) evaluated at the marginal utility level $D = U_x(t, x)$. The deflated dual coefficients are

$$\mu^D(t, y) = \alpha^\star\big(t, I(t, y)\big)\, y, \qquad \text{and} \qquad \sigma^D(t, y) = -y\, \theta_t + (I_d - \sigma_t^+ \sigma_t)\, \delta_x\big(t, I(t, y)\big), \qquad t \geq 0,\ y > 0.$$

Furthermore, the deflated dual process solves

$$dD_t^\star = \mu^D(t, D_t^\star)\, dt + \sigma^D(t, D_t^\star)^\top\, dW_t, \qquad t \geq 0, \tag{71}$$

which is the conjugate form of the marginal utility SDE (67), since $D^\star = Y^\star/\kappa^\star = U_x(\cdot, X^\star)$.

Assumption 3.13. *Bounds on the model inputs.* There exist nonnegative adapted processes K^0, K^1, K^2, K^3, K^4 such that, for every $T > 0$,

$$\int_0^T \big(|\theta_t|^2 + (K_t^1)^2 + (K_t^2)^2\big)\,dt < \infty, \qquad \int_0^T (K_t^4)^2\big(|\theta_t| + K_t^1\big)^2\,dt < \infty, \qquad \int_0^T \big(K_t^0 + K_t^3\big)\,dt < \infty,$$

$\mathbb{P}$-a.s., and such that, for $dt \otimes d\mathbb{P}$-a.e. (t,ω) and every $x > 0$, the following inequalities hold:

(A1) *Volatility characteristics.*

$$|\delta_x(t,x)| \le K_t^1\,|U_x(t,x)| \qquad \text{and} \qquad |(I_d - \sigma_t^+\sigma_t)\,\delta_{xx}(t,x)| \le K_t^2\,|U_{xx}(t,x)|, \tag{72}$$

(A2) *Consumption and relative risk tolerance.*

$$c^\star(t,x) \le K_t^3\,x \qquad \text{and} \qquad \left|\frac{U_x(t,x)}{x\,U_{xx}(t,x)}\right| \le K_t^4, \tag{73}$$

(A3) *Recursive discount.* The map $x \mapsto \alpha^\star(t,x)$ is continuously differentiable and

$$|\alpha^\star(t,x)| + \left|\frac{U_x(t,x)}{U_{xx}(t,x)}\,\partial_x\alpha^\star(t,x)\right| \le K_t^0. \tag{74}$$

We note that no separate Lipschitz condition is imposed on either the deflated dual coefficients or the primal feedback coefficients. For the dual coefficients, the reason is the following. At $x = I(t,y)$, it holds that $U_x(t,x) = y$, and thus the first bound in (A1) gives the linear growth of σ^D in y. In turn, the bound on $\alpha^\star$ in (A3) gives the linear growth of μ^D. Differentiating in y and using $I_y(t,y) = 1/U_{xx}(t, I(t,y))$, the bounds (A1) on $(I_d - \sigma_t^+\sigma_t)\delta_{xx}$ and (A3) on $\partial_x\alpha^\star$ give a Lipschitz bound in y on every finite horizon, in analogy with [4, Theorem 4.2(i)]. The deflated dual equation (71) has a unique strong solution, and its strict positivity is a consequence of the multiplicative form that the equation takes after dividing the coefficients by y. This is presented in detail in Step 1 of the proof.

For the optimal wealth SDE, uniqueness is likewise established rather than assumed. Step 3 of the proof shows that *every* strictly positive solution of the feedback wealth equation satisfies the same marginal flow identity (77), and therefore coincides with the process constructed from the dual flow through the inverse marginal field. It is this argument, rather than a Lipschitz estimate on $\sigma^\star$ and $c^\star$, that yields pathwise uniqueness.

The first bound in (A1) concerns the full gradient volatility and controls both projections at once: the orthogonal part $(I_d - \sigma_t^+\sigma_t)\delta_x$ drives the dual diffusion σ^D, while the projected part $\sigma_t^+\sigma_t\delta_x$ enters the optimal portfolio through (70). The bound (73) on $|U_x/(xU_{xx})|$ controls the relative risk tolerance and converts the wealth volatility coefficient bound on $\sigma_t^\star$ into a portfolio volatility coefficient bound on $\sigma_t^{\pi,\star}(x) = \sigma^\star(t,x)/x$, which is what Definition 2.1 requires.

Before stating the result, we introduce the two families of solutions that the theorem below connects, namely, the family of solutions of the deflated dual equation (71), indexed by their initial dual value $y > 0$, and the family of solutions of the optimal wealth equation, indexed by their initial wealth $x > 0$,

$$y \longmapsto D_t^\star(y) \;\text{ and }\; x \longmapsto X_t^\star(x), \qquad t \ge 0.$$

Theorem 3.14 (Admissibility and primal–dual consistency). Suppose that (U, F) is a normalized forward aggregator system satisfying Assumptions 2.9, 3.2 and 3.12 and the nondegeneracy condition (28), and that U satisfies the normalized forward recursive HJB SPDE (27). Then, the following assertions hold.

(1) *Connection between the two solution families.* For every $y > 0$, the deflated dual SDE (71) has a unique strictly positive strong solution $D^\star(y)$ with initial value y, defined and finite for all $t \ge 0$, and nondecreasing in its initial value. For $x > 0$, the process

$$X_t^\star(x) = I\big(t, D_t^\star(U_x(0,x))\big), \qquad t \ge 0, \tag{75}$$

is the unique strictly positive strong solution, defined and finite for all $t \geq 0$, of

$$dX_t^\star = \big(\sigma^\star(t, X_t^\star)^\top \theta_t - c^\star(t, X_t^\star)\big)\, dt + \sigma^\star(t, X_t^\star)^\top dW_t, \qquad t \geq 0, \qquad X_0^\star = x, \tag{76}$$

and it is nondecreasing in x. The two solution families satisfy the identity

$$U_x\big(t, X_t^\star(x)\big) = D_t^\star\big(U_x(0, x)\big), \qquad x > 0,\ t \geq 0. \tag{77}$$

(2) *Primal consistency.* For every $T > 0$,

$$\int_0^T \big(c^\star(t, X_t^\star) + |\sigma_t^{\pi,\star}(X_t^\star)|^2\big)\, dt < \infty \qquad \mathbb{P}\text{-a.s.}$$

Hence, the feedback control pair $\big(\pi^\star(t, X_t^\star), c^\star(t, X_t^\star)\big)_{t \geq 0}$ is in $\mathcal{A}$ and the process $Y^\star = \kappa^\star U_x(\cdot, X^\star)$ is in $\mathcal{Y}$. Furthermore, by Theorem 2.10, (U, F) is a consistent normalized forward recursive aggregator system and this control pair is optimal.

(3) *Dual consistency.* The conjugate field $\widetilde{U}$ satisfies the dual forward recursive HJB SPDE (41). For every $y > 0$, the equation

$$dY_t = Y_t\big(-\theta_t + \nu_t^{\mathrm{dual}}(Y_t)\big)^\top dW_t, \qquad Y_0 = y, \qquad t \geq 0,$$

has a unique strictly positive strong solution, and the resulting dual control $\nu^{\mathrm{dual}}(Y)$ is also in $\mathcal{A}^{\mathrm{dual}}$. Hence, $(\widetilde{U}, \widetilde{F})$ is dual consistent on $\mathcal{Y}$.

(4) *The two Fenchel equalities.* Suppose that Assumption 3.8 also holds and that $\alpha_t^\star = -F_u(t, c_t^\star, U_t^\star)$, $t \geq 0$, is in $\mathcal{A}^{\mathrm{disc}}$. Then, along the optimal control pair, we have

$$Y_t^\star = \kappa_t^\star U_x(t, X_t^\star), \qquad t \geq 0, \tag{78}$$

and we also have the equalities (55) and (56).

Proof. The proof is given in Appendix B. □

Corollary 3.15. Under the assumptions of Theorem 3.14(4), the bound (54) applies to every control pair in $\mathcal{A}$ and every density in $\mathcal{Y}$ forming an integrable triple with $\alpha^\star$, and it is attained at $\big((\pi^\star, c^\star), Y^\star\big)$.

Proof. The assertion follows easily from Proposition 3.9 and Theorem 3.14(4). □

We stress that Theorem 3.14 provides a sufficient construction. The identity (77) is necessary for every sufficiently smooth admissible optimum, but no general necessity claim is made for Assumption 3.13.

4 Forward Epstein–Zin recursive aggregator systems

This section develops the homothetic Epstein–Zin class. We start with the linear Uzawa benchmark and subsequently turn to the separable Epstein–Zin construction, their viability condition and optimal controls. The state price density and conjugate equation follow next, before the Merton and Heston examples. The notation would only be introduced when it is first used. Throughout the section the letter u is reserved for the deterministic wealth utility $u(x)$ of the multiplicative forms below, and the letter y for the dual state, except in the generic driver notation of Remark 4.21. Thus, we denote the continuation-utility argument in F and $\widetilde{F}$ by v.

Forward Uzawa utilities

Let $\beta : [0, \infty) \to (0, \infty)$ be a deterministic locally integrable function satisfying $\int_0^T \beta_t\, dt < \infty$, for each $T > 0$, and introduce the normalized aggregator on $\mathcal{U} = \mathbb{R}$,

$$F(t, c, v) = \frac{c^{1-\gamma}}{1-\gamma} - \beta_t\, v, \quad c > 0,\ v \in \mathbb{R},\ \gamma > 0,\ \gamma \neq 1.$$

Its Fenchel transform is

$$\widetilde{F}(t,d,v)=\frac{\gamma}{1-\gamma}\,d^{\,1-\frac{1}{\gamma}}-\beta_t\,v,\quad d>0,\ v\in\mathbb{R},$$

and the optimal feedback consumption control is $c^\star(t,x)=(U_x(t,x))^{-1/\gamma}$, $t\geq 0,\ x>0$.

In turn, the normalized forward recursive HJB SPDE (27) takes the form

$$\begin{aligned} dU(t,x)=\Big(&\frac{|U_x(t,x)\theta_t+\sigma_t^+\sigma_t\delta_x(t,x)|^2}{2\,U_{xx}(t,x)}-\frac{\gamma}{1-\gamma}\,(U_x(t,x))^{1-\frac{1}{\gamma}}+\beta_t\,U(t,x)\Big)dt\\ &+\delta(t,x)^\top dW_t,\quad t\geq 0,\ x>0, \end{aligned}\tag{79}$$

for a suitable initial condition $U(0,x)=u_0(x)$, $x>0$.

The corresponding dual forward SPDE, obtained by substituting $\widetilde{F}(t,d,v)$ above in (41), becomes

$$\begin{aligned} d\widetilde{U}(t,y)=\Bigg(&-\frac{\gamma}{1-\gamma}\,y^{\,1-\frac{1}{\gamma}}+\beta_t\big(\widetilde{U}(t,y)-y\,\widetilde{U}_y(t,y)\big)-\tfrac{1}{2}\,y^2\,\widetilde{U}_{yy}(t,y)\,|\theta_t|^2\\ &+y\,\theta_t^\top\widetilde{\delta}_y(t,y)+\frac{|(I_d-\sigma_t^+\sigma_t)\,\widetilde{\delta}_y(t,y)|^2}{2\,\widetilde{U}_{yy}(t,y)}\Bigg)\,dt+\widetilde{\delta}(t,y)^\top dW_t,\quad t\geq 0,\quad y>0, \end{aligned}\tag{80}$$

with $\widetilde{U}(0,y)=\widetilde{u}_0(y)$, $y>0$, the convex conjugate of the initial field u_0 of (15). We note that the discount model input β enters (80) linearly, through the term $\beta_t(\widetilde{U}-y\,\widetilde{U}_y)$. However, SPDE (80) *remains nonlinear* through the unspanned volatility term $|(I_d-\sigma_t^+\sigma_t)\widetilde{\delta}_y|^2/(2\widetilde{U}_{yy})$. It *becomes linear* in the non volatile case considered next.

In the special case of zero recursive volatility, $\delta(t,x)\equiv 0$, $t\geq 0$, $x>0$, SPDE (79) reduces to the PDE with random coefficients,

$$U_t(t,x)-\frac{U_x(t,x)^2|\theta_t|^2}{2\,U_{xx}(t,x)}+\frac{\gamma}{1-\gamma}\,(U_x(t,x))^{1-\frac{1}{\gamma}}-\beta_t\,U(t,x)=0,\qquad t\geq 0,\ x>0.\tag{81}$$

Since the aggregator is affine in y, the recursive criterion reduces to a discounted additive one. We note that β being deterministic is not needed for the Bellman SPDE (79), but it is needed for the explicit non volatile representation below. Indeed, if β were merely progressively measurable, the heat kernel formula would involve its future values which, however, may not be $\mathcal{F}_t$-measurable. Taking conditional expectations would restore the proper measurability but would, in general, introduce a martingale component.

When $\delta\equiv 0$, the model falls within the class of forward investment and consumption criteria studied by Källblad [10]. Its convex dual satisfies a linear equation which, after a suitable time rescaling, takes the form of the ill-posed inhomogeneous heat equation used below. The resulting criterion decomposes into an infinite horizon Merton type consumption component and a pure forward investment component. We stress that this linear structure does not extend to the forward Epstein–Zin recursive case, whose dual equation remains nonlinear through the term $\widetilde{U}-y\widetilde{U}_y$.

The convex conjugate

Equation (81) is quasilinear in U, but, as seen in [10], can be linearized by passing to the dual domain. Using the conjugacy identities (36), the non volatile dual field satisfies $d\widetilde{U}(t,y)=\widetilde{U}_t(t,y)\,dt$. Thus, SPDE (80) with $\widetilde{\delta}(t,y)\equiv 0$, $t\geq 0$, $y>0$, becomes

$$\widetilde{U}_t(t,y)+\frac{|\theta_t|^2}{2}\,y^2\,\widetilde{U}_{yy}(t,y)-\beta_t\big(\widetilde{U}(t,y)-y\,\widetilde{U}_y(t,y)\big)+\tfrac{\gamma}{1-\gamma}y^{\,1-\frac{1}{\gamma}}=0,\qquad t\geq 0,\ y>0,\tag{82}$$

which is the analogue of equation (18) in [10], but incorporating a discount factor. We stress that equation (82) is *linear* in $\widetilde{U}$.

Next, using a suitable transformation, we convert the problem to the one studied in [10]. Specifically, let

$$V(t,x)=e^{-\int_0^t\beta_s\,ds}\,U(t,x),\qquad t\geq 0,\ x>0.\tag{83}$$

From (81), it follows directly that V solves the random HJB equation in [10] *without* zeroth order term, namely,

$$V_t(t,x) - \frac{|\theta_t|^2}{2}\frac{V_x(t,x)^2}{V_{xx}(t,x)} + \frac{\gamma}{1-\gamma}e^{-\frac{1}{\gamma}\int_0^t \beta_s\,ds}V_x(t,x)^{1-\frac{1}{\gamma}} = 0, \qquad t \geq 0,\ x > 0.$$

Then, for the convex conjugate, we deduce that

$$\widetilde{U}(t,y) = e^{\int_0^t \beta_s\,ds}\widetilde{V}\big(t, e^{-\int_0^t \beta_s\,ds}y\big), \qquad t \geq 0,\ y > 0.$$

The field $\widetilde{V}$ satisfies the same linear dual equation as in [10] with a source depending on (t,y) only. Specifically,

$$\widetilde{V}_t(t,y) + \frac{|\theta_t|^2}{2}\,y^2\,\widetilde{V}_{yy}(t,y) + \frac{\gamma e^{-\frac{1}{\gamma}\int_0^t \beta_s\,ds}}{1-\gamma}y^{1-\frac{1}{\gamma}} = 0, \qquad t \geq 0,\ y > 0; \tag{84}$$

see equation (18) in [10].

Time rescaling and reduction to the ill-posed inhomogeneous heat equation

Following the arguments developed in [10], we introduce the time change,

$$M_t = \int_0^t \theta_s^\top dW_s \quad \text{and} \quad \Theta_t = \langle M\rangle_t = \int_0^t |\theta_s|^2\,ds, \qquad t \geq 0. \tag{85}$$

In this section, we work under the following standing assumption.

Assumption 4.1. There exists a deterministic, locally integrable function $a : [0,\infty) \to (0,\infty)$ such that

$$|\theta_t|^2 = a(t), \qquad \text{and} \qquad \Theta_t = \int_0^t a(s)\,ds \uparrow \infty, \qquad \text{as } t \uparrow \infty. \tag{86}$$

In other words, the "traded clock" is deterministic and strictly increasing, although the direction of θ_t may only remain progressively measurable. This restriction, together with β being deterministic, ensures that the future convolution product displayed below does not make the random coefficients anticipative. In turn, in analogy to the Definition 3.4 of [10], we introduce two auxiliary functions $H, H_c : \mathbb{R}\times[0,\infty) \to (0,\infty)$ defined analogously through the marginals

$$V_x\big(t, H(y,t)\big) = \exp\Big(-y + \tfrac{1}{2}\Theta_t\Big), \qquad \text{and} \qquad u^c_x\big(t, H_c(y,t)\big) = \exp\Big(-y + \tfrac{1}{2}\Theta_t\Big), \tag{87}$$

where $u^c(t,c) = e^{-\int_0^t \beta_s\,ds}\,c^{1-\gamma}/(1-\gamma)$ is the discounted felicity utility. We also write $H(y,t) = h(y,\Theta_t)$ and $H_c(y,t) = |\theta_t|^2 h_c(y,\Theta_t)$, which defines the pair (h,h_c) on the traded clock through the inverse of Θ, $T(\tau) = \Theta^{-1}_\tau$. In turn, the inhomogeneous term generated by the consumption part of the criterion is explicitly given by

$$h_c(y,\tau) = \frac{1}{a(T(\tau))}\exp\left(\frac{y - \tau/2 - \int_0^{T(\tau)}\beta_s\,ds}{\gamma}\right), \qquad y \in \mathbb{R},\ \tau \geq 0. \tag{88}$$

In particular, h_c is deterministic under the above assumptions. Transformation (87) turns the operator $\frac{1}{2}|\theta_t|^2y^2\partial_{yy}$ of (84) into a constant coefficient Laplacian, with the shift $\frac{1}{2}\Theta_t$ absorbing the accompanying first order drift. Working as in Lemma 3.5 in [10], (H,H_c) satisfy $H_t + \frac{1}{2}|\theta_t|^2H_{yy} + H_c = 0$. We obtain that the time changed pair (h,h_c) satisfies the *constant coefficient* ill-posed inhomogeneous heat equation

$$h_\tau(y,\tau) + \tfrac{1}{2}\,h_{yy}(y,\tau) + h_c(y,\tau) = 0, \qquad \tau \geq 0,\ y \in \mathbb{R}, \tag{89}$$

where $\tau = \Theta_t$ is the traded time scale and the subscript τ denotes differentiation in *that* clock. It is the transformation $H(y,t) = h(y,\Theta_t)$, $\partial_t H = |\theta_t|^2 h_\tau$, that removes the market coefficient $|\theta_t|^2$, yielding a diffusion coefficient equal to $\frac{1}{2}$. Because the Uzawa discount was absorbed into the inhomogeneous term u^c in (83), it appears only through h_c and (89) does not contain the zeroth order term. This is the ill-posed equation (20) in [10].

Heat kernel convolution and Widder representation

Let $k(\tau,z)=(2\pi\tau)^{-1/2}\exp(-z^2/2\tau)$, $\tau\geq 0$, $z\in\mathbb{R}$, be the fundamental solution of (89) in the homogeneous case, with the convention $k(z)=k(1,z)$, and let

$$(k\star h_c)(y,\tau)=\int_\tau^\infty\int_{-\infty}^\infty k(s-\tau,z-y)\,h_c(z,s)\,dz\,ds$$

denote its space-time convolution with the inhomogeneous consumption term. Under Assumption 4.1 and the standing assumption that β is deterministic, the function h_c is the deterministic function in (88). Assuming that $k\star h_c$ is finite, we choose a deterministic positive Borel measure ϖ on $[0,\infty)$ satisfying

$$\int_0^\infty e^{ry}\varpi(dr)<\infty,\qquad\text{for each } y\in\mathbb{R}.$$

By Theorem 3.7 and Corollary 3.8 in [10], the time-changed dual field admits the representation

$$h(y,\tau)=(k\star h_c)(y,\tau)+h^\varpi(y,\tau),\qquad\text{with } h^\varpi(y,\tau)\text{ equal to }\quad\int_0^\infty e^{\,ry-\frac{1}{2}r^2\tau}\,\varpi(dr),\tag{90}$$

with the term h^ϖ being the Widder positive measure representation of the positive solutions of the homogeneous ill-posed heat equation. We stress that h is the time-changed inverse marginal field, and not the dual utility itself. The dual utility is recovered by integrating the identity $-\widetilde{V}_y=I$. To make this integration valid also when ϖ has mass at $r=1$, we introduce

$$K_r(t,y)=\begin{cases}\dfrac{y^{1-r}\exp\big(\frac{1}{2}(r-r^2)\Theta_t\big)-1}{r-1}, & r\neq 1,\\ -\log y-\frac{1}{2}\Theta_t, & r=1.\end{cases}$$

Then, direct differentiation gives

$$-\partial_y K_r(t,y)=y^{-r}\exp\big(\tfrac{1}{2}(r-r^2)\Theta_t\big).\tag{91}$$

Next, we assume in addition that

$$J_t=\int_t^\infty\exp\bigg(-\frac{1}{\gamma}\int_0^s\beta_u\,du-\frac{\gamma-1}{2\gamma^2}(\Theta_s-\Theta_t)\bigg)\,ds<\infty,\qquad\text{for each } t\geq 0.\tag{92}$$

Choosing K_r fixes the integration constant in the dual variable. We easily deduce that the discounted dual field $\widetilde{V}$ decomposes as

$$\begin{aligned}\widetilde{V}(t,y)=&\int_t^\infty e^{-\frac{1}{\gamma}\int_0^s\beta_u\,du}\int_{-\infty}^\infty\widetilde{f}\Big(y\,e^{\sqrt{\Theta_s-\Theta_t}\,z-\frac{1}{2}(\Theta_s-\Theta_t)}\Big)k(z)\,dz\,ds\\&+\int_0^\infty K_r(t,y)\,\varpi(dr),\end{aligned}\tag{93}$$

where

$$\widetilde{f}(d)=\frac{\gamma}{1-\gamma}d^{1-\frac{1}{\gamma}},\qquad d>0,$$

is the Fenchel transform of $c^{1-\gamma}/(1-\gamma)$. The first term in (93) equals $\widetilde{f}(y)J_t$ and is therefore finite under (92). The original dual field is then recovered from

$$\widetilde{U}(t,y)=e^{\int_0^t\beta_s\,ds}\widetilde{V}\left(t,e^{-\int_0^t\beta_s\,ds}y\right).$$

This is the analogue of [10, formula (24)]. The definition of K_r fixes the sign imposed by the conjugacy convention (35) and also covers the case $r=1$ by continuity.

Under Assumption 4.1 and the determinism of β, the representation is adapted to the underlying filtration $(\mathcal{F}_t)_{t\geq 0}$. Its first term is the dual value of the infinite horizon Merton type consumption problem, whereas the Widder term corresponds to a pure forward investment criterion determined by the measure ϖ. Thus, the criterion separates into a consumption component and a forward investment component.

We note that β being a deterministic process is only a sufficient condition for the appropriate measurability requirements. More generally, both h_c and $h=k\star h_c+h^\varpi$ must be adapted to the time-changed filtration. Progressive measurability of β alone is insufficient because the convolution $k\star h_c$ depends on future values of the source. The regularity, range, and martingale conditions of [10, Theorem 3.13] are also required.

Forward Epstein–Zin recursive aggregators

We now introduce one of the main contributions in our work, which is the construction and detailed study of the forward analogue of the celebrated Epstein–Zin recursive utility.

Let (γ, η, λ) satisfy

$$\lambda = \frac{1-\gamma}{1-\frac{1}{\eta}}, \quad \text{with } \gamma > 0, \gamma \neq 1, \eta > 0, \eta \neq 1, \tag{94}$$

and set $\mathcal{U}_\gamma = (1-\gamma)\mathbb{R}_+^* = \{v \in \mathbb{R} : (1-\gamma)v > 0\}$, the range on which the powers $((1-\gamma)v)^{1-1/\lambda}$ are defined in $\mathbb{R}$. In analogy to Epstein and Zin [24], and its continuous time formulation in [2], we introduce the *normalized Epstein–Zin recursive aggregator*

$$f(c,v) = \frac{1}{1-\frac{1}{\eta}}\Big(c^{1-\frac{1}{\eta}}\,((1-\gamma)v)^{1-\frac{1}{\lambda}} - (1-\gamma)v\Big), \quad c > 0,\ v \in \mathcal{U}_\gamma. \tag{95}$$

The model parameters γ and η play distinct economic roles. Coefficient γ is the relative risk aversion of the homothetic wealth index, whereas η is the *elasticity of intertemporal substitution* (EIS). We state that the composite parameter λ satisfies

$$\frac{\eta}{\lambda} = \frac{\eta-1}{1-\gamma} \qquad \text{and} \qquad \lambda - 1 = \frac{1-\gamma\eta}{\eta-1}.$$

Thus, $\lambda = 1$ if and only if $\eta = 1/\gamma$. This is the additive CRRA boundary, at which the EIS is constrained to be the reciprocal of risk aversion. At this boundary,

$$f(c,v) = \frac{c^{1-\gamma}}{1-\gamma} - v,$$

so the aggregator becomes additive with respect to both the felicity and continuation utility. Away from this boundary, Epstein–Zin recursion permits attitudes toward intertemporal substitution and risk to vary independently (see e.g. [27, 24, 2]).

Next, we let $\Psi > 0$ be an $\mathcal{F}_t$-progressively measurable process and define

$$F(t,c,v) = \Psi_t\, f(c,v), \quad t \geq 0,\ c > 0,\ v \in \mathcal{U}_\gamma.$$

with f as in (95).

We note that in the classical Epstein–Zin case, the multiplier Ψ is a constant, whereas here it is allowed to be a progressively measurable modeling input. We keep the present notation, as all subsequent calculations rely on it.

The Fenchel transform of F is

$$\widetilde{F}(t,d,v) = \frac{d^{\,1-\eta}}{\eta-1}\,((1-\gamma)v)^{\,\eta\left(1-\frac{1}{\lambda}\right)}\,\Psi_t^\eta - \lambda\Psi_t\, v, \qquad t \geq 0,\ d > 0,\ v \in \mathcal{U}_\gamma, \tag{96}$$

and, in turn, the candidate optimal feedback consumption takes the form

$$c^\star(t,x) = \big((1-\gamma)U(t,x)\big)^{\frac{1-\eta\gamma}{1-\gamma}}\left(\frac{\Psi_t}{U_x(t,x)}\right)^\eta, \qquad t \geq 0,\ x > 0.$$

Combining the above with the normalized Bellman SPDE (27), we obtain the (normalized) *forward Epstein–Zin Bellman SPDE*,

$$\begin{aligned} dU(t,x) = \Bigg(&\frac{|U_x(t,x)\theta_t + \sigma_t^+\sigma_t\delta_x(t,x)|^2}{2\,U_{xx}(t,x)} - \frac{\Psi_t^\eta}{\eta-1}\,U_x(t,x)^{1-\eta}\big((1-\gamma)U(t,x)\big)^{\eta\left(1-\frac{1}{\lambda}\right)} \\ &+\lambda\,\Psi_t\,U(t,x)\Bigg)\,dt + \delta(t,x)^\top dW_t, \quad t \geq 0,\ x > 0, \end{aligned} \tag{97}$$

for a suitable initial condition $U(0,x) = u_0(x)$, $x > 0$.

Using (96) and (41), we then obtain the *dual forward Epstein–Zin recursive SPDE*. Here $\hat{u}(t,y)$ from (39) is the primal utility expressed in the dual state. We have

$$d\widetilde{U}(t,y) = \Bigg(-\frac{\Psi_t^\eta}{\eta-1}\, y^{1-\eta}\big((1-\gamma)\hat{u}(t,y)\big)^{\eta\left(1-\frac{1}{\lambda}\right)} + \lambda\,\Psi_t\,\hat{u}(t,y) - \tfrac{1}{2}\,y^2\,\widetilde{U}_{yy}(t,y)\,|\theta_t|^2 + y\,\theta_t^\top\widetilde{\delta}_y(t,y) \\ + \frac{|(I_d - \sigma_t^+\sigma_t)\,\widetilde{\delta}_y(t,y)|^2}{2\,\widetilde{U}_{yy}(t,y)} \Bigg) dt + \widetilde{\delta}(t,y)^\top dW_t, \quad t \geq 0,\ y > 0, \tag{98}$$

with $\widetilde{U}(0,y) = \widetilde{u}_0(y)$, $y > 0$, the convex conjugate of u_0.

Unlike the primal Bellman SPDE (97), the dual equation contains a nonlinear dependence on $\hat{u}(t,y)$ and, therefore, couples $\widetilde{U}$ with $\widetilde{U}_y$ in a nonlinear way. Because of this coupling, a direct linearization of the dual problem is not, in general, possible. In the next section, we circumvent this difficulty by working with a special but still rather general class which allows for closed form solution for the dual problem.

The forward separable recursive specification below provides a different reduction. In turn, factoring out the wealth dependence turns the forward Epstein–Zin SPDE into a scalar Bernoulli type equation for Φ. Proposition 4.2 shows that the power transformation $\Xi = \Phi^{\eta/\lambda}$ converts this equation into a linear SDE. Thus, we make the important observation that conjugacy linearizes the non-volatile Uzawa problem, whereas homothetic separation and a power transformation linearize the separable forward Epstein–Zin problem.

4.1 Separable forward Epstein–Zin recursive aggregator systems

We consider forward recursive aggregator pairs (U,F) of both the multiplicative and separable form

$$U(t,x) = \Phi_t u(x) \ \text{ and } \ F(t,c,v) = \Psi_t\, f(c,v), \quad t \geq 0,\ x > 0,\ c > 0,\ v \in \mathcal{U}_\gamma, \tag{99}$$

with

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0,\ \gamma > 0, \gamma \neq 1,$$

where Φ is a strictly positive Itô process and Ψ is a strictly positive $\mathcal{F}_t$-progressively measurable process. We impose no semimartingale dynamics on Ψ. These processes constitute a modeling input as follows. Specifically, we allow Ψ to be general under mild integrability conditions. For process Φ, its volatility process δ^Φ is a modeling input while, as we show, its drift is specified via the martingale and supermartingale conditions in the definition of forward recursive aggregators.

Because of the homothetic structure, the optimal feedback policies depend on wealth only through the relative consumption

$$\check{c}_t = \frac{c_t}{X_t^{\pi,c}}, \quad t \geq 0,$$

the fraction of wealth consumed per unit time. We, therefore, work with this quantity throughout the analysis of the separable case.

We first construct Φ explicitly from Ψ and δ^Φ, and then characterize consistency and determine the optimal policies. Corollary 4.8 verifies the conditions needed to apply the results of Section 3.

4.1.1 The processes Φ and Ψ

Let δ^Φ be a progressively measurable process, modeling the volatility of process Φ. Let $\varphi > 0$ be an initial multiplier (initial condition of Φ), set

$$m = \frac{\eta}{\lambda},$$

and define the auxiliary processes

$$\chi_t = \eta\Psi_t + \frac{1-\eta}{2\gamma}\left|\theta_t + \sigma_t^+\sigma_t\delta_t^\Phi\right|^2 + \frac{m(m-1)}{2}|\delta_t^\Phi|^2, \qquad t \geq 0, \tag{100}$$

and

$$\mathcal{E}_t = \exp\!\Bigg(\int_0^t \left(\eta\Psi_s + \frac{1-\eta}{2\gamma}\left|\theta_s + \sigma_s^+\sigma_s\delta_s^\Phi\right|^2 - \frac{\eta}{2\lambda}\,|\delta_s^\Phi|^2\right) ds + \frac{\eta}{\lambda}\int_0^t (\delta_s^\Phi)^\top dW_s\Bigg), \qquad t \geq 0. \tag{101}$$

We impose the following assumptions:

(V1) for each $T > 0$, $\int_0^T \Psi_s^\eta \, ds < \infty$ $\mathbb{P}$-a.s.;

(V2) for each $T > 0$, $\int_0^T (\Psi_s + |\delta_s^\Phi|^2) ds < \infty$ $\mathbb{P}$-a.s.;

(V3) it holds that

$$\mathbb{P}\left(\varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta \mathcal{E}_u^{-1} \, du > 0 \text{ for every } t \geq 0\right) = 1. \tag{102}$$

Under (V2) and the standing assumption (11), the exponent in (101) is finite on every finite horizon, so $\mathcal{E}$ is well defined and strictly positive. We note that no Novikov type condition is needed because $\mathcal{E}$ is used only as an integrating factor, and not as a martingale density. Assumption (V1) makes the integral in (V3) finite on every interval $[0, T]$, $\mathbb{P}$-a.s., and (V3) is the global positivity condition needed for the construction below.

Let process Φ admit the Itô decomposition

$$d\Phi_t = \Phi_t \big(b_t^\Phi \, dt + (\delta_t^\Phi)^\top dW_t\big), \qquad \Phi_0 = \varphi > 0, \tag{103}$$

with volatility δ^Φ being the modeling input presented earlier, and with drift process b_t^Φ given by

$$b_t^\Phi = \frac{1-\gamma}{1-\eta} \frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}} - \frac{1-\gamma}{2\gamma} \left|\theta_t + \sigma_t^+ \sigma_t \delta_t^\Phi\right|^2 + \lambda \Psi_t, \quad t \geq 0, \tag{104}$$

Then, the Itô decomposition of Φ is

$$d\Phi_t = \Phi_t \Big(\frac{1-\gamma}{1-\eta} \frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}} - \frac{1-\gamma}{2\gamma} \left|\theta_t + \sigma_t^+ \sigma_t \delta_t^\Phi\right|^2 + \lambda \Psi_t\Big) dt + \Phi_t (\delta_t^\Phi)^\top dW_t, \quad \Phi_0 = \varphi > 0, \qquad t \geq 0. \tag{105}$$

As anticipated, the form of drift in (104) is not postulated but derived. Theorem 4.4 shows that it is exactly the condition under which the separable pair (U, F) constitutes a forward recursive aggregator system.

The next result linearizes (105) and gives Φ in closed form.

Proposition 4.2. Let Φ solve (105) and set $\Xi_t = \Phi_t^m$, $t \geq 0$. Then,
i) if Φ is a strictly positive solution of (105), then, with $m = \eta/\lambda$ as above, Ξ is a strictly positive solution of the linear SDE

$$d\Xi_t = \Big(- \Psi_t^\eta + \chi_t \, \Xi_t\Big) dt + \frac{\eta}{\lambda} \, \Xi_t (\delta_t^\Phi)^\top dW_t, \quad t \geq 0, \tag{106}$$

with $\Xi_0 = \varphi^m$ and χ given by (100).

Conversely, a solution Ξ of (106) produces the solution $\Phi = \Xi^{1/m}$ of (105) only on a stochastic interval on which Ξ is strictly positive.

ii) with $\mathcal{E}$ as in (101), the process Φ is given by

$$\Phi_t = \left(\mathcal{E}_t \left(\varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta \, \mathcal{E}_u^{-1} \, du\right)\right)^{\lambda/\eta}, \qquad 0 \leq t < \tau, \tag{107}$$

where

$$\tau = \inf\left\{t \geq 0 : \varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta \mathcal{E}_u^{-1} \, du \leq 0\right\}. \tag{108}$$

Proof. We first derive the SDE satisfied by Ξ, and, in turn, solve it explicitly.
Step 1. Derivation of SDE for Ξ. Recalling that $m = \eta/\lambda$ and $\Xi_t = \Phi_t^m$, Itô's formula gives

$$\begin{aligned} d\Xi_t &= m \Phi_t^{m-1} \, d\Phi_t + \frac{m(m-1)}{2} \Phi_t^{m-2} \, d\langle \Phi \rangle_t \\ &= m \Xi_t \frac{d\Phi_t}{\Phi_t} + \frac{m(m-1)}{2} \Xi_t |\delta_t^\Phi|^2 \, dt. \end{aligned} \tag{109}$$

From (105),

$$\frac{d\Phi_t}{\Phi_t} = \left(\frac{1-\gamma}{1-\eta} \Psi_t^\eta \Phi_t^{-m} - \frac{1-\gamma}{2\gamma} |\theta_t + \sigma_t^+ \sigma_t \delta_t^\Phi|^2 + \lambda \Psi_t\right) dt + (\delta_t^\Phi)^\top dW_t,$$

which in view of (109) yields

$$
\begin{aligned}
d\Xi_t &= m\frac{1-\gamma}{1-\eta}\Psi_t^{\eta}\Xi_t\Phi_t^{-m}\,dt \\
&\quad + \Xi_t\left(-m\frac{1-\gamma}{2\gamma}|\theta_t+\sigma_t^{+}\sigma_t\delta_t^{\Phi}|^2 + m\lambda\Psi_t + \frac{m(m-1)}{2}|\delta_t^{\Phi}|^2\right)dt \\
&\quad + m\Xi_t(\delta_t^{\Phi})^{\top}dW_t.
\end{aligned}
\tag{110}
$$

Using that $\Xi_t\Phi_t^{-m}=1$ and the identities

$$
m\frac{1-\gamma}{1-\eta}=-1, \qquad m\lambda=\eta, \qquad -m(1-\gamma)=1-\eta,
$$

(110) gives

$$
d\Xi_t = \big(-\Psi_t^{\eta}+\chi_t\Xi_t\big)dt + \frac{\eta}{\lambda}\Xi_t(\delta_t^{\Phi})^{\top}dW_t,
$$

where χ_t is as in (100). This proves (106). The converse direction follows easily, provided Ξ remains strictly positive.
Step 2. Explicit solution of the linear equation. The process $\mathcal{E}$ in (101) satisfies

$$
\frac{d\mathcal{E}_t}{\mathcal{E}_t} = \chi_t\,dt + m(\delta_t^{\Phi})^{\top}dW_t, \qquad \mathcal{E}_0=1,
$$

and, therefore,

$$
d\mathcal{E}_t^{-1} = \mathcal{E}_t^{-1}\left((-\chi_t+m^2|\delta_t^{\Phi}|^2)dt - m(\delta_t^{\Phi})^{\top}dW_t\right).
$$

Thus,

$$
\begin{aligned}
d(\mathcal{E}_t^{-1}\Xi_t) &= \mathcal{E}_t^{-1}d\Xi_t + \Xi_t d\mathcal{E}_t^{-1} + d\langle\mathcal{E}^{-1},\Xi\rangle_t \\
&= -\mathcal{E}_t^{-1}\Psi_t^{\eta}\,dt.
\end{aligned}
$$

Using that $\Xi_0=\varphi^{\eta/\lambda}$, we derive

$$
\Xi_t = \mathcal{E}_t\left(\varphi^{\eta/\lambda} - \int_0^t \Psi_u^{\eta}\mathcal{E}_u^{-1}\,du\right).
$$

Finally, since $\Xi_t>0$ for all $0\le t<\tau$, we conclude that

$$
\Phi_t = \Xi_t^{\lambda/\eta}, \qquad 0\le t<\tau, \quad \mathbb{P}\text{-a.s.},
$$

which yields (107). □

We note that the squared norms in (104) and (106) have distinct origins. The first comes from the Hamiltonian and uses the projected component $\theta_t+\sigma_t^{+}\sigma_t\delta_t^{\Phi}=\sigma_t^{+}\sigma_t(\theta_t+\delta_t^{\Phi})$, by (12). The second is the quadratic-variation correction from Itô's formula applied to $\Xi=\Phi^m$ and uses $|\delta_t^{\Phi}|^2$.

Under (V1)–(V3), the process Φ given by (107) is well defined and strictly positive for all $t\ge 0$. Hence, $U(t,\cdot)$ takes values in $\mathcal{U}_\gamma$, and the pair (Φ,Ψ) is globally viable.

4.1.2 Viability and the classical infinite horizon parameter regimes

The fact that, in the forward setting, the resolution of the problem is treated for every $\lambda\neq 0$ deserves some clarification. In the classical infinite horizon Epstein–Zin case, different ranges of the composite parameter λ require different notions of solution and different techniques. These difficulties do not disappear in the forward formulation. Rather, they reappear as a global viability requirement, namely condition (V3).

In this subsection we keep assumptions (V1) and (V2) and treat (V3) as the condition to be analyzed. Indeed, the trajectories violating (V3) are included only to identify the range of global viability.

To compare notations with the existing literature, let R denote the relative risk aversion and let S denote the elasticity of intertemporal complementarity, which is the reciprocal of the EIS. We use the sans serif font for these two parameters in order to avoid any confusion with the stock price processes S of (10). The key parameter used in [7, 8, 9] is

$$\vartheta = \frac{1-\mathsf{R}}{1-\mathsf{S}}.$$

With $\mathsf{R} = \gamma$ and $\mathsf{S} = 1/\eta$, this parameter is exactly the λ herein. Away from the additive boundary $\lambda = 1$, the literature on the classical infinite horizon theory distinguishes three regimes. For $0 < \lambda < 1$, a utility process is associated uniquely with a broad class of consumption streams and in the generalized formulation of [8] with every nonnegative consumption stream. Bayraktar and Lawless [1] recently extended the infinite horizon analysis in this range to incomplete markets with stochastic investment opportunities. They give a variational characterization of the value function and include stochastic volatility examples. Their analysis is organized by the range of the parameter λ, and the difficulties differ across its range. If $\lambda > 1$, uniqueness fails in general, and the economically relevant optimal consumption process is selected through the notion of properness. When $\lambda < 0$, the risk aversion parameter and the elasticity of intertemporal complementarity lie on opposite sides of one. Hence, the usual infinite horizon utility index may display the "utility bubble" phenomenon emphasized in [7]. In the empirically important case $\gamma > 1$ and $\eta > 1$, Shigeta [21] showed that the economic sign problem disappears after an order-equivalent transformation. Existence, however, still requires a suitable admissible class of consumption streams. The boundary value $\lambda = 1$ corresponds to the additive CRRA aggregator.

In the forward setting developed in this section, the range of parameters for which the variational duality of Section 3.4 is available should be distinguished from the range for which the primal forward construction is available. Appendix C shows that the forward recursive aggregator is convex in its continuation value when $\gamma\eta \geq 1$ and concave if $\gamma\eta \leq 1$. The variational results of Section 3.4 apply under the convexity condition of Assumption 3.8. In the concave case, a corresponding variational formulation requires the additional conditions described there and is not used in the primal construction below. When $\gamma\eta = 1$ the aggregator is affine in its continuation value. Hence, the primal forward construction developed above covers every $\lambda \neq 0$, subject to viability.

The forward formulation reverses the direction in which the preference structure is specified. Instead of a terminal condition, it prescribes the initial field

$$U(0,x) = \varphi\, u_\gamma(x), \qquad u_\gamma(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad x > 0,$$

where u_γ is the wealth utility u of (99), with the exponent displayed for later comparison, and propagates this field forward. In contrast, the classical lifetime formulation must select a solution of a backward recursion without a terminal condition at infinity. Once (Ψ, δ^Φ) and φ are fixed, the linear equation for $\Xi = \Phi^{\eta/\lambda}$ has a unique solution up to its first exit from $(0,\infty)$. This new forward formulation clarifies why the same requirement covers all cases $\lambda \neq 0$. However, it does not imply that the classical infinite horizon existence questions, and the question of which solution of the backward recursion is to be singled out, have been resolved for every parameter range.

There also exists an exact link with the order-equivalent coordinate used in [21], as we discuss next. To this end, let

$$u_{1/\eta}(x) = \frac{x^{1-1/\eta}}{1-1/\eta}, \qquad x > 0,$$

and consider the increasing transformation $u_{1/\eta} \circ u_\gamma^{-1}$ to the homothetic forward field. Since

$$u_\gamma^{-1}\big(U(t,x)\big) = \Phi_t^{1/(1-\gamma)} x, \qquad t \geq 0,\ x > 0,$$

we obtain

$$\big(u_{1/\eta} \circ u_\gamma^{-1}\big)\big(U(t,x)\big) = \Phi_t^{(1-1/\eta)/(1-\gamma)} u_{1/\eta}(x) = \Phi_t^{1/\lambda} u_{1/\eta}(x), \qquad t \geq 0,\ x > 0. \tag{111}$$

Consequently,

$$\Xi_t = \Phi_t^{\eta/\lambda} = \big(\Phi_t^{1/\lambda}\big)^\eta, \qquad t \geq 0.$$

The Bernoulli variable used in Proposition 4.2 is therefore the η-th power of the multiplier of the order-equivalent utility index. Thus, the linearization and the order transformation are two expressions of the same change of coordinates. Specifically, when $\lambda < 0$, if Ξ vanishes in finite time, the transformed multiplier vanishes, whereas the original multiplier $\Phi = \Xi^{\lambda/\eta}$ diverges.

A "preference budget" interpretation of (V3). Recall $m = \eta/\lambda$ and set

$$\mathfrak{a}_t = \Psi_t^\eta \mathcal{E}_t^{-1}, \qquad \text{and} \qquad B_t = \varphi^m - \int_0^t \mathfrak{a}_s\, ds, \qquad t \geq 0. \tag{112}$$

Then, $\mathfrak{a} > 0$, and Proposition 4.2 gives

$$\mathcal{E}_t^{-1}\Phi_t^m = B_t, \qquad t \geq 0, \qquad dB_t = -\mathfrak{a}_t\, dt, \qquad \check{c}_t^\star = \frac{\mathfrak{a}_t}{B_t} = -\frac{d}{dt}\log B_t, \qquad \text{for } dt \otimes d\mathbb{P}\text{-a.e. } (t,\omega). \tag{113}$$

Process B is decreasing while $\mathfrak{a}$ is the instantaneous rate at which it decreases. This identity supports a "preference budget" interpretation: B_t represents the part of the initial preference budget φ^m that has not yet been consumed by the aggregator. Condition (V3) is then exactly the pathwise solvency constraint

$$\int_0^t \mathfrak{a}_s\, ds < \varphi^m, \quad \text{for each } t \geq 0. \tag{114}$$

Let

$$C_\infty = \int_0^\infty \mathfrak{a}_s\, ds, \qquad \text{and} \qquad B_\infty = \varphi^m - C_\infty.$$

There are three possibilities.

- If $C_\infty > \varphi^m$, then B reaches zero at a finite time. At that time Φ exits from its admissible domain and $\check{c}^\star$ explodes, and, hence, the forward system is not globally viable. This, however, is not possible under condition (V3).
- If $C_\infty = \varphi^m$, then $B_t > 0$, $t \geq 0$ and $B_t \downarrow 0$ as $t \to \infty$. In this critical case,

  $$B_t = \int_t^\infty \mathfrak{a}_s\, ds, \qquad \text{and} \qquad \int_0^\infty \check{c}_t^\star\, dt = \infty.$$

  The initial budget is completely "used" at infinity.
- If $C_\infty < \varphi^m$, then $B_\infty > 0$ and a positive recursive preference budget is never used. In this case,

  $$\int_0^\infty \check{c}_t^\star\, dt = \log\frac{\varphi^m}{B_\infty} < \infty.$$

  The residual amount B_∞ is a forward preference component which is not generated by future aggregator intensity.

The above trichotomy also makes the role of the initial multiplier φ precise. Since $m = \eta/\lambda$ has the same sign as λ, the solvency inequality $C_\infty \leq \varphi^m$ changes direction when it is expressed in terms of φ. If $\lambda > 0$, a larger φ provides a larger budget, while if $\lambda < 0$, a larger φ provides a smaller budget. In particular, when $\gamma > 1$ and $\eta > 1$, then $\lambda < 0$ and

$$\varphi^{\eta/\lambda} = \varphi^{-|\eta/\lambda|}.$$

A larger multiplier makes $U(0,x) = \varphi u_\gamma(x)$ more negative, and increases the initial consumption-to-wealth ratio

$$\check{c}_0^\star = \Psi_0^\eta \varphi^{-\eta/\lambda},$$

and reduces the amount of future preference intensity that the system can sustain. In this regime, $\varphi^{\eta/\lambda}$ may be interpreted as an initial "preference budget".

Finite-time viability versus transversality. Condition (V3) is sufficient for the forward random field and the feedback controls to be well defined on every finite horizon. However, it is not, by itself, a classical transversality condition. To see this, let τ_n be a localizing sequence for the criterion process and let $Z_s^\star = Z_s^{\pi^\star,c^\star}$, $s \geq 0$. Along the optimal control pair, the stopped local martingale property gives, for $0 \leq t \leq T$,

$$\begin{aligned} &U(t\wedge\tau_n, X_{t\wedge\tau_n}^\star) \\ &= \mathbb{E}_{t\wedge\tau_n}\left[U(T\wedge\tau_n, X_{T\wedge\tau_n}^\star) + \int_{t\wedge\tau_n}^{T\wedge\tau_n} F\big(s, c_s^\star, U(s, X_s^\star)\big)\, ds\right]. \end{aligned} \tag{115}$$

For every $T > 0$, the martingale optimality principle yields a finite horizon recursive problem with terminal field $U(T,\cdot)$. For an arbitrary admissible control pair, equality is replaced by an inequality. Removing localization requires class-D or uniform integrability type conditions. An infinite horizon interpretation also requires a transversality condition, which for $\lambda < 0$ is the transformed condition used in [21]. In the Merton model, $B_\infty = 0$ selects the stationary solution, whereas $B_\infty > 0$ yields a nonstationary forward solution. The condition $B_\infty = 0$ expresses exhaustion of the initial preference budget but does not by itself imply transversality, which also depends on $\mathcal{E}$, the optimal wealth, as well as the recursive discount.

4.1.3 Characterization and optimal policies

We first derive the pointwise properties of the aggregator and the resulting optimal relative consumption.

Proposition 4.3 (Aggregator and optimal consumption feedback). The forward Epstein–Zin recursive aggregator has the following properties.
(i) For every fixed (t, ω, v) with $v \in \mathcal{U}_\gamma$, the map $c \mapsto F(t, \omega, c, v)$ is strictly increasing, strictly concave, and satisfies the Inada conditions. Hence, Assumption 2.9 holds, and its Fenchel transform is (96).
(ii) Under the separable form (99), the optimal feedback consumption is proportional to wealth

$$\check{c}_t^\star = \frac{c^\star(t,x)}{x} = \frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad t \geq 0,\ x > 0. \tag{116}$$

Proof. Fix $t \geq 0$, $\omega \in \Omega$, and $v \in \mathcal{U}_\gamma$. Since $(1-\gamma)v > 0$, differentiation gives

$$F_c(t,c,v) = \Psi_t c^{-1/\eta}\big((1-\gamma)v\big)^{1-1/\lambda}$$

and

$$F_{cc}(t,c,v) = -\frac{\Psi_t}{\eta} c^{-1/\eta-1}\big((1-\gamma)v\big)^{1-1/\lambda}.$$

The first derivative is strictly positive while the second one is strictly negative. Moreover,

$$\lim_{c\downarrow 0} F_c(t,c,v) = \infty, \qquad \text{and} \qquad \lim_{c\uparrow\infty} F_c(t,c,v) = 0, \qquad t \geq 0.$$

The formulas in Appendix C show that F_c and F_u are continuous on $(0,\infty) \times \mathcal{U}_\gamma$. These properties prove all assertions concerning the map $c \mapsto F(t,\omega,c,v)$ and verify the remaining regularity in Assumption 2.9.

Let $d > 0$. The unique maximizer in the definition of $\widetilde{F}(t,d,v)$ solves $F_c(t,c,v) = d$. Therefore,

$$c = \left(\frac{\Psi_t\big((1-\gamma)v\big)^{1-1/\lambda}}{d}\right)^\eta, \qquad t \geq 0. \tag{117}$$

Substituting (117) into $F(t,c,v) - dc$ gives

$$\widetilde{F}(t,d,v) = \frac{d^{1-\eta}}{\eta-1}\big((1-\gamma)v\big)^{\eta(1-1/\lambda)}\Psi_t^\eta - \lambda\Psi_t v, \qquad t \geq 0,$$

and (96) follows.

Next, we evaluate the maximizer at

$$d = U_x(t,x) = \Phi_t x^{-\gamma}, \qquad \text{and} \qquad (1-\gamma)U(t,x) = \Phi_t x^{1-\gamma}, \qquad t \geq 0,\ x > 0.$$

Equation (117) yields

$$c^\star(t,x) = \left(\frac{\Psi_t(\Phi_t x^{1-\gamma})^{1-1/\lambda}}{\Phi_t x^{-\gamma}}\right)^\eta = \Psi_t^\eta \Phi_t^{-\eta/\lambda} x^{\eta((1-\gamma)(1-1/\lambda)+\gamma)}, \qquad t \geq 0,\ x > 0.$$

The definition of λ implies

$$(1-\gamma)(1-1/\lambda) + \gamma = \frac{1}{\eta}.$$

Consequently,

$$c^\star(t,x) = x\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad t \geq 0,\ x > 0,$$

which proves (116). $\square$

Theorem 4.4 (Separable characterization). Let (U,F) have the separable form (99), and Φ have dynamics as in (103).

(i) The field $U(t,x)=\Phi_t u(x)$ solves the forward Epstein–Zin recursive SPDE if and only if the drift of Φ is (104). In this case, the candidate feedback control processes are

$$\pi_t^\star = \frac{1}{\gamma}(\sigma_t\sigma_t^\top)^{-1}\left(\mu_t+\sigma_t\delta_t^\Phi\right), \qquad \text{and} \qquad \check{c}_t^\star = \frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad t\ge 0. \tag{118}$$

(ii) If the drift condition in part (i) and (V1)–(V3) hold, then (U,F) is a normalized forward recursive aggregator system on $\mathcal{U}_\gamma$, the feedback control pair $\left(\pi^\star,\check{c}^\star X^\star\right)$ is admissible and optimal, and the optimal wealth process is

$$X_t^\star = x\,\exp\biggl(\int_0^t\left((\pi_s^\star)^\top\sigma_s\theta_s-\check{c}_s^\star-\tfrac{1}{2}|(\pi_s^\star)^\top\sigma_s|^2\right)ds+\int_0^t(\pi_s^\star)^\top\sigma_s\,dW_s\biggr), \quad t\ge 0. \tag{119}$$

Proof. We prove the two assertions in three steps.

Step 1. Reduction of the Bellman SPDE to the equation for Φ. For every $t\ge 0$ and $x>0$,

$$U(t,x)=\Phi_t\frac{x^{1-\gamma}}{1-\gamma}.$$

Differentiation with respect to x gives

$$U_x(t,x)=\Phi_t x^{-\gamma}, \qquad \text{and} \qquad U_{xx}(t,x)=-\gamma\Phi_t x^{-\gamma-1}, \qquad t\ge 0,\ x>0. \tag{120}$$

For a fixed $x>0$, the identity $dU(t,x)=u(x)d\Phi_t$ and the decomposition (103) give

$$dU(t,x)=U(t,x)b_t^\Phi\,dt+U(t,x)(\delta_t^\Phi)^\top dW_t, \qquad t\ge 0.$$

Therefore,

$$b(t,x)=U(t,x)b_t^\Phi, \qquad \delta(t,x)=U(t,x)\delta_t^\Phi, \qquad \delta_x(t,x)=U_x(t,x)\delta_t^\Phi, \quad t\ge 0,\ x>0. \tag{121}$$

We compute the first nonlinear term in SPDE (97). Equations (120) and (121) imply

$$U_x(t,x)\theta_t+\sigma_t^+\sigma_t\delta_x(t,x)=\Phi_t x^{-\gamma}\bigl(\theta_t+\sigma_t^+\sigma_t\delta_t^\Phi\bigr), \quad t\ge 0,\ x>0.$$

Therefore,

$$\begin{aligned}
&\frac{|U_x(t,x)\theta_t+\sigma_t^+\sigma_t\delta_x(t,x)|^2}{2U_{xx}(t,x)}\\
&\quad=-\frac{\Phi_t x^{1-\gamma}}{2\gamma}|\theta_t+\sigma_t^+\sigma_t\delta_t^\Phi|^2\\
&\quad=-\frac{1-\gamma}{2\gamma}|\theta_t+\sigma_t^+\sigma_t\delta_t^\Phi|^2 U(t,x), \quad t\ge 0,\ x>0,
\end{aligned} \tag{122}$$

where in the last equality we used $\Phi_t x^{1-\gamma}=(1-\gamma)U(t,x)$.

For the second nonlinear term we work as follows. From (120),

$$\begin{aligned}
&U_x(t,x)^{1-\eta}\bigl((1-\gamma)U(t,x)\bigr)^{\eta(1-1/\lambda)}\\
&\quad=(\Phi_t x^{-\gamma})^{1-\eta}(\Phi_t x^{1-\gamma})^{\eta(1-1/\lambda)}\\
&\quad=\Phi_t^{1-\eta/\lambda}x^{1-\gamma}\\
&\quad=(1-\gamma)\Phi_t^{-\eta/\lambda}U(t,x), \quad t\ge 0,\ x>0.
\end{aligned}$$

The third equality follows from the definition of λ, which gives

$$-\gamma(1-\eta)+\eta(1-\gamma)(1-1/\lambda)=1-\gamma.$$

It, then, follows that

$$
\begin{aligned}
&-\frac{\Psi_t^\eta}{\eta-1}U_x(t,x)^{1-\eta}\big((1-\gamma)U(t,x)\big)^{\eta(1-1/\lambda)}\\
&=\frac{1-\gamma}{1-\eta}\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}U(t,x), \quad t\geq 0,\ x>0.
\end{aligned}\tag{123}
$$

Substituting (122) and (123) into (97), and comparing the resulting drift with $b(t,x)=U(t,x)b_t^\Phi$, gives

$$
U(t,x)b_t^\Phi = U(t,x)\left(\frac{1-\gamma}{1-\eta}\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}-\frac{1-\gamma}{2\gamma}|\theta_t+\sigma_t^+\sigma_t\delta_t^\Phi|^2+\lambda\Psi_t\right), \qquad t\geq 0,\ x>0.
$$

Using that $U(t,x)\neq 0$, dividing by $U(t,x)$ gives (104). Every computation above is reversible and we easily deduce the equivalence in part (i).

Step 2. Computation of the optimal feedback controls. The two ratios that enter the candidate optimal portfolio feedback control are

$$
-\frac{U_x(t,x)}{xU_{xx}(t,x)}=\frac{1}{\gamma} \qquad \text{and} \qquad \frac{\delta_x(t,x)}{U_x(t,x)}=\delta_t^\Phi, \qquad t\geq 0,\ x>0,
$$

where we used (120) and (121). From $(\sigma_t^\top)^+=(\sigma_t\sigma_t^\top)^{-1}\sigma_t$ and (30) we obtain

$$
\pi^\star(t,x)=\frac{1}{\gamma}(\sigma_t\sigma_t^\top)^{-1}\sigma_t(\theta_t+\delta_t^\Phi)=\frac{1}{\gamma}(\sigma_t\sigma_t^\top)^{-1}(\mu_t+\sigma_t\delta_t^\Phi), \qquad t\geq 0,\ x>0.
$$

The second equality follows from the fact that $\sigma_t\theta_t=\mu_t$, $t\geq 0$. The right-hand side is independent of x, and it is precisely the portfolio stated in (118). Proposition 4.3 gives

$$
c^\star(t,x)=x\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad t\geq 0,\ x>0,
$$

and, hence,

$$
\check{c}_t^\star=\frac{c^\star(t,x)}{x}=\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad t\geq 0,\ x>0,
$$

which proves the second formula in (118).

Step 3. Consistency, admissibility, and optimal wealth. Proposition 4.3 proves Assumption 2.9. Formulas (120) and (121), together with the inverse marginal

$$
I(t,y)=\left(\frac{\Phi_t}{y}\right)^{1/\gamma}, \qquad y>0,
$$

yield the required spatial regularity and the Inada conditions. Since Φ is a strictly positive Itô process, the homothetic forms of U and I inherit the random-field regularity of Assumption 3.2. Conditions (V1) and (V2) provide the required local integrability of their characteristics. Under (V1)–(V3), Corollary 4.8 verifies every bound in Assumption 3.13, including the integrability condition for which $K^4=1/\gamma$. Therefore, Theorem 3.14 implies that (U,F) is a normalized forward recursive aggregator system and that the feedback control pair is admissible and optimal.

It remains to solve the optimal wealth equation. Since $c_t^\star=X_t^\star\check{c}_t^\star$, equation (13) becomes

$$
\frac{dX_t^\star}{X_t^\star}=\big((\pi_t^\star)^\top\sigma_t\theta_t-\check{c}_t^\star\big)\,dt+(\pi_t^\star)^\top\sigma_t\,dW_t, \qquad t\geq 0.
$$

Applying Itô's formula gives

$$
d\log X_t^\star=\left((\pi_t^\star)^\top\sigma_t\theta_t-\check{c}_t^\star-\frac{1}{2}|(\pi_t^\star)^\top\sigma_t|^2\right)dt+(\pi_t^\star)^\top\sigma_t\,dW_t, \qquad t\geq 0.
$$

and (119) follows. ☐

Combining Theorem 4.4 with the explicit form of Φ obtained in Proposition 4.2 yields the optimal consumption in closed form.

Corollary 4.5. Assume that the drift condition (104) and (V1)–(V3) hold. Then, the optimal relative consumption rate is given by

$$\check{c}_t^\star = \Psi_t^\eta\, \mathcal{E}_t^{-1}\Big(\varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta\, \mathcal{E}_u^{-1}\, du\Big)^{-1}, \quad t \geq 0,$$

and the optimal consumption process is $c_t^\star = X_t^\star\, \check{c}_t^\star$. In particular, (V3) is the one that guarantees that $\check{c}^\star$ is finite and positive.

Remark 4.6. The optimal portfolio $\pi^\star$ in (118) has the same structure as the one in the classical homothetic setting, but depends on the volatility δ^Φ. We remind the reader that the latter is an exogenous modeling input rather than a quantity determined by the solution of the problem in a fixed horizon. The optimal relative consumption $\check{c}^\star$ depends nonlinearly on both Φ and Ψ.

Remark 4.7. Theorem 4.4 reduces the forward recursive problem to the one-dimensional equation (105) for Φ and to the viability conditions (V1)–(V3). Indeed, the strict positivity of Φ ensures that $U(t,\cdot) \in \mathcal{U}_\gamma$, while (V1) and (V2) yield the integrability of the model inputs and, through Corollary 4.8 below, the admissibility of the feedback control pair. These are the properties established in Section 3, for general setting.

The final step is to verify the conditions of Section 3 directly from the model processes (Ψ, δ^Φ). No integrability beyond (V1)–(V3) is required.

Corollary 4.8. Under the separable representation (99) with the drift condition (104) and (V1)–(V3), Assumption 3.13 holds and Theorem 3.14 applies. In particular, the feedback control pair in (118) is admissible.

Proof. Using that

$$U_x(t,x) = x^{-\gamma}\Phi_t, \qquad U_{xx}(t,x) = -\gamma\, x^{-\gamma-1}\Phi_t, \qquad \delta_x(t,x) = U_x(t,x)\,\delta_t^\Phi, \qquad t \geq 0,$$

the bounds in Assumption 3.13 hold with

$$K_t^1 = K_t^2 = |\delta_t^\Phi|, \qquad K_t^3 = \check{c}_t^\star = \frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad K_t^4 = \frac{1}{\gamma}, \qquad K_t^0 = |\alpha_t^\star|, \qquad t \geq 0,$$

where the discount $\alpha^\star$ satisfies

$$\alpha_t^\star = -F_u(t, c_t^\star, U_t^\star) = \lambda\Psi_t - (\lambda - 1)\,\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}}, \qquad t \geq 0,$$

We first note that $\partial_x \alpha^\star \equiv 0$. Moreover, the optimal feedback controls are linear in the state and are given by

$$\sigma^\star(t,x) = \frac{x}{\gamma}\sigma_t^+\sigma_t\big(\theta_t + \delta_t^\Phi\big), \qquad c^\star(t,x) = \check{c}_t^\star x, \qquad t \geq 0,\ x > 0.$$

It remains to check the corresponding integrability conditions. Let $T > 0$. By the standing market condition (11) on the market price of risk and (V2),

$$\int_0^T \big(|\theta_t|^2 + |\delta_t^\Phi|^2 + \Psi_t\big)\, dt < \infty, \qquad \mathbb{P}\text{-a.s.}$$

Hence,

$$\int_0^T (K_t^4)^2\big(|\theta_t| + K_t^1\big)^2\, dt \leq \frac{2}{\gamma^2}\int_0^T \big(|\theta_t|^2 + |\delta_t^\Phi|^2\big)\, dt < \infty.$$

We now consider the feedback consumption control. To this end, set

$$B_t = \varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta \mathcal{E}_u^{-1}\, du, \qquad t \geq 0,$$

which is the quantity multiplying $\mathcal{E}_t$ in the explicit formula (107) for Φ. By Corollary 4.5,

$$\check{c}_t^\star = -\frac{d}{dt}\log B_t, \qquad \text{for } dt\otimes d\mathbb{P}\text{-a.e. } (t,\omega).$$

Therefore,

$$\int_0^T \check{c}_t^\star\,dt = \log\left(\frac{\varphi^{\eta/\lambda}}{B_T}\right) < \infty, \qquad \mathbb{P}\text{-a.s.},$$

where the last inequality follows from (V3). Finally,

$$\int_0^T |\alpha_t^\star|\,dt \le |\lambda|\int_0^T \Psi_t\,dt + |\lambda-1|\int_0^T \check{c}_t^\star\,dt < \infty, \qquad \text{for each } T\ge 0.$$

Thus, the required integrability conditions hold on every finite horizon. □

4.1.4 Zero-volatility separable forward Epstein–Zin utilities

When $\delta^\Phi \equiv 0$, the process Φ has no martingale part, although it does not need to be monotone without further assumptions. The stochastic exponential (101) is of finite variation,

$$\mathcal{E}_t = \exp\!\left(\int_0^t \left(\eta\Psi_s + \frac{1-\eta}{2\gamma}\,|\theta_s|^2\right) ds\right), \quad t\ge 0,$$

and (107) gives

$$\Phi_t = \left(\mathcal{E}_t\left(\varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta\,\mathcal{E}_u^{-1}\,du\right)\right)^{\lambda/\eta}, \quad t\ge 0. \tag{124}$$

Direct calculations yield the optimal policies and their associated wealth process in closed form. Specifically, using that $\delta^\Phi \equiv 0$, the optimal portfolio becomes

$$\pi_t^\star = \frac{1}{\gamma}\,(\sigma_t\sigma_t^\top)^{-1}\mu_t, \quad t\ge 0,$$

and the optimal consumption

$$c_t^\star = \frac{\mathcal{E}_t^{-1}\,\Psi_t^\eta}{\varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta\,\mathcal{E}_u^{-1}\,du}\,X_t^\star, \quad t\ge 0.$$

We easily deduce that the related optimal wealth process is

$$X_t^\star = x\,\exp\!\left(\int_0^t \left((\pi_s^\star)^\top\sigma_s\theta_s - \check{c}_s^\star - \tfrac{1}{2}|(\pi_s^\star)^\top\sigma_s|^2\right) ds + \int_0^t (\pi_s^\star)^\top\sigma_s\,dW_s\right), \quad t\ge 0.$$

Remark 4.9. The time monotone specification in the forward Epstein–Zin recursive class we are developing is the natural forward analogue of the well established time monotone homothetic constructions in the forward performance literature. The optimal consumption-to-wealth ratio is the ratio of Ψ_t^η to the integrated factor $\mathcal{E}_t(\varphi^{\eta/\lambda} - \int_0^t \Psi_u^\eta\mathcal{E}_u^{-1}\,du)$.

When $\delta^\Phi \equiv 0$ the hedging part is eliminated. The portfolio is then governed by risk aversion alone through the multiplier $1/\gamma$, while the EIS affects the timing of consumption through the evolution of Φ and the ratio $\Psi^\eta/\Phi^{\eta/\lambda}$. This is the setting in which the separation between risky investment and intertemporal substitution is most transparent.

4.2 State price density

Corollary 4.10. Under the assumptions of Theorem 4.4(ii), the recursive discount rate satisfies

$$\begin{aligned} F_u(t, c_t^\star, U_t^\star) &= \widetilde{F}_u\big(t, U_x(t, X_t^\star), U(t, X_t^\star)\big) \\ &= (\lambda - 1)\check{c}_t^\star - \lambda \Psi_t \\ &= (\lambda - 1)\frac{\Psi_t^\eta}{\Phi_t^{\eta/\lambda}} - \lambda \Psi_t, \qquad t \geq 0. \end{aligned} \tag{125}$$

The state price density (66) generated by the primal optimum becomes

$$Y_t^\star = U_x(t, X_t^\star) \exp\left(-\int_0^t \left(\lambda \Psi_s + (1-\lambda)\frac{\Psi_s^\eta}{\Phi_s^{\eta/\lambda}}\right) ds\right), \qquad t \geq 0. \tag{126}$$

Remark 4.11. In (126), the exponent of the discount factor separates into the Ψ-driven time-discount term $-\lambda\Psi$ and the correction term $(\lambda-1)\check{c}^\star$. The latter vanishes in the additive case $\lambda = 1$ but, otherwise, depends on the optimal consumption-to-wealth ratio. The sign of the correction follows that of $\lambda - 1$. Since $\lambda = 1$ is equivalent to $\eta = 1/\gamma$, this correction is precisely how the separation between the EIS and risk aversion is reflected in the pricing kernel. When $\lambda = 1$, it disappears, and the state price density reduces to the marginal utility discounted only by the Ψ-term. Away from this case, the optimal consumption also enters the recursive discount. In the homothetic representation, we have $U_x(t, X_t^\star) = \Phi_t (X_t^\star)^{-\gamma}$, and, thus, γ governs the direct exposure of marginal utility to wealth, whereas η affects the consumption policy and the recursive discount.

Remark 4.12 (Comparison with Matoussi and Xing). The identity (66) has the same structural form as the utility-gradient formula of Matoussi and Xing [17] for the classical (backward) Epstein–Zin problem in a finite horizon $[0, T]$. In their notation, in which f denotes the aggregator itself and therefore plays the role of F herein, the state price density selected by optimality satisfies (cf. [17, Corollary 3.7, Equation (3.11)])

$$D_t^\star \ \propto\ f_c(c_t^\star, U_t^{c^\star}) \exp\left(\int_0^t f_u(c_s^\star, U_s^{c^\star})\, ds\right), \qquad t \in [0, T].$$

At the optimum, the first order condition gives $f_c(c^\star, U^\star) = U_x$ while the envelope identity yields $f_u(c^\star, U^\star) = F_u = \widetilde{F}_u$. Thus, in both formulas, the state price density is given by the marginal utility multiplied by the exponential of the derivative of the aggregator with respect to u.

The backward and forward pricing kernels are nonetheless different objects. Specifically, in the backward problem the discount factor $\exp(\int f_u\, ds)$ is evaluated along the solution of a BSDE pinned by a given, specific terminal condition. In contrast, in the forward setting, it is evaluated along the forward SDE (**??**) for Φ, whose drift is determined by the market itself through self-generation.

4.3 Duality in forward Epstein–Zin recursive aggregator systems

The homothetic structure of $U(t,x)$ reduces the argument $\widetilde{U}(t,y) - y\,\widetilde{U}_y(t,y)$ inside $\widetilde{F}$ to a scalar multiple of $\widetilde{U}$. Then, the dual value process of Section 3.3 becomes a self-contained Epstein–Zin recursive problem in $\widetilde{U}$, with an explicit dual aggregator g and the dual-consistency property of Definition 3.5.

Proposition 4.13 (forward Epstein–Zin dual factor). Let $U(t,x) = \Phi_t\, u(x)$, $x > 0$, $t \geq 0$, be as in (99), with Φ as in (103) and with drift b^Φ satisfying (104), or equivalently, let U solve (97). Set $\widetilde{\Phi}_t = \Phi_t^{1/\gamma}$, $t \geq 0$.

(i) Conjugate field. The forward recursive convex conjugate is of separable form,

$$\widetilde{U}(t,y) = \frac{\gamma}{1-\gamma}\widetilde{\Phi}_t y^{(\gamma-1)/\gamma}, \qquad t \geq 0,\ y > 0. \tag{127}$$

Moreover, (39) becomes

$$\hat{u}(t,y) = \frac{1}{\gamma}\widetilde{U}(t,y), \qquad t \geq 0,\ y > 0. \tag{128}$$

(ii) Dual aggregator. The dual forward Epstein–Zin equation depends on $\widetilde{U}$ only, with generator

$$g(t,y,v) = \widetilde{F}\left(t, y, \frac{v}{\gamma}\right) = \frac{y^{1-\eta}}{\eta - 1}\left(\frac{1-\gamma}{\gamma}v\right)^{\eta(1-1/\lambda)} \Psi_t^\eta - \frac{\lambda}{\gamma}\Psi_t v, \qquad t \geq 0,\ y > 0,\ (1-\gamma)v > 0. \tag{129}$$

(iii) Dual-factor dynamics. The factor process $\widetilde{\Phi}$ satisfies

$$d\widetilde{\Phi}_t = \widetilde{\Phi}_t\big(b_t^{\widetilde{\Phi}}\, dt + (\delta_t^{\widetilde{\Phi}})^\top dW_t\big), \qquad \delta_t^{\widetilde{\Phi}} = \frac{1}{\gamma}\,\delta_t^{\Phi}, \qquad t \ge 0, \tag{130}$$

with

$$b_t^{\widetilde{\Phi}} = \frac{\lambda}{\gamma}\Psi_t - \frac{1-\gamma}{\gamma(\eta-1)}\Psi_t^{\eta}\widetilde{\Phi}_t^{-\gamma\eta/\lambda} - \frac{1-\gamma}{2\gamma^2}|\theta_t|^2 - \frac{1-\gamma}{\gamma}\theta_t^\top\delta_t^{\widetilde{\Phi}} + \frac{1-\gamma}{2}|(I_d - \sigma_t^+\sigma_t)\delta_t^{\widetilde{\Phi}}|^2, \qquad t \ge 0. \tag{131}$$

Proof. Fix $t \ge 0$ and $y > 0$. The function that is maximized in the definition of $\widetilde{U}(t, y)$ is

$$x \longmapsto \Phi_t \frac{x^{1-\gamma}}{1-\gamma} - xy, \qquad x > 0.$$

Since it is strictly concave, its unique maximizer is

$$I(t, y) = \left(\frac{\Phi_t}{y}\right)^{1/\gamma}, \qquad t \ge 0,\ y > 0.$$

Thus,

$$\begin{aligned}
\widetilde{U}(t, y) &= \Phi_t \frac{I(t, y)^{1-\gamma}}{1-\gamma} - yI(t, y) \\
&= \frac{\Phi_t^{1/\gamma} y^{(\gamma-1)/\gamma}}{1-\gamma} - \Phi_t^{1/\gamma} y^{(\gamma-1)/\gamma} \\
&= \frac{\gamma}{1-\gamma}\Phi_t^{1/\gamma} y^{(\gamma-1)/\gamma}, \qquad t \ge 0,\ y > 0,
\end{aligned}$$

and (127) follows. Differentiating the last expression with respect to y gives

$$\widetilde{U}_y(t, y) = -\Phi_t^{1/\gamma} y^{-1/\gamma}, \qquad t \ge 0,\ y > 0,$$

and, thus,

$$\begin{aligned}
\widetilde{U}(t, y) - y\widetilde{U}_y(t, y) &= \left(\frac{\gamma}{1-\gamma} + 1\right)\Phi_t^{1/\gamma} y^{(\gamma-1)/\gamma} \\
&= \frac{1}{\gamma}\widetilde{U}(t, y), \qquad y > 0,\ t \ge 0,
\end{aligned}$$

and (128) follows. Substituting

$$\hat{u}(t, y) = \widetilde{U}(t, y) - y\widetilde{U}_y(t, y) = \frac{1}{\gamma}\widetilde{U}(t, y), \qquad t \ge 0,\ y > 0,$$

into (96) gives, for every v in the range of $\widetilde{U}$,

$$\begin{aligned}
g(t, y, v) &= \widetilde{F}\left(t, y, \frac{v}{\gamma}\right) \\
&= \frac{y^{1-\eta}}{\eta-1}\left(\frac{1-\gamma}{\gamma}v\right)^{\eta(1-1/\lambda)}\Psi_t^{\eta} - \frac{\lambda}{\gamma}\Psi_t v, \qquad t \ge 0,\ y > 0,
\end{aligned}$$

and (129) follows.

It remains to compute the dynamics of $\widetilde{\Phi}_t = \Phi_t^{1/\gamma}$. Itô's formula gives

$$\frac{d\widetilde{\Phi}_t}{\widetilde{\Phi}_t} = \left(\frac{b_t^{\Phi}}{\gamma} + \frac{1}{2\gamma}\left(\frac{1}{\gamma} - 1\right)|\delta_t^{\Phi}|^2\right)dt + \frac{1}{\gamma}(\delta_t^{\Phi})^\top dW_t, \tag{132}$$

and, thus,

$$\delta_t^{\widetilde{\Phi}} = \frac{1}{\gamma}\delta_t^{\Phi}, \qquad t \geq 0.$$

To identify the drift, we substitute (104) into (132), and we use

$$\Phi_t^{-\eta/\lambda} = \widetilde{\Phi}_t^{-\gamma\eta/\lambda} \qquad \text{and} \qquad \delta_t^{\Phi} = \gamma\delta_t^{\widetilde{\Phi}}, \qquad t \geq 0,$$

together with (12). Expanding the square gives

$$\begin{aligned}
&-\frac{1-\gamma}{2\gamma^2}|\theta_t + \gamma\sigma_t^+\sigma_t\delta_t^{\widetilde{\Phi}}|^2 + \frac{1-\gamma}{2}|\delta_t^{\widetilde{\Phi}}|^2 \\
&= -\frac{1-\gamma}{2\gamma^2}|\theta_t|^2 - \frac{1-\gamma}{\gamma}\theta_t^\top\delta_t^{\widetilde{\Phi}} + \frac{1-\gamma}{2}|(I_d - \sigma_t^+\sigma_t)\delta_t^{\widetilde{\Phi}}|^2, \qquad t \geq 0.
\end{aligned}$$

Combining this identity with the remaining terms and (132) yields (131), and the drift of $\widetilde{\Phi}$ is fully specified. □

Remark 4.14. The generator $g(t, y, v) = \widetilde{F}(t, y, v/\gamma)$ in (129) is the forward self-generating analogue of the finite-horizon dual recursive generator of Matoussi and Xing [17]. The difference is that here the dual factor $\widetilde{\Phi}$ evolves forward in time through (130) and (131), while the analogous object in [17] is fully specified by an exogenous terminal condition. The state price density generated by the primal optimum in Theorem 3.12 is $Y_t^\star = \widetilde{\Phi}_t^\gamma\,\kappa_t^\star\,(X_t^\star)^{-\gamma} = \Phi_t\,\kappa_t^\star\,u_x(X_t^\star)$, $t \geq 0$. In this homothetic class, the feedback control (44) is independent of y, and the density $Y^{\nu^{\text{dual}}}$ of Theorem 3.7, with an exogenous initial condition $Y_0^{\nu^{\text{dual}}} = U_x(0, x)$, coincides with $Y^\star$.

Remark 4.15 (Bernoulli variable under duality). The Bernoulli variable is unchanged by conjugacy, in that $\widetilde{\Phi}_t^{\gamma\eta/\lambda} = \Phi_t^{\eta/\lambda} = \Xi_t$, $t \geq 0$. Thus, the primal and dual reductions use the same linear state variable.

4.4 The Black and Scholes market: explicit closed form solutions

Consider a market with constant coefficients: $\mu \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^{n\times d}$ of full rank. The market price of risk is then constant $\bar{\theta}$, given by

$$\bar{\theta} = \sigma^\top(\sigma\sigma^\top)^{-1}\mu \in \mathbb{R}^d, \qquad \theta_t \equiv \bar{\theta}, \qquad t \geq 0.$$

Fix a constant $\bar{\Psi} > 0$ and set $\Psi_t \equiv \bar{\Psi}$ and $\delta_t^{\Phi} \equiv \bar{\delta}^{\Phi} = 0$, $t \geq 0$.
Furthermore, let the constant

$$\bar{\chi} = \eta\bar{\Psi} + \frac{1-\eta}{2\gamma}\,|\bar{\theta}|^2,$$

(see Proposition 4.2). Then, representation (124) takes, on the positive lifetime interval $0 \leq t < \tau$ of (108), the following form:

i) if $\bar{\chi} \neq 0$,

$$\Phi_t = \left(e^{\bar{\chi}t}\varphi^{\eta/\lambda} - \frac{\bar{\Psi}\eta}{\bar{\chi}}\left(e^{\bar{\chi}t} - 1\right)\right)^{\lambda/\eta}, \qquad 0 \leq t < \tau, \tag{133}$$

ii) if $\bar{\chi} = 0$,

$$\Phi_t = \left(\varphi^{\eta/\lambda} - \bar{\Psi}\eta t\right)^{\lambda/\eta}, \qquad 0 \leq t < \tau.$$

In the case where $\bar{\chi} > 0$ and $\varphi^{\eta/\lambda} \geq \bar{\Psi}\eta/\bar{\chi}$, the optimal investment and relative consumption controls are

$$\pi_t^\star = \frac{1}{\gamma}\,(\sigma\sigma^\top)^{-1}\mu \qquad \text{and} \qquad \check{c}_t^\star = \frac{\bar{\Psi}\eta}{e^{\bar{\chi}t}\varphi^{\eta/\lambda} + \frac{\bar{\Psi}\eta}{\bar{\chi}}\,(1 - e^{\bar{\chi}t})}, \qquad t \geq 0. \tag{134}$$

We give more information on the origin of the two inequalities below.

Remark 4.16. The optimal portfolio process $\pi^\star$ in (134) is constant and resembles the classical Merton proportion determined by the relative risk aversion γ and the market price of risk $\bar{\theta}$. In contrast, the optimal consumption rate is not constant, but rather a deterministic function that depends on the various parameters $\lambda, \eta, \varphi, \bar{\Psi}$ and $\bar{\chi}$. We stress, however, that the forward and the backward Merton problems do not coincide for an arbitrary value of the initial multiplier. The critical initial value identified below removes the residual forward component and produces the stationary Merton candidate. Other viable initial values generate non stationary forward consumption paths. This benchmark gives the clearest direct separation of the two Epstein–Zin preference parameters. Specifically, risk aversion determines the risky allocation through the factor $1/\gamma$, whereas the EIS affects the consumption path through η and the composite parameter λ. The coefficient $\bar{\chi}$ defined above is not an additional preference parameter and should not be confused with the positive Uzawa discount β used earlier. Indeed, it is the growth coefficient of the linearized factor $\Xi = \Phi^{\eta/\lambda}$. It aggregates the consumption input, investment opportunities, risk aversion, and the EIS. Consequently, even in this simple market setting, the optimal portfolio exhibits a transparent, direct dependence on γ, while the level and dynamics of the optimal consumption process reflect both parameters through the forward consistency condition.

4.4.1 Infinite horizon viability of the pair (Φ, Ψ)

Using the notations of Section 4.1, we have

$$C_\infty = \int_0^\infty \mathfrak{a}_t \, dt = \begin{cases} \dfrac{\bar{\Psi}^\eta}{\bar{\chi}}, & \bar{\chi} > 0, \\ +\infty, & \bar{\chi} \leq 0. \end{cases} \tag{135}$$

It follows that (V3) holds on $[0, \infty)$ if and only if

$$\bar{\chi} > 0 \quad \text{and} \quad \varphi^{\eta/\lambda} \geq \frac{\bar{\Psi}^\eta}{\bar{\chi}}. \tag{136}$$

The three budget regimes of Section 4.1 now have an explicit form. Specifically, if

$$\varphi^{\eta/\lambda} < \frac{\bar{\Psi}^\eta}{\bar{\chi}},$$

then Ξ_t reaches zero in finite time. Consequently, Φ_t decreases to zero if $\lambda > 0$ and increases to $+\infty$ if $\lambda < 0$. If, on the other hand,

$$\varphi^{\eta/\lambda} = \frac{\bar{\Psi}^\eta}{\bar{\chi}}, \tag{137}$$

then

$$B_t = \frac{\bar{\Psi}^\eta}{\bar{\chi}} e^{-\bar{\chi} t} \qquad \text{and} \qquad \Xi_t = \frac{\bar{\Psi}^\eta}{\bar{\chi}}, \qquad t \geq 0,$$

and therefore

$$\Phi_t = \left(\frac{\bar{\Psi}^\eta}{\bar{\chi}} \right)^{\lambda/\eta}, \qquad \check{c}_t^\star = \bar{\chi}, \qquad t \geq 0.$$

The factor process Φ and the optimal relative consumption process are both stationary. Finally, if

$$\varphi^{\eta/\lambda} > \frac{\bar{\Psi}^\eta}{\bar{\chi}},$$

then

$$B_\infty = \varphi^{\eta/\lambda} - \frac{\bar{\Psi}^\eta}{\bar{\chi}} > 0.$$

In this case, the forward aggregator is globally viable, but it retains a residual forward preference component. $\check{c}_t^\star$ decreases to zero and the factor Φ_t is nonstationary.

As noted in the general case above, the dependence on the initial multiplier changes with the sign of λ. If $\lambda > 0$, condition (136) imposes a lower bound on φ, while, if $\lambda < 0$, it imposes an upper bound. Thus, in the range $\gamma > 1$, $\eta > 1$, and $\lambda < 0$ studied by [21], increasing φ moves the forward system toward a finite time preference insolvency rather than away from such an insolvency.

4.4.2 Relation with the classical (backward) Merton solution

The optimal portfolio in (134) resembles the classical Merton proportion. The forward construction allows every initial multiplier satisfying (136), whereas the critical value (137) yields the stationary Merton candidate. At this value, Φ and $\check{c}^\star$ are constant and $B_\infty = 0$. A strict inequality in (136), namely, a strictly positive residual preference budget, yields a nonstationary forward solution with $B_\infty > 0$. Thus, condition (V3) imposes initial viability rather than terminal selection and is not itself a transversality condition.

4.4.3 The role of the volatility δ^Φ

While $(\mu, \sigma, \bar{\theta})$ are constant market parameters, the modeling input $\delta^\Phi_t \not\equiv 0$, $t \geq 0$, turns the preference factor Φ into a random process and, as a result, the optimal portfolio obtains the hedging correction $\pi^\star_t - \frac{1}{\gamma}(\sigma\sigma^\top)^{-1}\mu = \frac{1}{\gamma}(\sigma\sigma^\top)^{-1}\sigma\,\delta^\Phi_t$, $t \geq 0$, through the $\sigma^+_t\sigma_t\delta_x$ term in the Hamiltonian. The Heston model below provides one of many possible ways to specify the modeling input δ^Φ based on a particular market structure. Specifically, setting $\Phi_t = \bar{\varphi}(\Gamma_t)$ yields a non-trivial δ^Φ_t (as it follows from Itô's formula) and, in turn, a state-dependent hedging component in the optimal portfolio process.

4.5 Heston stochastic volatility

We conclude with an incomplete market model with Heston type stochastic volatility. To this end, we let $W = (W^S, W^\perp)$ be an $(n+1)$-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

The variance factor Γ follows the Cox–Ingersoll–Ross SDE

$$d\Gamma_t = b(\bar{\kappa} - \Gamma_t)\,dt + a\sqrt{\Gamma_t}\big(\rho^\top dW^S_t + \sqrt{1-|\rho|^2}\,dW^\perp_t\big), \qquad t \geq 0, \qquad \Gamma_0 > 0, \tag{138}$$

with $b, \bar{\kappa}, a > 0$, $\rho \in \mathbb{R}^n$, $|\rho| \leq 1$, satisfying the Feller condition

$$2b\bar{\kappa} \geq a^2, \tag{139}$$

where Γ_0 is deterministic.

The price processes of the risky assets $S = (S_1, ..., S_n)$ satisfy

$$dS_t = \mathrm{diag}(S_t)\big(\sigma_0\, q\,\Gamma_t\,dt + \sqrt{\Gamma_t}\,\sigma_0\,dW^S_t\big), \tag{140}$$

with $\sigma_0 \in \mathbb{R}^{n\times n}$ being invertible and $q \in \mathbb{R}^n$.

The asset volatility matrix is then given by $\sigma_t = \sqrt{\Gamma_t}\,(\sigma_0, 0) \in \mathbb{R}^{n\times(n+1)}$, the market price of risk θ satisfies

$$\theta_t = \sigma^\top_t(\sigma_t\sigma^\top_t)^{-1}\mu_t = \sqrt{\Gamma_t}\,\bar{\theta}, \qquad \bar{\theta} = (q, 0)^\top \in \mathbb{R}^{n+1}, \qquad |\theta_t|^2 = \Gamma_t\,|q|^2, \tag{141}$$

and the traded subspace projection is $\sigma^+_t\sigma_t = \mathrm{diag}(I_n, 0)$, so that $I_{n+1} - \sigma^+_t\sigma_t$ projects onto the unspanned direction $W^\perp$.

We choose a twice continuously differentiable function $\bar{\varphi} : (0,\infty) \to (0,\infty)$, to be determined below, and a constant $\bar{\Psi} > 0$, and introduce

$$U(t,x) = \bar{\varphi}(\Gamma_t)u(x) \qquad \text{and} \qquad F(t,c,u) = \bar{\Psi}\,f(c,u), \qquad t \geq 0,\ x > 0, \tag{142}$$

with $u(x) = x^{1-\gamma}/(1-\gamma)$ and f as in (95).

A direct application of Itô's formula gives

$$\begin{aligned} b^\Phi_t &= \frac{b(\bar{\kappa} - \Gamma_t)\bar{\varphi}'(\Gamma_t) + \frac{1}{2}a^2\Gamma_t\bar{\varphi}''(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}, \qquad \text{and} \\ \delta^\Phi_t &= \frac{a\sqrt{\Gamma_t}\,\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\big(\rho, \sqrt{1-|\rho|^2}\big)^\top, \qquad t \geq 0. \end{aligned} \tag{143}$$

Substituting (143) and (141) into the scalar drift condition (104) of Theorem 4.4 and using

$$|\sigma^+_t\sigma_t\delta^\Phi_t|^2 = \left(\frac{a\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\right)^2\Gamma_t|\rho|^2, \qquad t \geq 0,$$

we eliminate the common factor $\bar{\varphi}(\Gamma_t)$ and obtain the stationary singular ODE

$$\begin{aligned} \frac{a^2 x}{2}\,\bar{\varphi}''(x) + \Big(b(\bar{\kappa}-x) + \tfrac{(1-\gamma)\,a\,q^\top \rho}{\gamma}\,x\Big)\bar{\varphi}'(x) + \frac{(1-\gamma)a^2|\rho|^2 x}{2\gamma}\,\frac{\bar{\varphi}'(x)^2}{\bar{\varphi}(x)} \\ + \Big(\tfrac{(1-\gamma)|q|^2}{2\gamma}\,x - \lambda\bar{\Psi}\Big)\bar{\varphi}(x) \;=\; \frac{1-\gamma}{1-\eta}\,\bar{\Psi}^\eta\,\bar{\varphi}(x)^{1-\eta/\lambda}. \end{aligned} \tag{144}$$

The substitution $\bar{h} = \bar{\varphi}^{\,\alpha_\rho}$ with $\alpha_\rho = 1 + (1-\gamma)|\rho|^2/\gamma$ transforms (144) into the semilinear ODE

$$\frac{a^2 x}{2}\,\bar{h}''(x) + \Big(b(\bar{\kappa}-x) + \tfrac{(1-\gamma)\,a\,q^\top \rho}{\gamma}\,x\Big)\bar{h}'(x) + \alpha_\rho\Big(\tfrac{(1-\gamma)|q|^2}{2\gamma}\,x - \lambda\bar{\Psi}\Big)\bar{h}(x) = \frac{\alpha_\rho(1-\gamma)}{1-\eta}\,\bar{\Psi}^\eta\,\bar{h}(x)^{1-\eta/(\alpha_\rho\lambda)}. \tag{145}$$

The affine regime below is inspired by the analysis of Seiferling and Seifried [28].

Assumption 4.17 (Affine forward Epstein–Zin regime). The model parameters satisfy $\gamma > 1$, $q \neq 0$,

$$\frac{\eta}{\lambda} = \alpha_\rho = \frac{\gamma + (1-\gamma)|\rho|^2}{\gamma}, \tag{146}$$

$$\widehat{\kappa} = b - \frac{(1-\gamma)\,a\,q^\top \rho}{\gamma} > 0, \qquad \widehat{c} = \alpha_\rho\,\frac{(1-\gamma)\,|q|^2}{2\gamma}, \tag{147}$$

and the Feller condition (139).

Assumption 4.17 identifies an affine solvability regime for the stationary ODE (144). Note that we make no claim that it represents all possible solutions for this problem. Since $\gamma > 1$ and $|\rho| \le 1$, one has $\alpha_\rho \in (0,1]$, and (146) forces $\eta = 1 - \alpha_\rho(\gamma-1) \in (0,1)$, which requires $\alpha_\rho(\gamma-1) < 1$. Moreover, $\widehat{c} < 0$, since $\gamma > 1$ and $q \neq 0$.

This affine restriction also has a transparent preference interpretation. Under Assumption 4.17,

$$\eta = 1 - \alpha_\rho(\gamma-1) = 2 - \gamma + \frac{(\gamma-1)^2}{\gamma}|\rho|^2, \qquad \eta - \frac{1}{\gamma} = -\frac{(\gamma-1)^2}{\gamma}(1-|\rho|^2).$$

Consequently, $0 < \eta \le 1/\gamma < 1$ and $0 < \lambda \le 1$. Equality $\eta = 1/\gamma$ occurs when $|\rho| = 1$, in which case the variance shock, that is, the Brownian noise driving the variance process Γ in (138), is fully spanned by the traded assets, and the coefficient multiplying the unspanned Brownian motion $W^\perp$ in the state price density (154) vanishes. If $|\rho| < 1$, the model contains unspanned volatility risk and the affine restriction requires $\eta < 1/\gamma$, or, equivalently, $\lambda < 1$. This connection between spanning and preferences is a property of the explicit affine solvability regime, and not a general restriction of the forward Epstein–Zin recursive preferences.

In particular, when $|\rho| < 1$ then $\gamma\eta < 1$, so the forward Epstein–Zin aggregator is concave, rather than convex, in its continuation value; see Appendix C. Therefore, the convex variational results of Section 3.4 do not apply in this incomplete affine setting. The state price density in Theorem 4.19 is obtained directly from the discounted marginal utility identity of Theorem 3.12, which, we recall, does not require either convexity or concavity in the continuation value. At the limiting case $|\rho| = 1$, the variance shock is fully spanned, and one has $\gamma\eta = 1$. The aggregator becomes affine in its continuation value.

Proposition 4.18 (Stationary Markovian Heston solution). Under Assumption 4.17, the stationary ODE (144) admits the following explicit solution.

(i) Linear equation. The transformed equation for $\bar{h}$ becomes

$$\frac{a^2 x}{2}\,\bar{h}''(x) + \Big(b(\bar{\kappa}-x) + \tfrac{(1-\gamma)\,a\,q^\top \rho}{\gamma}\,x\Big)\bar{h}'(x) + \alpha_\rho\Big(\tfrac{(1-\gamma)|q|^2}{2\gamma}\,x - \lambda\bar{\Psi}\Big)\bar{h}(x) = \frac{\alpha_\rho(1-\gamma)}{1-\eta}\,\bar{\Psi}^\eta. \tag{148}$$

(ii) Resolvent. One solution is given by

$$\bar{h}(x) = k\int_0^\infty e^{A(s)-\mathcal{R}(s)x}\,ds, \qquad k = \frac{\alpha_\rho\,(\gamma-1)}{1-\eta}\,\bar{\Psi}^\eta > 0, \tag{149}$$

where

$$\dot{\mathcal{R}}(s) = -\widehat{\kappa}\,\mathcal{R}(s) - \tfrac{1}{2}a^2\mathcal{R}(s)^2 - \widehat{c}, \qquad \mathcal{R}(0) = 0, \tag{150}$$

and

$$\dot{A}(s) = -b\bar{\kappa}\,\mathcal{R}(s) - \eta\bar{\Psi}, \qquad A(0) = 0.$$

(iii) Properties. The function $\bar{h}$ is positive and twice continuously differentiable on $(0, \infty)$, with

$$-\mathcal{R}_\infty \leq \frac{\bar{h}'(x)}{\bar{h}(x)} \leq 0 \qquad \text{and} \qquad \mathcal{R}_\infty = \frac{-\widehat{\kappa} + \sqrt{\widehat{\kappa}^2 - 2a^2\widehat{c}}}{a^2} > 0. \tag{151}$$

Furthermore, the function $\bar{\varphi} = \bar{h}^{1/\alpha_\rho}$ is a positive and twice continuously differentiable solution of (144), and $\bar{\varphi}^{-\eta/\lambda} = 1/\bar{h}$ is locally bounded.

Proof. We first study the two deterministic functions $\mathcal{R}$ and A that enter the resolvent representation and, in turn, verify directly that the resolvent solves the required differential equation.

Step 1. Properties of $\mathcal{R}$ and A. By (146), we have $\frac{\eta}{\alpha_\rho \lambda} = 1$. Consequently, the right-hand side of (145) is the constant $\frac{\alpha_\rho(1-\gamma)}{1-\eta}\bar{\Psi}^\eta = -k$, which is negative since $\gamma > 1$ and $\eta \in (0, 1)$. Thus, (145) reduces to the linear equation (148).

Set

$$Q(r) = \frac{a^2}{2}r^2 + \widehat{\kappa}r + \widehat{c}.$$

The assumptions $\widehat{c} < 0$ and $\widehat{\kappa} > 0$ imply that Q has one negative and one positive root. Its positive root is given by

$$\mathcal{R}_\infty = \frac{-\widehat{\kappa} + \sqrt{\widehat{\kappa}^2 - 2a^2\widehat{c}}}{a^2}.$$

Equation (150) can be written as follows; it holds that

$$\dot{\mathcal{R}}(s) = -Q(\mathcal{R}(s)), \qquad \mathcal{R}(0) = 0, \qquad s \geq 0.$$

For every $r \in [0, \mathcal{R}_\infty)$, $Q(r) < 0$. It follows that $\mathcal{R}$ is increasing as long as $\mathcal{R}(s) < \mathcal{R}_\infty$. Uniqueness for the Riccati equation prevents $\mathcal{R}$ from crossing the stationary solution $s \mapsto \mathcal{R}_\infty$. Therefore,

$$0 \leq \mathcal{R}(s) < \mathcal{R}_\infty \quad \text{for each } s \geq 0, \qquad \text{and} \qquad \lim_{s\to\infty} \mathcal{R}(s) = \mathcal{R}_\infty. \tag{152}$$

The definition of A gives

$$A(s) = -\int_0^s \left(b\bar{\kappa}\mathcal{R}(r) + \eta\bar{\Psi}\right) dr, \qquad s \geq 0.$$

Since $\mathcal{R}(r) \geq 0$, we have

$$A(s) \leq -\eta\bar{\Psi}s, \qquad s \geq 0.$$

Hence, for every $x > 0$ and $s \geq 0$,

$$0 < e^{A(s)-\mathcal{R}(s)x} \leq e^{-\eta\bar{\Psi}s}, \qquad s \geq 0.$$

The function on the right-hand side above is integrable on $(0, \infty)$. The resolvent in (149) is therefore finite and strictly positive.

Step 2. Regularity and logarithmic derivative. The bound in (152) and the preceding exponential estimate allow us to differentiate twice under the integral sign, obtaining, for each $x > 0$,

$$\bar{h}'(x) = -k\int_0^\infty \mathcal{R}(s)e^{A(s)-\mathcal{R}(s)x}\,ds \qquad \text{and}$$

$$\bar{h}''(x) = k\int_0^\infty \mathcal{R}(s)^2 e^{A(s)-\mathcal{R}(s)x}\,ds.$$

Thus, $\bar{h}$ is twice continuously differentiable. Moreover,

$$\frac{\bar{h}'(x)}{\bar{h}(x)} = -\frac{\int_0^\infty \mathcal{R}(s)e^{A(s)-\mathcal{R}(s)x}\,ds}{\int_0^\infty e^{A(s)-\mathcal{R}(s)x}\,ds}, \qquad x > 0.$$

The quotient on the right-hand side is the negative of a weighted average of the values $\mathcal{R}(s)$. Since $0 \le \mathcal{R}(s) \le \mathcal{R}_\infty$, $s \ge 0$, we obtain

$$-\mathcal{R}_\infty \le \frac{\bar{h}'(x)}{\bar{h}(x)} \le 0, \qquad x > 0,$$

and (151) follows.

Step 3. Derivation of the differential equation. Let

$$K_s(x) = e^{A(s) - \mathcal{R}(s)x}, \qquad s \ge 0,\ x > 0.$$

Then,

$$\partial_x K_s(x) = -\mathcal{R}(s) K_s(x), \qquad \partial_{xx} K_s(x) = \mathcal{R}(s)^2 K_s(x), \qquad s \ge 0,\ x > 0.$$

Substitution into the differential operator on the left-hand side of (148) yields

$$\begin{aligned}
&\frac{a^2 x}{2}\partial_{xx}K_s(x) + \big(b\bar{\kappa} - \widehat{\kappa}x\big)\partial_x K_s(x) + \big(\widehat{c}x - \eta\bar{\Psi}\big)K_s(x) \\
&\quad = \Big[-b\bar{\kappa}\mathcal{R}(s) - \eta\bar{\Psi} + x\big(\tfrac{a^2}{2}\mathcal{R}(s)^2 + \widehat{\kappa}\mathcal{R}(s) + \widehat{c}\big)\Big]K_s(x) \\
&\quad = \big(\dot{A}(s) - \dot{\mathcal{R}}(s)x\big)K_s(x) \\
&\quad = \partial_s K_s(x), \qquad s \ge 0,\ x > 0,
\end{aligned}$$

where, in the second equality, we used the defining equations for A and $\mathcal{R}$. Since $\bar{h}(x) = k\int_0^\infty K_s(x)\,ds$, the same differential operator applied to $\bar{h}$ is equal to

$$\begin{aligned}
k\int_0^\infty \partial_s K_s(x)\,ds &= k\Big(\lim_{s\to\infty} K_s(x) - K_0(x)\Big) \\
&= -k \\
&= \frac{\alpha_\rho(1-\gamma)}{1-\eta}\bar{\Psi}^\eta.
\end{aligned}$$

Here, $K_0(x) = 1$, while the exponential estimate in Step 1 gives $\lim_{s\to\infty} K_s(x) = 0$. This proves that $\bar{h}$ solves (148).

Finally, if we define $\bar{\varphi} = \bar{h}^{1/\alpha_\rho}$, which inverts the transformation $\bar{h} = \bar{\varphi}^{\alpha_\rho}$ which was used to derive (145), then $\bar{\varphi}$ is a positive, twice continuously differentiable solution of (144). Moreover, (146) gives

$$\bar{\varphi}^{-\eta/\lambda} = \bar{h}^{-1}, \qquad x > 0.$$

The right-hand side is locally bounded because $\bar{h}$ is continuous and strictly positive on $(0,\infty)$. □

Theorem 4.19 (Forward recursive utility under Heston volatility). Assume the Heston market specification (138)–(141) and recall Assumption 4.17. Let $\bar{\varphi}$ be the positive stationary Markovian solution in Proposition 4.18. Then, the pair (U, F) in (142) is a consistent forward aggregator system on $\mathcal{U}_\gamma$. The optimal control policies are

$$\pi_t^\star = \frac{1}{\gamma}(\sigma_0\sigma_0^\top)^{-1}\sigma_0\Big(q + a\rho\,\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\Big), \qquad \check{c}_t^\star = \frac{\bar{\Psi}^\eta}{\bar{\varphi}(\Gamma_t)^{\eta/\lambda}}, \qquad t \ge 0. \tag{153}$$

The state price density $Y^\star$ satisfies

$$\frac{dY_t^\star}{Y_t^\star} = -\sqrt{\Gamma_t}\,q^\top dW_t^S + \sqrt{1-|\rho|^2}\,a\sqrt{\Gamma_t}\,\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\,dW_t^\perp, \qquad Y_0^\star = U_x(0,x), \qquad t \ge 0. \tag{154}$$

Equivalently,

$$Y_t^\star = \bar{\varphi}(\Gamma_t)(X_t^\star)^{-\gamma}\kappa_t^\star \qquad \text{and} \qquad \kappa_t^\star = \exp\left(-\int_0^t \big((1-\lambda)\check{c}_s^\star + \lambda\bar{\Psi}\big)\,ds\right), \qquad t \ge 0.$$

Proof. We verify the drift equation, the viability conditions, the feedback controls, and the state price density separately.

Step 1. Local characteristics of Φ. Set

$$\Phi_t = \bar{\varphi}(\Gamma_t), \qquad \varphi = \Phi_0 = \bar{\varphi}(\Gamma_0), \qquad t \geq 0.$$

Applying Itô's formula and using (138) gives

$$\begin{aligned} d\Phi_t = &\left(b(\bar{\kappa} - \Gamma_t)\bar{\varphi}'(\Gamma_t) + \frac{a^2\Gamma_t}{2}\bar{\varphi}''(\Gamma_t) \right) dt \\ &+ a\sqrt{\Gamma_t}\bar{\varphi}'(\Gamma_t)\left(\rho^\top dW_t^S + \sqrt{1-|\rho|^2}\, dW_t^\perp \right), \qquad t \geq 0. \end{aligned}$$

Dividing the drift and the Brownian coefficient by $\Phi_t = \bar{\varphi}(\Gamma_t)$ yields

$$\begin{aligned} b_t^\Phi &= \frac{b(\bar{\kappa} - \Gamma_t)\bar{\varphi}'(\Gamma_t) + \frac{a^2\Gamma_t}{2}\bar{\varphi}''(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}, \qquad t \geq 0, \qquad \text{and} \\ \delta_t^\Phi &= \frac{a\sqrt{\Gamma_t}\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\left(\rho, \sqrt{1-|\rho|^2} \right)^\top, \qquad t \geq 0, \end{aligned}$$

and (143) follows.

Step 2. Verification of the scalar drift equation. The projection onto the traded Brownian directions removes the last component of δ_t^Φ. Hence

$$\sigma_t^+\sigma_t\delta_t^\Phi = \frac{a\sqrt{\Gamma_t}\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}(\rho, 0)^\top, \qquad t \geq 0.$$

Since $\theta_t = \sqrt{\Gamma_t}(q,0)^\top$, we obtain

$$\theta_t + \sigma_t^+\sigma_t\delta_t^\Phi = \sqrt{\Gamma_t}\left(q + a\rho\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}, 0 \right)^\top, \qquad t \geq 0.$$

Therefore,

$$\begin{aligned} \left|\theta_t + \sigma_t^+\sigma_t\delta_t^\Phi\right|^2 &= \Gamma_t \left| q + a\rho\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)} \right|^2 \\ &= \Gamma_t|q|^2 + 2a\Gamma_t q^\top\rho\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)} + a^2\Gamma_t|\rho|^2\frac{\bar{\varphi}'(\Gamma_t)^2}{\bar{\varphi}(\Gamma_t)^2}. \end{aligned}$$

We now substitute the above expression and the drift from Step 1 into the scalar drift equation (104). After multiplying by $\bar{\varphi}(\Gamma_t)$, the drift condition is equivalent to

$$b(\bar{\kappa} - \Gamma_t)\bar{\varphi}'(\Gamma_t) + \frac{a^2\Gamma_t}{2}\bar{\varphi}''(\Gamma_t) = \frac{1-\gamma}{1-\eta}\bar{\Psi}^\eta\bar{\varphi}(\Gamma_t)^{1-\eta/\lambda} - \frac{1-\gamma}{2\gamma}\Gamma_t\bar{\varphi}(\Gamma_t)\left| q + a\rho\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)} \right|^2 + \lambda\bar{\Psi}\bar{\varphi}(\Gamma_t), \qquad t \geq 0.$$

Rearranging terms and expanding the squared norm show that this identity is precisely (144) evaluated at $x = \Gamma_t$, where

$$\begin{aligned} &\frac{a^2x}{2}\bar{\varphi}''(x) + \left(b(\bar{\kappa} - x) + \frac{(1-\gamma)aq^\top\rho}{\gamma}x \right)\bar{\varphi}'(x) + \frac{(1-\gamma)a^2|\rho|^2x}{2\gamma}\frac{\bar{\varphi}'(x)^2}{\bar{\varphi}(x)} \\ &+ \left(\frac{(1-\gamma)|q|^2}{2\gamma}x - \lambda\bar{\Psi} \right)\bar{\varphi}(x) = \frac{1-\gamma}{1-\eta}\bar{\Psi}^\eta\bar{\varphi}(x)^{1-\eta/\lambda}, \qquad x > 0. \end{aligned}$$

By Proposition 4.18, $\bar{\varphi}$ solves this equation on $(0,\infty)$. Since $\Gamma_t > 0$, $t \geq 0$, the scalar drift condition (104) follows. Therefore, Theorem 4.4 (i) implies that $U(t,x) = \bar{\varphi}(\Gamma_t)u(x)$ solves the forward Epstein–Zin recursive SPDE (97).

Step 3. Viability and admissibility. The process $\Psi_t = \bar{\Psi}$ is strictly positive and constant, and, hence, it satisfies (V1) and the part of (V2) that concerns Ψ. Proposition 4.18 gives

$$\frac{\bar{\varphi}'(x)}{\bar{\varphi}(x)} = \frac{1}{\alpha_\rho}\frac{\bar{h}'(x)}{\bar{h}(x)}, \qquad x > 0.$$

The logarithmic derivative on the right-hand side is bounded by (151). Under the Feller condition, Γ_t is strictly positive and has continuous paths. Thus, on every finite time interval, Γ has a finite pathwise maximum and a strictly positive pathwise minimum. Then, the formula for δ_t^Φ in Step 1 shows that δ^Φ is pathwise bounded on every finite time interval, and the remaining part of (V2) follows.

Using that $\Phi_t = \bar{\varphi}(\Gamma_t)$ solves (105), Proposition 4.2 gives, for $0 \le t < \tau$,

$$\bar{\varphi}(\Gamma_0)^{\eta/\lambda} - \int_0^t \bar{\Psi}^\eta \mathcal{E}_u^{-1}\, du = \Phi_t^{\eta/\lambda}\mathcal{E}_t^{-1}, \qquad 0 \le t < \tau,$$

where τ is the positivity lifetime of Φ introduced in (108). Suppose that $\tau < \infty$. By continuity and the local finiteness and strict positivity of $\mathcal{E}$, letting $t \uparrow \tau$ yields

$$0 = \Phi_\tau^{\eta/\lambda}\mathcal{E}_\tau^{-1} = \bar{\varphi}(\Gamma_\tau)^{\eta/\lambda}\mathcal{E}_\tau^{-1},$$

which contradicts the strict positivity of $\bar{\varphi}(\Gamma_\tau)$ and $\mathcal{E}_\tau$. Thus, $\tau = \infty$, and (V3) follows.

Proposition 4.18 also gives

$$\bar{\varphi}(x)^{-\eta/\lambda} = \bar{h}(x)^{-1}, \qquad x > 0.$$

This function is locally bounded on $(0,\infty)$. Since each path of Γ remains in a compact subset of $(0,\infty)$ on a finite time interval, the optimal relative consumption

$$\check{c}_t^\star = \bar{\Psi}^\eta \bar{\varphi}(\Gamma_t)^{-\eta/\lambda}, \qquad t \ge 0,$$

is pathwise bounded on any finite time interval. The same is true for the discount rate

$$\alpha_t^\star = (1-\lambda)\check{c}_t^\star + \lambda\bar{\Psi}, \qquad t \ge 0.$$

Together with the pathwise bounds for Γ, δ^Φ, and θ, these estimates verify the input bounds of Assumption 3.13 with the constants specified in Corollary 4.8. In turn, Proposition 4.3 and the homothetic relations in the proof of Theorem 4.4 verify Assumptions 2.9 and 3.2, while Step 2 verifies the HJB SPDE. Theorem 3.14 implies that the feedback control pair is admissible and optimal.

Step 4. Feedback controls and state price density. Substituting the Heston model parameters in (118) gives

$$\begin{aligned}\pi_t^\star &= \frac{1}{\gamma}\left(\Gamma_t\sigma_0\sigma_0^\top\right)^{-1}\left(\Gamma_t\sigma_0 q + a\Gamma_t\sigma_0\rho\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\right)\\ &= \frac{1}{\gamma}(\sigma_0\sigma_0^\top)^{-1}\sigma_0\left(q + a\rho\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\right), \qquad t \ge 0.\end{aligned}$$

Similarly, we obtain

$$\check{c}_t^\star = \frac{\bar{\Psi}^\eta}{\bar{\varphi}(\Gamma_t)^{\eta/\lambda}}, \qquad t \ge 0,$$

and (153) follows.

For the dual control process, (65) and the homothetic identity $\delta_x(t,x)/U_x(t,x) = \delta_t^\Phi$ imply

$$\begin{aligned}\nu_t^\star &= (I_{n+1} - \sigma_t^+\sigma_t)\delta_t^\Phi\\ &= \left(0, \sqrt{1-|\rho|^2}\, a\sqrt{\Gamma_t}\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\right)^\top, \qquad t \ge 0.\end{aligned}$$

Therefore, (69) gives

$$\begin{aligned}\frac{dY_t^\star}{Y_t^\star} &= (-\theta_t + \nu_t^\star)^\top dW_t\\ &= -\sqrt{\Gamma_t}\, q^\top dW_t^S + \sqrt{1-|\rho|^2}\, a\sqrt{\Gamma_t}\frac{\bar{\varphi}'(\Gamma_t)}{\bar{\varphi}(\Gamma_t)}\, dW_t^\perp, \qquad t \ge 0,\end{aligned}$$

and (154) follows. Finally,

$$U_x(t, X_t^\star) = \bar{\varphi}(\Gamma_t)(X_t^\star)^{-\gamma}, \qquad t \ge 0.$$

Combining this identity with Theorem 3.12 and Corollary 4.10 gives

$$Y_t^\star = \bar\varphi(\Gamma_t)(X_t^\star)^{-\gamma}\kappa_t^\star \qquad \text{and} \qquad \kappa_t^\star = \exp\left(-\int_0^t \big((1-\lambda)\check c_s^\star + \lambda\bar\Psi\big)\,ds\right), \qquad t\ge 0,$$

and we easily conclude. □

Remark 4.20. The optimal portfolio process in (153) decomposes into the Merton component $\frac{1}{\gamma}(\sigma_0\sigma_0^\top)^{-1}\sigma_0 q$ and an intertemporal hedging component proportional to $\rho\,\bar\varphi'(\Gamma_t)/\bar\varphi(\Gamma_t)$. The consumption-to-wealth ratio depends on Γ_t through $\bar\varphi(\Gamma_t)^{-\eta/\lambda}$; and the orthogonal part $\sqrt{1-|\rho|^2}\,a\sqrt{\Gamma_t}\,\bar\varphi'/\bar\varphi$ of $Y^\star$ in (154) is the unspanned Heston component generated by discounted marginal utility (Theorem 3.12). The myopic demand is scaled directly by $1/\gamma$ and does not depend on η. The EIS parameter η nevertheless affects the stationary solution $\bar\varphi$ through (144). It therefore enters the portfolio indirectly through the intertemporal hedging term, and affects the optimal consumption directly through $\bar\Psi^\eta/\bar\varphi^{\eta/\lambda}$. Thus the risk aversion parameter γ governs the immediate market price of risk exposure, whereas the EIS parameter η operates through consumption substitution and the hedge against changes in the forward preferences.

Remark 4.21. The nonlinear term in (144) contains $\bar\varphi^{-\eta/\lambda}$, and the logarithmic transform $g=\log\bar\varphi$ is not invariant under additive shifts $g\mapsto g+C$. Therefore, the usual ergodic BSDE reduction is not natural here. An infinite-horizon BSDE with a level-dependent monotone driver is a more plausible representation away from the affine locus. Indeed, let

$$\mathcal{L}^{\widehat\kappa}v(x) = \frac{a^2x}{2}v''(x) + \big(b\bar\kappa - \widehat\kappa x\big)v'(x), \qquad x>0,$$

and set $z=a\sqrt{x}\,g'(x)$. Dividing (144) by $\bar\varphi$, we formally deduce that g solves

$$\mathcal{L}^{\widehat\kappa}g(x) + \mathfrak{f}\big(x,g(x),a\sqrt{x}\,g'(x)\big)=0, \qquad x>0,$$

where

$$\mathfrak{f}(x,y,z) = \frac{\alpha_\rho}{2}|z|^2 + \frac{(1-\gamma)|q|^2}{2\gamma}x - \lambda\bar\Psi - \frac{1-\gamma}{1-\eta}\bar\Psi^\eta e^{-(\eta/\lambda)y}. \tag{155}$$

The nonlinear term has the exact monotonicity property, in that

$$\partial_y\mathfrak{f}(x,y,z) = -\bar\Psi^\eta e^{-(\eta/\lambda)y} < 0, \tag{156}$$

since $\eta/\lambda=(\eta-1)/(1-\gamma)$. Thus, the recursive term entirely determines the level that the standard ergodic formulation would not determine. However, the monotonicity is not uniform since its modulus degenerates in one tail of y. Moreover, the driver has quadratic growth in z and an unbounded linear dependence on the CIR state. Therefore, establishing an infinite horizon representation requires a suitable Lyapunov or properness class. The existing infinite horizon BSDE representation of homothetic forward criteria in [14] does not directly cover this combination, and we leave this problem for future research.

The constant coefficient case makes the associated selection mechanism transparent. When

$$\dot\Xi_t = -\bar\Psi^\eta + \bar\chi\Xi_t, \qquad \bar\chi>0, \qquad t\ge 0,$$

the critical point $\Xi^{\mathrm{crit}}=\bar\Psi^\eta/\bar\chi$ is an unstable equilibrium of this forward equation, in the sense that any trajectory started away from it diverges as $t\uparrow\infty$. If, however, the same deterministic equation is considered backward from infinity, then boundedness selects this critical solution; indeed, it is the only solution that remains bounded on $[0,\infty)$. The other viable forward trajectories have the form

$$\Xi_t = \Xi^{\mathrm{crit}} + Ce^{\bar\chi t}, \qquad C>0, \qquad t\ge 0,\ \bar\chi>0,$$

and carry a residual preference component. Hence the bounded backward construction selects the critical solution $B_\infty=0$, whereas the forward formulation also retains the residual solutions $B_\infty>0$. In the nonlinear Heston case this "stable–unstable" interpretation may serve as a prospective selection principle rather than a direct sum decomposition, since both Z and δ^Φ depend endogenously on the solution.

The dual stationary equation for $\widetilde\Phi_t=\bar\varphi(\Gamma_t)^{1/\gamma}$ is obtained from (130)–(131) by setting $\widetilde\Phi_t=\bar\varphi(\Gamma_t)^{1/\gamma}$.

Appendix A: Regular random fields and detailed proofs for the forward recursive utilities

We first recall the regularity classes used in Section 2 and then summarize the stochastic calculus details underlying the verification and normalization results. We keep the time and state arguments visible throughout. For example, when a random field is evaluated along a wealth process, we write, $U_x(t, X_t)$.

Regular random fields

We follow Kunita [12] and the formulation in [13, Appendix A]. Let $G(t, x)$, $t \geq 0$, $x > 0$, be a progressive random field, $m \in \mathbb{N}_0$ and $\varepsilon \in (0, 1]$. We use the convention $\partial_x^0 G = G$.

The field G is called $\mathcal{C}^m$-regular if, $\mathbb{P}$-almost surely, the map $x \mapsto G(t, x)$ is m-times continuously differentiable for each $t \geq 0$. It is called $\mathcal{C}^{m,\varepsilon}$-regular if, on the same almost-sure event, $\partial_x^m G(t, \cdot)$ is locally ε-Hölder continuous for each $t \geq 0$. For every compact set $K \subset (0, \infty)$ and $t \geq 0$, define

$$\|G\|_{m,\varepsilon:K}(t) = \sup_{x \in K} \frac{|G(t,x)|}{1+|x|} + \sum_{j=1}^{m} \sup_{x \in K} |\partial_x^j G(t,x)| + \sup_{\substack{x,y \in K \\ x \neq y}} \frac{|\partial_x^m G(t,x) - \partial_x^m G(t,y)|}{|x-y|^{\varepsilon}}.$$

A $\mathcal{C}^{m,\varepsilon}$-regular field G belongs to $\mathcal{K}_{\mathrm{loc}}^{m,\varepsilon}$ if, for every compact $K \subset (0, \infty)$ and each $T > 0$,

$$\int_0^T \|G\|_{m,\varepsilon:K}(t)\, dt < \infty \qquad \mathbb{P}\text{-a.s.},$$

and it belongs to $\overline{\mathcal{K}}_{\mathrm{loc}}^{m,\varepsilon}$ if

$$\int_0^T \|G\|_{m,\varepsilon:K}(t)^2\, dt < \infty.$$

For vector-valued random fields, these requirements are understood componentwise.

Let now $U(t, x)$, $t \geq 0$, $x > 0$, be a progressively measurable Itô semimartingale random field with decomposition

$$dU(t,x) = b(t,x)\, dt + \delta(t,x)^{\top} dW_t, \qquad t \geq 0,\ x > 0.$$

The field U is called a $\mathcal{K}_{\mathrm{loc}}^{m,\varepsilon}$-semimartingale random field if $U(0, \cdot)$ is $\mathcal{C}^{m,\varepsilon}$-regular and

$$\int_0^{\cdot} b(s,\cdot)\, ds \in \mathcal{K}_{\mathrm{loc}}^{m,\varepsilon} \qquad \text{and} \qquad \int_0^{\cdot} \delta(s,\cdot)^{\top} dW_s \in \overline{\mathcal{K}}_{\mathrm{loc}}^{m,\varepsilon}.$$

If U is a $\mathcal{K}_{\mathrm{loc}}^{m,\varepsilon}$-semimartingale random field, then, for every $\varepsilon' < \varepsilon$,

$$(b, \delta) \in \mathcal{K}_{\mathrm{loc}}^{m,\varepsilon'} \times \overline{\mathcal{K}}_{\mathrm{loc}}^{m,\varepsilon'}.$$

Conversely, if (b, δ) is in $\mathcal{K}_{\mathrm{loc}}^{m,\varepsilon} \times \overline{\mathcal{K}}_{\mathrm{loc}}^{m,\varepsilon}$ then U is a $\mathcal{K}_{\mathrm{loc}}^{m,\varepsilon'}$-semimartingale random field for every $\varepsilon' < \varepsilon$.

In particular, if U is $\mathcal{C}^2$-regular and a $\mathcal{K}_{\mathrm{loc}}^{1,\varepsilon}$-semimartingale random field, then, the Itô–Ventzel formula applies along every continuous Itô process X with values in $(0, \infty)$ and gives, for $t \geq 0$,

$$dU(t, X_t) = b(t, X_t)\, dt + \delta(t, X_t)^{\top} dW_t + U_x(t, X_t)\, dX_t + \frac{1}{2} U_{xx}(t, X_t)\, d\langle X \rangle_t + \delta_x(t, X_t)^{\top} d\langle W, X \rangle_t.$$

Proof of Theorem 2.6

Proof. Fix an admissible control pair $(\pi, c) \in \mathcal{A}$ and write $X_t = X_t^{\pi,c}$. We first compute the dynamics of the forward recursive criterion process. We, then, consider a specific family of stopping times to verify the local consistency.

Step 1. Dynamics along an admissible wealth process. The Itô–Ventzel formula applied to the random field U along X gives

$$\begin{aligned} dU(t, X_t) &= b(t, X_t)\, dt + \delta(t, X_t)^{\top} dW_t \\ &\quad + U_x(t, X_t)\, dX_t + \frac{1}{2} U_{xx}(t, X_t)\, d\langle X \rangle_t + \delta_x(t, X_t)^{\top} d\langle W, X \rangle_t, \qquad t \geq 0. \end{aligned}$$

This yields

$$\begin{aligned} dU(t,X_t) = \Big(& b(t,X_t) + U_x(t,X_t)\big(X_t\pi_t^\top \sigma_t\theta_t - c_t\big) \\ &+ \frac{1}{2}U_{xx}(t,X_t)\big|X_t\sigma_t^\top \pi_t\big|^2 + X_t\pi_t^\top \sigma_t\delta_x(t,X_t)\Big)\,dt \\ &+ \Big(\delta(t,X_t) + U_x(t,X_t)X_t\sigma_t^\top \pi_t\Big)^\top dW_t, \qquad t \geq 0. \end{aligned}$$

After adding the aggregator term, we obtain

$$dV_t^{\pi,c} = H(t,X_t,\pi_t,c_t)\,dt + (Z_t^{\pi,c})^\top dW_t, \qquad t \geq 0, \tag{157}$$

where

$$\begin{aligned} H(t,X_t,\pi_t,c_t) = & \, b(t,X_t) + U_x(t,X_t)\big(X_t\pi_t^\top \sigma_t\theta_t - c_t\big) \\ &+ \frac{1}{2}U_{xx}(t,X_t)\big|X_t\sigma_t^\top \pi_t\big|^2 + X_t\pi_t^\top \sigma_t\delta_x(t,X_t) \\ &+ F\big(t,c_t,U(t,X_t),Z(t,X_t,\pi_t)\big), \qquad t \geq 0. \end{aligned}$$

Here $Z(t,x,\pi)$ is the pointwise map introduced in Section 2.2, so that its value along the control pair is $Z(t,X_t,\pi_t) = Z_t^{\pi,c}$, with $Z^{\pi,c}$ as in (16).

Step 2. Local integrability. Fix $T > 0$. Since X is continuous and strictly positive, its range on $[0,T]$ is a compact subset of $(0,\infty)$, $\mathbb{P}$-a.s. The local regularity of U and the bounds on the local characteristics established in Appendix A therefore imply that $\sup_{0\leq t\leq T}\big(|U_x(t,X_t)| + |U_{xx}(t,X_t)|\big) < \infty$ and $\int_0^T \big(|b(t,X_t)| + |\delta(t,X_t)|^2 + |\delta_x(t,X_t)|^2\big)\,dt < \infty$, $\mathbb{P}$-a.s.

The admissibility of the control pair (π,c) gives

$$\int_0^T \left(c_t + |\pi_t^\top \sigma_t|^2\right)\,dt < \infty.$$

In addition, the standing market assumption gives

$$\int_0^T |\theta_t|^2\,dt < \infty.$$

Since $|\pi_t^\top \sigma_t\theta_t| \leq |\pi_t^\top \sigma_t|\,|\theta_t|$, $t \geq 0$, the Cauchy–Schwarz inequality shows that the corresponding product is integrable on $[0,T]$. Thus, all terms in the Hamiltonian other than the aggregator are pathwise integrable on $[0,T]$.

Step 3. The local supermartingale property. We assume that (π,c) also satisfies (17). For $n \geq 1$, define

$$\tau_n = \inf\left\{ t \geq 0 \;\middle|\; X_t \notin \left(\frac{1}{n},n\right) \text{ or } \int_0^t |Z_s^{\pi,c}|^2\,ds \geq n \text{ or } \int_0^t |H(s,X_s,\pi_s,c_s)|\,ds \geq n \right\}.$$

The local integrability established in Step 2 and the form of $Z_t^{\pi,c}$ imply

$$\int_0^T |Z_s^{\pi,c}|^2\,ds < \infty \qquad \text{and} \qquad \int_0^T |H(s,X_s,\pi_s,c_s)|\,ds < \infty,$$

$\mathbb{P}$-a.s. for every $T > 0$. Since X is continuous and strictly positive, it follows that τ_n increases to infinity a.s.

On $[0,\tau_n]$, the stochastic integral in (157) is a square integrable martingale. The stopped finite variation term has total variation bounded by n, and $V_{\cdot\wedge\tau_n}^{\pi,c}$ is integrable. Moreover, (23) holds and, hence,

$$H(t,X_t,\pi_t,c_t) \leq 0, \qquad t \geq 0,$$

for $dt \otimes d\mathbb{P}$-a.e. (t,ω). Therefore, $V_{\cdot\wedge\tau_n}^{\pi,c}$ is a supermartingale. Since $\tau_n \uparrow \infty$, the process $V^{\pi,c}$ is a local supermartingale. This proves condition i) of Definition 2.3.

Step 4. Equality along the feedback control pair. Let $X_t^\star$ be the wealth generated by the candidate optimal feedback control pair. Assumption ii) of the theorem and (22) give

$$H\left(t, X_t^\star, \pi^\star(t, X_t^\star), c^\star(t, X_t^\star)\right) = 0, \qquad t \geq 0, \tag{158}$$

for $dt \otimes d\mathbb{P}$-a.e. (t, ω).

All other terms in H are locally integrable by Step 2. Hence, the vanishing of the Hamiltonian in (158) also gives (17) along the feedback control pair.

We now see that equation (157) has zero drift along the feedback control pair. The same stopping times make the stopped criterion process a true martingale. Therefore, $V^{\pi^\star, c^\star}$ is a local martingale. This proves condition ii) of Definition 2.3. Both consistency conditions hold, and the feedback control pair is optimal. ☐

Proof of Proposition 2.8

Proof. We first note that the control class $\mathcal{A}$ is unchanged, since Definition 2.1 involves only the market coefficients and the wealth equation, and makes no reference to the pair (U, F). Next, we prove that the transformation preserves consistency in the forward direction, which would then imply (i) and (ii). In turn, we apply the inverse transformation, and verify the spatial shape of the transformed utility.

Step 1. Transformation of the criterion dynamics. Fix $(\pi, c) \in \mathcal{A}$ satisfying (17), and set

$$X_t = X_t^{\pi,c}, \qquad u_t = U(t, X_t), \qquad z_t = Z_t^{\pi,c}, \qquad t \geq 0.$$

Since $V^{\pi,c}$ is a continuous local supermartingale, there exist a continuous local martingale M and a continuous non-increasing adapted process D, with $D_0 = 0$, such that

$$V_t^{\pi,c} = V_0^{\pi,c} + M_t + D_t, \qquad t \geq 0.$$

The computation in Theorem 2.6 identifies that

$$M_t = \int_0^t z_s^\top dW_s, \qquad t \geq 0,$$

and, thus, we have that

$$du_t = z_t^\top dW_t - F(t, c_t, u_t, z_t)\, dt + dD_t, \qquad t \geq 0. \tag{159}$$

Along an optimal control pair, $D \equiv 0$.

By definition, $\bar{U}(t, X_t) = \varphi(u_t)$. Itô's formula gives

$$d\bar{U}(t, X_t) = \varphi'(u_t)\, du_t + \frac{1}{2}\varphi''(u_t)|z_t|^2\, dt, \qquad t \geq 0. \tag{160}$$

Next, we substitute (159) and use the equality

$$F(t, c_t, u_t, z_t) = f(t, c_t, u_t) + \frac{1}{2}A(u_t)|z_t|^2, \qquad t \geq 0,$$

into (160). Then, the coefficient of $|z_t|^2$ becomes $\frac{1}{2}\left(\varphi''(u_t) - A(u_t)\varphi'(u_t)\right)$ and vanishes since $\varphi''(u) = A(u)\varphi'(u)$. Therefore, we have

$$\begin{aligned} d\bar{U}(t, X_t) &= \varphi'(u_t) z_t^\top dW_t - \varphi'(u_t) f(t, c_t, u_t)\, dt + \varphi'(u_t)\, dD_t \\ &= \varphi'(u_t) z_t^\top dW_t - \bar{F}\big(t, c_t, \bar{U}(t, X_t)\big)\, dt + \varphi'(u_t)\, dD_t. \end{aligned}$$

The stochastic integral is a continuous local martingale. Since $\varphi'(u_t) > 0$ and D is non-increasing, the process

$$\int_0^t \varphi'(u_s)\, dD_s, \qquad t \geq 0,$$

is non-increasing. It, then, follows that the transformed criterion is a local supermartingale, $t \geq 0$, and we easily conclude.

Step 2. Local integrability. On a finite time interval, the continuous process $u_t = U(t, X_t)$, has compact range. Since, on the other hand, φ' is continuous and strictly positive, both $\varphi'(u_t)$ and $1/\varphi'(u_t)$ are bounded on that interval. The continuity of A also makes $A(u_t)$ bounded on a finite time interval. Moreover, $\int_0^T |z_t|^2\,dt < \infty$. Therefore, the identity

$$f(t,c_t,u_t) = F(t,c_t,u_t,z_t) - \frac{1}{2}A(u_t)|z_t|^2, \qquad t \geq 0,$$

implies

$$\int_0^T |f(t,c_t,u_t)|\,dt \leq \int_0^T |F(t,c_t,u_t,z_t)|\,dt - \frac{1}{2}\int_0^T A(u_t)|z_t|^2\,dt < \infty, \qquad t \geq 0.$$

The same identity proves that the local integrability of $f(t,c_t,u_t)$ implies the local integrability of F. Hence,

$$\int_0^T |F(t,c_t,u_t,z_t)|\,dt < \infty \quad \Longleftrightarrow \quad \int_0^T |f(t,c_t,u_t)|\,dt < \infty.$$

On the other hand,

$$\bar{F}\big(t,c_t,\varphi(u_t)\big) = \varphi'(u_t)f(t,c_t,u_t), \qquad t \geq 0.$$

Therefore, the upper and lower pathwise bounds for $\varphi'(u_t)$ yield that

$$\int_0^T |\bar{F}(t,c_t,\bar{U}(t,X_t))|\,dt < \infty \quad \Longleftrightarrow \quad \int_0^T |f(t,c_t,u_t)|\,dt < \infty, \qquad \text{for each } T > 0.$$

Combining the two equivalences proves that the required local integrability can be obtained in both directions between the original and the normalized systems.

Step 3. The inverse transformation. Let $\psi = \varphi^{-1}$. Differentiating $\varphi(\psi(\bar{u})) = \bar{u}$ gives

$$\begin{aligned}
\psi'(\bar{u}) &= \frac{1}{\varphi'(\psi(\bar{u}))}, \\
\psi''(\bar{u}) &= -\frac{\varphi''(\psi(\bar{u}))}{\varphi'(\psi(\bar{u}))^3} \\
&= -A(\psi(\bar{u}))\psi'(\bar{u})^2.
\end{aligned}$$

The martingale coefficient of $\bar{U}(t,X_t) = \varphi(u_t)$ is $\bar{z}_t = \varphi'(u_t)z_t$, $t \geq 0$. Applying Itô's formula to $U(t,X_t) = \psi(\bar{U}(t,X_t))$, and using the above formula for ψ'', produces the term $-\frac{1}{2}A(u_t)|z_t|^2$. This term eliminates the quadratic part of F, while $\psi' > 0$ preserves the sign of the non-increasing finite variation term. Together with Step 2, this proves consistency in the inverse direction and preserves the equality case.

Step 4. Monotonicity and concavity. Fix $t \geq 0$. Since φ is strictly increasing, the strict monotonicity of $U(t,\cdot)$ implies that of $\bar{U}(t,\cdot)$. For $x_1 \neq x_2$ and $a \in (0,1)$, strict concavity of $U(t,\cdot)$ and monotonicity and concavity of φ give

$$\begin{aligned}
\bar{U}\big(t, ax_1 + (1-a)x_2\big) &> \varphi\big(aU(t,x_1) + (1-a)U(t,x_2)\big) \\
&\geq a\bar{U}(t,x_1) + (1-a)\bar{U}(t,x_2),
\end{aligned}$$

and, thus, $\bar{U}(t,\cdot)$ is strictly concave. We note that this argument does not require the pointwise condition $U_{xx} < 0$.
□

Proof of Theorem 2.10

Proof. We verify the two Hamiltonian inequalities in Theorem 2.6 and determine when they hold as equalities.

To this end, let $x > 0$, $\pi \in \mathbb{R}^n$, $c > 0$, and for $dt \otimes d\mathbb{P}$-a.e. (t,ω), define

$$v = x\sigma_t^\top \pi \in \operatorname{Im}\sigma_t^\top, \qquad t \geq 0,$$

and

$$\Lambda(t,x) = U_x(t,x)\theta_t + \sigma_t^+\sigma_t\delta_x(t,x), \qquad t \geq 0.$$

Since v belongs to $\operatorname{Im}\sigma_t^\top$, we have

$$v^\top\delta_x(t,x) = v^\top\sigma_t^+\sigma_t\delta_x(t,x), \qquad t \geq 0,\ x > 0.$$

Therefore, the Hamiltonian can be written as

$$H(t,\omega,x,\pi,c) = b(t,x) + \frac{1}{2}U_{xx}(t,x)|v|^2 + v^\top \Lambda(t,x) + F(t,c,U(t,x)) - U_x(t,x)c, \qquad t \geq 0,\ x > 0. \tag{161}$$

Step 1. The investment term. The equation (27) gives that the drift satisfies

$$b(t,x) = \frac{|\Lambda(t,x)|^2}{2U_{xx}(t,x)} - \widetilde{F}\big(t, U_x(t,x), U(t,x)\big), \qquad t \geq 0,\ x > 0.$$

Substituting the above into the Hamiltonian (161) separates the portfolio term from the consumption one. Specifically, the former becomes

$$\frac{|\Lambda(t,x)|^2}{2U_{xx}(t,x)} + \frac{1}{2}U_{xx}(t,x)|v|^2 + v^\top \Lambda(t,x) = \frac{1}{2}U_{xx}(t,x)\left|v + \frac{\Lambda(t,x)}{U_{xx}(t,x)}\right|^2, \qquad t \geq 0,\ x > 0.$$

Since $U_{xx}(t,x) < 0$, this expression is non-positive and becomes equal to zero if and only if

$$v = -\frac{\Lambda(t,x)}{U_{xx}(t,x)}, \qquad t \geq 0,\ x > 0. \tag{162}$$

The vector $\Lambda(t,x)$ belongs to $\operatorname{Im}\sigma_t^\top$. Because σ_t is of full row rank, (162) determines a unique portfolio. In turn, solving $x\sigma_t^\top \pi = v$ gives

$$\begin{aligned}\pi &= -\frac{1}{xU_{xx}(t,x)}(\sigma_t^\top)^+\Lambda(t,x)\\ &= -\frac{U_x(t,x)}{xU_{xx}(t,x)}(\sigma_t^\top)^+\left(\theta_t + \frac{\delta_x(t,x)}{U_x(t,x)}\right), \qquad t \geq 0,\end{aligned}$$

and (30) follows.

Step 2. The consumption term. Set $d = U_x(t,x)$, $t \geq 0$, $x > 0$. Then, $d > 0$ by strict monotonicity and concavity. The definition of Fenchel transform gives, for every $c > 0$,

$$F(t,c,U(t,x)) - dc \leq \widetilde{F}(t,d,U(t,x)), \qquad t \geq 0,\ x > 0.$$

Then,

$$F(t,c,U(t,x)) - U_x(t,x)c - \widetilde{F}\big(t,U_x(t,x),U(t,x)\big) \leq 0, \qquad t \geq 0,\ x > 0.$$

Assumption 2.9 and (29) show that equality holds if and only if

$$c = F_c^{-1}\big(t, U_x(t,x), U(t,x)\big) = c^\star(t,x), \qquad t \geq 0,\ x > 0.$$

Step 3. Application of Theorem 2.6. Steps 1 and 2 prove $H(t,\omega,x,\pi,c) \leq 0$ for every $\pi \in \mathbb{R}^n$ and $c > 0$. Equality holds at $\big(\pi,c\big) = \big(\pi^\star(t,x), c^\star(t,x)\big)$ and, thus, these two statements are exactly (23) and (22).

The optimal feedback portfolio is progressively measurable because it is constructed from the progressively measurable fields $U_x(t,x)$, $U_{xx}(t,x)$, and $\delta_x(t,x)$ together with the predictable processes σ_t and θ_t. The optimal feedback consumption control is progressively measurable because $F_c^{-1}(t,d,u)$ is measurable in (t,ω) and continuous in (d,u), while $U_x(t,x)$ and $U(t,x)$ are progressively measurable.

The theorem assumes that the wealth equation with the candidate optimal feedback controls has a unique strictly positive strong solution and that the resulting feedback control pair is admissible. All the hypotheses of Theorem 2.6 are therefore satisfied. Its conclusion proves consistency and optimality. □

Appendix B: Technical lemmas and detailed proofs for the duality theory

We present the technical lemmas and stochastic calculus computations underlying Proposition 3.4, Theorem 3.7, Proposition 3.10, Theorem 3.12, and Theorem 3.14. Throughout, we work systematically with the dual domain, using the convex conjugate $\widetilde{U}$, its consumption analogue $\widetilde{F}$, and the inverse marginal $I(t,y)$. For transparency, we retain all time, state, and dual arguments in the calculations.

Lemma 3.3 (Conjugacy identities and characteristics). Under Assumption 3.2, the convex dual $\widetilde{U}$ defined in (35) is a semimartingale random field and satisfies (36)–(38).

Proof. We first establish the deterministic conjugacy identities and, in turn, compute the semimartingale characteristics.

Step 1. The Legendre point and its derivatives. Fix (t,ω) in the a.s. event on which Assumption 3.2 holds, let $y>0$. The strictly concave map $x \mapsto U(t,x) - xy$ has the unique maximizer $x = I(t,y)$, characterized by

$$U_x(t, I(t,y)) = y, \qquad y > 0. \tag{163}$$

Thus,

$$\widetilde{U}(t,y) = U(t, I(t,y)) - yI(t,y), \qquad y>0,\ t\geq 0. \tag{164}$$

Differentiating (164) and using (163) gives

$$\widetilde{U}_y(t,y) = U_x(t,I(t,y))I_y(t,y) - I(t,y) - yI_y(t,y) = -I(t,y), \qquad y>0,\ t\geq 0.$$

Differentiating (163) yields

$$U_{xx}(t,I(t,y))I_y(t,y) = 1.$$

$$I_y(t,y) = \frac{1}{U_{xx}(t,I(t,y))}, \qquad \widetilde{U}_{yy}(t,y) = -I_y(t,y) = -\frac{1}{U_{xx}(t,I(t,y))},$$

which are the identities in (36).

Step 2. The diffusion coefficient of $I(t,y)$. For fixed $y>0$, we write the semimartingale decomposition

$$dI(t,y) = \mu^I(t,y)\,dt + \rho^I(t,y)^\top dW_t, \qquad t\geq 0,\ y>0.$$

Differentiating (14) with respect to x gives

$$dU_x(t,x) = b_x(t,x)\,dt + \delta_x(t,x)^\top dW_t, \qquad t\geq 0,\ x>0.$$

Recalling (163) and using the Itô–Ventzel formula therefore gives

$$\begin{aligned}
0 &= b_x(t,I(t,y))\,dt + \delta_x(t,I(t,y))^\top dW_t \\
&\quad + U_{xx}(t,I(t,y))\,dI(t,y) + \frac{1}{2}U_{xxx}(t,I(t,y))\,d\langle I(\cdot,y)\rangle_t \\
&\quad + \delta_{xx}(t,I(t,y))^\top d\langle W, I(\cdot,y)\rangle_t, \qquad t\geq 0,\ y>0.
\end{aligned}$$

Its Brownian coefficient vanishes, and hence

$$\delta_x(t,I(t,y)) + U_{xx}(t,I(t,y))\rho^I(t,y) = 0, \qquad t\geq 0,\ y>0,$$

which implies

$$\begin{aligned}
\rho^I(t,y) &= -\frac{\delta_x(t,I(t,y))}{U_{xx}(t,I(t,y))}, \\
d\langle I(\cdot,y)\rangle_t &= \frac{|\delta_x(t,I(t,y))|^2}{U_{xx}(t,I(t,y))^2}\,dt, \\
d\langle W, I(\cdot,y)\rangle_t &= -\frac{\delta_x(t,I(t,y))}{U_{xx}(t,I(t,y))}\,dt.
\end{aligned} \tag{165}$$

Step 3. Characteristics of $\widetilde{U}(t,y)$. From (164) and the Itô–Ventzel formula, we obtain

$$\begin{aligned}
d\widetilde{U}(t,y) &= dU(t,I(t,y)) - y\,dI(t,y) \\
&= \left(b(t,I(t,y)) + \frac{1}{2}U_{xx}(t,I(t,y))|\rho^I(t,y)|^2 + \delta_x(t,I(t,y))^\top \rho^I(t,y)\right)dt \\
&\quad + \delta(t,I(t,y))^\top dW_t.
\end{aligned}$$

Substituting $\rho^I(t,y) = -\delta_x(t,I(t,y))/U_{xx}(t,I(t,y))$ yields

$$d\widetilde{U}(t,y) = \left(b(t,I(t,y)) - \frac{|\delta_x(t,I(t,y))|^2}{2U_{xx}(t,I(t,y))}\right)dt + \delta(t,I(t,y))^\top dW_t, \qquad t \geq 0,\ y > 0,$$

and (37) follows.

Step 4. Relation between the primal and dual volatilities. Differentiating $\widetilde{\delta}(t,y) = \delta(t,I(t,y))$ with respect to y gives

$$\widetilde{\delta}_y(t,y) = \delta_x(t,I(t,y))I_y(t,y) = \frac{\delta_x(t,I(t,y))}{U_{xx}(t,I(t,y))}, \qquad t \geq 0,\ y > 0.$$

Using that $U_{xx}(t,I(t,y)) = -1/\widetilde{U}_{yy}(t,y)$ gives

$$\delta_x(t,I(t,y)) = -\frac{\widetilde{\delta}_y(t,y)}{\widetilde{U}_{yy}(t,y)},$$

and (38) follows. The identities in Step 1 hold pathwise, while (37) and (38), obtained in Steps 2–4, hold for $dt \otimes d\mathbb{P}$-a.e. (t,ω) and every $y > 0$, as stated above. ☐

Proof of Proposition 3.4

Proof. Fix (t,ω) in a set of full $dt \otimes d\mathbb{P}$-measure on which (37) and (38) hold. Let $y > 0$, and set $P_t = \sigma_t^+\sigma_t$, $Q_t = I_d - P_t$, and $x = I(t,y)$. At $x = I(t,y)$, the conjugacy identities give that

$$U_x(t,x) = y \qquad \text{and} \qquad U(t,x) = \hat{u}(t,y), \qquad t \geq 0,\ y > 0.$$

Using (27), (37), $P_t\theta_t = \theta_t$, and $|\delta_x|^2 = |P_t\delta_x|^2 + |Q_t\delta_x|^2$, we obtain

$$\begin{aligned}
\widetilde{b}(t,y) &= b(t,x) - \frac{|\delta_x(t,x)|^2}{2U_{xx}(t,x)} \\
&= -\widetilde{F}\big(t,y,\hat{u}(t,y)\big) + \frac{y^2|\theta_t|^2 + 2y\theta_t^\top\delta_x(t,x) - |Q_t\delta_x(t,x)|^2}{2U_{xx}(t,x)} \\
&= -\widetilde{F}\big(t,y,\hat{u}(t,y)\big) - \frac{1}{2}y^2\widetilde{U}_{yy}(t,y)|\theta_t|^2 + y\theta_t^\top\widetilde{\delta}_y(t,y) + \frac{|Q_t\widetilde{\delta}_y(t,y)|^2}{2\widetilde{U}_{yy}(t,y)}, \qquad t \geq 0,\ y > 0,
\end{aligned}$$

and (40) follows. The volatility identity is the first identity in (37). ☐

Proof of Theorem 3.7

Proof. We compute the drift of the dual criterion, verify its sign, and identify the case where equality is satisfied.

Step 1. Characteristics of the state price density. Fix $\nu \in \mathcal{A}^{\mathrm{dual}}$, write $Y = Y^\nu$, and set $P_t = \sigma_t^+\sigma_t$ and $Q_t = I_d - P_t$. From (33),

$$d\langle Y\rangle_t = Y_t^2\big(|\theta_t|^2 + |\nu_t|^2\big)dt \qquad \text{and} \qquad d\langle W, Y\rangle_t = Y_t(-\theta_t + \nu_t)dt, \qquad t \geq 0,$$

since $\theta_t^\top\nu_t = 0$, for $dt \otimes d\mathbb{P}$-a.e. (t,ω).

Step 2. Dynamics of the dual criterion. Under Assumption 3.2, the local characteristics $(\widetilde{b},\widetilde{\delta})$ of $\widetilde{U}$ computed in (37) are continuously differentiable in y with locally Hölder derivatives, which are compositions of (b,δ) and their spatial derivatives with the field I. Therefore, the Itô–Ventzel formula gives

$$\begin{aligned}
d\widetilde{U}(t,Y_t) = \Big(&\widetilde{b}(t,Y_t) + \tfrac{1}{2}\widetilde{U}_{yy}(t,Y_t)\,Y_t^2\big(|\theta_t|^2 + |\nu_t|^2\big) \\
&+ Y_t\,\widetilde{\delta}_y(t,Y_t)^\top\big(-\theta_t + \nu_t\big)\Big)dt + \big(\widetilde{Z}_t^\nu\big)^\top dW_t, \qquad t \geq 0.
\end{aligned}$$

The volatility process in the above expansion yields

$$\widetilde{Z}_t^\nu = \widetilde{\delta}(t,Y_t) + \widetilde{U}_y(t,Y_t)Y_t\big(-\theta_t + \nu_t\big), \qquad t \geq 0.$$

Together with (43) and (40), we have

$$d\mathcal{Z}_t^\nu = (\widetilde{Z}_t^\nu)^\top dW_t + \left(\frac{|Q_t\widetilde{\delta}_y(t,Y_t)|^2}{2\widetilde{U}_{yy}(t,Y_t)} + \frac{1}{2}Y_t^2\widetilde{U}_{yy}(t,Y_t)|\nu_t|^2 + Y_t\widetilde{\delta}_y(t,Y_t)^\top\nu_t\right)dt$$
$$= (\widetilde{Z}_t^\nu)^\top dW_t + \frac{|Q_t\widetilde{\delta}_y(t,Y_t) + Y_t\widetilde{U}_{yy}(t,Y_t)\nu_t|^2}{2\widetilde{U}_{yy}(t,Y_t)}\,dt, \qquad t \geq 0. \tag{166}$$

Here $Q_t\nu_t = \nu_t$. Moreover,

$$\widetilde{U}_{yy}(t,y) = -\frac{1}{U_{xx}(t,I(t,y))} > 0, \qquad t \geq 0,\ y > 0.$$

Hence the drift in (166) is nonnegative and vanishes if and only if

$$\nu = -\frac{(I_d - \sigma_t^+\sigma_t)\widetilde{\delta}_y(t,y)}{y\widetilde{U}_{yy}(t,y)},$$

and (44) follows.

For later use, we denote the nonnegative drift in (166) by

$$g_t^\nu = \frac{|Q_t\widetilde{\delta}_y(t,Y_t) + Y_t\widetilde{U}_{yy}(t,Y_t)\nu_t|^2}{2\widetilde{U}_{yy}(t,Y_t)}, \qquad t \geq 0.$$

Step 3. Integrability of the running term. We write

$$d\widetilde{U}(t,Y_t) = a_t^\nu\,dt + (\widetilde{Z}_t^\nu)^\top dW_t, \qquad t \geq 0.$$

Then, equation (166) gives, for $dt \otimes d\mathbb{P}$-a.e. (t,ω),

$$a_t^\nu + \widetilde{F}(t,Y_t,\hat{u}(t,Y_t)) = g_t^\nu \geq 0 \qquad \text{and} \qquad \widetilde{F}(t,Y_t,\hat{u}(t,Y_t))^- \leq (a_t^\nu)^+, \qquad t \geq 0. \tag{167}$$

For every $T > 0$, localization to the compact range of the positive continuous process Y and the random field bounds in Appendix A give $\int_0^T |a_t^\nu|dt < \infty$ $\mathbb{P}$-a.s. Thus (167) yields that the negative part is integrable on every finite horizon. Therefore, condition (42) may fail only through the positive part.

Step 4. Localization. Consider $\nu \in \mathcal{A}^{\text{dual}}$ satisfying (42), and set, for $n \geq 1$,

$$\tau_n = \inf\Big\{t \geq 0 : Y_t \notin \big(\tfrac{1}{n}, n\big) \text{ or } \int_0^t \big|\widetilde{Z}_s^\nu\big|^2\,ds \geq n \text{ or } \int_0^t g_s^\nu\,ds \geq n\Big\}, \qquad t > 0.$$

Fix $T > 0$. Since Y is continuous and strictly positive, its range on $[0,T]$ is a compact subset of $(0,\infty)$, $\mathbb{P}$-a.s.

The local regularity of $\widetilde{U}_y$ and the integrated bound on $\widetilde{\delta} = \delta(\cdot, I(\cdot,\cdot))$ along this range, together with the pathwise square-integrability of θ and ν, yield $\int_0^T |\widetilde{Z}_s^\nu|^2\,ds < \infty$, $\mathbb{P}$-a.s.

Moreover, Step 3 gives $\int_0^T |a_s^\nu|\,ds < \infty$, $\mathbb{P}$-a.s. Therefore, (167) and (42) imply that $\int_0^T g_s^\nu\,ds < \infty$, $\mathbb{P}$-a.s.

Hence, we must have that none of the three conditions defining τ_n is reached before T for sufficiently large n. Since $T > 0$ is arbitrary, $\tau_n \uparrow \infty$, $\mathbb{P}$-a.s.

On each $[0,\tau_n]$, the stopped stochastic integral $\int_0^{\cdot\wedge\tau_n}(\widetilde{Z}_s^\nu)^\top dW_s$ is a square integrable martingale, while the stopped finite variation part is bounded by n in total variation, and thus $\mathcal{Z}_{\cdot\wedge\tau_n}^\nu$ is integrable. Since $g^\nu \geq 0$, the process $\mathcal{Z}^\nu$ has nonnegative drift. Thus, $\mathcal{Z}_{\cdot\wedge\tau_n}^\nu$ is a submartingale for every n. Letting $n \uparrow \infty$ shows that $\mathcal{Z}^\nu$ is a local submartingale. This proves the first requirement of the consistency condition in Definition 3.5(i).

Step 5. Equality at the dual optimal density. Let $Y^{\nu^{\text{dual}}}$ be the solution assumed in the theorem. By (44), the completed square vanishes along this density, $g_t^{\nu^{\text{dual}}} = 0$, for $dt \otimes d\mathbb{P}$-a.e. (t,ω). Then, from equation (167), we deduce that

$$\widetilde{F}\big(t, Y_t^{\nu^{\text{dual}}}, \hat{u}(t, Y_t^{\nu^{\text{dual}}})\big) = -a_t^{\nu^{\text{dual}}}, \qquad t \geq 0,$$

and thus (42) holds. Using the same localization as in Step 4, $\mathcal{Z}^{\nu^{\text{dual}}}$ is a local martingale. This proves the second requirement for the dual consistency condition in Definition 3.5(ii), so the conjugate pair is dual consistent, with dual optimal control ν^{dual}. □

Proof of Proposition 3.10

Proof. We first prove the pointwise Fenchel equality and then compute the complete dynamics of $\mathcal{Z}^{\alpha^\star,\nu}$.

Step 1. The pointwise Fenchel equality. Fix (t,ω) in a set of full $dt\otimes d\mathbb{P}$-measure on which Assumptions 2.9 and 3.7 hold. Let $D>0$ and write $\hat u=\hat u(t,D)$, $\alpha^\star=\alpha^\star(t,D)$ and $\hat c=F_c^{-1}(t,D,\hat u)$. Then,

$$\alpha^\star=-F_u(t,\hat c,\hat u)\qquad\text{and}\qquad F_c(t,\hat c,\hat u)=D.$$

Furthermore, the convexity in u and concavity in c give, for each $c>0$,

$$\begin{aligned}\widehat F(t,\hat c,\alpha^\star)&=F(t,\hat c,\hat u)+\alpha^\star\hat u,\\ \widehat F(t,c,\alpha^\star)-Dc&\le F(t,c,\hat u)+\alpha^\star\hat u-Dc\\ &\le \widetilde F(t,D,\hat u)+\alpha^\star\hat u.\end{aligned}$$

Both inequalities become equalities at $c=\hat c$. Taking the supremum over $c>0$ proves (62).

Step 2. Characteristics of the deflated state price density. Write $D_t=Y_t^\nu/\kappa_{0,t}^{\alpha^\star}$ and $\kappa_t=\kappa_{0,t}^{\alpha^\star}$. Since $d\kappa_t=-\alpha_t^\star\kappa_t dt$ and $dY_t^\nu=Y_t^\nu(-\theta_t+\nu_t)^\top dW_t$,

$$dD_t=\alpha_t^\star D_t dt+D_t(-\theta_t+\nu_t)^\top dW_t,\qquad d\langle D\rangle_t=D_t^2(|\theta_t|^2+|\nu_t|^2)dt,$$

and $d\langle W,D\rangle_t=D_t(-\theta_t+\nu_t)dt$.

Step 3. Dynamics of the envelope. Set $Q_t=I_d-\sigma_t^+\sigma_t$ and $\hat u_t=\hat u(t,D_t)$. By (61), the finite variation term of (52) converges absolutely on every finite horizon and, thus, $\mathcal{Z}^{\alpha^\star,\nu}$ is a well defined continuous process for $t\ge 0$. The Itô–Ventzel formula and the product rule give

$$\begin{aligned}d\mathcal{Z}_t^{\alpha^\star,\nu}&=dM_t^\nu+\kappa_t\Big(\widetilde b(t,D_t)+\alpha_t^\star D_t\widetilde U_y(t,D_t)-\alpha_t^\star\widetilde U(t,D_t)\\ &\qquad+\tfrac12 D_t^2\widetilde U_{yy}(t,D_t)\big(|\theta_t|^2+|\nu_t|^2\big)+D_t\widetilde\delta_y(t,D_t)^\top(-\theta_t+\nu_t)+G(t,D_t,\alpha_t^\star)\Big)dt\\ &=dM_t^\nu+\kappa_t\Big(\widetilde b(t,D_t)+\widetilde F(t,D_t,\hat u_t)+\tfrac12 D_t^2\widetilde U_{yy}(t,D_t)\big(|\theta_t|^2+|\nu_t|^2\big)\\ &\qquad\qquad+D_t\widetilde\delta_y(t,D_t)^\top(-\theta_t+\nu_t)\Big)dt\\ &=dM_t^\nu+\kappa_t\Big(\frac{|Q_t\widetilde\delta_y(t,D_t)|^2}{2\widetilde U_{yy}(t,D_t)}+\tfrac12 D_t^2\widetilde U_{yy}(t,D_t)|\nu_t|^2+D_t\widetilde\delta_y(t,D_t)^\top\nu_t\Big)dt\\ &=dM_t^\nu+\kappa_t\frac{|Q_t\widetilde\delta_y(t,D_t)+D_t\widetilde U_{yy}(t,D_t)\nu_t|^2}{2\widetilde U_{yy}(t,D_t)}\,dt,\end{aligned}$$

where we use (62) in the second equality, (40) in the third, and the fact that

$$dM_t^\nu=\kappa_t\big(\widetilde\delta(t,D_t)+D_t\widetilde U_y(t,D_t)(-\theta_t+\nu_t)\big)^\top dW_t,\qquad t\ge 0.$$

On every finite horizon, localization to the compact range of the positive process D, together with the local random field bounds and the integrability of $\alpha^\star$, θ, and ν, makes the stopped process M^ν square integrable and the displayed finite variation term finite. By (28) and (36), $\widetilde U_{yy}(t,D_t)=-1/U_{xx}(t,I(t,D_t))>0$, so the finite variation term in (63) is nonnegative and $\mathcal{Z}^{\alpha^\star,\nu}$ is a local submartingale. If it is a local martingale, however, its finite variation term is a continuous local martingale of finite variation vanishing at the origin and, hence, identically zero. Since $\widetilde U_{yy}>0$, this occurs exactly when (64) holds. Conversely, (64) makes the term vanish. This proves (63) and the stated equivalence. ☐

Lemma 3.11 (Marginal utility dynamics). Under Assumption 3.2 and the hypotheses of Theorem 2.10, we include the associated wealth process $X^\star$ and set $U_t^\star=U(t,X_t^\star)$ and $c_t^\star=c^\star(t,X_t^\star)$, $t\ge 0$. Then, (67) and (68) hold, with the orthogonal control $\nu^\star$ defined in (65).

Proof. Set

$$P_t=\sigma_t^+\sigma_t,\qquad Q_t=I_d-P_t,\qquad \Lambda(t,x)=U_x(t,x)\theta_t+\delta_x(t,x),\qquad t\ge 0.$$

In the formulas below, all fields without an explicit state argument are evaluated at $(t, X_t^\star)$. We set

$$p_t = P_t \Lambda(t, X_t^\star), \qquad \sigma_t^\star = -\frac{p_t}{U_{xx}}, \qquad dX_t^\star = \big((\sigma_t^\star)^\top \theta_t - c_t^\star\big)dt + (\sigma_t^\star)^\top dW_t, \qquad t \geq 0.$$

The envelope identities (29) and $F_c(t, c_t^\star, U_t^\star) = U_x$ yield

$$b_x = \Lambda^\top \theta_t + \frac{p_t^\top \delta_{xx}}{U_{xx}} - \frac{|p_t|^2 U_{xxx}}{2U_{xx}^2} + c_t^\star U_{xx} - F_u(t, c_t^\star, U_t^\star) U_x, \qquad t \geq 0. \tag{168}$$

The Itô–Ventzel formula gives

$$\begin{aligned} dU_x(t, X_t^\star) = &\Big(b_x + U_{xx}\big((\sigma_t^\star)^\top \theta_t - c_t^\star\big) + \tfrac{1}{2} U_{xxx} |\sigma_t^\star|^2 + \delta_{xx}^\top \sigma_t^\star\Big) dt \\ &+ \big(\delta_x + U_{xx}\sigma_t^\star\big)^\top dW_t, \qquad t \geq 0. \end{aligned} \tag{169}$$

For the drift in (169), (168) implies

$$\begin{aligned} &b_x + U_{xx}\big((\sigma_t^\star)^\top \theta_t - c_t^\star\big) + \tfrac{1}{2} U_{xxx} |\sigma_t^\star|^2 + \delta_{xx}^\top \sigma_t^\star \\ &= \Lambda^\top \theta_t + \frac{p_t^\top \delta_{xx}}{U_{xx}} - \frac{|p_t|^2 U_{xxx}}{2U_{xx}^2} + c_t^\star U_{xx} - F_u(t, c_t^\star, U_t^\star) U_x \\ &\quad - \Lambda^\top \theta_t - c_t^\star U_{xx} + \frac{|p_t|^2 U_{xxx}}{2U_{xx}^2} - \frac{p_t^\top \delta_{xx}}{U_{xx}} \\ &= -F_u(t, c_t^\star, U_t^\star) U_x, \qquad t \geq 0. \end{aligned}$$

Its diffusion coefficient is given by

$$\delta_x + U_{xx}\sigma_t^\star = \delta_x - p_t = -U_x \theta_t + Q_t \delta_x = U_x(-\theta_t + \nu_t^\star), \qquad \nu_t^\star = \frac{Q_t \delta_x}{U_x}, \qquad t \geq 0.$$

Thus, the proof of (67) is finished, and $\nu_t^\star \in (\operatorname{Im} \sigma_t^\top)^\perp$ for $dt \otimes d\mathbb{P}$-a.e. (t, ω).

For $T > 0$, the paths of $X^\star$ remain in a compact subset of $(0, \infty)$ on $[0, T]$, and, furthermore, the continuous process $U_x(t, X_t^\star)$ achieves a strictly positive minimum there. Therefore, the standing bounds imply

$$\int_0^T |\nu_t^\star|^2 dt < \infty, \qquad \int_0^T \big|F_u(t, c_t^\star, U_t^\star) U_x(t, X_t^\star)\big| dt < \infty \qquad \mathbb{P}\text{-a.s.}$$

Dividing the second integrand by the pathwise lower bound for $U_x(t, X_t^\star)$ proves (68). ☐

Proof of Theorem 3.12

Proof. We apply the product rule to the discounted marginal utility and, then, verify the optimal feedback relation.

Step 1. Dynamics of the discounted marginal utility. Set $L_t = U_x(t, X_t^\star)$ and $a_t = F_u(t, c_t^\star, U_t^\star)$, $t \geq 0$. By (66) and Lemma 3.11,

$$d\kappa_t^\star = a_t \kappa_t^\star dt \qquad \text{and} \qquad dL_t = -a_t L_t dt + L_t(-\theta_t + \nu_t^\star)^\top dW_t, \qquad t \geq 0.$$

Since $\kappa^\star$ has finite variation,

$$\begin{aligned} dY_t^\star &= d(\kappa_t^\star L_t) = \kappa_t^\star dL_t + L_t d\kappa_t^\star \\ &= \kappa_t^\star L_t(-\theta_t + \nu_t^\star)^\top dW_t = Y_t^\star(-\theta_t + \nu_t^\star)^\top dW_t, \qquad t \geq 0, \qquad Y_0^\star = U_x(0, x), \end{aligned}$$

and (69) follows. The integrability properties in (68) show simultaneously that $\kappa^\star$ is finite and strictly positive on finite horizons and that $\nu^\star$ is an admissible dual control. Since $\nu_t^\star \in (\operatorname{Im} \sigma_t^\top)^\perp$, the process $Y^\star$ has the state price density form (33) and, thus, $Y^\star \in \mathcal{Y}$.

Step 2. The feedback relation along the optimum. Suppose that Assumption 3.8 holds and that $\alpha^\star$ is admissible. Since $D_t^\star = U_x(t, X_t^\star)$, $t \geq 0$, we obtain

$$\hat{u}(t, D_t^\star) = U_t^\star, \qquad F_c^{-1}(t, D_t^\star, \hat{u}(t, D_t^\star)) = c_t^\star, \qquad \alpha^\star(t, D_t^\star) = -F_u(t, c_t^\star, U_t^\star) = \alpha_t^\star, \qquad t \geq 0,$$

and the feedback relation (60) follows. Next, we show that (61) also holds, so that $Y^\star$ admits $\alpha^\star$ as a feedback discounting map. To this end, evaluating (62) at $(D^\star_t, \alpha^\star_t)$, and using that $c^\star_t$ is the maximizer of the Fenchel transform $\widetilde{F}(t, D^\star_t, U^\star_t)$, we deduce that

$$G(t, D^\star_t, \alpha^\star_t) = F(t, c^\star_t, U^\star_t) - D^\star_t c^\star_t + \alpha^\star_t U^\star_t, \qquad t \geq 0.$$

Fix $T > 0$. The first term is integrable on $[0, T]$ by (17), which holds automatically along the optimum, as noted in Remark 2.5. For the second term, $\int_0^T c^\star_t \, dt < \infty$ by Definition 2.1 and $D^\star$ is pathwise bounded on $[0, T]$ by continuity. For the third term, $\int_0^T |\alpha^\star_t| \, dt < \infty$ by (68) and the fact that $U^\star$ is pathwise bounded on $[0, T]$. Since $\kappa^\star$ is continuous and strictly positive, we have that

$$\int_0^T \kappa^\star_t \big|G(t, D^\star_t, \alpha^\star_t)\big| \, dt < \infty \qquad \mathbb{P}\text{-a.s.}$$

Using (38), evaluated at $D^\star_t = U_x(t, X^\star_t)$,

$$-\frac{(I_d - \sigma_t^+ \sigma_t)\widetilde{\delta}_y(t, D^\star_t)}{D^\star_t \widetilde{U}_{yy}(t, D^\star_t)} = \frac{(I_d - \sigma_t^+ \sigma_t)\delta_x(t, X^\star_t)}{U_x(t, X^\star_t)} = \nu^\star_t, \qquad t \geq 0,$$

and (64) follows. In turn, Proposition 3.10 shows that $\mathcal{Z}^{\alpha^\star, \nu^\star}$ is a local martingale. □

Proof of Theorem 3.14

Proof. We establish the assertions in several steps. To ease the presentation, we occasionally rewrite some formulae.

Step 1. Well-posedness of the deflated dual SDE. By Assumption 3.2,

$$U_x(t, I(t, y)) = y \qquad \text{and} \qquad I_y(t, y) = \frac{1}{U_{xx}(t, I(t, y))}, \qquad t \geq 0, \; y > 0.$$

The bounds in (72) imply

$$\begin{aligned} &|\sigma^D(t, y)| \leq |y|\big(|\theta_t| + K^1_t\big), \qquad |\sigma^D_y(t, y)| \leq |\theta_t| + K^2_t, \\ &\sigma^D_y(t, y) = -\theta_t + \frac{(I_d - \sigma_t^+ \sigma_t)\, \delta_{xx}(t, I(t, y))}{U_{xx}(t, I(t, y))}, \qquad t \geq 0, \; y > 0, \end{aligned}$$

and (74) gives

$$\begin{aligned} &|\mu^D(t, y)| \leq K^0_t |y|, \qquad |\mu^D_y(t, y)| \leq K^0_t, \\ &\mu^D_y(t, y) = \alpha^\star(t, I(t, y)) + \frac{U_x(t, I(t, y))}{U_{xx}(t, I(t, y))}\, \partial_x \alpha^\star(t, I(t, y)), \qquad t \geq 0, \; y > 0, \text{ respectively.} \end{aligned}$$

The derivative bounds hold for every $y > 0$, and, thus, the mean value theorem yields that the coefficients of (71) are Lipschitz in y on $(0, \infty)$ and of linear growth, with adapted constants that are pathwise L^1- and L^2-integrable on $[0, T]$. This is the exact analogue of the uniform Lipschitz property in [4, Theorem 4.2(i)]. Thus, standard localization gives a unique maximal strong solution of (71) with values in $(0, \infty)$. The linear growth bounds rule out explosion to ∞ on a finite time interval.

It remains to exclude exit from zero. To this end, let τ_0 be the first time at which $D^\star_t(y)$ reaches zero. On $[0, \tau_0)$, Itô's formula gives

$$\begin{aligned} d \log D^\star_t(y) = &\left(\frac{\mu^D(t, D^\star_t(y))}{D^\star_t(y)} - \frac{1}{2} \frac{|\sigma^D(t, D^\star_t(y))|^2}{D^\star_t(y)^2} \right) dt \\ &+ \frac{\sigma^D(t, D^\star_t(y))}{D^\star_t(y)} \, dW_t, \qquad t \geq 0. \end{aligned}$$

On $[0, \tau_0)$,

$$\left| \frac{\mu^D(t, D^\star_t(y))}{D^\star_t(y)} \right| \leq K^0_t \qquad \text{and} \qquad \left| \frac{\sigma^D(t, D^\star_t(y))}{D^\star_t(y)} \right| \leq |\theta_t| + K^1_t, \qquad t \geq 0.$$

Thus, the finite variation part of $\log D^\star(y)$ has finite total variation and its martingale part has finite quadratic variation on every finite horizon, $\mathbb{P}$-a.s. Hence, $\log D_t^\star(y)$ cannot diverge to $-\infty$ at a finite time, and thus $\tau_0 = \infty$ $\mathbb{P}$-a.s. Continuity and pathwise uniqueness imply that two solutions driven by the same Brownian motion cannot cross. We easily deduce that $y \mapsto D_t^\star(y)$ is nondecreasing.

Step 2. Construction of the primal wealth by inverse marginality. For $x > 0$, define

$$X_t^\star(x) = I\big(t, D_t^\star(U_x(0,x))\big), \qquad t \geq 0. \tag{170}$$

By Assumption 3.2(ii), the Inada limits make $I(t,\cdot)$ map $(0,\infty)$ onto $(0,\infty)$, so $X^\star(x)$, $x > 0$, is strictly positive. Since $D^\star(U_x(0,x))$ is continuous and I is jointly continuous, the paths of $X^\star(x)$ are continuous and therefore remain in a compact subset of $(0,\infty)$ on every finite horizon.

We now use the inverse marginal identity (77) rather than a separate Lipschitz theorem for the primal coefficients. We give the inverse marginal calculation explicitly. For fixed $y > 0$, we write

$$dI(t,y) = \mu^I(t,y)\,dt + \rho^I(t,y)^\top dW_t, \qquad t \geq 0.$$

Applying the Itô–Ventzel formula to the identity $U_x(t, I(t,y)) = y$, as in (165), gives

$$\begin{aligned}
\rho^I(t,y) &= -\frac{\delta_x}{U_{xx}}, \\
\mu^I(t,y) &= -\frac{b_x}{U_{xx}} - \frac{U_{xxx}|\delta_x|^2}{2U_{xx}^3} + \frac{\delta_{xx}^\top \delta_x}{U_{xx}^2}.
\end{aligned} \tag{171}$$

Here each derivative on the right is evaluated at $(t, I(t,y))$. The spatial derivatives needed below are

$$I_y = \frac{1}{U_{xx}}, \qquad I_{yy} = -\frac{U_{xxx}}{U_{xx}^3}, \qquad \rho_y^I = -\frac{\delta_{xx}}{U_{xx}^2} + \frac{\delta_x U_{xxx}}{U_{xx}^3}. \tag{172}$$

Set $D_t = D_t^\star(U_x(0,x))$ and $X_t = I(t, D_t)$, $t \geq 0$. At the point (t, X_t), let

$$P_t = \sigma_t^+ \sigma_t, \qquad Q_t = I_d - P_t, \qquad p_t = U_x(t,X_t)\theta_t + P_t \delta_x(t,X_t), \qquad t \geq 0.$$

Since $D_t = U_x(t, X_t)$, the diffusion coefficient in (71) is

$$\sigma^D(t, D_t) = -U_x(t,X_t)\theta_t + Q_t \delta_x(t, X_t), \qquad t \geq 0.$$

Applying the Itô–Ventzel formula to $I(t, D_t)$, and using (171) and (172), its volatility coefficient becomes

$$\begin{aligned}
\rho^I(t,D_t) + I_y(t,D_t)\sigma^D(t,D_t) &= -\frac{\delta_x}{U_{xx}} + \frac{-U_x\theta_t + Q_t\delta_x}{U_{xx}} \\
&= -\frac{p_t}{U_{xx}} = \sigma^\star(t, X_t).
\end{aligned}$$

The spatial derivative of the Bellman drift, computed in Lemma 3.11, is given by

$$b_x = (U_x\theta_t + \delta_x)^\top \theta_t + \frac{p_t^\top \delta_{xx}}{U_{xx}} - \frac{|p_t|^2 U_{xxx}}{2U_{xx}^2} + c^\star U_{xx} - F_u U_x, \qquad t \geq 0.$$

On the other hand,

$$\mu^D(t,D_t) = -F_u U_x, \qquad \delta_x - \sigma^D(t,D_t) = p_t, \qquad p_t^\top \theta_t = (U_x\theta_t + \delta_x)^\top \theta_t, \qquad t \geq 0,$$

and, thus, the Itô–Ventzel drift of $I(t, D_t)$ yields

$$\begin{aligned}
&\mu^I + I_y \mu^D + \tfrac{1}{2} I_{yy} |\sigma^D|^2 + (\rho_y^I)^\top \sigma^D \\
&= \frac{-b_x + \mu^D}{U_{xx}} + \frac{\delta_{xx}^\top (\delta_x - \sigma^D)}{U_{xx}^2} + \frac{U_{xxx}}{2U_{xx}^3}\big(-|\delta_x|^2 - |\sigma^D|^2 + 2\delta_x^\top \sigma^D\big) \\
&= \frac{-b_x - F_u U_x}{U_{xx}} + \frac{p_t^\top \delta_{xx}}{U_{xx}^2} - \frac{|p_t|^2 U_{xxx}}{2U_{xx}^3} \\
&= -\frac{(U_x \theta_t + \delta_x)^\top \theta_t}{U_{xx}} - c^\star = \sigma^\star(t, X_t)^\top \theta_t - c^\star(t, X_t), \qquad t \geq 0.
\end{aligned}$$

Consequently,

$$dX_t^\star = \big(\sigma^\star(t, X_t^\star)^\top \theta_t - c^\star(t, X_t^\star)\big)dt + \sigma^\star(t, X_t^\star)^\top dW_t, \qquad t \geq 0.$$

Therefore, (170) is a strong solution of (76) on $[0, \infty)$. Since $U_x(0, \cdot)$ and $I(t, \cdot)$ are strictly decreasing and $D_t^\star(\cdot)$ is nondecreasing, the map $x \mapsto X_t^\star(x)$ is nondecreasing.

Step 3. Pathwise uniqueness and the inverse marginal identity. From (170) we easily deduce that

$$U_x\big(t, X_t^\star(x)\big) = D_t^\star\big(U_x(0, x)\big), \qquad t \geq 0. \tag{173}$$

Next, we prove uniqueness without assuming any local Lipschitz continuity of $\sigma^\star$ or $c^\star$. To this end, let $\bar{X}$ be any strictly positive solution of (76) on the stochastic interval $[0, \zeta)$, where ζ is its exit time from $(0, \infty)$. The calculation in the proof of Lemma 3.11 is local and therefore applies before global admissibility has been established. Together with the fact that $\alpha^\star = -F_u$, we deduce that the process $U_x(t, \bar{X}_t)$ satisfies, on $[0, \zeta)$,

$$\begin{aligned}
dU_x(t, \bar{X}_t) &= \alpha^\star(t, \bar{X}_t) U_x(t, \bar{X}_t) dt \\
&\quad + U_x(t, \bar{X}_t)\Big(-\theta_t + \frac{(I_d - \sigma_t^+ \sigma_t)\delta_x(t, \bar{X}_t)}{U_x(t, \bar{X}_t)}\Big)^\top dW_t, \qquad t \geq 0.
\end{aligned}$$

We now define the process D by $D_t = U_x(t, \bar{X}_t)$, $t \in [0, \zeta)$. The above equation is then precisely equation (71), with initial value $D_0 = U_x(0, x)$. Its coefficients were shown in Step 1 to be locally Lipschitz and of linear growth, and, therefore, this equation has a unique strong solution. Since $D^\star(U_x(0, x))$ is, by construction, the solution of (71) started at $U_x(0, x)$, we conclude that

$$U_x(t, \bar{X}_t) = D_t^\star(U_x(0, x)), \qquad t \in [0, \zeta),$$

and applying $I(t, \cdot)$ yields $\bar{X}_t = X_t^\star(x)$ on $[0, \zeta)$. Since $D^\star(U_x(0, x))$ is continuous and strictly positive and I is jointly continuous, $X^\star(x)$ remains in a compact subset of $(0, \infty)$ on every finite horizon. Hence $\bar{X}$ cannot exit $(0, \infty)$ in finite time, so $\zeta = \infty$ $\mathbb{P}$-a.s. and $\bar{X} = X^\star(x)$. This completes the proof of (1).

Step 4. Admissibility in the sense of Definition 2.1. From (70), the contraction property of the orthogonal projection $\sigma_t^+ \sigma_t$, and (72) and (73), we obtain

$$|\sigma_t^{\pi,\star}(X_t^\star)| = \frac{|\sigma^\star(t, X_t^\star)|}{X_t^\star} \leq \big(|\theta_t| + K_t^1\big) \left|\frac{U_x(t, X_t^\star)}{X_t^\star U_{xx}(t, X_t^\star)}\right| \leq K_t^4\big(|\theta_t| + K_t^1\big), \qquad t \geq 0.$$

Hence,

$$\int_0^T |\sigma_t^{\pi,\star}(X_t^\star)|^2 \, dt < \infty,$$

by the integrability requirement on K^4 in Assumption 3.13. Furthermore,

$$\int_0^T c^\star(t, X_t^\star) \, dt \leq \int_0^T K_t^3 X_t^\star \, dt < \infty,$$

since $X^\star$ is continuous on $[0, T]$ and K^3 is integrable. It, then, follows that $\big(\pi^\star(t, X_t^\star), c^\star(t, X_t^\star)\big)_{t \geq 0} \in \mathcal{A}$.

Step 5. Consistency. Multiplying (173) with $\kappa^\star$ yields $Y^\star = \kappa^\star U_x(\cdot, X^\star) \in \mathcal{Y}$ as in Theorem 3.12. All hypotheses of Theorem 2.10 are then satisfied, and we deduce that (U, F) is a normalized forward recursive utility and that the feedback control pair is optimal. This proves (2).

Step 6. Dual verification. From Proposition 3.4, $\widetilde{U}$ satisfies the dual forward recursive HJB SPDE. The conjugacy identities give

$$\nu_t^{\mathrm{dual}}(y) = \frac{(I_d - \sigma_t^+ \sigma_t)\delta_x(t, I(t,y))}{y}, \qquad t \geq 0,\ y > 0.$$

Hence, the state price density equation controlled by ν^{dual} has the same diffusion coefficient as in (71), but with zero drift. The Lipschitz, growth, and positivity arguments of Step 1 then imply a unique strictly positive strong solution for every initial value $y > 0$. Moreover, (72) yields $|\nu_t^{\mathrm{dual}}(y)| \leq K_t^1$, $t \geq 0$, from which the admissibility of ν^{dual} follows. Theorem 3.7 now gives dual consistency on $\mathcal{Y}$. This proves (3).

Step 7. The two Fenchel equalities along the optimum. Equation (78) holds by definition, and $Y^\star \in \mathcal{Y}$ by Step 5. Under Assumption 3.8, Theorem 3.12 gives the feedback relation, which is the definition (66). Equation (55) follows from the fact that $D_t^\star = U_x(t, X_t^\star)$, $t \geq 0$. Equation (56) also holds, since $c_t^\star$ is the maximizer in the supremum that defines G, and (62) holds at $(D_t^\star, \alpha_t^\star)$.

We have $\int_0^T |\alpha_t^\star| dt \leq \int_0^T K_t^0 dt < \infty$ by (74). Thus, if $\widehat{F}(t, c, \alpha_t^\star) > -\infty$ for every $c > 0$, the discount $\alpha^\star$ belongs to $\mathcal{A}^{\mathrm{disc}}$, and Step 2 of the proof of Theorem 3.12 shows that $Y^\star$ admits $\alpha^\star$ as a feedback discount, so that (61) holds along the optimum. Along the optimal control pair, equality in the continuation utility variable gives

$$\widehat{F}(t, c_t^\star, \alpha_t^\star) = F(t, c_t^\star, U(t, X_t^\star)) + \alpha_t^\star U(t, X_t^\star), \qquad t \geq 0,$$

and hence

$$\int_0^T \kappa_t^\star \big|\widehat{F}(t, c_t^\star, \alpha_t^\star)\big| dt < \infty \qquad \mathbb{P}\text{-a.s. for every } T > 0.$$

Thus, the triple $\big(\alpha^\star, (\pi^\star, c^\star), Y^\star\big)$ is integrable in the sense of (53). Then, Proposition 3.10 yields the local martingale property, while Proposition 3.9 gives (54). The two Fenchel equalities show that this bound is attained along $\big((\pi^\star, c^\star), Y^\star\big)$, and (4) follows. $\square$

Appendix C: Properties of the forward Epstein–Zin recursive aggregator

We collect here the analytic properties of the aggregator f introduced in (95), which are used in Proposition 4.3 and in the proof of Theorem 4.4.

Partial derivatives and convexity

Set

$$z = (1-\gamma)v, \qquad p = 1 - \frac{1}{\lambda} = \frac{1/\eta - \gamma}{1-\gamma}, \qquad v \in \mathcal{U}_\gamma.$$

Then, (95) becomes

$$f(c, v) = \frac{1}{1 - 1/\eta}\left(c^{1-1/\eta} z^p - z\right), \qquad c > 0,\ v \in \mathcal{U}_\gamma,$$

and, in turn,

$$f_c(c, v) = c^{-1/\eta} z^p \qquad \text{and} \qquad f_{cc}(c, v) = -\frac{1}{\eta} c^{-1/\eta - 1} z^p, \qquad c > 0,\ v \in \mathcal{U}_\gamma.$$

For the derivative with respect to v, we use

$$\frac{(1-\gamma)p}{1 - 1/\eta} = \lambda - 1, \qquad \frac{1-\gamma}{1-1/\eta} = \lambda, \qquad p - 1 = -\frac{1}{\lambda}.$$

It follows that

$$f_u(c, v) = \frac{1}{1-1/\eta}\left(c^{1-1/\eta} p z^{p-1}(1-\gamma) - (1-\gamma)\right) = (\lambda - 1) c^{1-1/\eta} z^{-1/\lambda} - \lambda, \qquad c > 0,\ v \in \mathcal{U}_\gamma.$$

Differentiating this identity once more with respect to v, and using that

$$(\lambda - 1)\left(-\frac{1}{\lambda}\right)(1-\gamma) = \gamma - \frac{1}{\eta},$$

gives

$$f_{uu}(c, v) = \left(\gamma - \frac{1}{\eta}\right) c^{1-1/\eta} z^{-(\lambda+1)/\lambda}, \qquad c > 0,\ v \in \mathcal{U}_\gamma.$$

Finally, differentiating $f_c(c,v) = c^{-1/\eta} z^p$ with respect to v gives

$$f_{cu}(c,v) = c^{-1/\eta} p(1-\gamma) z^{p-1} = \left(\frac{1}{\eta} - \gamma\right) c^{-1/\eta} z^{-1/\lambda}, \qquad c > 0,\ v \in \mathcal{U}_\gamma.$$

Thus, for all $c > 0$ and $(1-\gamma)v > 0$, we have

$$\begin{aligned}
f_c(c,v) &= c^{-1/\eta} \big((1-\gamma)v\big)^{\frac{1/\eta-\gamma}{1-\gamma}}, \\
f_{cc}(c,v) &= -\tfrac{1}{\eta}\, c^{-1/\eta-1} \big((1-\gamma)v\big)^{\frac{1/\eta-\gamma}{1-\gamma}}, \\
f_u(c,v) &= (\lambda-1)\, c^{1-1/\eta} \big((1-\gamma)v\big)^{\frac{1/\eta-1}{1-\gamma}} - \lambda, \\
f_{uu}(c,v) &= \big(\gamma - \tfrac{1}{\eta}\big)\, c^{1-1/\eta} \big((1-\gamma)v\big)^{-(\lambda+1)/\lambda}, \\
f_{cu}(c,v) &= \big(\tfrac{1}{\eta} - \gamma\big)\, c^{-1/\eta} \big((1-\gamma)v\big)^{-1/\lambda}.
\end{aligned}$$

Because $c > 0$ and $z > 0$, we deduce

$$f_c(c,v) > 0, \qquad f_{cc}(c,v) < 0.$$

Moreover,

$$\operatorname{sign}(f_{cu}(c,v)) = \operatorname{sign}\left(\frac{1}{\eta} - \gamma\right),$$

and

$$\operatorname{sign}(f_{uu}(c,v)) = \operatorname{sign}\left(\gamma - \frac{1}{\eta}\right).$$

Hence, f is strictly increasing and strictly concave in c. It is convex in v when $\gamma\eta \geq 1$ and concave if $\gamma\eta \leq 1$.

For completeness, we provide the determinant of the Hessian,

$$f_{cc}(c,v) f_{uu}(c,v) - f_{cu}(c,v)^2 = \gamma \left(\frac{1}{\eta} - \gamma\right) c^{-2/\eta} z^{-2/\lambda}, \qquad c > 0,\ v \in \mathcal{U}_\gamma.$$

Since $f_{cc}(c,v) < 0$, the Hessian is negative semidefinite if and only if $1/\eta - \gamma \geq 0$. Therefore, f is jointly concave if and only if $\gamma\eta \leq 1$.

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