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CONSUMPTION-INVESTMENT MODELS WITH CONSTRAINTS*

THALEIA ZARIPHOPOULOU†

Abstract. The paper examines a general investment and consumption problem for a single agent who consumes and invests in a riskless asset and a risky one. The objective is to maximize the total expected discounted utility of consumption. Trading constraints, limited borrowing, and no bankruptcy are binding, and the optimization problem is formulated as a stochastic control problem with state and control constraints. It is shown that the value function is the unique smooth the associated Hamilton–Jacobi–Bellman equation and the optimal consumption and portfolios are provided in feedback form.

Key words. dynamic programming, Bellman equation, viscosity solutions, state constraints, mathematical finance, investment and consumption models

AMS subject classifications. 49L20, 49L25, 90A09, 60H10

Introduction. This paper treats a general consumption and investment problem for a single agent. The investor consumes wealth X_t at a nonnegative rate C_t and distributes it between two assets continuously in time. One asset is a *bond*, i.e., a riskless security with instantaneous rate of return r. The other asset is a *stock* whose value is driven by a Wiener process.

The objective is to maximize the *total expected (discounted) utility from consumption* over an infinite trading horizon and the *total expected utility both from consumption and terminal wealth* in the case of finite horizon. The investor faces the following *trading constraints:* Wealth must stay nonnegative, i.e., bankruptcy never occurs, moreover, the amount π_t invested in stock must not exceed an exogenous function $f(X_t)$ of the wealth at any time t. The function f represents general borrowing constraints, which are frequently binding in practice, such as in portfolio insurance models with prespecified liability flow, models with nontraded assets, stochastic income and/or uninsurable risks, etc. The possibility of imposing short-selling constraints, which amounts to requiring $g(x_t) \leq \pi_t$ for some exogenous function g, is addressed in detail in §1. Finally, the agent is a "small investor," in that his or her decisions do not affect the asset prices and he or she does not pay transaction fees when trading.

This financial model gives rise to a stochastic control problem with control variables *consumption rate* C_t and *portfolio vector* (π_t^0, π_t), where π_t^0 and π_t are the amount of wealth invested in bond and stock, respectively. The state variable X_t is the total wealth at time t. Finally, the value function is the maximum total expected discounted utility.

The goal of this paper is to determine the value functions of these control problems, to examine how smooth they are, and to characterize the optimal policies. The basic tools come from the theory of partial differential equations, in particular the theory of viscosity solutions for second-order partial differential equations and elliptic regularity. We first show that the value functions are the unique constrained viscosity solutions of the associated Hamilton–Jacobi–Bellman (HJB) equation. Then we prove that viscosity solutions of these equations are smooth. Finally, we obtain an explicit feedback form for the optimal policies (C^*, π^*).

The paper is organized as follows: In §1 we describe the model and we give a summary of the history of consumption—investment models in continuous-time finance. Sections 2–5 deal with the infinite horizon model. More precisely, in §2 we describe basic properties of the value function, and in §3 we characterize the value function as a constrained viscosity

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† Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, Massachusetts 01609.

solution of the HJB equation. Moreover, in §4 we prove that the value function is the unique constrained solution of the HJB equation. In §5, we show that the value function is also a smooth solution of this equation and we provide the optimal policies. Finally, in §6 we state results for the finite horizon model.

1. We consider a market with two assets: A *bond* and a *stock*. The price P_t^0 of the bond is given by

$$\begin{aligned} &dP_t^0 = rP_t^0\,dt && (t \geq 0)\\ &P_0^0 = p_0, && (p_0 > 0), \end{aligned} \tag{1.1}$$

where $r > 0$ is the *interest rate*. The price P_t of the stock satisfies

$$\begin{aligned} &dP_t = bP_t\,dt + \sigma P_t\,dW_t && (t \geq 0)\\ &P_0 = p, && (p > 0), \end{aligned} \tag{1.2}$$

where b is the *mean rate of return*, σ is the *dispersion coefficient* and the process $W_.$, which represents the source of uncertainty in the market, is a standard Brownian motion defined on the underlying probability space (Ω, F, P). We will denote by F_t the augmentation under P of $F_t^W = \sigma(W_s : 0 \leq s \leq t)$ for $0 < t < +\infty$. The interest rate r, the mean rate of return b, and the dispersion coefficient σ are assumed to be constant with $\sigma \neq 0$ and $b > r > 0$.

The total current *wealth* $X_t = \pi_t^0 + \pi_t$ is the state variable and π_t^0 and π_t are the amount of wealth invested in bond and stock, respectively; X_t evolves (see [40]) according to the equation

$$\begin{aligned} &dX_t = rX_t\,dt + (b-r)\pi_r\,dt - C_t\,dt + \sigma\pi_t\,dW_t \qquad (t \geq 0)\\ &X_0 = x, \qquad (x \in [0, +\infty)) \end{aligned} \tag{1.3}$$

where x is the *initial endowment* of the investor.

The *control process* are the consumption rate C_t and the portfolio π_t. To state their properties we introduce the following sets:

$$\begin{aligned} \mathcal{L}_+ = \Big\{ z_t : z_t \text{ is } F_t\text{-progressively measurable process, } z_t \geq 0 \text{ a.s. } \forall t \geq 0\\ \text{and } \int_o^t z_s\,ds < +\infty \text{ a.s. } \forall t \geq 0 \Big\}\\ \mathcal{M} = \Big\{ z_t : z_t \text{ is } F_t\text{-progressively measurable process}\\ \text{and } \int_0^t z_s^2\,ds < +\infty \text{ a.s. } \forall t \geq 0 \Big\}. \end{aligned}$$

The set $\mathcal{A}_x$ of *admissible controls* for $x \in [0, +\infty)$ consists of all pairs (C, π) such that:

(i) $C \in \mathcal{L}_+$,

(ii) $\pi \in \mathcal{M}$.

Moreover; $\pi_t \leq f(X_t)$ almost surely for all $t \geq 0$, where the function $f : [0, +\infty) \to [0, +\infty)$ has the following properties:

$$\begin{aligned} &f \text{ is increasing, concave, } \quad f(0) \geq 0 \quad \text{and}\\ &|f(x) - f(y)| \leq K|x - y| \quad \forall x, y \geq 0 \end{aligned} \tag{1.4}$$

(iii) $X_t \geq 0$ almost surely for all $t \geq 0$, where X_t is the trajectory given by the state equation (1.3) using the controls (C, π).

The function f represents the borrowing constraints that the investor must meet; these constraints are present in models with prespecified liabilities such as problems of management of funds as well as in models with uninsurable risks. The possibility of short-selling constraints, i.e., $g(x) \le \pi$, is not examined in this paper for the following reasons: First, if $g \le 0$, the short-selling constraints can be removed because the model is of constant coefficients with $b > r$ (see, for example, [40] and [8]). Second, if $0 < g(x) \le \pi$ this only facilitates the analysis presented here and therefore this case is not discussed.

All the results in this paper hold for the case $f \equiv \infty$, which was studied in [18], provided that some of the arguments in what follows are slightly modified. We will not pursue this any further in this paper unless it is necessary for the study of the $f \equiv \infty$ case. On the other hand, we will occasionally use some results of [18] only to facilitate the presentation and avoid lengthy arguments.

The *total expected discounted utility* J coming from consumption is given by

$$J(x, C, \pi) = E \int_0^{+\infty} e^{-\beta t} U(C_t)\, dt$$

with $(C, \pi) \in \mathcal{A}_x$, where Eg denotes the expectation of g with respect to the probability measure P, $\beta > 0$ is a *discount factor* such that

$$\beta > r, \tag{1.5}$$

and U is the *utility function,* which is assumed to have the following properties:

$$\begin{aligned}&U \text{ is a strictly increasing, concave } C^2(0, +\infty) \text{ function such that}\\ &U(c) \le M(1+c)^\gamma \quad \text{with } 0 < \gamma < 1 \quad \text{and} \quad M > 0,\\ &U(0) \ge 0, \quad \lim_{c \to 0} U'(c) = +\infty, \quad \lim_{c \to \infty} U'(c) = 0.\end{aligned} \tag{1.6}$$

The *value function* is given by

$$v(x) = \sup_{\mathcal{A}_x} E \int_0^{+\infty} e^{-\beta t} U(C_t)\, dt. \tag{1.7}$$

To guarantee that the value function is well defined when U is unbounded, we assume that

$$\beta > r\gamma + \gamma(b-r)/\sigma^2(1-\gamma).$$

The above condition yields that the value function which corresponds to $f \equiv +\infty$ and $U(c) = M(1+c)^\gamma$, and thereby all value functions, are finite (see [18]).

The goal is to characterize v as a classical solution of the HJB equation, associated with the control problem, and use the regularity of v to provide the optimal policies.

We now state the main results.

THEOREM 1.1. *The value function v is the unique $C^2((0, +\infty)) \cap C([0, +\infty))$ solution of*

$$\beta v = \max_{\pi \le f(x)} \left[\tfrac{1}{2}\sigma^2 \pi^2 v_{xx} + (b-r)\pi v_x\right] + \max_{c \ge 0}[-c v_x + U(c)] + r x v_x \tag{1.8}$$

in the class of concave functions.

THEOREM 1.2. *The optimal policies C_t^* and π_t^* are given in the feedback form $C_t^* = c^*(X_t), \pi_t^* = \pi^*(X_t)$ where*

$$c^*(x) = (U')^{-1}(v_x(x)) \quad \textit{and} \quad \pi^*(x) = \min\left\{ f(x), -\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)} \right\}.$$

We continue with a brief discussion of the history of the model.

The single agent consumption-portfolio problem was first investigated by Merton in 1969 and 1971 ([28], [29]). He assumed that the returns of asset prices in perfect markets satisfy the "geometric Brownian motion" hypothesis and he considered utility functions belonging to the hyperbolic absolute risk aversion (HARA) family, i.e., $U(c) = 1 - \gamma/\gamma[\beta c/1 - \gamma + \eta]^{\gamma}$. Under these assumptions, he found explicit formulae for the optimal consumption and portfolio in both the finite and infinite horizon case. Moreover, he showed that the optimal policies are linear functions of the current wealth if and only if the utility function belongs to the HARA family.

In Merton's work, the portfolio is unconstrained, which means that unlimited borrowing and short selling are allowed. Moreover, the consumption process has to stay nonnegative and bankruptcy should never occur. Extra restrictions on the parameters β, γ, and η were later imposed by Merton [30] and Sethi and Taksar [34] to meet the above feasibility conditions.

Another important contribution is the work of Karatzas et al. [18], which is a continuation of work initiated by Lehoczky, Sethi, and Shreve [25]. Reference [18] examines a model with constant coefficients when borrowing and short selling are allowed (i.e., $f \equiv \infty$) and provides solutions of the Bellman equation in closed form. The possibility of bankruptcy is treated in this paper as well as in Sethi and Taksar [33]. The special case of a finite horizon model with constant market coefficients is examined by the same authors in [19]. The fact that borrowing and short selling are allowed is used strongly in [19] (see also [4]) to "linearize" the fully nonlinear Bellman equation to get a system of two linear parabolic equations. Solving these linear equations, they obtain a closed-form solution of the HJB equation.

The Bellman equation can be also linearized when only short-selling constraints are imposed; such a model was studied by Shreve and Xu [35], [36] and Xu [39] in a finite horizon setting in incomplete markets. Such linearization cannot be done if general borrowing constraints are imposed, which is the case we treat in this paper.

A different approach to studying investment-consumption problems with constraints in continuous-time finance was introduced by the author in [40], which studies an investment consumption model with borrowing and short-selling constraints, i.e., $0 \leq \pi_t \leq X_t$. This new approach is based on the theory of viscosity solutions of nonlinear first- and second-order partial differential equations and appears to be flexible enough to handle a wide variety of problems with constraints and related asymptotic problems, e.g., convergence of numerical schemes, asymptotic behavior, etc.

The asymptotic behavior of the value function and the optimal policies for the model with constraints and different interest rates were examined by Fleming and Zariphopoulou in [13]. Moreover, numerical results for the optimal policies and the value function were obtained by Fitzpatrick and Fleming in [10]. A consumption-investment model with leverage constraints (i.e., $f(x) = x + L, L > 0$) was examined by Vila and Zariphopoulou in [38].

Finally, a martingale representation technology has been used by Pliska [32], Cox and Huang [4], Pages [31], and Karatzas, Lehoczky, and Shreve [19] to study optimal portfolio and consumption policies in models with general market coefficients. Moreover, the case of incomplete markets with short-selling constraints in the finite horizon setting has been examined by He and Pearson [15], Xu [39], Shreve and Xu [35], [36], and in the absence of constraints by Karatzas et al. [20].

After this paper was submitted, the author received a paper by Cvitanic and Karatzas

[7]. This paper uses martingale and convex duality methods to study a finite horizon model with nonconstant coefficients and constrained portfolio policies but with utility functions which are more restrictive than the ones used in this paper; in particular, they only consider the case of utility functions with Arrow–Pratt index less than one.

2. In this section we derive some basic properties of the value function.

PROPOSITION 2.1. *The value function v is concave and strictly increasing.*

Proof. The concavity of v is an immediate consequence of the concavity of the utility function U and the fact that if $(C^1,\pi^1) \in \mathcal{A}_{x_1}$, $(C^2,\pi^2) \in \mathcal{A}_{x_2}$, and $\lambda \in (0,1)$, then $(\lambda C^1 + (1-\lambda)C^2, \lambda\pi^1 + (1-\lambda)\pi^2) \in \mathcal{A}_{\lambda x_1 + (1-\lambda)x_2}$; the latter follows from the linear dependence of the dynamics (1.3) with respect to the controls and the state variable.

That v is increasing follows from the observation that $\mathcal{A}_{x_1} \subset \mathcal{A}_{x_2}$ if $x_1 \leqq x_2$. If v is not strictly increasing, then it must be constant on an interval, which, by concavity, has to be of the form $[x_0,\infty)$ for some $x_0 \geqq 0$, i.e., there must exist $x_0 \in [0,+\infty)$ such that $v(x) = v(x_0)$, for all $x \geqq x_0$. In this case, fix $\epsilon > 0$ and choose $(C^\epsilon, \pi^\epsilon) \in \mathcal{A}_{x_0}$ such that

$$v(x_0) \leqq E\int_0^{+\infty} e^{-\beta t}U(C_t^\epsilon)\,dt + \epsilon.$$

If

$$x_1 > \max\left(x_0, \frac{U^{-1}\left[\beta\left(E\int_0^{+\infty} e^{-\beta t}U(C_t^\epsilon)\,dt + \epsilon\right)\right]}{r}\right),$$

the policy $(\overline{C},\overline{\pi}) = (rx_1, 0)$ is in $\mathcal{A}_{x_1}$. Therefore

$$v(x_0) < \frac{1}{\beta}U(rx_1) = E\int_0^{+\infty} e^{-\beta t}U(rx_1)\,dt \leqq v(x_1),$$

which contradicts our assumption. □

PROPOSITION 2.2. *The value function v is uniformly continuous on $\overline{\Omega} = [0,\infty)$ and $v(0) = U(0)/\beta$.*

Proof. Since $(0,0) \in \mathcal{A}_0, v(0) \geqq U(0)/\beta$. On the other hand, $v \leqq u$ in $[0,+\infty)$, where u is the value function with $f \equiv +\infty$ studied in [16]. Since (cf. [18]) $u(0) = U(0)/\beta$ and $u \in C([0,+\infty))$, it follows that $v(0) = U(0)/\beta$ and v is continuous at $x = 0$. The continuity of v in $(0,+\infty)$ follows from concavity.

Finally, since v is uniformly continuous on compact subsets of $\overline{\Omega}$, we remark that its uniform continuity on $\overline{\Omega}$ follows from the fact that, by concavity, v is Lipschitz continuous in $[a,+\infty)$ with Lipschitz constant of order $1/a$ for every $a > 0$. □

PROPOSITION 2.3. *The value function satisfies $v(x) \leq 0(x^\gamma)$ as $x \to +\infty$.*

Proof. Since $v \leq u$ on $\overline{\Omega}$, where u is the value function with $f \equiv +\infty$ and $U(c) = M(1+x)^\gamma$, we only need to check this upper bound for u.

On the other hand, a direct modification of the proof of Theorem 4.5 in [13] yields that if $U \sim c^\gamma$ (as $c \to \infty$), then $u \sim x^\gamma$ (as $x \to \infty$). □

We conclude this section by stating (for a proof see [1], [26]) a fundamental property of the value function known as the *Dynamic Programming Principle.*

PROPOSITION 2.4. *If θ is a stopping time (i.e., a nonnegative $\mathcal{F}$-measurable random variable) then*

$$v(x) = \sup_{\mathcal{A}_x} E\left[\int_0^\theta e^{-\beta t}U(C_t)\,dt + e^{-\beta\theta}v(X_\theta)\right] \; (x \in \overline{\Omega}). \tag{2.1}$$

3. In this section we show that the value function v is a constrained viscosity solution of the HJB equation associated with the underlying stochastic control problem. The characterization of v as a constrained viscosity solution is natural because of the presence of the state $(X_t \geq 0)$ and control $(\pi_t \leq f(X_t))$ constraints.

The notion of viscosity solution was introduced by Crandall and Lions [6] for first-order and by Lions [27] for second-order equations. For a general overview of the theory we refer to the *User's Guide* by Crandall, Ishii, and Lions [5].

Next we recall the notion of contrained viscosity solutions, which was introduced by Soner [37] and Capuzzo-Dolcetta and Lions [3] for first-order equations (see also Ishii and Lions [16] and Katsoulakis [21]). To this end, consider a nonlinear second-order partial differential equation of the form

$$F(x, u, u_x, u_{xx}) = 0 \quad \text{in } \Omega, \tag{3.1}$$

where Ω is an open subset of $\mathbb{R}$ and $F : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and (degenerate) elliptic, i.e.,

$$F(x, t, p, X + Y) \leq F(x, t, p, X) \quad \text{if } Y \geqq 0.$$

DEFINITION 3.1. A continuous function $u : \overline{\Omega} \longrightarrow \mathbb{R}$ is a *constrained viscosity solution* of (3.1) if and only if

(i) u is a *viscosity subsolution* of (3.1) on $\overline{\Omega}$, i.e., if for any $\varphi \in C^2(\overline{\Omega})$ and any maximum point $x_0 \in \Omega$ of $u - \varphi$,

$$F(x_0, u(x_0), \varphi_x(x_0), \varphi_{xx}(x_0)) \leq 0;$$

and

(ii) u is a *viscosity supersolution* of (3.1) in Ω, i.e., if for any $\varphi \in C^2(\overline{\Omega})$ and any minimum point $x_0 \in \Omega$ of $u - \varphi$,

$$F(x_0, u(x_0), \varphi_x(x_0), \varphi_{xx}(x_0)) \geq 0.$$

Remark 1. We say that $u \in C(\overline{\Omega})$ is a viscosity solution of (3.1) in Ω if and only if it is both sub- and supersolution in Ω.

Remark 2. As a matter of fact, we can extend the definition of viscosity subsolutions (respectively, supersolutions) for upper-semicontinuous (respectively, lower-semicontinuous) functions.

THEOREM 3.1. *The value function v is a constrained viscosity solution of* (1.8) *on* $\overline{\Omega}$.

The fact that, in general, value functions of control problems and differential games turn out to be viscosity solutions of the associated partial differential equations is a direct consequence of the principle of dynamic programming and the definition of viscosity solutions (see, for example, Lions [26], Evans and Souganidis [9], Fleming and Souganidis [12], etc.). The main difficulty, however, in the problem at hand is that the consumption rates and the portfolios are not uniformly bounded. This gives rise to some serious complications in the proofs of the results of the aforementioned papers. To overcome these difficulties we need to introduce a number of approximations of the original problem and make use repeatedly of the *stability* properties of viscosity solutions.

Proof of Theorem 3.1. We first show that v is a viscosity supersolution of (1.8) in Ω.

Let $\varphi \in C^2(\overline{\Omega})$ and $x_0 \in \Omega$ be a minimum of $v - \varphi$; without any loss of generality, we may assume that

$$v(x_0) = \varphi(x_0) \quad \text{and} \quad v \geq \varphi \quad \text{in } \Omega. \tag{3.2}$$

We need to show that

$$\beta v(x_0) \geq \max_{\pi \leq f(x_0)} \left[\tfrac{1}{2}\sigma^2\pi^2\varphi_{xx}(x_0) + (b-r)\pi\varphi_x(x_0)\right] + \max_{c \geq o}[-c\varphi_x(x_0) + U(c)] + rx_0\varphi_x(x_0). \tag{3.3}$$

To this end, at $(C,\pi) \in \mathcal{A}_{x_0}$ such that $C_t = C_0, \pi_t = \pi_0 \leq f(x_0)$, for all $t \geq 0$. The dynamic programming principle, together with (3.2), yields

$$v(x_0) \geq E\left[\int_0^\theta e^{-\beta t}U(C_0)\,dt + e^{-\beta\theta}\varphi(X_\theta)\right] \tag{3.4}$$

where $X.$ is the trajectory given by (1.3) using the controls (C_0, π_0) and starting at x_0 and $\theta = \min(\tau, \frac{1}{n})$, with $n > 0$ and $\tau = \inf\{t \geq 0 : X_t = 0\}$.

On the other hand, applying Itô's lemma to $g(t, X_t) = e^{-\beta t}\varphi(X_t)$, we get

$$E[e^{-\beta\theta}\varphi(X_\theta)] = v(x_0) + E\int_0^\theta e^{-\beta t}\Big[-\beta\varphi(X_t) + \frac{1}{2}\sigma^2\pi_0^2\varphi_{xx}(X_t) + (b-r)\pi_0\varphi_x(X_t) - C_0\varphi_x(X_t) + rX_t\varphi_x(X_t)\Big]\,dt.$$

Combining the above equality with (3.4) and using standard estimates from the theory of stochastic differential equations (see [14]), we get

$$E\int_0^\theta \Big[-\beta v(x_0) + \frac{1}{2}\sigma^2\pi_0^2\varphi_{xx}(x_0) + (b-r)\pi_0\varphi_x(x_0) - C_0\varphi_x(x_0) + U(C_0) + rx_0\varphi_x(x_0)\Big]\,ds + E\int_0^\theta h(s)\,ds \leq 0,$$

where $h(s) = 0(s)$. Dividing both sides by $E(\theta)$ and passing to the limit as $n \to \infty$ yields

$$\beta v(x_0) \geq \left[\tfrac{1}{2}\sigma^2\pi_0^2\varphi_{xx}(x_0) + (b-r)\pi_0\varphi_x(x_0)\right] + [-C_0\varphi_x(x_0) + U(C_0)] + rx_0\varphi_x(x_0),$$

for every pair of constant controls $(C_0, \pi_0), C_0 \geq 0$, and $\pi_0 \leq f(x_0)$; inequality (3.3) then follows easily.

We next show that v is a viscosity subsolution of (1.8) on $\overline{\Omega}$.

We first approximate v by a sequence of functions $(v_\epsilon^{N,n})$ defined by

$$v_\epsilon^{N,n}(x) = \sup_{\mathcal{A}_{N,n}} E\int_0^{+\infty} e^{-\beta t}\left[U(C_t) - \frac{1}{\epsilon}p(X_t)\right]dt, \qquad (x \in \mathbb{R})$$

for $\epsilon > 0, N > 0, n > 0$, and $p(x) = \max(0, -x)$. The set of admissible policies $\mathcal{A}_{N,n}$ consists of all pairs (C,π) such that

(i) $C \in \mathcal{L}_+$ and $C_t \leq N$ almost surely for all $t \geq 0$,

(ii) $\pi \in \mathcal{M}$ and $-n \leq \pi_t \leq \hat{f}(X_t)$ almost surely for all $t \geq 0, n > 0$ where the function $\hat{f} : \mathbb{R} \to \mathbb{R}$ (denoted for convenience in the sequel by f) satisfies (1.4) and coincides with f on $[0, +\infty)$;

(iii) X_t is the trajectory given by the state equation (1.3) using the controls (C,π) and starting at $x \in \mathbb{R}$.

It follows from the dynamic programming principle and the definition of viscosity solution (see [27]), that $v_\epsilon^{N,n}$ is a viscosity solution of

$$\beta v_\epsilon^{N,n} = \max_{-n \leq \pi \leq f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 v_{\epsilon,xx}^{N,n} + (b-r)\pi v_{\epsilon,x}^{N,n}\right] + \max_{0 \leq c \leq N}[-cv_{\epsilon,x}^{N,n} + U(c)] + rxv_{\epsilon,x}^{N,n} - \frac{1}{\epsilon}p(x) \qquad (x \in \mathbb{R}).$$

We next observe that as $n \to \infty$,

$$v_\epsilon^{N,n} \to v_\epsilon^N, \text{ locally uniformly in } \mathbb{R},$$

(see [22, Chap. 6]) where

$$v_\epsilon^{N(x)} = \sup_{\mathcal{A}_N} E \int_0^{+\infty} e^{-\beta t} \left[U(C_t) - \frac{1}{\epsilon} p(X_t) \right] dt \qquad (x \in \mathbb{R})$$

and the set $\mathcal{A}_N$ of admissible policies is defined in the same way as $\mathcal{A}_{N,n}$, but without a lower bound on π.

It is immediate that

$$v_\epsilon^N \leqq \frac{U(N)}{\beta} \quad \text{in } \mathbb{R} \tag{3.5}$$

and

$$v^N \leqq v_\epsilon^N \quad \text{on } [0, +\infty), \tag{3.6}$$

where, for $x \in [0, +\infty)$,

$$v^N(X) = \sup_{\mathcal{A}_{x,N}} E \int_0^{+\infty} e^{-\beta t} U(C_t)\, dt \tag{3.7}$$

and

$$\mathcal{A}_{x,N} = \{(C, \pi) \in \mathcal{A}_x : C_t \leqq N \text{ a.s. } \forall t \geqq 0\}.$$

Moreover, the v_ϵ^N's are increasing and concave with respect to x. Both properties follow as in Proposition 2.1.

Finally, the stability property of viscosity solutions (see [27, Prop. I.3]) yields that v_ϵ^N is a viscosity solution of

$$\begin{aligned} \beta v_\epsilon^N = \max_{\pi \le f(x)} \left[\tfrac{1}{2} \sigma^2 \pi^2 v_{\epsilon,xx}^N + (b - r) \pi v_{\epsilon,x}^N \right] + \max_{0 \le c \le N} [-c v_{\epsilon,x}^N + U(c)] \\ + r x v_{\epsilon,x}^N - \tfrac{1}{\epsilon} p(x) \qquad (x \in \mathbb{R}). \end{aligned} \tag{3.8}$$

In the sequel we look at the behavior of the v_ϵ^N's on $[0, +\infty)$ as $\epsilon \to 0$. Since the only available bounds on the v_ϵ^N's are the ones stated above, we employ the limsup operation introduced by Barles and Perthame [2]. To this end, we define

$$v^{N,*}(x) = \limsup_{y \to x, \epsilon \to 0} v_\epsilon^N(y) \qquad (x \in [0, +\infty))$$

and we claim that

(i) $v^{N,*}$ is an upper semicontinuous viscosity subsolution on $\overline{\Omega}$ of

$$\begin{aligned} \beta v^{N,*} = \max_{\pi \le f(x)} \left[\tfrac{1}{2} \sigma^2 \pi^2 v_{xx}^{N,*} + (b - r) \pi v_x^{N,*} \right] \\ + \max_{0 \le c \le N} [-c v_x^{N,*} + U(c)] + r x v_x^{N,*} \qquad (x \in [0, +\infty)); \end{aligned} \tag{3.9}$$

and

(ii) $v^{N,*} = v^N$ on $\overline{\Omega}$.

We first observe that $v^{N,*}$ is increasing and concave on $\overline{\Omega}$. The first property is an immediate consequence of the definition. For the concavity we argue as follows: The concavity of $v^{N,*}$ in Ω follows from the fact that, since the v_ϵ^N's are concave in Ω and uniformly bounded on $\overline{\Omega}$, they converge, as $\epsilon \to 0$, locally uniformly to a concave function which actually coincides with $v^{N,*}$. It remains to show that

$$v^{N,*}((1-\lambda)x) \geq \lambda v^{N,*}(0) + (1-\lambda)v^{N,*}(x) \tag{3.10}$$

for $\lambda \in (0,1)$ and $x > 0$.

Let (ϵ_n) and $(y_n) \in \mathbb{R}$ be sequences such that, as $n \to \infty, \epsilon_n \to 0, y_n \to 0$, and $v^{N,*}(0) = \limsup_{y_n \to 0, \epsilon \to 0} v_\epsilon^N(y_n)$. The concavity of v_ϵ^N yields

$$v_{\epsilon_n}^N(\lambda y_n + (1-\lambda)x) \geq \lambda v_{\epsilon_n}^N(y_n) + (1-\lambda)v_{\epsilon_n}^N(x). \tag{3.11}$$

On the other hand,

$$v^{N,*}(x) = \lim_{\epsilon \to 0} v_\epsilon^N(x) \qquad (x \in (0, +\infty)). \tag{3.12}$$

Indeed, let $x \in [x_1, x_2]$ with $x_1 > 0$. The concavity of v_ϵ^N's and (3.5) yields that the v_ϵ^N are locally Lipschitz on $[x_1, x_2]$ with Lipschitz constant L independent of ϵ, i.e.,

$$v_\epsilon^N(y) \leq v_\epsilon^N(x) + L|y-x| \qquad (x, y \in [x_1, x_2]);$$

therefore

$$\limsup_{\epsilon \to 0, y \to x} v_\epsilon^N(y) \leq \limsup_{\epsilon \to 0} v_\epsilon^N(x).$$

Moreover,

$$\limsup_{\epsilon \to 0} v_\epsilon^N(x) = \lim_{\epsilon \to 0} v_\epsilon^N(x)$$

since the v_ϵ^N's are increasing in ϵ. Combining the last inequalities we get

$$v^{N,*}(x) \leq \lim_{\epsilon \to 0} v_\epsilon^N(x)$$

which, together with the definition of $v^{N,*}$ yields (3.12).

We now observe that, for n large enough, $\lambda y_n + (1-\lambda)x > a$, for some $a > 0$. Sending $n \to \infty$ in (3.11) and using the properties of $(\epsilon_n), (y_n)$, and (3.12) we conclude.

We continue with the proof of (3.9). We need to examine the following cases.

Case 1. $f \equiv \infty$.

Let $\varphi \in C^2(\overline{\Omega})$ and assume that $v^{N,*} - \varphi$ has a maximum at 0, which can be assumed to be strict. We need to show

$$\beta v^{N,*}(0) \leq \max_\pi \left[\tfrac{1}{2}\sigma^2\pi^2\varphi_{xx}(0) + (b-r)\pi\varphi_x(0)\right] + \max_{0 \leq c \leq N}[-c\varphi_x(0) + U(c)]. \tag{3.13}$$

First observe that the concavity and monotonicity of $v^{N,*}$ imply $\varphi_x(0) > 0$. Inequality (3.5), along with the fact that the max with respect to π in (3.13) is unconstrained, implies that (3.13) holds if $\varphi_{xx}(0) \geqq 0$. It remains to prove (3.13) if $\varphi_{xx}(0) < 0$. To this end, we first extend φ to $\mathbb{R}^-$ in $C^2(\mathbb{R})$ so that for some $\alpha > 0$,

$$\varphi_{xx}(x) < 0 \qquad (-\alpha \leq x \leq 0)$$

and

$$v_\epsilon^N(-\alpha) \leq v_\epsilon^N(0) - \varphi(0) + \varphi(-\alpha) - \alpha.$$

Let x_ϵ be a maximum point of $v_\epsilon^N - \varphi$ over $[-\alpha, \alpha]$. If $x_\epsilon = -\alpha$, the choice of φ together with (3.6) yields

$$v_\epsilon^N(-\alpha) - \varphi(-\alpha) \leq v_\epsilon^N(0) - \varphi(0) - \alpha$$

which is a contradiction. Moreover, 0 being a strict maximum of $v^{N,*} - \varphi$ yields $x_\epsilon \neq \alpha$ for ϵ small enough. Since v_ϵ^N is viscosity solution of (3.8) we have

$$\begin{aligned} \frac{1}{\epsilon}p(x_\epsilon) \leq \max_\pi &\left[\tfrac{1}{2}\sigma^2\pi^2\varphi_{xx}(x_\epsilon) + (b-r)\pi\varphi_x(x_\epsilon)\right] \\ &+ \max_{0 \leq c \leq N}[-c\varphi_x(x_\epsilon) + U(c)] + rx_\epsilon\varphi_x(x_\epsilon) - \beta v_\epsilon^N(x_\epsilon). \end{aligned} \tag{3.14}$$

We next observe that the right-hand side of the above inequality is finite since $\varphi_{xx}(x_\epsilon) < 0, \varphi_x(x_\epsilon) > 0$ and $-v_\epsilon^N(x_\epsilon) < +\infty$, where the latter follows from

$$v_\epsilon^N(x_\epsilon) - \varphi(x_\epsilon) \geq v_\epsilon^N(0) - \varphi(0) \geq v^N(0) - \varphi(0).$$

Let $\overline{x}$ be a limit (along subsequence) of the x_ϵ's. The definitions of p and (3.14) yield $\overline{x} \geq 0$. Actually, $\overline{x} = 0$.

Indeed,

$$v_\epsilon^N(x_\epsilon) - \varphi(x_\epsilon) \geq v_\epsilon^N(0) - \varphi(0)$$

and therefore

$$v^{N,*}(\overline{x}) - \varphi(\overline{x}) \geq v^{N,*}(0) - \varphi(0)$$

which yields $\overline{x} = 0$, since 0 is a strict maximum.

Moreover,

$$\lim_{\epsilon \to 0} v_\epsilon^N(x_\epsilon) = v^{N,*}(0).$$

Indeed, $\limsup_{\epsilon\to 0} v_\epsilon^N(x_\epsilon) \leq v^{N,*}(0)$. On the other hand, if $\limsup_{\epsilon\to 0} v_\epsilon^N(x_\epsilon) < v^{N,*}(0)$, then $v^{N,*}(0) - \varphi(0) > \limsup_{\epsilon\to 0}[v_\epsilon^N(x_\epsilon) - \varphi(x_\epsilon)]$, which is a contradiction. Finally, passing to the limit in (3.14) as $\epsilon \to 0$, we get (3.13).

Working similarly, we show that $v^{N,*}$ is a viscosity subsolution of (1.8) in $(0, +\infty)$.

It remains to show that

$$v^{N,*} = v^N \quad \text{on } [0, +\infty). \tag{3.15}$$

Since $v^{N,*}$ and v^N are, respectively, viscosity subsolution of (1.8) on $[0, +\infty)$ and supersolution in $(0, +\infty)$, a comparison result similar to Theorem 4.1 (easily modified for the case the consumption rates are uniformly bounded) implies

$$v^{N,*} \leqq v^N \quad \text{on } [0, +\infty) \tag{3.16}$$

which together with (3.6) yields (3.15) in $(0, +\infty)$.

Finally, the upper semicontinuity of $v^{N,*}$ implies $v^{N,*}(0) = v^N(0)$.

Case 2. $f < +\infty$.

In view of the analysis above, we only have to examine the case $\varphi_{xx}(0) = 0$, i.e., we need to show

$$\beta v^{N,*}(0) \leq (b-r)f(0)\varphi_x(0) + \max_{0 \leq c \leq N} [-c\varphi_x(0) + U(c)]$$

where we used that $\varphi_x(0) > 0$. We first observe

$$v^{N,*}(0) = \frac{U(0)}{\beta}.$$

This follows from the fact that $U(0)/\beta \leq v^{N,*}(0) \leq v_\infty^N(0)$ and $v_\infty^N(0) \leq u(0) = U(0)/\beta$, where v_∞^N is given by (3.7) for $f \equiv \infty$.

Using that $\max_{0 \leq c \leq N}[-c\varphi_x(0) + U(c)] \geq U(0)$, (3.17), and $\varphi_x(0) > 0$, we conclude.

We now conclude the proof of the theorem.

In view of the stability properties of viscosity solutions, to conclude the proof of the theorem we only need to establish that as $N \to \infty$,

$$v^N \to v, \text{ locally uniformly on } \overline{\Omega}.$$

To this end, fix $x \in \overline{\Omega}, \epsilon > 0$, and choose $(C^\epsilon, \pi^\epsilon) \in \mathcal{A}_x$ such that

$$v(x) \leq E \int_0^{+\infty} e^{-\beta t} U(C_t^\epsilon)\, dt + \epsilon. \tag{3.17}$$

From the definitions of $\mathcal{A}_{x,N}$ and $\mathcal{A}_x$ we have that $(C^\epsilon \wedge N, \pi) \in \mathcal{A}_{x,N}$. Moreover, since U is increasing and nonnegative, the monotone convergence theorem yields

$$\lim_{N\to\infty} E \int_0^{+\infty} e^{-\beta t} U(C_t^\epsilon \wedge N)\, dt = E \int_0^{+\infty} e^{-\beta t} U(C_t^\epsilon)\, dt$$

which, combined with (3.17) and the definitions of v^N and v, gives

$$v^N(x) \leq v(x) \leq E \int_0^{+\infty} e^{-\beta t} U(C_t^\epsilon \wedge N)\, dt + 2\epsilon \leq v^N(x) + 2\epsilon \quad \text{for } N \geq N(\epsilon).$$

Therefore, $v^N \to v$ as $N \to \infty$, for each $x \in \overline{\Omega}$. On the other hand, since v^N increases with respect to N and v is continuous, Dini's theorem implies that $v^N \to v$ locally uniformly on $\overline{\Omega}$. □

4. In this section we present a comparison result for constrained viscosity solutions of (1.8). Comparison results for a large class of boundary problems were given by Ishii and Lions [16]. The equation on hand, however, does not satisfy some of the assumptions in [16], in view of the fact that the controls are not uniformly bounded. It is therefore necessary to modify some of the arguments of Theorem II.2 of [16] to take care of these difficulties. For completeness we present the whole proof; we rely, however, on some basic facts which are analyzed in [14].

THEOREM 4.1. *If u is an upper-semicontinuous concave viscosity subsolution of* (1.8) *on $\overline{\Omega}$ and v is a bounded from below, sublinearly growing, uniformly continuous on $\overline{\Omega}$, and locally Lipschitz in Ω supersolution of* (1.8) *on Ω, then $u \leq v$ on $\overline{\Omega}$.*

Before we begin with the proof of the theorem, we observe that (1.8) can be written as

$$\beta u = G(x, u_x, u_{xx}) \tag{4.1}$$

where $G : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$G(x,p,A) = \max_{\pi \le f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 A + (b-r)\pi p\right] + \max_{c \ge 0}[-cp + U(c)] + rxp.$$

An important ingredient of the proof of Theorem 4.1 is the sup- and inf-*convolution* approximation of u and v, respectively. Next we recall their definitions and summarize their main properties. For a general discussion about sup- and inf-convolution as well as their use in proving comparison results for second-order PDEs we refer to Lasry and Lions [24], Jensen, Lions, and Souganidis [17], and Ishii and Lions [16].

For $\epsilon > 0$ the ϵ sup-convolution of u is defined by

$$u^\epsilon(x) = \sup_{y \in \Omega}\left\{ u(y) - \frac{1}{\epsilon}|x-y|^2 \right\} \quad \forall x \in \overline{\Omega}, \tag{4.2}$$

and, similarly, the ϵ inf-convolution of v by

$$v_\epsilon(x) = \inf_{z \in \Omega}\left\{ v(z) + \frac{1}{\epsilon}|x-z|^2 \right\} \quad \forall x \in \overline{\Omega}. \tag{4.3}$$

It follows that the sup and inf in the definitions of u^ϵ and v_ϵ are actually taken for

$$|x-y| \le C\sqrt{\epsilon} \quad \text{and} \quad |x-z| \le C\sqrt{\epsilon}, \tag{4.4}$$

where $C = C(x)$ depends on the coefficients of the sublinear growth of u and v.

Moreover

(i) u^ϵ is a viscosity subsolution of

$$F_\epsilon(x, u^\epsilon, u^\epsilon_x, u^\epsilon_{xx}) = 0 \quad \text{in } \Omega_\epsilon,$$

where $F^\epsilon(x,t,p,A) = \min\{\beta t - G(y,p,A) : |y-x| \le C\sqrt{\epsilon}\}$ and $\overline{\Omega}_\epsilon = \{x \in \Omega : x \ge C\sqrt{\epsilon}\}$; and

(ii) v_ϵ is a viscosity supersolution of

$$F^\epsilon(x, v_\epsilon, v_{\epsilon,x}, v_{\epsilon,xx}) = 0 \quad \text{in } \Omega_\epsilon,$$

where

$$F^\epsilon(x,t,p,A) = \max\{\beta t - G(y,p,A) : |y-x| \le C\sqrt{\epsilon}\}.$$

Proof of Theorem 4.1. We present the proof of the theorem for the case $f < +\infty$. The case $f \equiv \infty$ is discussed at the end of the section.

We argue by contradiction, i.e., we assume that

$$\sup_{x \in \overline{\Omega}}[u(x) - v(x)] > 0. \tag{4.5}$$

Then for sufficiently small $\theta > 0$

$$\sup_{x\in\overline{\Omega}}[u(x) - v(x) - \theta x] > 0. \tag{4.6}$$

Indeed, if not, there would be a sequence $\theta_n \downarrow 0$ such that $\sup_{x\in\overline{\Omega}}[u(x) - v(x) - \theta_n x] \leq 0$, which in turn would yield $\sup_{x\in\overline{\Omega}}[u(x) - v(x)] \leq 0$, contradicting (4.5).

Since u has, by concavity, sublinear growth and v is bounded from below, there exists $\overline{x} \in \overline{\Omega}$ such that

$$\sup_{x\in\overline{\Omega}}[u(x) - v(x) - \theta x] = u(\overline{x}) - v(\overline{x}) - \theta\overline{x}. \tag{4.7}$$

Next, for $\delta > 0$ and $\eta > 0$ we define $\varphi : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ by

$$\varphi(x, y) = u(x) - v(y) - \left|\frac{y - x}{\delta} - 4\eta\right|^4 - \theta x$$

and observe that for each fixed η, φ attains its maximum at a point (x_0, y_0) such that for δ small and some $l = l(\theta) > 0$,

$$|y_0 - x_0| \leq l\delta. \tag{4.8}$$

Indeed, φ is bounded and

$$\sup_{\overline{\Omega}\times\overline{\Omega}} \varphi(x, y) \geq \varphi(x, x + 4\eta\delta) \geq u(\overline{x}) - v(\overline{x}) - \theta\overline{x} - \omega_v(k\eta\delta) \tag{4.9}$$

where ω_v is the modulus of continuity of v and $k > 0$. Using (4.6) and (4.7) we get

$$\sup_{\overline{\Omega}\times\overline{\Omega}} \varphi(x, y) > 0 \tag{4.10}$$

for δ and η sufficiently small.

Next, let (x_n, y_n) be a maximizing sequence for φ and observe that

$$u(x_n) - v(y_n) - \theta x_n \geq \left|\frac{y_n - x_n}{\delta} - 4\eta\right|^4 \quad \text{as } n \to \infty.$$

The last inequality, combined with the fact that u has sublinear growth, implies (4.8).

On the other hand, the choice of (x_n, y_n) and (4.10) yields that the sequence (x_n) and, in view of the above observation, (y_n) are bounded as $n \to \infty$. Hence, along subsequences, (x_n, y_n) converges to a maximum point of φ, which we denote by (x_0, y_0).

We next fix δ small enough and we consider for $\epsilon \in (0, 1)$ the function

$$\varphi^\epsilon(x, y) = u^\epsilon(x) - v_\epsilon(y) - \left|\frac{y - x}{\delta} - 4\eta\right|^4 - \theta x$$

where u^ϵ and v_ϵ are, respectively, the ϵ sup- and inf-convolutions of u and v given by (4.3) and (4.4).

In the sequel we need to study separately the cases $\overline{x} > 0$ and $\overline{x} = 0$.

Case A. $\overline{x} > 0$.

If δ is small enough it follows that the point (x_0, y_0) lies in a fixed compact subset of $\Omega \times \Omega$. Moreover, the function φ^ϵ achieves its maximum at a point that we denote by

$(\overline{x}_\epsilon, \overline{y}_\epsilon)$, which lies in $\Omega_\epsilon \times \Omega_\epsilon$ (see [16]). Since $\beta > 0$ we can apply Proposition II.3 of [14], according to which there exist $X_\epsilon, Y_\epsilon \in \mathbb{R}$ such that

$$F_\epsilon(\overline{x}_\epsilon, u^\epsilon(\overline{x}_\epsilon), w_x(\overline{x}_\epsilon), \overline{y}_\epsilon) + \theta, X_\epsilon) \leq 0, \qquad F^\epsilon(\overline{y}_\epsilon, v_\epsilon(\overline{y}_\epsilon), -w_y(\overline{x}_\epsilon, \overline{y}_\epsilon), -Y_\epsilon) \geq 0$$

and

$$\begin{pmatrix} X_\epsilon & 0 \\ 0 & Y_\epsilon \end{pmatrix} \leq \begin{pmatrix} w_{xx}(\overline{x}_\epsilon, \overline{y}_\epsilon) & w_{xy}(\overline{x}_\epsilon, \overline{y}_\epsilon) \\ w_{yx}(\overline{x}_\epsilon, \overline{y}_\epsilon) & w_{yy}(\overline{x}_\epsilon, \overline{y}_\epsilon) \end{pmatrix}$$

where $w(x,y) = |(y-x)/\delta - 4\eta|^4$. Therefore

$$F_\epsilon\left(\overline{x}_\epsilon, u^\epsilon(\overline{x}_\epsilon), -\frac{4}{\delta}\left(\frac{\overline{y}_\epsilon - \overline{x}_\epsilon}{\delta} - 4\eta\right)^3 + \theta, X_\epsilon\right) \leq 0, \tag{4.11}$$

$$F^\epsilon\left(\overline{y}_\epsilon, v_\epsilon(\overline{y}_\epsilon), -\frac{4}{\delta}\left(\frac{\overline{y}_\epsilon - \overline{x}_\epsilon}{\delta} - 4\eta\right)^3, -Y_\epsilon\right) \geq 0. \tag{4.12}$$

Also,

$$\begin{pmatrix} X_\epsilon & 0 \\ 0 & Y_\epsilon \end{pmatrix} \leq \frac{12}{\delta^2}\left(\frac{\overline{y}_\epsilon - \overline{x}_\epsilon}{\delta} - 4\eta\right)^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and therefore

$$X_\epsilon + Y_\epsilon \leq 0. \tag{4.13}$$

We next observe that there exists a constant $c \geq 0$ such that

$$Y_\epsilon \geq c. \tag{4.14}$$

We argue by contradiction. Let us assume that there exists a subsequence (Y_{ϵ_n}) such that $\lim_{\epsilon_n \to 0} Y\epsilon_n = Y < 0$. From the definition of F_ϵ we have

$$\begin{aligned}\beta v_\epsilon(\overline{y}\epsilon) \geq \max_{\pi \leq f(\hat{y}_\epsilon)} &\left[-\tfrac{1}{2}\sigma^2\pi^2 Y_{\epsilon_n} - (b-r)\pi w_y(\overline{x}_\epsilon, \overline{y}_\epsilon)\right] \\ &+ \max_{c \geq 0}[c w_y(\overline{x}_\epsilon, \overline{y}_\epsilon) + U(c)] - r\hat{y}_\epsilon w_y(\overline{x}_\epsilon, \overline{y}_\epsilon),\end{aligned}$$

for some $\hat{y}_\epsilon \in \Omega$ such that $|\overline{y}_\epsilon - \hat{y}_\epsilon| \leq C\sqrt{\epsilon}$. Sending $\epsilon_n \downarrow 0$ and using that $v_\epsilon(\overline{y}_\epsilon) \leq U(N)/\beta$, we get a contradiction.

Therefore, there exists $Y \in \mathbb{R}_0^+$ or $Y = +\infty$ such that $\lim_{\epsilon \to 0} Y_\epsilon = Y$ (along a subsequence). Moreover, (4.13) and (4.14) imply that there exists $X \in \mathbb{R}_0^-$ or $X = -\infty$ such that $\lim_{\epsilon \to 0} X_\epsilon = X$ (along a subsequence).

Sending $\epsilon \to 0$, inequalities (4.11) and (4.12) yield (see [16])

$$\begin{aligned}\beta u(x_0) \leq \max_{\pi \leq f(x_0)} &\left[\tfrac{1}{2}\sigma^2\pi^2 X + (b-r)\pi(w_x(x_0, y_0) + \theta)\right] \\ &+ g(w_x(x_0, y_0) + \theta) + r x_0(w_x(x_0, y_0) + \theta)\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\beta v(y_0) \geq \max_{\pi \leq f(y_0)} &\left[-\tfrac{1}{2}\sigma^2\pi^2 Y + (b-r)\pi w_x(x_0, y_0)\right] \\ &+ g(w_x(x_0, y_0)) + r y_0 w_x(x_0, y_0)\end{aligned} \tag{4.16}$$

where we used that $w_x(x_0, y_0) = -w_y(x_0, y_0)$ and $g(p) = \max_{c \geq 0}[-cp + U(c)]$.

We now look at the following cases.

Case (i). $X = -\infty$. Inequalities (4.15) and (4.16) yield

$$\beta u(x_0) \leq g(w_x(x_0, y_0) + \theta) + r x_0(w_x(x_0, y_0) + \theta)$$

and

$$\beta v(y_0) \geq g(w_x(x_0, y_0)) + r y_0 w_x(x_0, y_0).$$

Therefore,

$$\beta(u(x_0) - v(y_0) - \theta x_0) \leq r(x_0 - y_o) w_x(x_0, y_0) \tag{4.17}$$

where we used that g is a decreasing function and (1.5).

Case (ii). $X \leq 0$. From (1.4), (4.8), and (4.15) we get

$$\begin{aligned}\beta u(x_0) \leq &\max_{\pi \leq f(y_0) + Kl(\theta)\delta} \left[\tfrac{1}{2}\sigma^2 \pi^2 X + (b - r)\pi w_x(x_0, y_0)\right] + g(w_x(x_0, y_0) + \theta) \\ &+ r x_0(w_x(x_0, y_0) + \theta) + (b - r)\theta f(x_0)\end{aligned}$$

which, combined with (4.16), gives

$$\begin{aligned}\beta[u(x_0) - v(y_0) - \theta_{x_0}] \leq &\max_{\pi \leq f(y_0) + Kl(\theta)\delta} \left[\tfrac{1}{2}\sigma^2 \pi^2 X + (b - r)\pi w_x(x_0, y_0)\right] \\ &- \max_{\pi \leq f(y_0)} \left[\tfrac{1}{2}\sigma^2 \pi^2 y + \pi w_x(x_0, y_0)\right] \\ &+ r(x_0 - y_0) w_x(x_0, y_0) + (b - r)\theta f(x_0).\end{aligned} \tag{4.18}$$

In the sequel we will need the following two lemmas.

LEMMA 4.1. *Let $p > 0$ and $X \leq 0, Y \geq 0$ be such that*

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq A \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4.19}$$

with $A > 0$. Then

$$\begin{aligned}&\max_{\pi \leq a_1} \left[\tfrac{1}{2}\sigma^2 \pi^2 X + (b - r)\pi p\right] - \max_{\pi \leq a_2} \left[-\tfrac{1}{2}\sigma^2 \pi^2 Y + (b - r)\pi p\right] \\ &\leq \omega((a_1 - a_2)^2 A + (a_1 - a_2)p)\end{aligned} \tag{4.20}$$

where $a_1 > a_2$ and $\omega : [0, +\infty) \to [0, +\infty)$ is uniformly continuous with $\omega(0) = 0$.

LEMMA 4.2. *For fixed $\eta > 0$ and $\theta > 0$, the following holds:*

$$\lim_{\delta \downarrow 0} \left| \frac{y_0 - x_0}{\delta} - 4\eta \right| = 0. \tag{4.21}$$

Moreover,

$$\lim_{\theta \downarrow 0} \lim_{\delta \downarrow 0} \theta f(x_0(\theta, \delta)) = 0. \tag{4.22}$$

We now conclude the proof of the theorem and next give the proof of the lemmas. First observe that (4.17) gives

$$\beta(u(\overline{x}) - v(\overline{x}) - \theta \overline{x}) \leq \left[-4r\left(\frac{y_0 - x_0}{\delta}\right)\left(\frac{y_0 - x_0}{\delta} - 4\eta\right)^3 + \omega_v(k\eta\delta) + \left(\frac{y_0 - x_0}{\delta} - 4\eta\right)\right]^4.$$

Sending $\delta \to 0$ and using (4.21) we contradict (4.6).

Next, we use (4.18) and Lemma 4.2 with

$$A = \frac{12}{\delta^2}\left(\frac{y_0 - x_0}{\delta} - 4\eta\right)^2, \quad a_1 = f(y_0) + Kl(\theta)\delta, \quad a_2 = f(y_0)$$

and

$$p = -\frac{4}{\delta}\left(\frac{y_0 - x_0}{\delta} - 4\eta\right)^3.$$

Note that from the definition of g and (4.16) we must have $p = w_x(x_0, y_0) > 0$. We get

$$\beta[u(\overline{x}) - v(\overline{x}) - \theta\overline{x}] \leq \omega\left(12K^2l^2(\theta)\left(\frac{y_0 - x_0}{\delta} - 4\eta\right)^2 + 4Kl(\theta)\left|\frac{y_0 - x_0}{\delta} - 4\eta\right|^3\right) + 4r\left|\frac{y_0 - x_0}{\delta}\right|\left|\frac{y_0 - x_0}{\delta} - 4\eta\right|^3 + (b-r)\theta f(x_0) + \omega_v(k\eta\delta) + \left(\frac{y_0 - x_0}{\delta} - 4\eta\right)^4. \tag{4.23}$$

We now use (4.21) and (4.22) and we send first $\delta \downarrow 0$, then $\theta \downarrow 0$, and last $\eta \downarrow 0$ to contradict (4.3).

Case B. $\overline{x} = 0$.

Since the proof follows along the lines of Theorem VI.5 in [16], modified with arguments similar to the ones in Case A, we only present the main steps.

First, we work as in Theorem VI.5 in [16], with $h \equiv -\infty$ and w as before, to get the existence of $X_\epsilon, Y_\epsilon \in \mathbb{R}$ such that

$$\beta u^\epsilon(\overline{x}_\epsilon) \leq G(\widehat{x}_\epsilon, w_x(\overline{x}_\epsilon, \overline{y}_\epsilon) + \theta, X_\epsilon),$$

$$\beta v_\epsilon(\overline{y}_\epsilon) \geq G(\widehat{y}_\epsilon, -w_y(\overline{x}_\epsilon, \overline{y}_\epsilon), -y_\epsilon),$$

and

$$\begin{bmatrix} X_\epsilon & 0 \\ 0 & Y_\epsilon \end{bmatrix} \leq \begin{bmatrix} w_{xx}(\overline{x}_\epsilon, \overline{y}_\epsilon) & w_{xy}(\overline{x}_\epsilon, \overline{y}_\epsilon) \\ w_{yx}(\overline{x}_\epsilon, \overline{y}_\epsilon) & w_{yy}(\overline{x}_\epsilon, \overline{y}_\epsilon) \end{bmatrix}$$

for some $\widehat{x}_\epsilon$, $\widehat{y}_\epsilon \in \mathbb{R}$, where $|\widehat{x}_\epsilon - \overline{x}_\epsilon| \leq C\sqrt{\epsilon}$ and $|\widehat{y}_\epsilon - y_\epsilon| \leq C\sqrt{\epsilon}$ for some positive constant C independent of ϵ, δ, and η.

Next, working as in Case A we pass to the limit as $\epsilon \downarrow 0$ and using Lemma 3.2 we derive (4.23) for $\overline{x} = 0$. Finally, we use (4.21) and (4.22) and we send first $\theta \downarrow 0$, then $\delta \downarrow 0$ and last $\eta \downarrow 0$ to conclude.

Proof of Lemma 4.1. We first observe that (4.18) yields

$$X + Y \leq 0. \tag{4.24}$$

Let π_1^* and π_2^*, satisfying $\pi_1^* \leq a_1$ and $\pi_2^* \leq a_2$, be the points where the constrained maxima of $[\frac{1}{2}\sigma^2\pi^2 X + (b-r)\pi p]$ and $[-\frac{1}{2}\sigma^2\pi^2 Y + (b-r)\pi p]$ are, respectively, achieved. We look at the following cases.

Case 1. $\pi_1^* = a_1$ and $\pi_2^* = a_2$. Then

$$\begin{aligned} &\max_{\pi \leq a_1}\left[\tfrac{1}{2}\sigma^2\pi^2 X + (b-r)\pi p\right] - \max_{\pi \leq a_2}\left[-\tfrac{1}{2}\sigma^2\pi^2 Y + (b-r)\pi p\right] \\ &= \tfrac{1}{2}\sigma^2(a_1^2 X + a_2^2 Y) + (b-r)p(a_1 - a_2). \end{aligned} \tag{4.25}$$

Multiplying both sides of (4.19) by the positive definite matrix

$$\Sigma = \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix},$$

and taking the trace, yields

$$\tfrac{1}{2}(a_1^2 X + a_2^2 Y) \leq \tfrac{1}{2}(a_1 - a_2)^2 A$$

which, combined with (4.25), gives (4.20).

Case 2.

$$\pi_1^* = -\frac{(b-r)p}{\sigma^2 X} \qquad \text{and} \qquad \pi_2^* = \frac{(b-r)p}{\sigma^2 Y}.$$

Then, the maxima are unconstrained and we easily get

$$\max_\pi \left[\tfrac{1}{2}\sigma^2\pi^2 X + (b-r)\pi p\right] - \max_\pi \left[-\tfrac{1}{2}\sigma^2\pi^2 Y + (b-r)\pi p\right] \leq \max_\pi \left[\tfrac{1}{2}\sigma^2\pi^2 (X+Y)\right] = 0$$

where we used (4.24).

Case 3. $\pi_1^* = -(b-r)p/\sigma^2 X < a_1$ and $\pi_2^* = a_2$. Then

$$\max_{\pi \leq a_1} \left[\frac{1}{2}\sigma^2\pi^2 X + (b-r)\pi p\right] = -\frac{(b-r)^2 p^2}{2\sigma^2 X} < \frac{(b-r)a_1}{2} p$$

and

$$\max_{\pi \leq a_2} \left[-\frac{1}{2}\sigma^2\pi^2 Y + (b-r)\pi p\right] = -\frac{1}{2}\sigma^2 a_2^2 Y + (b-r)a_2 p.$$

If $Y = 0$, (4.20) follows immediately. If $Y > 0$, then by assumption, $(b-r)p/\sigma^2 Y > a_2$. Therefore,

$$\begin{aligned} \frac{(b-r)}{2} a_1 p - \left[-\frac{1}{2}\sigma^2 a_2^2 Y + (b-r)a_2 p\right] &= \frac{(b-r)}{2} p(a_1 - a_2) + \frac{1}{2}\sigma^2 a_2^2 Y - \frac{(b-r)}{2} a_2 p \\ &= \frac{(b-r)}{2} p(a_1 - a_2) + \frac{1}{2} a_2[\sigma^2 a_2 Y - (b-r)p] \\ &< \frac{(b-r)}{2} p(a_1 - a_2). \end{aligned}$$

Case 4. $\pi_1^* = a_1$ and $\pi_2^* = (b-r)p/\sigma^2 Y$. This case is easily reduced to Case 2. □

Proof of Lemma 4.2. Relation (4.21) follows directly from (4.7) and (4.8). To show (4.22), since f is Lipschitz, it suffices to show $\lim_{\theta\downarrow 0}\lim_{\delta\downarrow 0}\theta x_0(\theta,\delta) = 0$. Indeed, from (4.7) we have

$$\text{(4.26)} \quad u(x_0) - v(x_0) - \theta x_0(\theta,\delta) \geq [u(\overline{x}) - v(\overline{x}) - \theta\overline{x}] - \omega_v(kl(\theta)\delta) - \omega_v(k\eta\delta)$$

which in turn implies (for fixed θ) $\sup_{\delta>0} x_0(\theta,\delta) < +\infty$.

Therefore, there exists $\overline{x}_0(\theta)$ such that $\lim_{\delta\to 0} x_0(\theta,\delta) = \overline{x}_0(\theta)$. The limit here is taken along subsequences which, to simplify notation, we denote the same way as the whole family. By sending $\delta \to 0$, (4.26) combined with (4.5) implies

$$\text{(4.27)} \qquad u(\overline{x}_0) - v(\overline{x}_0) - \theta\overline{x}_0 \geq [u(x) - v(x) - \theta x] \quad \forall x \in \overline{\Omega}.$$

We now send $\theta \to 0$. If $\lim_{\theta\downarrow 0} \theta \bar{x}_0 = \alpha \neq 0$, again along subsequences, (4.27) yields $\sup_{\overline{\Omega}}[u-v] - \alpha \geq \sup_{\overline{\Omega}}[u-v]$ which implies $\sup_{\overline{\Omega}}[u-v] \leq 0$, which contradicts (4.5). □

Remark. In the case $f \equiv \infty$, we assume

$$\sup[u(x) - v(x) - \theta x^\delta] > 0$$

for some $\delta \in (\gamma, 1)$, and we argue as before.

5. In this section we show that the value function is a smooth solution of the Hamilton–Jacobi–Bellman equation and we characterize the optimal policies.

THEOREM 5.1. *The value function v is the unique continuous on* $[0, +\infty)$ *and twice continuously differentiable in* $(0, +\infty)$ *solution of* (1.8) *in the class of concave functions.*

Before we go into the details of the proof of the theorem we describe the main ideas. We will work in intervals $(x_1, x_2) \subset [0, +\infty)$ and show that v solves a uniformly elliptic HJB equation in (x_1, x_2) with boundary conditions $v(x_1)$ and $v(x_2)$. Standard elliptic regularity theory (cf. Krylov [23]) and the uniqueness result about viscosity solutions will yield that v is smooth in (x_1, x_2).

We next explain how we come up with the uniformly elliptic HJB equation. Formally, according to the constraints, the optimal π^* is either $f(x)$, if

$$-\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)} \geq f(x) \quad \text{or} \quad -\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)}, \quad \text{if} \quad -\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)} \leq f(x).$$

In the second case, we want to get a positive lower bound of π^* in $[x_1, x_2]$. Since v_x is nonincreasing and strictly positive, it is bounded from below away from zero. Therefore, it suffices to find a lower bound for v_{xx}.

An important feature of the proof is the approximation of v by a family of smooth functions (v^ϵ) which are solutions of a suitably regularized equation. Next, we define the v^ϵ's and discuss their main properties.

Let W_t^1 be a Wiener process which is independent of W_t and is defined on some probability space (Ω^1, F^1, P^1). We consider the process $\overline{W}_t = (W_t, W_t^1)$, which is a Wiener process, on $(\Omega \times \Omega^1, \overline{F}, P \times P^1)$ and $\overline{F} = \sigma(F \times F^1)$ where $\sigma(F \times F^1)$ is the smallest σ-algebra which contains $F \times F^1$. Let ϵ be a positive number. A real process $(C^\epsilon, \pi^\epsilon)$ which is F_t-progressively measurable is called an *admissible* policy if:

(i) $C_t^\epsilon \geq 0$ a.s. $\forall t \geq 0$ and $\int_0^{+\infty} C_s^\epsilon ds < +\infty$ a.s.;

(ii) $\int_0^{+\infty} (\pi_s^\epsilon)^2 ds < +\infty$ a.s. and $\pi_t^\epsilon \leq f(X_t^\epsilon)$ a.s. $\forall t \geq 0$ where f satisfies (1.4);

(iii) $X_t^\epsilon \geq 0$ a.s. $\forall t \geq 0$, where X_t^ϵ is the trajectory given by the state equation

$$\begin{cases} dX_t^\epsilon = [rX_t^\epsilon + (b-r)\pi_t^\epsilon - C_t^\epsilon]dt + \sigma\pi_t^\epsilon dW_t + \sigma\epsilon X_t^\epsilon dW_t^1 (t > 0) \\ X_0^\epsilon = x \qquad (x\epsilon[0, +\infty)) \end{cases}$$

using the controls $(C^\epsilon, \pi^\epsilon)$.

We denote by $\mathcal{A}_x^\epsilon$ the set of admissible policies. We define the value function v^ϵ by

$$v^\epsilon(x) = \sup_{\mathcal{A}_x^\epsilon} E\int_0^{+\infty} e^{-\beta t} U(C_t^\epsilon)dt,$$

where U is the usual utility function. Using arguments similar to those in Propositions 2.1 and 2.2 we can prove that v^ϵ is concave, strictly increasing in x, and uniformly continuous on $\overline{\Omega}$.

Using a variation of Theorems 3.1 and 4.1, we have that the value function v^ϵ is the unique constrained viscosity solution on $\overline{\Omega}$ of the equation

$$\beta v^\epsilon = \max_{\pi \leq f(x)} \left[\frac{1}{2}\sigma^2(\pi^2 + \epsilon^2 x^2)v^\epsilon_{xx} + (b-r)\pi v^\epsilon_x\right] + \max_{c \geq 0}[-cv^\epsilon_x + U(c)] + rxv^\epsilon_x. \tag{5.1}$$

We next consider a sequence (v^ϵ_L) with

$$v^\epsilon_L(x) = \sup_{\mathcal{A}^{\epsilon,L}} E\int_0^{+\infty} e^{-\beta t}U(C_t)dt \qquad (x \in \mathbb{R}).$$

The set $\mathcal{A}^{\epsilon,L}$ of admissible policies consists of pairs $(C^{\epsilon,L}, \pi^{\epsilon,L})$ such that $C^{\epsilon,L}, \pi^{\epsilon,L}$ are F_t-progressively measurable satisfying (i), (ii), and also $-L \leq \pi^{\epsilon,L}_t$ almost surely for all $t \geq 0$.

Working as in Theorems 3.1 and 4.1, we get that v^ϵ_L is the unique constrained viscosity solution of $\overline{\Omega}$ of

$$\begin{aligned}\beta v^\epsilon_L = &\max_{-L \leq \pi \leq f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 v^\epsilon_{L,xx} + (b-r)\pi v^\epsilon_{L,x}\right]\\ &+ \max_{c \geq 0}[-cv^\epsilon_{L,x} + U(c)] + rxv^\epsilon_{L,x}\end{aligned} \tag{5.2}$$

and, also, the unique viscosity solution (see [16]) of

$$\begin{aligned}\beta u = &\max_{-L \leq \pi \leq f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 u_{xx} + (b-r)\pi u_x\right]\\ &+ \max_{c \geq 0}[-cu_x + U(c)] + rxu_x (x\epsilon[x_1, x_2])\\ u(x_1) &= v^\epsilon_L(x_1), \qquad u(x_2) = v^\epsilon_L(x_2).\end{aligned} \tag{5.3}$$

On the other hand, (5.3) admits a unique smooth solution u which is also the unique viscosity solution; therefore, v^ϵ_L is smooth which, together with the fact that v^ϵ_L is increasing and concave, yields that v^ϵ_L is also smooth solution of

$$\begin{aligned}\beta u = &\max_{0 \leq \pi \leq f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 u_{xx} + (b-r)\pi u_x\right]\\ &+ \max_{c \geq 0}[-cu_x + U(c)] + rxu_x\\ u(x_1) &= v^\epsilon_L(x_1), \qquad u(x_2) = v^\epsilon_L(x_2).\end{aligned}$$

We next observe that there exists w concave such that $v^\epsilon_L \to w$, as $L \to \infty$, locally uniformly in $\overline{\Omega}$. Therefore, w is a constrained viscosity solution of (5.1) and, by uniqueness of viscosity solution, it coincides with v^ϵ. This also implies that v^ϵ is a viscosity solution of

$$\begin{aligned}\beta u = &\max_{0 \leq \pi \leq f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 u_{xx} + (b-r)\pi u_x\right]\\ &+ \max_{c \geq 0}[-cu_x + U(c)] + rxu_x\\ u(x_1) &= v^\epsilon(x_1), \quad u(x_2) = v^\epsilon(x_2).\end{aligned} \tag{5.4}$$

Equation (5.4) admits a unique smooth solution which is the unique viscosity solution. Therefore v^ϵ is smooth.

Proof. Consider an interval $[x_1, x_2] \subset [0, +\infty)$ and let $[\overline{x}_1, \overline{x}_2]$ and $[X_1, X_2]$ with $X_1 > 0$ be such that $[x_1, x_2] \subset [\overline{x}_1, \overline{x}_2] \subset [X_1, X_2]$. Since v is concave and increasing, its first and second derivatives exist almost everywhere. Without any loss of generality, we can assume that $v_x(X_1)$ and $v_x(X_2)$ exist. The reason for this will become apparent in the sequel.

We are now going to prove that the optimal portfolio π^ϵ of the approximating problem is bounded from below by a positive number which is independent of ϵ (of course it may depend on (x_1, x_2)). To this end, it suffices to show that

$$-\frac{b-r}{\sigma^2}\frac{v_x^\epsilon}{v_{xx}^\epsilon} \geqq c > 0 \quad \text{on } [x_1, x_2].$$

We first show that $v^\epsilon \to v$ locally uniformly on $\overline{\Omega}$. Indeed, v^ϵ and v_x^ϵ are bounded locally by u and u/x where u is the value function with $f(x) = +\infty$ and $\epsilon = 0$. This follows from the fact that $v^\epsilon \leq u^\epsilon \leq u$ (where u^ϵ is the value function with $f(x) = +\infty$ and $\epsilon > 0$), which can be proved by using a similar comparison result as in Theorem 4.1. Therefore, there exists a subsequence $\{v^{\epsilon_n}\}$ and a function w such that $v^{\epsilon_n} \to w$ locally uniformly in $\overline{\Omega}$. Moreover, an argument similar to the proof of Proposition 2.2 yields that $\lim_{x\to 0} v^\epsilon(x) = U(0)/\beta$ uniformly in ϵ, therefore $w(0) = U(0)/\beta$.

Using a variation of Proposition I.3 in [27] we get that w is a constrained viscosity of (1.8). Moreover, since w is concave, using Theorem 4.1 we get that it is the unique constrained viscosity solution of (1.8). Therefore, we conclude that all subsequences have the same limit which coincides with v. Using that $v^\epsilon \to v$ locally uniformly and the fact that v^ϵ and v are concave, we get that $\lim_{\epsilon\to 0} v_x^\epsilon(x_0) = v_x(x_0)$ at any point x_0 where $v_x(x_0)$ exists. Taking into account that v_x^ϵ is nondecreasing in $[X_1, X_2]$, we conclude that there exist positive constants $R_1 = R_1([X_1, X_2])$ and $R_2 = R_2([X_1, X_2])$ such that

$$R_1 \leq v_x^\epsilon(x) \leq R_2 \quad \text{on } [x_1, x_2]. \tag{5.5}$$

We next show that there exists a constant $R = R([X_1, X_2])$ such that

$$|v_{xx}^\epsilon(x)| \geq R \quad \text{on } [x_1, x_2]. \tag{5.6}$$

To this end, let $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ be as follows:

(i) $\zeta \in C_0^\infty$ (i.e., ζ is a smooth function with compact support);

(ii) $\zeta \equiv 1$ on $[x_1, x_2], \zeta \equiv 0$ on $\mathbb{R}\backslash[\overline{x}_1, \overline{x}_2]$, and $0 \leq \zeta \leq 1$ otherwise;

(iii) $|\zeta_x| \leq M\zeta^p, |\zeta_{xx}| \leq M\zeta^p$ with $0 < p < 1$ and $M > 0$.

From now on, for simplicity we suppress the ϵ-notation. We next consider a function $Z : [X_1, X_2] \to \mathbb{R}$ given by $Z(x) = \zeta^2 v_{xx}^2 + \lambda v_x^2 - \mu v$, where λ and μ are positive constants to be chosen later. We are interested in looking at the maximum of Z on $[X_1, X_2]$. The following cases can happen.

Case 1. The function Z attains its maximum at a point $x_0 \notin$ supp ζ. Then using (ii) and that $v > 0, v_x > 0$, we get

$$v_{xx}^2(x) \leq \lambda v_x^2(x_0) + \mu v(x) \quad \forall x \in [x_1, x_2]$$

which implies (5.6).

Case 2. The function Z attains its maximum at the point $x_0 \in$ supp ζ. In this case we have

$$Z_x(x_0) = 0 \quad \text{and} \quad Z_{xx}(x_0) \leq 0 \tag{5.7}$$

where

$$Z_x = 2\zeta\zeta_x v_{xx}^2 + 2\zeta^2 v_{xx}v_{xxx} + 2\lambda v_x v_{xx} - \mu v_x \tag{5.8}$$

and

$$\begin{aligned} Z_{xx} = 2\zeta_x^2 v_{xx}^2 &+ 2\zeta\zeta_{xx}v_{xx}^2 + 8\zeta\zeta_x v_{xx}v_{xxx} + 2\zeta^2 v_{xxx}^2 \\ &+ 2\zeta^2 v_{xx}v_{xxxx} + 2\lambda v_{xx}^2 + 2\lambda v_x v_{xxx} - \mu v_{xx}. \end{aligned} \tag{5.9}$$

We examine each of the following cases separately.

Case 2(a).

$$x_0 \in A_1 = \left\{ x \in [x_1, x_2] : -\frac{b-r}{\sigma^2}\frac{v_x^\epsilon(x)}{v_{xx}^\epsilon(x)} < f(x) \right\}.$$

In this case the Bellman equation has the form

$$\beta v = -\gamma\frac{v_x^2}{v_{xx}} + \frac{1}{2}\epsilon^2\sigma^2x^2v_{xx} - v_x I(v_x) + U(I(v_x)) + rxv_x \tag{5.10}$$

with $\gamma = (b-r)^2/2\sigma^2$. Here we used that $\max_{c\geq 0}[-cp + U(c)] = -pI(p) + U(I(p))$ with $I = (U')^{-1}$. After differentiating (5.10) twice and rearranging the terms we get:

$$\begin{aligned} -\gamma\frac{v_x^2 v_{xxxx}}{v_{xx}^2} - \frac{1}{2}\epsilon^2\sigma^2x^2v_{xxx} - \gamma\frac{v_x^2v_{xxx}^2}{v_{xx}^3} &= -\beta v_{xx} \\ &+ v_{xx}[2r - 2\gamma + \epsilon^2\sigma^2] + 2\gamma\frac{v_x v_{xxx}}{v_{xx}} - 3\gamma\frac{v_x^2 v_{xxx}^2}{v_{xx}^3} \\ &+ v_{xxx}[rx - I(v_x) + 2\epsilon^2\sigma^2 x] - v_{xx}^2 I'(v_x). \end{aligned} \tag{5.11}$$

On the other hand, (5.7) yields

$$-\left[\gamma\frac{v_x^2(x_0)}{v_{xx}^2(x_0)} + \frac{1}{2}\epsilon^2\sigma^2x_0^2\right] Z_{xx}(x_0) \geq 0$$

which, combined with (5.9), yields

$$\begin{aligned} -\left[\gamma\frac{v_x^2}{v_{xx}} + \frac{1}{2}\epsilon^2\sigma^2x^2\right] Z_{xx} &= 2\zeta^2 v_{xx}\left[-\gamma\frac{v_x^2 v_{xxxx}}{v_{xx}^2} - \frac{1}{2}\epsilon^2\sigma^2x^2v_{xxxx} - \gamma\frac{v_x^2v_{xxx}^2}{v_{xx}^3}\right] \\ &+ 2\lambda v_x\left[-\gamma v_x - \frac{1}{2}\epsilon^2\sigma^2x^2v_{xx} - \gamma\frac{v_x^2v_{xxx}}{v_{xx}^2}\right] + \mu\left[\gamma\frac{v_x^2}{v_{xx}} + \frac{1}{2}\epsilon^2\sigma^2x^2v_{xx}\right] \\ &- 2\gamma\zeta_x^2v_x^2 - \epsilon^2\sigma^2x^2\zeta_x^2v_{xx}^2 - 2\gamma v_x^2\zeta\zeta_{xx} - x^2\epsilon^2\sigma^2\zeta\zeta_{xx}v_{xx}^2 - 8\gamma\zeta\zeta_x\frac{v_x^2v_{xxx}}{v_{xx}} \\ &- 4x^2\epsilon^2\sigma^2\zeta\zeta_x v_{xx}v_{xxx} - \epsilon^2\sigma^2x^2\zeta^2v_{xxx}^2 - \lambda x^2\epsilon^2\sigma^2v_{xx}^2 \geq 0, \end{aligned} \tag{5.12}$$

where all the expressions are evaluated at x_0.

Using (5.10), (5.11), and

$$\begin{aligned} -\gamma v_x - \frac{1}{2}\epsilon^2\sigma^2x^2v_{xxx} - \gamma\frac{v_x^2v_{xxx}}{v_{xx}^2} &= v_{xx}[rx - I(v_x) + \epsilon^2\sigma^2x] \\ +v_x(r - 3\gamma) - \beta v_x & \end{aligned} \tag{5.13}$$

which follows from differentiating (5.10) once and rearranging terms, we obtain from (5.12):

$$\begin{aligned}
&-2\beta\zeta^2 v_{xx}^2 + 2\zeta^2(2r-2\gamma+\epsilon^2\sigma^2)v_{xx}^2 + 4\gamma\zeta^2 v_x v_{xxx} - 6\gamma\zeta^2\frac{v_x^2 v_{xxx}^2}{v_{xx}^2}\\
&\quad + 2\zeta^2 v_{xx}v_{xxx}[rx - I(v_x) + 2\epsilon^2\sigma^2 x] - 2\zeta^2 v_{xx}^3 I'(v_x)\\
&\quad + 2\lambda v_x v_{xx}[rx - I(v_x) + \epsilon^2\sigma^2 x] + 2\lambda(r-3\gamma)v_x^2 - 2\lambda\beta v_x^2 + 2\gamma\mu\frac{v_x^2}{v_{xx}}\\
&\quad + \mu v_x[-rx + I(v_x)] - \mu U(I(v_x))\\
&\quad + \beta\mu v - 2\gamma\zeta_x^2 v_x^2 - \epsilon^2\sigma^2 x^2\zeta_x^2 v_{xx}^2 - 2\gamma v_x^2\zeta\zeta_{xx} - \epsilon^2\sigma^2x^2\zeta\zeta_{xx}v_{xx}^2 - 8\gamma\zeta\zeta_x\frac{v_x^2 v_{xxx}}{v_{xx}}\\
&\quad - 4x^2\epsilon^2\sigma^2\zeta\zeta_x v_{xx}v_{xxx} - \epsilon^2\sigma^2x^2\zeta_x^2 v_{xxx}^2 - \lambda x^2\epsilon^2\sigma^2 v_{xx}^2 \geq 0.
\end{aligned}$$

A further calculation yields that at x_0,

$$\begin{aligned}
&(rx - I(v_x) + \epsilon^2\sigma^2 x)(2\zeta^2 v_{xx}v_{xxx} + 2\lambda v_x v_{xx} - \mu v_x) + 2\zeta^2\epsilon^2\sigma^2 x v_{xx}v_{xxx}\\
&\quad + \epsilon^2\sigma^2\mu x v_x - 2\beta\zeta^2 v_{xx}^2 - 4\left(\gamma - r - \frac{\epsilon^2\sigma^2}{2}\right)\zeta^2 v_{xx}^2 + 4\gamma\zeta^2 v_x v_{xxx}\\
&\quad - 6K\zeta^2\frac{v_x^2 v_{xxx}^2}{v_{xx}^2} - 2\lambda\beta v_x^2 - 2\lambda(3\gamma - r)v_x^2 + \mu\beta v + 2\gamma\mu\frac{v_x^2}{v_{xx}} - \mu U(I(v_x))\\
&\quad - 2\gamma\zeta_x^2 v_x^2 - 2\gamma\zeta\zeta_{xx}v_x^2 - 8\gamma\zeta\zeta_x\frac{v_x^2 v_{xxx}}{v_{xx}} + \epsilon^2\sigma^2x^2[4\zeta\zeta_x|v_{xx}|v_{xxx} - \zeta^2 v_{xxx}^2\\
&\quad - \lambda v_{xx}^2] - \epsilon^2\sigma^2x^2\zeta_x^2 v_{xx}^2 - \epsilon^2\sigma^2x^2\zeta\zeta_{xx}v_{xx}^2 \geq 2\zeta^2 v_{xx}^3 I'(v_x).
\end{aligned} \tag{5.14}$$

If $A(x) = 2\zeta^2 v_{xx}v_{xxx} + 2\lambda v_x v_{xx} - \mu v_{x'}$ (5.7) and (5.8) imply

$$A(x_0) = -2\zeta(x_0)\zeta_x(x_0)v_{xx}^2(x_0).$$

Then (5.14) becomes

$$\begin{aligned}
&-2\zeta\zeta_x v_{xx}^2(rx - I(v_x) + \epsilon^2\sigma^2 x) - 2\beta\zeta^2 v_{xx}^2 - 2\lambda\beta v_x^2 + \mu\beta v - 2\lambda(3\gamma - r)v_x^2\\
&-2\gamma\mu\frac{v_x^2}{|v_{xx}|} - \mu U(I(v_x)) - 2\gamma\zeta_x^2 v_x^2 - \epsilon^2\sigma^2x^2\zeta_x^2 v_{xx}^2 - \epsilon^2\sigma^2x^2\zeta\zeta_{xx}v_{xx}^2 - 2\gamma\zeta\zeta_{xx}v_x^2\\
&+\left[-4\left(\gamma - r - \frac{\epsilon^2\sigma^2}{2}\right)\zeta^2 v_{xx}^2 + 4\gamma\zeta^2 v_x v_{xxx} - 6\gamma\zeta^2\frac{v_x^2 v_{xxx}^2}{v_{xx}^2} + 8\gamma\zeta\zeta_x\frac{v_x^2 v_{xxx}}{|v_{xx}|}\right]\\
&+\epsilon^2\sigma^2x^2\left[\left(4\zeta\zeta_x - \frac{2\zeta^2}{x}\right)|v_{xx}|v_{xxx} - \zeta^2 v_{xxx}^2 - \lambda v_{xx}^2\right] + \epsilon^2\sigma^2\mu x v_x\\
&\geq 2\zeta^2 v_{xx}^3 I'(v_x).
\end{aligned} \tag{5.15}$$

Let

$$B(x) = -4\left(\gamma - r - \frac{\epsilon^2\sigma^2}{2}\right)\zeta^2 v_{xx}^2 + 4\gamma\zeta^2 v_x v_{xxx} - 6\gamma\zeta^2\frac{v_x^2 v_{xxx}^2}{v_{xx}^2} + 8\gamma\zeta\zeta_x\frac{v_x^2 v_{xxx}}{|v_{xx}|}$$

and

$$C(x) = \left(4\zeta\zeta_x - \frac{2\zeta^2}{x}\right)|v_{xx}|v_{xxx} - \zeta^2 v_{xxx}^2 - \lambda v_{xx}^2.$$

Let ϵ sufficiently small and $\theta \in (0, \frac{3}{4})$. Using the Cauchy–Schwartz inequality we get

$$B(x_0) \leq C\left(\frac{\zeta_x^2(x_0)}{\theta}v_x^2(x_0) + \zeta^2(x_0)v_{xx}^2(x_0)\right) \tag{5.16}$$

for some positive constant C.

A similar argument yields

$$C(x_0) \le v_{xx}^2(x_0)\left[\left(2\zeta_x(x_0) - \frac{\zeta(x_0)}{x_0}\right)^2 - \lambda\right]. \tag{5.17}$$

We next choose λ so that $\lambda > 4\zeta_x^2 + (2/x_1^2)$ on $[\overline{x}_1, \overline{x}_2]$. Then $C(x_0) \le 0$. If we leave out all the negative terms in (5.15) and use (5.16) and (5.17) we get

$$\begin{aligned} & v_{xx}^2(x_0)[-2\zeta(x_0)\zeta_x(x_0)[rx_0 - I(v_x(x_0))) + 2\epsilon^2\sigma^2 x_0] \\ & -\epsilon^2 x_0^2\sigma^2\zeta(x_0)\zeta_{xx}(x_0) + C\zeta^2(x_0)] \\ & + v_x^2(x_0)\left[C\frac{\zeta_x^2(x_0)}{\theta} - 2\gamma\zeta(x_0)\zeta_{xx}(x_0)\right] \\ & + \mu[\beta v(x_0) + 2\epsilon\sigma^2 x_0 v_x(x_0))] \\ & \ge 2\zeta(x_0)^2 v_{xx}^3(x_0) I'(v_x(x_0)). \end{aligned} \tag{5.18}$$

We now return to the ϵ-notation. From (5.5) we have

$$I(v_x^\epsilon(x_0)) \le I(R_1) \tag{5.19}$$

and

$$|I'(v_x^\epsilon(x_0))| \le |I'(R_2)|. \tag{5.20}$$

From (5.18), (5.19), and (5.20) we get that for some constants k_1, k_2, k_3, and k_4,

$$v_{xx}^\epsilon(x_0)^2[k_1\zeta(x_0)|\zeta_x(x_0)| + k_2\zeta(x_0)|\zeta_{xx}(x_0)| + C\zeta^2(x_0)] + k_4 \ge k_3\zeta(x_0)^2|v_{xx}^\epsilon(x_0)|^3.$$

Using property (iii) of ζ with $p = \frac{1}{3}$ we obtain

$$C_1 v_{xx}^\epsilon(x_0)^2[\zeta(x_0)^2 + \zeta(x_0)^{1+1/3}] + k \ge k_3\zeta(x_0)^2|v_{xx}^\epsilon(x_0)|^3 \tag{5.21}$$

for some $C_1 > 0$. Now if $w(x_0) = \zeta^2(x_0)|v_{xx}^\epsilon(x_0)|^3$, then $w(x_0)^{2/3} = \zeta(x_0)^{4/3}|v_{xx}^\epsilon(x_0)|^2$. In view of property (ii) of ζ, (5.21) yields

$$2C_1 w(x_0)^{2/3} + \overline{k} \ge k_3 w(x_0),$$

for some $\overline{k} > 0$ and, therefore,

$$w(x_0) \le N$$

where $N = N(C_1, \overline{K}, K_3)$ is independent of ϵ.

Thus

$$(v_{xx}^\epsilon)^2 + \lambda(v_x^\epsilon)^2 - \mu v^\epsilon \le N^{2/3} + \lambda(v_x^\epsilon(x_0))^2 - \mu v^\epsilon(x_0) \text{ on } [x_1, x_2],$$

i.e., there exists a constant L_1, independent of ϵ, such that

$$|v_{xx}^\epsilon| \le L_1 \text{ on } [x_1, x_2].$$

Case 2b.

$$x_0 \epsilon A_2 = \left\{x\epsilon[x_1, x_2] : -\frac{b-r}{\sigma^2}\frac{v_x^\epsilon(x)}{v_{xx}^\epsilon(x)} \ge f(x)\right\}.$$

In this case,

$$|v_{xx}^{\epsilon}(x_0)| \le L_2 \quad \text{on } [x_1, x_2] \quad \text{where } L_2 = \frac{b-r}{\sigma^2}\frac{R_2}{f(x_1)}.$$

Therefore,

$$|v_{xx}^{\epsilon}| \le R \quad \text{on } [x_1, x_2]. \tag{5.22}$$

where $R = \max(L_1, L_2)$, independent of ϵ.

Combining (5.5) and (5.22) we see that

$$-\frac{b-r}{\sigma^2}\frac{v_x^{\epsilon}}{v_{xx}^{\epsilon}} \ge B > 0 \quad \text{on } [x_1, x_2] \quad \text{with } B = \frac{b-r}{\sigma^2}\frac{R_1}{R}.$$

Let us now consider the equation

$$\begin{aligned} \beta u = &\max_{B \le \pi \le f(x)} \left[\tfrac{1}{2}\sigma^2(\pi^2 + \epsilon^2 x^2)u_{xx} + (b-r)\pi u_x\right] \\ &+ \max_{c \ge 0}[-cu_x + U(c)] + rxu_x \\ u(x_1) = v^{\epsilon}&(x_1), \quad u(x_2) = v^{\epsilon}(x_2) \quad (x\epsilon[x_1, x_2]). \end{aligned} \tag{5.23}$$

In view of the above analysis, we know that v^{ϵ} solves (5.23). Let $\epsilon \to 0$. Since $v^{\epsilon} \to v$, locally uniformly, v is a viscosity solution of

$$\begin{aligned} \beta u = &\max_{B \le \pi \le f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 u_{xx} + (b-r)\pi u_x\right] \\ &+ \max_{c \ge 0}[-cu_x + U(c)] + rxu_x (x\epsilon[x_1, x_2]) \\ u(x_1) = v&(x_1), \quad u(x_2) = v(x_2). \end{aligned} \tag{5.24}$$

On the other hand, (4.21) admits a unique smooth solution u (see [23]) which is the unique viscosity solution (see [16, Thm. II.2]); therefore v is smooth. □

THEOREM 5.2. *The feedback optimal controls C^* and π^* are given by*

$$c^*(x) = I(v_x(x)) \quad \textit{and} \quad \pi^*(x) = \min\left\{-\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)}, f(x)\right\} \quad \textit{for } x > 0.$$

The state equation (1.3) *has a strong unique solution X_t^*, corresponding to $C_t^* = c^*(X_t^*)$ and $\pi_t^* = \pi^*(X_t^*)$ and starting at $x > 0$ at $t = 0$, which is unique in probability law up to the first time τ such that $X_\tau^* = 0$.*

Proof. The formulae for π^* and C^* follow from a standard verification theorem (see [11]) and the equation. We now show that π^* and C^* are locally Lipschitz functions of x. It is clear that v_x is locally Lipschitz because in any compact set K there exists a constant $C = C(K)$ such that $|v_{xx}| \le C(K), (x\epsilon K)$. Therefore C^* is locally Lipschitz. Moreover, from the Bellman equation we have that

$$v_{xx} = H(x, v, v_x)$$

where

$$H(x, v, v_x) = \frac{2[\beta v - (b-r)f(x)v_x + v_x I(v_x) - U(I(v_x)) + rxv_x]}{\sigma^2 f(x)^2}$$

$$\text{if} \quad -\frac{b-r}{\sigma^2}\frac{v_x}{v_{xx}} \ge f(x)$$

and

$$H(x, v, v_x) = -\frac{b-r}{2\sigma^2} \frac{v_x^2}{\beta v - rxv_x + v_x I(v_x) - U(I(v_x))} \quad \text{if} \quad -\frac{b-r}{\sigma^2}\frac{v_x}{v_{xx}} < f(x).$$

Since v_x is locally Lipschitz we get that v_{xx} is also locally Lipschitz. Therefore (see Gikhman and Skorohod [14]) equation (1.3) has a unique strong solution X_t^* in probability law up to the first time τ such that $X_\tau^* = 0$. □

6. In this section we discuss the finite horizon model and we state results about the value function and the optimal policies.

The investor starts at time $t \in [0, T)$ with an initial endowment x, consumes at rate C_s and invests π_s^0 (respectively, π_s) amount of money in bond (respectively, in stock) for $t \leq s \leq T$. The prices of the bond and the stock satisfy the same equations as in the infinite horizon case. The wealth of the investor $X_s = \pi_s^0 + \pi_s (t \leq s \leq T)$ satisfies the state equation

$$\begin{aligned} &dX_s = rX_s ds + (b-r)\pi_s ds - C_s ds + \sigma\pi_s dW_s \qquad (t \leq s \leq T) \\ &X_t = x \qquad (x > 0). \end{aligned} \tag{6.1}$$

The agent faces the same constraints as in the infinite horizon case. In other words, the wealth must stay nonnegative; the agent cannot consume at a negative rate and must meet borrowing constraints ($\pi_s \leq f(X_s)$ for $t \leq s \leq T$).

The *total utility* coming both from consumption and terminal wealth is

$$J(x, t, C, \pi) = E\left[\int_t^T U(C_s)ds + \Phi(X_T)\right]$$

where U is the usual utility function and Φ is the *bequest* function which is typically concave, increasing, and smooth.

The *value function* is

$$\mathcal{A}(x, t) = \sup_{\mathcal{A}_{(x,t)}} J(x, t, C, \pi)$$

where $\mathcal{A}_{(x,t)}$ is the set of admissible controls.

In the sequel we state the main theorems. Since the proofs are modifications of the ones given in the previous sections they are not presented here.

THEOREM 6.1. *The value function is the unique continuous on $\Omega \times [t, T]$ and $C^{2,1}(\Omega \times (t, T))$ solution of*

$$\begin{aligned} &v_t + \max_{\pi \leq f(x)} \left[\tfrac{1}{2}\sigma^2\pi^2 v_{xx} + (b-r)\pi v_x\right] + \max_{c \geq 0}[-cv_x + U(c)] + rxv_x = 0 \\ &v(x, T) = \Phi((x) \end{aligned} \tag{6.2}$$

in the class of concave (with respect to the space variable x) functions.

THEOREM 6.2. *The feedback optimal controls C^* and π^* are given by $C_t^* = c^*(X_t^*, t)$ and $\pi_t^* = \pi^*(X_t^*, t)$ where $C^*(x, t) = (U')^{-1}(v_x(x, t))$ and*

$$\pi^*(x, t) = \min\left\{-\frac{b-r}{\sigma^2}\frac{v_x(x, t)}{v_{xx}(x, t)}, f(x)\right\} \quad \textit{for } x > 0.$$

REFERENCES

[1] R. S. Bellman, *Dynamic Programming,* Princeton University Press, Princeton, NJ, 1957.

[2] G. Barles and B. Perthame, *Exit time problems in optimal control and the vanishing viscosity method,* SIAM J. Control Optim., 26 (1988), pp. 1133–1148.

[3] I. Capuzzo–Dolcetta and P.-L. Lions, *Hamilton–Jacobi equations and state-constraints problems,* preprint.

[4] J. Cox and C. Huang, *Optimal consumption and portfolio policies when asset prices follow a diffusion process,* J. Economic Theory, 49 (1989), pp. 33–83.

[5] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations,* preprint.

[6] M. G. Crandall and P.-L. Lions, *Viscosity solutions of Hamilton–Jacobi equations,* Trans. AMS, 277 (1983), pp. 1–42.

[7] J. Cvitanic and I. Karatzas, *Convex duality in constrained portfolio optimization,* Department of Statistics, Columbia University, New York, preprint.

[8] D. Duffie, W. Fleming, and T. Zariphopoulou, *Hedging in incomplete markets with HARA utility,* Business School, Stanford University, Stanford, CA, preprint.

[9] L. C. Evans and P.E. Souganidis, *Differential games and representation for solutions of Hamilton–Jacobi–Isaacs equations,* Indiana U. Math. J., 33 (1984), pp. 773–797.

[10] B. G. Fitzpatrick and W.H. Fleming, *Numerical methods for an optimal investment–consumption model,* Math. Oper. Res., 16 (1991), pp. 823–841.

[11] W. H. Fleming and R.W. Rishel, *Deterministic and Stochastic Optimal Control,* Springer-Verlag, Berlin, New York, 1975.

[12] W. H. Fleming and P.E. Souganidis, *On the existence of value functions of two player, zero-sum stochastic differential games,* Indiana U. Math. J., 38 (1989), pp. 293–314.

[13] W. H. Fleming and T. Zariphopoulou, *An optimal investment-consumption model with borrowing,* Math. Oper. Res., 16 (1991), pp. 802–822.

[14] I. I. Gikhman and A.V. Skorohod, *Stochastic Differential Equations,* Springer-Verlag, Berlin, New York, 1972.

[15] H. He and N.D. Pearson, *Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case,* J. Econom. Theory, 54 (1991), 259–304.

[16] H. Ishii and P.-L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations,* J. Differential Equations, 83 (1990), pp. 26–78.

[17] R. Jensen, P.-L. Lions, and P.E. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*, Proc. AMS, 102 (1988), pp. 975–978.

[18] I. Karatzas, J. Lehoczky, S. Sethi, and S. Shreve, *Explicit solution of a general consumption-investment problem,* Math. Oper. Res., 11 (1986), pp. 261–294.

[19] I. Karatzas, J. Lehoczky, and S. Shreve, *Optimal portfolio and consumption decisions for a small investor on a finite horizon,* SIAM J. Control Optim., 25 (1987), pp. 1557–1586.

[20] I. Karatzas, J. Lehoczky, S. Shreve, and G.-L. Xu, *Martingale and duality methods for utility maximization in an incomplete market,* SIAM J. Control Optim., 29 (1991), pp. 702–730.

[21] M. Katsoulakis, *State-constraints problems for second order fully nonlinear degenerate partial differential equations,* preprint.

[22] N. V. Krylov, *Controlled Diffusion Processes,* Springer-Verlag, Berlin, New York, 1980.

[23] ———, *Nonlinear Elliptic and Parabolic Equations of the Second Order,* D. Reidel, Dordrecht, Holland, 1987.

[24] J. M. Lasry and P.-L. Lions, *A remark on regularization in Hilbert spaces,* Israel J. Math., 55 (1986), pp. 257–266.

[25] J. Lehoczky, S. Sethi, and S. Shreve, *Optimal consumption and investment policies allowing consumption constraints and bankruptcy,* Math. Oper. Res., 8 (1983), pp. 613–636.

[26] P.-L. Lions, *Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations, Part 1,* Comm. Partial Differential Equations, 8 (1983), pp. 1101–1174.

[27] ———, *Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations, Part 2,* Comm. Partial Differential Equations, 8 (1983), pp. 1229–1276.

[28] R. C. Merton, *Lifetime portfolio selection under uncertainty: The continuous-time case,* Rev. Econom. Statist., 51 (1969), pp. 247–257.

[29] ———, *Optimum consumption and portfolio rules in a continuous-time model,* J. Econom. Theory 3 (1971), 373–413.

[30] ———, *Erratum,* J. Econom. Theory, 6 (1973), pp. 213–214.

[31] H. Pages, *Optimal consumption and portfolio policies when markets are incomplete,* MIT mimeo, Massachusetts Institute of Technology, Cambridge, MA, 1987.

[32] S. A. Pliska, *Stochastic calculus model of continuous trading: Optimal portfolios,* Math. Oper. Res., 11 (1986), pp. 371–382.

[33] S. SETHI AND M. TAKSAR, *Infinite horizon investment consumption model with a nonterminal bankruptcy,* FSU Stat. Report No. M-750,

[34] ———, *A Note on Merton's optimum consumption and portfolio rules in a continuous-time model,* FSU Statistics Report No. M746,

[35] S. SHREVE AND G.-L. XU, *A duality method for optimal consumption and investment under short-selling prohibition, Part 1: General market coefficients,* Ann. Appl. Probability, 1 (1991), pp. 87–112.

[36] ———, *A duality method for optimal consumption and investment under short-selling prohibition, Part 2: Constant market coefficients,* Ann. Appl. Probability 1 (1991), pp. 52–69.

[37] H. M SONER, *Optimal Control with state-space constraints,* I, SIAM J. Control Optim., 24 (1986), pp. 552–561.

[38] J.-L. VILA AND T. ZARIPHOPOULOU, *Optimal consumption and portfolio choice with borrowing constraints,* J. Econom. Theory, submitted.

[39] G. XU, *A duality approach to stochastic portfolio/consumption decision problem in a continuous time market with short-selling restriction,* Ph. thesis, Carnegie Mellon University, Pittsburgh, PA, 1988.

[40] T. ZARIPHOPOULOU, *Optimal investment-consumption models with constraints,* Ph. thesis, Brown University, Providence, RI, 1988.