Indifference Prices and Related Measures

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Abstract The traditional approach towards derivative pricing consists of dynamically replicating a future liability by trading the assets on which that liability is written. However, the assumption that one can trade the assets is often rather restrictive. In some cases, say of options on commodities or funds, one can at best trade another correlated asset. In others, as in the case of basket options, even when one can trade the basket components, for efficiency reasons one may still prefer to use a correlated index for pricing and hedging. Due to the departure from the traditional assumptions of valuation by replication and no arbitrage considerations, one needs to review the pricing and hedging methodologies to accommodate the above situations. A utility-based approach is developed herein for the specification of indifference price of claims written on non-traded assets. The pricing mechanism is based upon the parity between the maximal utilities, with and without employing the derivative. The residual amount gained from granting the option, which renders the investor impartial towards these two scenarios, is called the indifference price. Under exponential risk preferences such a price can be calculated by a nonlinear transformation of a solution to a linear parabolic equation. The transformation is independent of the risk preferences and only depends on the correlation between the traded and the non-traded risky assets. The equation is associated with a diffusion process whose dynamics are in turn identified by solving a relevant Hamilton-Jacobi-Bellman equation. The new diffusion turns out to be a drift-modified diffusion of the original one modelling the level of the non-traded asset. The drift modification corresponds to a new measure, referred to as the indifference measure, which depends on the correlation and the Sharpe ratio of the traded risky asset.

Key words: Non-traded assets, Utility maximization, HJB equations, indifference prices.

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1 Introduction

The purpose of this document is to develop and analyze pricing and risk management methodology for derivative instruments written on assets that cannot be traded. The level of the latter can be fully observed across time but it is not feasible to create a hedging portfolio using the asset. Therefore, the market is incomplete and alternatives to the arbitrage pricing must be developed in order to specify the appropriate price concept and to define the related risk management.

If it is not possible to hedge all risk, it seems natural to go beyond calculation of the price as the expectation of the discounted payoff, under a measure determined by a certain optimality criteria; and to also consider the variance of the relevant random variables in order to quantify the additional risk. This general approach is known as mean-variance hedging. The analysis can then be based on the self-financing trading strategies with the aim of minimizing the tracking error at the terminal date only (see, for example Duffie and Richardson (1991)). Alternatively, one can start by enlarging the class of trading strategies to allow for an additional transfer of funds. This means that the usual assumption that a trading strategy should be self-financing is simply abandoned. The aim of this approach is to focus on the minimization of the future risk exposure at any time, and not only at the terminal date. This method of hedging in incomplete markets originates from work by Follmer and Sondermann (1986). Both ideas draw on the concept of arbitrage based pricing and generalizations of the classical Black-Scholes model. There is extensive literature on the topic, we refer the interested reader to Musiela and Rutkowski (1997) and the references therein.

A very different approach to pricing and risk management is based on utility maximization. The underlying idea aims to incorporate an investor's attitude towards the risk that cannot be eliminated. From this perspective, the utility-based pricing method carries many characteristics and a lot of insight from the seminal work of Merton (1969) on stochastic models of expected utility maximization. The concept of derivative price which takes into account transaction cost was introduced by Hodeges and Neuberger (1989). It was further extended and analyzed by a number of authors, see, among others, Davis et al (1993), Davis and Zariphopoulou (1995), Barles and Soner (1998), Constantinides and Zariphopoulou (1999). In a different model setup, similar to the one considered herein, Davis (1999) (2000) formulates and studies the pricing and hedging problem, considering the basis risk as the source of market incompleteness. He analyses the underlying optimization problem via its dual and shows that the price satisfies an associated Hamilton-Jacobi-Bellman (HJB) equation. Moreover, he derives the hedging strategy, which depends on the levels of the traded and non-traded assets, on the model parameters, and on the derivative of the price with respect to the level of the non-traded asset. However, he does not provide a solution to the HJB equation.

Assuming, as in Davis (2000), exponential risk preferences but following a different path, we give in Theorem 2.1 a closed form formula for the price, as a nonlinear transformation of a solution to a linear second order parabolic equation. It turns out that the price is not determined with respect to the risk neutral measure as it is the case in a complete model setting. Neither it is determined with respect to the measure describing the historical behavior of the non-traded asset. In fact, it refers to an indifference measure which is defined as the closest to the risk neutral one and, at the same time, capable of measuring the unhedgeable risk, by being defined on the filtration of the Brownian motion used for the modelling of the non-traded asset dynamics. More specifically, in the nested complete Black-Scholes model one has to identify the Radon-Nikodym density of the unique martingale measure with respect to the historical one. This density is then projected, by calculation of the conditional expectation under the historical measure, on to the filtration of the second Brownian motion which introduces the incompleteness. Theorem 2.2 provides the details. Naturally, our incomplete model turns into a complete Black-Scholes model, when the correlation between the traded and the non-traded asset increases. We conclude Section 2 with the convergence analysis of our pricing mechanism. Theorem 2.3 shows that the indifference price converges to the Black-Scholes one.

Having identified three different measures that are relevant to our model, the historical, risk neutral and indifference, we continue to study their relationships in Section 3. In particular, in Theorem 3.1 we give estimates of the total variation distances between them. Note that, even though the utility-based price is already defined, we still lack a coherent mechanism to define the associated hedging strategies. Complete models promise to cover all risk in a derivative product via dynamic implementation of the appropriate replicating and self-financing hedging strategy. Incomplete models seem to be more realistic. They acknowledge that all risk cannot be hedged. The 'total risk' contains both, hedgeable and unhedgeable components. The main issues are: how to isolate the components optimally, with respect to what criteria; and how to manage the risk of each of them separately. The hedgeable component of risk can be managed in the traditional way, whereas the unhedgeable component must be dealt with differently. For example, one may use the diversification argument well known to the insurance industry, coupled with the reserves and return on equity calculations. However, in order to develop such a framework, one may need to look at the price and the associated hedging methodology as being, specifically related to a given portfolio as opposed to the price and the hedge in complete models which are related to the market portfolio. We do not address these issues here. Rather, we concentrate on the comparative analysis with the Black-Scholes model. To this end, in Theorem 3.2, we derive, under the historical measure, the payoff decomposition in terms of its indifference price, the price changes of the traded assets, and the unhedgeable risk component. The second part of Section 3 is dedicated to alternative probabilistic representation of the indifference price. Motivated by the main characteristic of the arbitrage-free representation of derivative prices in complete markets, namely, their representation as expectations of the discounted payoff under the appropriate measure, we provide a similar representation in Theorem 3.3. We conclude Section 3 by identifying in Theorem 3.4 a martingale measure for the forward indifference price process. The quasi-linear equation (3.24) for the indifference price corresponds to the HJB equation derived in Davis (2000). The optimal trading strategy and the payoff decomposition coincide as well, however, our results are more explicit because we give a closed form solution to the HJB equation.

In Section 4 we continue the comparative analysis with the Black-Scholes model. We begin with the analogue to the Black-Scholes delta, that is sensitivity of the price to the changes in the level of the non-traded asset. Note that the optimal policy (3.20) and the optimal controls (3.29) and (3.31) depend on the derivative of the indifference price with respect to the non-traded asset level. In Theorem 4.1 we derive a closed form formula for this derivative. If the above mentioned optimal policies and controls satisfy additional integrability conditions then certain local martingales analyzed in Section 3 become martingales. These sufficient conditions are given in Theorem 4.2. In theorem 4.3 we analyze the price dependence on the risk aversion, showing monotonicity and calculating a closed form formula for the first derivative. We conclude Section 4 with the closed form formulae for the derivatives of the price with respect to the correlation (Theorem 4.4) and the Sharpe ratio (Theorem 4.5). Throughout we benefit from the price representation in closed form, as given in Theorem 2.1.

Invariance of the pricing scheme on the monotonic transformations of the levels of the non-traded asset and its implications for the model specification and integrability conditions are discussed briefly in Section 5. To simplify the arguments, we consider a time homogeneous case. A class of Gauss-Markov processes, containing a mean reverting Ornstein-Uhlenbeck process, appears to be a natural class of models to represent possibly transformed levels of the non-traded asset.

2 The indifference price and related pricing measures

We assume a dynamic market setting with two risky assets, namely a stock that can be traded and a non-traded asset on which a European-type claim is written. We model the assets as diffusion processes denoted by S and Y, respectively.

The tradable asset's price is a log-normal diffusion satisfying

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dW_s^1, \quad t \leq s, \\ \\ S_t = S > 0. \end{cases} \tag{2.1}$$

The level of the non-traded asset is given by

$$\begin{cases} dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s, \quad t \leq s, \\ \\ Y_t = y \in \mathbb{R}. \end{cases} \tag{2.2}$$

The processes W_s^1 and W_s are standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})$,where $\mathcal{F}_s$ is the augmented σ-algebra generated by

$\left(W_u^1, W_u, \;\; 0 \leq u \leq s\right)$. The Brownian motions are correlated with correlation $\rho \in (-1,1)$. Assumptions on the drift and diffusion coefficients $b(\cdot,\cdot)$ and $a(\cdot,\cdot)$, respectively, are such that the above equation has a unique strong solution.

We also assume that a riskless bond B with maturity T is available for trading, yielding constant interest rate r which satisfies $0 < r < \mu$. Clearly, the bond price is given by

$$B_s = e^{-r(T-s)}, \quad t \leq s \leq T. \tag{2.3}$$

The derivative to be priced is of European type with bounded payoff $g\left(Y_T\right)$, at expiration T. A larger class of payoffs can be considered for more specific choices of the non-traded asset dynamics.

The valuation method developed herein is based on the comparison of maximal expected utility payoffs corresponding to investment opportunities with and without involving the derivative. In both situations, trading occurs in the time horizon $[t,T]$, $0 \leq t \leq T$, and only between the two traded assets, i.e., riskless bond B and the risky asset S. The investor starts, at time t, with initial endowment x and rebalances his portfolio holdings by dynamically choosing the investment allocations , say π_s^0 and π_s, $t \leq s \leq T$, in the bond and the risky asset, respectively. It is assumed throughout that no intermediate consumption nor infusion of exogenous funds are allowed. The current wealth, defined by

$$X_s = \pi_s^0 + \pi_s, \quad t \leq s \leq T, \tag{2.4}$$

satisfies the controlled diffusion equation

$$\begin{cases} dX_s = rX_s ds + (\mu - r)\pi_s ds + \sigma \pi_s dW_s^1, \quad t \leq s \leq T, \\ \\ X_t = x, \end{cases} \tag{2.5}$$

which is derived via (2.1) and the assumptions on the bond dynamics (see, for example, Merton (1969)). It is worth noticing that the traded asset prices do not affect directly the wealth evolution. This is a direct consequence of the log-normality assumptions on their dynamics. Moreover, the linear budget constraint (2.4) allows one to work with a single control variable π_s. The latter is deemed admissible if it is $\mathcal{F}_s$-progressively measurable and satisfies the integrability condition $E\int_t^T \pi_s^2 ds < \infty$. The set of admissible controls (also referred to as *policies*) is denoted by $\mathcal{Z}$.

The individual risk preferences are modelled via an exponential utility function

$$U(x) = -e^{-\gamma x}, x \in \mathbb{R}, \tag{2.6}$$

with the risk aversion parameter $\gamma > 0$.

Next, we introduce three stochastic optimization problems via which the indifference prices of the writer and the buyer will be constructed. The first problem arises in the classical *Merton model* of optimal investment, namely

$$V\left(x,t\right) = \sup_{\mathcal{Z}} E\left(-e^{-\gamma X_T} \left| X_t = x\right.\right). \tag{2.7}$$

In this model, the investor seeks to maximize the expected utility of terminal wealth without taking into account the European claim.

It is now assumed that a European derivative, with the aforementioned payoff $g(Y_T)$ at maturity T, is written/bought at time t and no trading of the asset Y, given by (2.2), is allowed in the time interval $[t, T]$. Moreover, no trading of the derivative is allowed after its inscription/purchase. Following the investment policy π the writer and the buyer have individual expected utility payoffs, say J^w and J^b, defined respectively by

$$J^w(x, y, t; \pi) = E\left(-e^{-\gamma(X_T - g(Y_T))}/X_t = x, Y_t = y\right),$$

and

$$J^b(x, y, t; \pi) = E\left(-e^{-\gamma(X_T + g(Y_T))}/X_t = x, Y_t = y\right).$$

The above quantities reflect the common risk preferences but the individual liability/payoff for the writer/buyer of the claim.

The writer's and the buyer's value functions, denoted by u^w and u^b, are defined as

$$u^w(x, y, t) = \sup_{\mathcal{Z}} J^w(x, y, t; \pi) \tag{2.8}$$

and

$$u^b(x, y, t) = \sup_{\mathcal{Z}} J^b(x, y, t; \pi). \tag{2.9}$$

A fundamental assumption is that both the writer and the buyer optimize over the same set of admissible policies $\mathcal{Z}$. Moreover, the traditional non-bankruptcy constraint $X_s \geqslant 0$ a.e., $t \leq s \leq T$, is not imposed due to the fact that the exponential utility allows for negative wealth levels. The removal of this constraint is not just a technical point that eases the analysis. In fact, it severely affects the set of admissible strategies of the writer/buyer of the derivative and, therefore, has a critical effect on the specification of the claim prices (see for example, the discussion in Constantinides and Zariphopoulou (1999) and Zariphopoulou (2001)).

Now we are ready to define the indifference prices. The *indifference writer's*, respectively *buyer's, price* of the European claim $g(Y_T)$, is defined as the function $h^w \equiv h^w(x, y, t)$, respectively $h^b \equiv h^b(x, y, t)$, such that the investor is indifferent towards the following two scenarios: optimize the utility payoff without employing the derivative and optimize his utility payoff taking into account, on the one hand, the *liability,* resp. *payoff,* $g(Y_T)$ at expiration T, and on the other, the *compensation* $h^w(x, y, t)$, resp. *cost* $h^b(x, y, t)$ at time of inscription t. Therefore, the indifference prices h^w and h^b must satisfy for all x, y, t

$$V(x, t) = u^w(x + h^w(x, y, t), y, t) \tag{2.10}$$

and

$$V(x, t) = u^b\left(x - h^b(x, y, t), y, t\right), \tag{2.11}$$

where u^w and u^b are defined in (2.8) and (2.9), respectively.

The above definition allows for derivative prices that depend on the initial wealth, as reflected in their x-argument. At first sight, this might look as an undesirable pricing feature given the universality of prices in complete markets. As will become apparent from the calculations below, the exponential utility choice induces prices that are wealth independent and, therefore, price universality is preserved, at least within the class of European claims. Wealth independence, however, is not generally achieved across different utility functions and/or in the presence of admissibility constraints. This difficulty is alleviated by relaxing the notion of indifference prices to the one of *reservation prices.* The latter prices are defined as wealth independent pricing bounds for which (2.10) and (2.11) hold as inequalities (see, for example, Constantinides and Zariphopoulou (1999), (2001) and Munk (2000)).

It is also worth observing that not all the payoffs may be incorporated in the analysis, in the sense that the associated solutions might not be well defined. Technical conditions on the payoff function are then required to produce meaningful prices. Intuitively speaking, unbounded payoffs generating unhedgeable liability may be allocated infinite prices in the case of exponential preferences. This may be seen as one of the arguments in favor of exponential utility. Indeed, one may prefer a pricing mechanism which penalizes situations in which very large losses may occur by allocating infinite prices to them. Also, the internal risk control departments may want to cap the maximal possible loss by constraining the class of payoffs.

In what follows, we construct the writer's indifference price by first calculating the value function V and u^w and, subsequently, using the pricing condition (2.10). The buyer's indifference price is calculated using similar arguments and therefore its detailed derivation is omitted. To facilitate the presentation we firstly assume that $r = 0$ and, we skip the w-superscript. The latter is reinstated in Theorem 2.1, where the general case $r \neq 0$ is treated and both prices are presented.

There is a vast array of literature on the specification of V and on the study of value functions in complete market settings. Generally speaking, these problems are analyzed either via martingale methods or via arguments originating from stochastic control theory and non-linear partial differential equations, namely the Hamilton-Jacobi-Bellman (HJB) equations (for the two general approaches, we refer the technically oriented reader to the monograph of Karatzas and Shreve (1998) and the review article by Zariphopoulou (2000), respectively). For value functions of problems in incomplete markets, like the optimization problems (2.8) and (2.9), martingale methods seem to produce limited results and the analysis via the HJB equation appears more appealing. To maintain a concise presentation, we choose to analyze both V and u following the second approach, in spite of the fact that the arguments for V are well known.

It is well established that due to their non-linearities, general HJB equations are effectively analyzed using a weak class of solutions, namely the viscosity solutions. This class has been successfully used in a number of stochastic optimization problems arising in the areas of asset pricing and portfolio management (see, among others, Davis et al. (1993), Duffie and Zariphopoulou (1993), Shreve and Soner (1994), Barles and Soner (1998) and the review articles by Zariphopoulou (2000), (2001)). The key component of the problems at hand is that V and u turn out to be unique viscosity solutions of their HJB equations. Therefore, they are readily constructed once candidate solutions are found, provided that the candidate solutions preserve the viscosity property. The specific to our model setup cases of HJB equations may also be analysed without making reference to the viscosity solutions, but following more direct argument as in Davis (2000). We choose not to go into technical arguments on existence and uniqueness of solutions since they are similar to those established in Duffie and Zariphopoulou (1993) or in Davis (2000).

Below, we present the relevant HJB equations and we produce closed form solutions. To this end, we start with the construction of the Merton's problem (2.7) value function V which is expected to satisfy the HJB equation

$$\begin{cases} V_t + \max_\pi \left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \mu\pi V_x\right) = 0, \\ V(x,T) = -e^{-\gamma x}. \end{cases} \tag{2.12}$$

The form of the terminal data and the fact that the coefficients in (2.12) are independent of the state x (recall $r = 0$), suggest a candidate solution of the separable form

$$V(x,t) = -e^{-\gamma x} f(t). \tag{2.13}$$

In fact, direct calculations in (2.12) show that the function (2.13) solves the HJB equation provided that f satisfies

$$\begin{cases} f_t = \frac{1}{2}\frac{\mu^2}{\sigma^2} f \\ f(T) = 1. \end{cases}$$

Therefore, $f(t) = e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)}$which, in turn, yields that

$$V(x,t) = -e^{-\gamma x} e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)}. \tag{2.14}$$

The next task is to specify the writer's value function. The associated HJB equation turns out to be

$$\begin{aligned} u_t + \max_\pi \left(\frac{1}{2}\sigma^2\pi^2 u_{xx} + \rho\sigma a(y,t)\pi u_{xy} + \mu\pi u_x\right) \\ + \frac{1}{2}a^2(y,t) u_{yy} + b(y,t) u_y = 0, \end{aligned} \tag{2.15}$$

with the terminal condition

$$u(x,y,T) = -e^{-\gamma(x-g(y))}.$$

Evaluating the HJB equation at the maximum

$$\pi = \pi^*(x,y,t) = -\rho\frac{a(y,t)}{\sigma}\frac{u_{xy}}{u_{xx}} - \frac{\mu}{\sigma^2}\frac{u_x}{u_{xx}}, \tag{2.16}$$

yields

$$u_t - \frac{1}{2}\frac{\left(\rho\sigma a\left(y,t\right)u_{xy} + \mu u_x\right)^2}{\sigma^2 u_{xx}} + \frac{1}{2}a^2\left(y,t\right)u_{yy} + b\left(y,t\right)u_y = 0. \tag{2.17}$$

Following similar arguments as before, we postulate a solution of a separable form, namely

$$u\left(x,y,t\right) = -e^{-\gamma x}F\left(y,t\right). \tag{2.18}$$

We are next going to determine the factor F. Using (2.18) and (2.17) yields

$$\begin{aligned} F_t + \frac{1}{2}a^2\left(y,t\right)F_{yy} + \Big(b\left(y,t\right) - \rho\frac{\mu}{\sigma}a\left(y,t\right)\Big)F_y \\ -\frac{1}{2}\rho^2 a^2\left(y,t\right)\frac{F_y^2}{F} = \frac{1}{2}\frac{\mu^2}{\sigma^2}F. \end{aligned} \tag{2.19}$$

The terminal condition for F, easily recovered from $u\left(x,y,T\right)$, is

$$F\left(y,T\right) = e^{\gamma g(y)}. \tag{2.20}$$

The above quasi-linear equation reduces to a *linear* one when $\rho = 0$. Consequently, only the risk preferences but not the presence of the traded asset affect the price in this case. For non-zero values of ρ in $(-1,1)$, a linear equation still appears but via a power transformation, namely

$$F\left(y,t\right) = v\left(y,t\right)^{\delta}, \tag{2.21}$$

with v solving a linear equation and δ being a constant to be determined. In this sense, we say that F is a *distorted* solution of a linear partial differential equation. As the calculations below demonstrate, the representation of F via solutions of linear equations is not a mere technical step that facilitates the analysis. On the contrary, distortions seem to be a natural vehicle bridging linear pricing rules of complete markets to non-linear pricing devices that emerge in incomplete market settings.

To determine δ, we use (2.19) and its derivatives calculated from (2.21) to get

$$\delta v_t v^{\delta-1} + \frac{1}{2}a^2\left(y,t\right)\left(\delta v_{yy}v^{\delta-1} + \delta\left(\delta-1\right)v_y^2 v^{\delta-2}\right)$$

$$+\Big(b\left(y,t\right) - \rho\frac{\mu}{\sigma}a\left(y,t\right)\Big)\delta v_y^{\delta-1} - \frac{1}{2}\rho^2 a^2\left(y,t\right)\frac{\delta^2 v^{2(\delta-1)}v_y^2}{v^{\delta}} = \frac{1}{2}\frac{\mu^2}{\sigma^2}v^{\delta}.$$

The above equation is still quasi-linear but of the form

$$\begin{aligned} v_t + \frac{1}{2}a^2\left(y,t\right)v_{yy} + \Big(b\left(y,t\right) - \rho\frac{\mu}{\sigma}a\left(y,t\right)\Big)v_y \\ +\frac{1}{2}a^2\left(y,t\right)\frac{v_y^2}{v}\left(\left(\delta-1\right) - \rho^2\delta\right) = \frac{1}{2\delta}\frac{\mu^2}{\sigma^2}v. \end{aligned} \tag{2.22}$$

The non-linearities are readily removed if we choose

$$\delta = \frac{1}{1-\rho^2}. \tag{2.23}$$

In this case, v solves the linear parabolic PDE

$$v_t + \frac{1}{2}a^2(y,t)\,v_{yy} + \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right)v_y = \frac{1-\rho^2}{2}\frac{\mu^2}{\sigma^2}v, \tag{2.24}$$

with the terminal condition

$$v(y,T) = e^{\gamma\left(1-\rho^2\right)g(y)}.$$

We are now ready to construct the value function u. Define the probability measure

$$\widetilde{\mathbb{P}}(A) = E\left(\exp\left(-\rho\frac{\mu}{\sigma}W_T - \frac{1}{2}\rho^2\frac{\mu^2}{\sigma^2}T\right)I_A\right), \quad A \in \mathcal{F}_T^W, \tag{2.25}$$

under which the process

$$\widetilde{W}_s = W_s + \rho\frac{\mu}{\sigma}s, \quad 0 \le s \le T$$

is a Brownian motion. Then, under $\widetilde{\mathbb{P}}$, the dynamics of the process Y are given by

$$\begin{cases} dY_s = \left(b(Y_s,s) - \rho\frac{\mu}{\sigma}a(Y_s,s)\right)dt + a(Y_s,s)\,d\widetilde{W}_s, \quad t \le s \le T, \\ Y_t = y, \end{cases}$$

and hence it is a diffusion with the infinitesimal generator

$$\frac{\partial}{\partial t} + \frac{1}{2}a^2(y,t)\frac{\partial^2}{\partial y^2} + \left(b(y,t) - \rho\frac{\mu}{\sigma}a(y,t)\right)\frac{\partial}{\partial y}.$$

Using the Feynman-Kac representations of solutions to (2.24), we deduce that

$$v(y,t) = E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T) - \frac{1}{2}\left(1-\rho^2\right)\frac{\mu^2}{\sigma^2}(T-t)}\,|Y_t = y\right), \tag{2.26}$$

under the appropriate integrability condition on the payoff. Combining (2.14), (2.21) and (2.26) yields the value function u evaluated at x, y, t, namely

$$u = -e^{-\gamma x}\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T) - \frac{1}{2}\left(1-\rho^2\right)\frac{\mu^2}{\sigma^2}(T-t)}\,|Y_t = y\right)\right)^{\frac{1}{1-\rho^2}}. \tag{2.27}$$

Taking into account the explicit formulae for the value functions V and u as well as the pricing condition (2.10), we are ready to derive the *writer's indifference price*. To this end, we rewrite (2.10) in the form

$$V\left(x-h\left(x,y,t\right),t\right)=u\left(x,y,t\right)$$

and we use (2.14) and (2.27). The price h must then satisfy

$$e^{-\frac{1}{2}\frac{\mu^{2}}{\sigma^{2}}(T-t)}e^{-\gamma(x-h(x,y,t))}$$
$$=e^{-\frac{1}{2}\frac{\mu^{2}}{\sigma^{2}}(T-t)}e^{-\gamma x}\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^{2}\right)g(Y_{T})}\,|Y_{t}=y\right)\right)^{\frac{1}{1-\rho^{2}}},$$

which, in turn yields, that h is *independent of* x and given by

$$h\left(y,t\right)=\frac{1}{\gamma\left(1-\rho^{2}\right)}\ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^{2}\right)g(Y_{T})}\,|Y_{t}=y\right)\right).$$

Next, we consider the case $r>0$. Given that the second traded asset is a bond B of maturity T, with the price process $B_s=e^{-r(T-s)}$, we can denominate wealth in the forward rather than the spot units. To this end, we define the *forward wealth process* $\widetilde{X}$ by

$$\widetilde{X}_s=e^{r(T-s)}X_s,\quad t\leqslant s\leqslant T.$$

Using (2.5), we deduce that $\widetilde{X}$ satisfies

$$d\widetilde{X}_s=\left(\mu-r\right)\widetilde{\pi}_s ds+\sigma\widetilde{\pi}_s dW_s^1,\quad t\leqslant s\leqslant T,\tag{2.28}$$

where

$$\widetilde{\pi}_s=e^{r(T-s)}\pi_s$$

is the forward value of the amount π_s, invested in the risky asset at time s. The value function of the original Merton's problem, when expressed in forward wealth units, is defined as

$$\widetilde{V}\left(\widetilde{x},t\right)=\sup_{\mathcal{Z}}E\left(-e^{-\gamma\widetilde{X}_T}\,\middle|\,\widetilde{X}_t=\widetilde{x}\right).$$

The solution can be directly derived from the case $r=0$ by replacing μ with $\mu-r$, yielding

$$\widetilde{V}\left(\widetilde{x},t\right)=-e^{-\gamma\widetilde{x}}e^{-\frac{1}{2}\frac{(\mu-r)^{2}}{\sigma^{2}}(T-t)}.$$

Obviously, the levels of forward and spot wealth are related by a static arbitrage argument giving $\widetilde{x}=e^{r(T-t)}x$. Consequently, we can also write $\widetilde{V}$ as a function of x and t, namely,

$$\widetilde{V}\left(x,t\right)=-e^{-\gamma e^{r(T-t)}x}e^{-\frac{1}{2}\frac{(\mu-r)^{2}}{\sigma^{2}}(T-t)}.\tag{2.29}$$

Similarly, the value function of the writer's optimization problem, expressed in the forward wealth units, is given by

$$\widetilde{u}\left(\widetilde{x},y,t\right)=\sup_{\mathcal{Z}}E\left(-e^{-\gamma\left(\widetilde{X}_T-g(Y_T)\right)}\,\middle|\,\widetilde{X}_t=\widetilde{x},Y_t=y\right),$$

which, after modifying the analysis previously done for the case $r=0$ gives

$$\widetilde{u}\left(\widetilde{x},y,t\right)=-e^{-\gamma\widetilde{x}}\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)-\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\left(1-\rho^2\right)(T-t)}\,|Y_t=y\right)\right)^{\frac{1}{1-\rho^2}}.$$

Likewise, $\tilde{u}$ evaluated at x,y,t is given by

$$\tilde{u}=-e^{-\gamma e^{r(T-t)}x}\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)-\frac{1}{2}\left(1-\rho^2\right)\frac{(\mu-r)^2}{\sigma^2}(T-t)}\,|Y_t=y\right)\right)^{\frac{1}{1-\rho^2}}, \tag{2.30}$$

where $\widetilde{\mathbb{P}}$, referred to in the future as the *indifference measure*, is given by

$$\widetilde{\mathbb{P}}\left(A\right)=E\left(\exp\left(-\rho\frac{\mu-r}{\sigma}W_T-\frac{1}{2}\rho^2\frac{\left(\mu-r\right)^2}{\sigma^2}T\right)I_A\right),\quad A\in\mathcal{F}_T^W. \tag{2.31}$$

Equating $\widetilde{V}\left(x-h\left(x,y,t\right),t\right)$ to $\widetilde{u}\left(x,y,t\right)$, yields the writer's indifference price

$$h^w\left(y,t\right)=e^{-r(T-t)}\frac{1}{\gamma\left(1-\rho^2\right)}\ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t=y\right)\right). \tag{2.32}$$

The derivation of the buyer's price follows the same arguments as in the writer's case and it is left to the reader. It turns out that the buyer's price is given by

$$h^b\left(y,t\right)=-e^{-r(T-t)}\frac{1}{\gamma\left(1-\rho^2\right)}\ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{-\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t=y\right)\right). \tag{2.33}$$

In the theorem below we summarize the main results so far.

Theorem 2.1. *Assuming that the preferences are modelled by an exponential utility function, the writer's and the buyer's indifference prices of the bounded European claim $g\left(Y_T\right)$ written on the non-traded asset Y, which is correlated with the traded asset S, are given by (2.32) and (2.33), respectively. The indifference measure $\widetilde{\mathbb{P}}$ is given by (2.31), and the dynamics, for $t\leq s\leq T$, of the process Y under $\widetilde{\mathbb{P}}$ are given by*

$$dY_s=\left(b\left(Y_s,s\right)-\rho\frac{\mu-r}{\sigma}a\left(Y_s,s\right)\right)ds+a\left(Y_s,s\right)d\widetilde{W}_s,\quad Y_t=y,$$

where the process

$$\widetilde{W}_s=W_s+\rho\frac{\mu-r}{\sigma}s,\quad 0\leq s\leq T,$$

is a Brownian motion.

Observe that the indifference measure $\widetilde{\mathbb{P}}$, and not the historical measure $\mathbb{P}$ is used in the price calculation. The intuitive reasons for it can be understood in the following way. When there is no correlation between the traded asset and the non-traded one upon which an option is written, the presence of the traded asset is irrelevant from the perspective of the risk to be managed. As a result, the pricing takes place under the historical measure $\mathbb{P}$. On the other hand, when there is a perfect correlation between the traded and non-traded assets, the model is complete and the risk management should be carried out under the risk neutral martingale measure. Put another way, the degree of incompleteness introduced by the second Brownian motion is directly related to the level of correlation between the traded and non-traded assets. Intuitively, the measure $\widetilde{\mathbb{P}}$ should be the *closest* measure (in some sense) to the risk neutral one and, at the same time, it should be defined on the filtration of the Brownian motion used for modelling of the non-traded asset dynamics.

More specifically, consider a reduced model in which one trades the asset S and the bond B but the non-traded asset Y is not taken into account. Evidently, such a model is complete and the measure

$$\mathbb{P}^*(A) = E\left(\exp\left(-\frac{\mu - r}{\sigma}W_T^1 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}T\right)I_A\right), \quad A \in \mathcal{F}_T^{W^1}, \tag{2.34}$$

where $\mathcal{F}_T^{W^1}$ is the augmented σ- algebra generated by $W_s^1, 0 \leq s \leq T$, is the unique martingale measure. In the full model, the dynamics of the non-traded asset Y are defined by another Brownian motion W which is correlated with the Brownian motion W^1. The Radon-Nikodym density of the measure $\mathbb{P}^*$ with respect to the historical measure $\mathbb{P}$, given by,

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma}W_T^1 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}T\right)$$

is a $\mathcal{F}_T^{W^1}$ measurable random variable. Let $\mathcal{F}_T^W$ stand for the augmented σ-algebra generated by $W_s, 0 \leq s \leq T$. The conditional expectation

$$E\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}\,|\mathcal{F}_T^W\right) = E\left(\exp\left(-\frac{\mu - r}{\sigma}W_T^1 - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}T\right)|\mathcal{F}_T^W\right),$$

is the *closest* in $\mathbf{L}^2$ sense to $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ and $\mathcal{F}_T^W$ measurable positive random variable with expectation equal to 1. Therefore, it can be used as a density with respect to the historical measure $\mathbb{P}$, of a measure which satisfies the intuitive arguments presented above. In order to calculate the conditional expectation of $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ given $\mathcal{F}_T^W$, one needs to identify the conditional distribution of W_T^1, given the trajectory $W_s, 0 \leq s \leq T$. This distribution turns out to be normal with the conditional mean ρW_T and the conditional variance $\left(1 - \rho^2\right)T$. Hence it coincides with the conditional distribution of W_T^1 given W_T. This result can be deduced by a direct calculation of the conditional distribution of

W_T^1 given a finite set of increments of the Brownian motion W and then passing to the limit. Indeed, the conditional mean of W_T^1 given $W_{s_1}, ..., W_{s_n}$, for $0 = s_0 < s_1, ..., < s_n = T$, coincides with the conditional mean of W_T^1 given the increments $W_{s_{i+1}} - W_{s_i}, i = 0, 1, ..., n-1$ which by the normal correlation theorem can be written as

$$E\left(W_T^1 \left| W_{s_{i+1}} - W_{s_i}, i = 0, 1, ..., n-1\right.\right)$$

$$= \rho \sum_{i=0}^{n-1} \frac{T \wedge s_{i+1} - T \wedge s_i}{s_{i+1} - s_i} \left(W_{s_{i+1}} - W_{s_i}\right) = \rho W_T.$$

The conditional variance can be calculated in the same way giving

$$Var\left(W_T^1 \left| W_{s_{i+1}} - W_{s_i}, i = 0, 1, ..., n-1\right.\right)$$

$$= T - \rho^2 \sum_{i=0}^{n-1} \frac{\left(T \wedge s_{i+1} - T \wedge s_i\right)^2}{\left(s_{i+1} - s_i\right)^2} \left(s_{i+1} - s_i\right) = \left(1 - \rho^2\right) T.$$

The σ- algebra generated by W_s, $0 \leq s \leq T$ is the limit of the σ- algebras generated by the increments and hence the result follows. Consequently,

$$E\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \left| \mathcal{F}_T^W \right.\right) = \exp\left(-\rho \frac{\mu - r}{\sigma} W_T - \frac{1}{2}\rho^2 \frac{\left(\mu - r\right)^2}{\sigma^2} T\right)$$

and, therefore, the indifference measure also satisfies

$$\widetilde{\mathbb{P}}\left(A\right) = E\left(E\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \left| \mathcal{F}_T^W \right.\right) I_A\right), \quad A \in \mathcal{F}_T^W. \tag{2.35}$$

Theorem 2.2. *The indifference measure* $\widetilde{\mathbb{P}}$ *used in the formulae (2.32) and (2.33) satisfies the following property*

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = E\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \left| \mathcal{F}_T^W \right.\right),$$

where $\mathbb{P}^*$*, given in (2.34), is the unique martingale measure of the reduced complete model in which one trades the asset* S *and the bond* B*, and where the non-traded asset* Y *is not taken into account.*

We conclude this section by studying the case $\rho^2 = 1$. Intuitively speaking, as $\rho^2 \rightarrow 1$, the market becomes complete and, therefore, the indifference price is expected to turn into the Black-Scholes one. Arbitrage-free arguments can then be applied directly and the utility methodology becomes redundant. Even though intuition is clear, it is not obvious that the pricing mechanism is robust as we move from the incomplete market setting to the complete one. In what follows, we concentrate on the writer's price and, we explore the restrictive behavior of the relevant value functions and the induced prices. To facilitate presentation, we parametrize all quantities involved by ρ, namely, the value function $u^{(\rho)}$, the factor $F^{(\rho)} = \left(v^{(\rho)}\right)^{\frac{1}{1-\rho^2}}$, the indifference writer's price $h^{(\rho)}$, and the indifference measure $\widetilde{\mathbb{P}}^{(\rho)}$. The pricing condition (2.10) is then rewritten as

$$\widetilde{V}(x,t) = \widetilde{u}^{(\rho)}\left(x + h^{(\rho)}(x,y,t), y, t\right),$$

where ρ now indicates the dependence on the correlation, while $\widetilde{V}$ and $\widetilde{u}^{(\rho)}$ are given by (2.29) and (2.30), respectively. It yields the limiting price once we compute the limit of $u^{(\rho)}$ as $\rho \mapsto 1$ (the case $\rho \mapsto -1$ is treated similarly). In the course of the previous analysis, we have demonstrated that $\widetilde{u}^{(\rho)}(x,y,t) = -e^{-\gamma e^{r(T-t)}x} F^{(\rho)}(y,t)$ with $F^{(\rho)}$ solving the quasi-linear equation (2.19) with μ replaced by $\mu - r$. This yields

$$h^{(\rho)}(y,t) = e^{-r(T-t)} \frac{1}{\gamma}\left(\ln F^{(\rho)}(y,t) + \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}(T-t)\right). \tag{2.36}$$

The key idea for the convergence of $F^{(\rho)}$ comes from the stability properties of viscosity solutions (see Lions (1983)). Because these technical arguments are well known by now, we only describe the principal steps (rigorous arguments for a related problem of optimal consumption may be found in Theorem 3.2 of Zariphopoulou (1992)). To this end, we observe that $F^{(\rho)}$ is uniformly bounded with respect to ρ. This follows from the fact that $F^{(\rho)} = \left(v^{(\rho)}\right)^{\frac{1}{1-\rho^2}}$ and the probabilistic representation (2.26) of $v^{(\rho)}$, under the appropriate integrability conditions on the payoff. Therefore, $F^{(\rho)}$ converges as $\rho \mapsto 1$ along subsequences.

On the other hand, equation (2.19) converges uniformly in ρ, to the equation

$$F_t + \frac{1}{2}a^2(y,t) F_{yy} + \left(b(y,t) - \frac{\mu - r}{\sigma} a(y,t)\right) F_y \tag{2.37}$$

$$-\frac{1}{2}a^2(y,t)\frac{F_y^2}{F} = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} F,$$

with the terminal condition

$$F(y,T) = e^{\gamma g(y)}.$$

Standard arguments yield that the above problem admits a unique viscosity solution, say F, which is thus readily identified with the limit of all subsequences of $F^{(\rho)}$. Note that the above arguments were applied directly to $F^{(\rho)}$ and not to its representation via the distorted solutions $v^{(\rho)}$ and the distortion $\frac{1}{1-\rho^2}$. The point of interest in what follows is the new type of distortion that emerges when passing to the complete market equation. Indeed, equation (2.37) is solved by applying an exponential, rather than a power, transformation; as a matter of fact, the latter fails to work since $\rho \mapsto 1$, the non-linearities in (2.24) cannot be eliminated by any choice of the parameter δ in (2.21). We represent the solution of (2.37) as

$$F(y,t) = e^{v(y,t)}.$$

It is evident that v must solve

$$v_t + \frac{1}{2}a^2(y,t)\,v_{yy} + \left(b(y,t) - \frac{\mu - r}{\sigma}a(y,t)\right)v_y = \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2},$$

with the terminal condition $v(y,T) = \gamma g(y)$. Employing the probabilistic representation of v and the above transformation yields

$$F(y,t) = e^{E_{\widetilde{\mathbb{P}}^{(1)}}(\gamma g(Y_T)|Y_t = y) - \frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)},$$

with $\widetilde{\mathbb{P}}^{(1)}$ defined in (2.31) for $\rho = 1$ which coincides with the risk neutral measure (2.34). Passing to the limit in (2.36) yields the Black-Scholes price

$$h(y,t) = e^{-r(T-t)}E_{\widetilde{\mathbb{P}}^{(1)}}\left(g(Y_T)|\, Y_t = y\right).$$

Similar results are readily obtained for the case $\rho \to -1$, as well as, for the limiting cases of the buyer's indifference price. The latter turns out, naturally, to be equal to the writer's price. Given that one can view $g(Y_T)$ as a claim written on the traded asset S, hence, the model is complete, the indifference measure $\widetilde{\mathbb{P}}^{(1)}$ reduces to the unique martingale measure $\mathbb{P}^*$. Indeed, in this case the density $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ is measurable with respect to $\mathcal{F}_T^W$ and therefore, using representation (2.35) we get for all $A \in \mathcal{F}_T^W$

$$\widetilde{\mathbb{P}}^{(1)}(A) = E\left(\frac{d\mathbb{P}^*}{d\mathbb{P}}I_A\right) = \mathbb{P}^*(A) \text{ for } \rho^2 = 1.$$

Subsequently, this restricts the drift of the process Y. This is to say that one must have

$$\frac{b(y,t) - ry}{a(y,t)} = \rho\frac{\mu - r}{\sigma}, \quad \text{for} \quad \rho^2 = 1.$$

We state the results below.

Theorem 2.3. *In the perfectly correlated case, and under the assumption that the excess return per unit of risk is the same for both the traded and non-traded asset, i.e., when* $\rho^2 = 1$ *and*

$$\frac{b(y,t) - ry}{a(y,t)} = \rho\frac{\mu - r}{\sigma},$$

the writer's and buyer's prices of the claim $g(Y_T)$ *coincide and are given by the formula*

$$h(y,t) = e^{-r(T-t)}E_{\widetilde{\mathbb{P}}^{(1)}}\left(g(Y_T)|\, Y_t = y\right),$$

where $\widetilde{\mathbb{P}}^{(1)} = \mathbb{P}^*$ *is the unique martingale measure defined in (2.34).*

3 Payoff decomposition and price representation

In this section, we continue the comparative analysis of the pricing methodology based on the concept of indifference with the arbitrage free pricing approach of a nested complete Black-Scholes model. We explore to what extent the price we derive via the utility mechanism preserves the fundamental characteristics of the Black-Scholes model. We concentrate on the following two cornerstones of the classical theory, namely, the martingale representation theorem and the related payoff decomposition, as well as the price representation as the expectation of the discounted payoff. Recall that in complete models both payoff decomposition and price calculation are done under the unique martingale measure $\mathbb{P}^*$, defined in (2.34). On the other hand, in our framework, it is the indifference measure $\widetilde{\mathbb{P}}$, defined in (2.31), that is used for the price calculation. Note, however, that the indifference price can also be computed with respect to an extension to the σ-algebra $\mathcal{F}_T$ of the measure $\mathbb{P}^*$. Namely, let from now on (keeping the same notation) $\mathbb{P}^*$ stands for a measure extended from $\mathcal{F}_T^{W^1}$ to $\mathcal{F}_T$ by

$$\mathbb{P}^*(A) = E\left(\exp\left(-\frac{\mu-r}{\sigma}W_T^1 - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T\right)I_A\right), \quad A \in \mathcal{F}_T. \tag{3.1}$$

Then, because $\widetilde{\mathbb{P}}$ is obviously the restriction to $\mathcal{F}_T^W$ of $\mathbb{P}^*$ extended to $\mathcal{F}_T$, Theorem 2.1 remains valid when one replaces $\widetilde{\mathbb{P}}$ with $\mathbb{P}^*$. We simply chose, and will continue to do so whenever possible, to give probabilistic representation of our results with respect to a measure defined on a minimal σ-algebra.

Recall that, in a complete model setting, the price is essentially equal to what it costs to manufacture the option payoff. In other words, thanks to the martingale representation theorem, the payoff is equal to the price plus the proceeds from trading the stock and the bond due to the execution of the self-financing and replicating strategy. Consequently, all risk can be hedged completely by taking positions in the market, with the price being uniquely determined. In incomplete models, however, not all risk can be hedged. The 'total risk' contains both, hegeable and unhegeable components. As a result, one would expect the payoff to be decomposed as a sum of the following three components: the price plus the wealth generated by the hedge execution plus the accumulated residual risk. This section provides such a decomposition under the historical measure $\mathbb{P}$. As expected, when the correlation increases to 1, the residual risk decreases to 0, and the decomposition converges to the one of the Black-Scholes model.

Note that the historical measure $\mathbb{P}$ plays an important role in our analysis, in contrast to the case of complete models, where the pricing and risk management are carried out under the measure $\mathbb{P}^*$. The historical data are used to identify the appropriate model for the dynamics of the non-traded asset. The correlation between the traded and non-traded asset is also estimated historically. Finally, specification of the parameter $\frac{\mu-r}{\sigma}$, which is in fact well known to the funds management industry and often referred to as *Sharpe ratio,* depends entirely on the assessment of the actual market conditions. It seems therefore useful to obtain estimates of the distances between the three measures, in terms of the parameters that define them. We choose to work with the total variation distance defining the strongest (norm) topology on the set of probability measures. Recall that for two probability measures, say, $\mathbb{P}$ and $\mathbb{Q}$, defined on the σ-algebra $\mathcal{G}$, the total variation distance $d(\mathbb{P},\mathbb{Q})$ is given by

$$d(\mathbb{P},\mathbb{Q}) = \sup_{A\in\mathcal{G}} \left|\mathbb{P}(A) - \mathbb{Q}(A)\right|. \tag{3.2}$$

We begin with the measures $\mathbb{P}^*$ and $\mathbb{P}$. Note that, using the definition (3.1), we have for all $A \in \mathcal{F}_T$

$$\begin{aligned}
\left|\mathbb{P}^*(A) - \mathbb{P}(A)\right| &= \left| E\left(\left(\exp\left(-\frac{\mu-r}{\sigma}W_T^1 - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T\right) - 1\right) I_A\right)\right| \\
&\leq E\left(\left|\exp\left(-\frac{\mu-r}{\sigma}W_T^1 - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}T\right) - 1\right| I_A\right) \\
&= E\left(\left|\exp\left(-\theta X - \frac{1}{2}\theta^2\right) - 1\right| I_A\right),
\end{aligned}$$

where $\theta = \frac{\mu-r}{\sigma}\sqrt{T}$ and X is a standard normal variable. Straightforward calculations of the last term lead to the following estimate

$$d(\mathbb{P}^*,\mathbb{P}) \leq 2\left(N\left(\frac{1}{2}\frac{\mu-r}{\sigma}\sqrt{T}\right) - N\left(-\frac{1}{2}\frac{\mu-r}{\sigma}\sqrt{T}\right)\right),$$

where

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du. \tag{3.3}$$

The distance between $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ can be estimated in the same way using (2.31). We have for all $A \in \mathcal{F}_T^W$

$$d\left(\widetilde{\mathbb{P}},\mathbb{P}\right) \leq 2\left(N\left(\frac{1}{2}\left|\rho\right|\frac{\mu-r}{\sigma}\sqrt{T}\right) - N\left(-\frac{1}{2}\left|\rho\right|\frac{\mu-r}{\sigma}\sqrt{T}\right)\right).$$

Finally, for the measures $\widetilde{\mathbb{P}}$ and $\mathbb{P}^*$, we obtain for all $A \in \mathcal{F}_T^W$

$$\begin{aligned}
&\left|\widetilde{\mathbb{P}}(A) - \mathbb{P}^*(A)\right| \\
&\leq E_{\mathbb{P}^*}\left|\exp\left(-\rho\frac{\mu-r}{\sigma}W_T + \frac{\mu-r}{\sigma}W_T^1 + \frac{1}{2}\left(1-\rho^2\right)\frac{(\mu-r)^2}{\sigma^2}T\right) - 1\right|.
\end{aligned}$$

But the distribution of $-\rho\frac{\mu-r}{\sigma}W_T + \frac{\mu-r}{\sigma}W_T^1$ under $\mathbb{P}^*$ is normal with the mean $-\left(1-\rho^2\right)\frac{(\mu-r)^2}{\sigma^2}T$ and the variance $\left(1-\rho^2\right)\frac{(\mu-r)^2}{\sigma^2}T$. Consequently, we obtain

$$d\left(\widetilde{\mathbb{P}},\mathbb{P}^*\right) \leq 2\left(N\left(\frac{1}{2}\sqrt{1-\rho^2}\frac{\mu-r}{\sigma}\sqrt{T}\right) - N\left(-\frac{1}{2}\sqrt{1-\rho^2}\frac{\mu-r}{\sigma}\sqrt{T}\right)\right).$$

Before proceeding any further, we summarize the previous results.

Theorem 3.1. *Let $d(\mathbb{P},\mathbb{Q})$, defined in (3.2), stands for the total variation distance between two probability measures $\mathbb{P}$ and $\mathbb{Q}$, and let the function D be given by*

$$D(x) = 2\left(N\left(\frac{1}{2}x\right) - N\left(-\frac{1}{2}x\right)\right) = \sqrt{\frac{2}{\pi}}\left(x - \frac{x^3}{24} + \frac{x^5}{640}\cdots\right),$$

where N, is defined in (3.3). Then we have the following estimates

$$d(\mathbb{P}^*,\mathbb{P}) \le D\left(\frac{\mu - r}{\sigma}\sqrt{T}\right),$$

$$d\left(\widetilde{\mathbb{P}},\mathbb{P}\right) \le D\left(|\rho|\,\frac{\mu - r}{\sigma}\sqrt{T}\right),$$

$$d\left(\widetilde{\mathbb{P}},\mathbb{P}^*\right) \le D\left(\sqrt{1-\rho^2}\frac{\mu - r}{\sigma}\sqrt{T}\right),$$

where $\mathbb{P},\widetilde{\mathbb{P}}$ and $\mathbb{P}^*$ are, respectively, the historical measure, the indifference measure and the unique martingale measure of the reduced model. Moreover, $\widetilde{\mathbb{P}} = \mathbb{P}^* = \mathbb{P}$ when $\frac{\mu-r}{\sigma} = 0$, $\widetilde{\mathbb{P}} = \mathbb{P}$ when $\rho = 0$, and $\widetilde{\mathbb{P}} = \mathbb{P}^*$ when $\rho^2 = 1$.

Now, we derive the payoff decomposition which will yield the optimal hedging strategy. We then derive the price representation as the expected value of an appropriately modified payoff. Using the stochastic control arguments, we represent the price as the value function of a stochastic maximization problem of expected terminal payoff. In contradiction to the complete market setting this payoff does not coincide with the derivative's payoff but rather with the former minus a certain cost factor. We obtain an analytic expression for the control which attains the supremum. Finally, we show which control applied to the non-traded asset induces a measure under which the discounted indifference prices are martingales. It turns out that all three controls used in our comparative study and the sensitivity analysis later on are of the form

$$A(t)\,a(y,t)\,h_y(y,t) + B(t), \tag{3.4}$$

where $A(t)$ and $B(t)$ are smooth deterministic functions of t. The technical integrability conditions which the above controls need to satisfy are related to the choices of the option payoffs and the non-traded asset dynamics. These, together with examples of the payoffs and dynamics, are given in the next section. For the moment, we assume that all necessary integrability conditions are satisfied and we only highlight the related constraints.

The analysis below corresponds to the case $r = 0$ with the results for $r \neq 0$ being derived and discussed afterwards. Because the indifference price is extracted from the arguments of the relevant value functions (see (2.10) and (2.11)), we expect the price process to be directly related to the optimally controlled state wealth process with and without employing the derivative contract. So we consider the writer's optimal wealth process X_s^*, $t \le s \le T$ evaluated at the optimal portfolio process Π_s^*, $t \leq s \leq T$. The optimal control is provided in the feedback form

$$\pi^* (x, y, t) = \frac{\rho}{\sigma} a (y, t) h_y (y, t) + \frac{\mu}{\sigma^2 \gamma}. \tag{3.5}$$

Therefore,

$$\Pi_s^* = \frac{\rho}{\sigma} a (Y_s, s) h_y (Y_s, s) + \frac{\mu}{\sigma^2 \gamma} \tag{3.6}$$

with its optimality following from the regularity properties of the value function u^w and classical verification results (see, Fleming and Soner, Chapter VI (1993)). The writer's optimal wealth solves

$$dX_s^* = \mu \Pi_s^* ds + \sigma \Pi_s^* dW_s^1, \quad t \leq s \leq T, \tag{3.7}$$

with initial condition $X_t^* = x + h (y, t)$, which reflects the compensation received at the contract's inscription. Respectively, the optimal wealth process $X_s^{0,*}$, $t \leq s \leq T$, of the classical Merton problem (2.7) is given by

$$dX_s^{0,*} = \mu \Pi_s^{0,*} ds + \sigma \Pi_s^{0,*} dW_s^1, \quad t \leq s \leq T, \tag{3.8}$$

with $\Pi_s^{0,*} = \frac{\mu}{\sigma^2 \gamma}$ and initial condition $X_t^{0,*} = x$. The optimal policy is directly computed from the first order conditions in (2.12) and straightforward optimality arguments. Alternatively, it may be derived directly from the writer's optimization problem for the degenerate payoff $g \equiv 0$. In fact, one can see that in this case, $h \equiv 0$ is the unique solution to (2.10) and Π_s^* in (3.6) reduces to $\Pi_s^{0,*}$.

The process

$$L_s = X_s^* - X_s^{0,*}, \ \ t \leq s \leq T, \quad L_t = h (y, t),$$

represents the *residual optimal wealth* generated due to the derivative contract. Its dynamics follow from (3.6), (3.7) and (3.8), namely

$$\begin{aligned} dL_s &= \mu \left(\Pi_s^* - \Pi_s^{0,*} \right) ds + \sigma \left(\Pi_s^* - \Pi_s^{0,*} \right) dW_s^1 \\ &= \frac{\rho}{\sigma} a (Y_s, s) h_y (Y_s, s) \left(\mu ds + \sigma dW_s^1 \right), \end{aligned} \tag{3.9}$$

and hence L is a local martingale under the measure $\mathbb{P}^*$ and a martingale subject to the appropriate integrability conditions.

Next, we introduce the indifference price process

$$H_s = h (Y_s, s), \quad t \leq s \leq T, \tag{3.10}$$

and we derive its dynamics. Recall that

$$h (y, t) = \frac{1}{\gamma (1 - \rho^2)} \ln w (y, t), \tag{3.11}$$

and w is the solution to the following Cauchy problem

$$\begin{cases} w_t + \frac{1}{2}a^2\left(y,t\right)w_{yy} + \left(b\left(y,t\right) - \rho\frac{\mu}{\sigma}a\left(y,t\right)\right)w_y = 0 \\ w\left(y,T\right) = e^{\gamma\left(1-\rho^2\right)g(y)}. \end{cases} \tag{3.12}$$

It easily follows that h solves the quasi-linear equation

$$\begin{cases} h_t + \frac{1}{2}a^2\left(y,t\right)h_{yy} + b\left(y,t\right)h_y + a\left(y,t\right)\phi\left(y,t,h_y\right) = 0, \\ h\left(y,T\right) = g\left(y\right), \end{cases} \tag{3.13}$$

where

$$\phi\left(y,t,h_y\right) = \frac{1}{2}\gamma\left(1-\rho^2\right)a\left(y,t\right)h_y^2 - \rho\frac{\mu}{\sigma}h_y.$$

Consequently

$$dH_s = -a\left(Y_s,s\right)\phi\left(Y_s,s,h_y\left(Y_s,s\right)\right)ds + a\left(Y_s,s\right)h_y\left(Y_s,s\right)dW_s, \tag{3.14}$$

with initial condition $H_t = h\left(y,t\right).$

Subsequently, we define the *residual risk process*

$$R_s = L_s - H_s, \quad t \leq s \leq T, \quad R_t = 0.$$

A key observation, justifying calling R_s the residual risk is that, under market completeness, $R_s = 0$ for all $t \leq s \leq T$. In this case, the residual wealth process reduces naturally to the derivative price process, and represents the wealth that needs to be put aside in order to hedge the derivative liability in (2.8).

Comparison of the residual wealth dynamics (3.9) with the price dynamics (3.14) yields

$$\begin{aligned} dR_s = dL_s - dH_s &= -\sqrt{1-\rho^2}a\left(Y_s,s\right)h_y\left(Y_s,s\right)dW_s^{\perp} \\ &+\frac{1}{2}\gamma\left(1-\rho^2\right)a^2\left(Y_s,s\right)h_y^2\left(Y_s,s\right)ds, \end{aligned} \tag{3.15}$$

where the process $W^{\perp}$ is defined by

$$W_s^{\perp} = \frac{1}{\sqrt{1-\rho^2}}W_s - \frac{\rho}{\sqrt{1-\rho^2}}W_s^1, \quad t \leq s \leq T.$$

Clearly $W^{\perp}$ is a Brownian motion orthogonal to W^1 and as such should naturally be linked to the *unhedged risk components*. Indeed, the difference between the terminal residual wealth and the payoff is given by

$$\begin{aligned} L_T - g\left(Y_T\right) = &-\sqrt{1-\rho^2}\int_t^T a\left(Y_s,s\right)h_y\left(Y_s,s\right)dW_s^{\perp} \\ &+\frac{1}{2}\gamma\left(1-\rho^2\right)\int_t^T a^2\left(Y_s,s\right)h_y^2\left(Y_s,s\right)ds. \end{aligned}$$

Note that under the historical measure the residual risk at expiration has a positive expected value and that one can write the following representation of the claim

$$g\left(Y_T\right) = h\left(Y_t, t\right) + \int_t^T \frac{\rho}{\sigma} a\left(Y_s, s\right) h_y\left(Y_s, s\right) \frac{dS_s}{S_s} \tag{3.16}$$

$$+\sqrt{1-\rho^2} \int_t^T a\left(Y_s, s\right) h_y\left(Y_s, s\right) dW_s^{\perp} - \frac{1}{2}\gamma\left(1-\rho^2\right) \int_t^T a^2\left(Y_s, s\right) h_y^2\left(Y_s, s\right) ds.$$

The first term in (3.16) is the indifference price. The integrand in the second represents the *hedge* one should put into the traded asset. Indeed $\Pi_s^* - \Pi_s^{0,*}$ is the optimal residual amount invested into the traded asset due to the presence of an option and hence

$$\frac{\Pi_s^* - \Pi_s^{0,*}}{S_s} \tag{3.17}$$

is the optimal number of shares of a correlated asset to be held in the portfolio. The last two terms quantify the risk that cannot be hedged. Note that when $\rho = 0$ there is no distortion, the pricing is done under the historical measure, and the optimal policy is the same as in the classical Merton problem. Also, when $\rho = 1$, $b\left(y,t\right) = \mu y$, $a\left(y,t\right) = \sigma y$ and $g\left(y\right) = \left(y - K\right)^+$, the integrand in the second term reduces to the usual delta of the Black-Scholes model.

It is worth noting that a martingale emerges from a *preference-adjusted exponential* of the residual risk process. Indeed, direct calculations yield that the process

$$Z_s = e^{-\gamma R_s}, \quad t \leq s \leq T, \quad Z_t = 1,$$

satisfies

$$dZ_s = Z_s \gamma \sqrt{1-\rho^2} a\left(Y_s, s\right) h_y\left(Y_s, s\right) dW_s^{\perp}, \quad t \leq s \leq T, \tag{3.18}$$

and hence, is a local martingale (and a martingale under the appropriate integrability conditions).

We go on, as before, to consider the case $r > 0$. Namely, we denominate the relevant quantities in the forward units and use the above results replacing μ with $\mu - r$. The forward writer's indifference price process is defined by

$$\widetilde{H}_s = \widetilde{h}\left(Y_s, s\right), \ \ t \leq s \leq T, \tag{3.19}$$

where the function $\widetilde{h}$ is defined in (3.11) and the function w is the solution to (3.12) with μ replaced by $\mu - r$, or, alternatively, $\widetilde{h}\left(y, s\right) = e^{r(T-s)} h\left(y, s\right)$ and the function h is defined in (2.32). The process

$$\widetilde{L}_s = \widetilde{X}_s^* - \widetilde{X}_s^{0,*}, \ \ t \leq s \leq T,$$

of the forward residual optimal wealth satisfies

$$d\widetilde{L}_s = \left(\mu - r\right)\left(\widetilde{\Pi}_s^* - \widetilde{\Pi}_s^{0,*}\right) ds + \sigma\left(\widetilde{\Pi}_s^* - \widetilde{\Pi}_s^{0,*}\right) dW_s^1,$$

where

$$\widetilde{\Pi}_s^* = \frac{\rho}{\sigma} a\left(Y_s, s\right) \widetilde{h}_y\left(Y_s, s\right) + \frac{\mu - r}{\sigma^2 \gamma}, \tag{3.20}$$

$$\widetilde{\Pi}_s^{0,*} = \frac{\mu - r}{\sigma^2 \gamma},$$

and $\widetilde{h}$ as in (3.19). The forward price process of the traded asset

$$\widetilde{S}_s = e^{r(T-s)} S_s$$

satisfies

$$d\widetilde{S}_s = \widetilde{S}_s \left(\left(\mu - r \right) ds + \sigma dW_s^1 \right)$$

and hence we can also write that

$$d\widetilde{L}_s = \frac{\rho}{\sigma} a\left(Y_s, s\right) \widetilde{h}_y\left(Y_s, s\right) \frac{d\widetilde{S}_s}{\widetilde{S}_s}. \tag{3.21}$$

Moreover, the residual risk process

$$\widetilde{R}_s = \widetilde{L}_s - \widetilde{H}_s, \quad t \le s \le T,$$

satisfies (3.15) with L, H, and h_y replaced with $\widetilde{L}, \widetilde{H}$ and $\widetilde{h}_y$, respectively. Consequently, we have derived a forward unit analogue of formula (3.16) in which h is replaced by $\widetilde{h}$, S is replaced by $\widetilde{S}$ and μ is replaced by $\mu - r$ as well as the analogue of (3.18), where Z_s is replaced with

$$\widetilde{Z}_s = e^{-\gamma \widetilde{R}_s}, \quad t \le s \le T.$$

In terms of the spot prices we obtain the following representation of the payoff

$$g\left(Y_T\right) = e^{r(T-t)} h\left(Y_t, t\right) + \int_t^T \frac{\rho}{\sigma} a\left(Y_s, s\right) e^{r(T-s)} h_y\left(Y_s, s\right) \frac{d\left(e^{r(T-s)} S_s\right)}{e^{r(T-s)} S_s}$$
$$+ \sqrt{1-\rho^2} \int_t^T a\left(Y_s, s\right) e^{r(T-s)} h_y\left(Y_s, s\right) dW_s^{\perp}$$
$$- \frac{1}{2} \gamma \left(1 - \rho^2\right) \int_t^T a^2\left(Y_s, s\right) e^{2r(T-s)} h_y^2\left(Y_s, s\right) ds,$$

where the function h is defined in (2.32). Moreover, we deduce that the ratio

$$\widetilde{Z}_s = e^{-\gamma e^{r(T-s)} R_s}, \quad t \le s \le T, \tag{3.22}$$

satisfies

$$d\widetilde{Z}_s = \widetilde{Z}_s \gamma \sqrt{1-\rho^2} a\left(Y_s, s\right) e^{r(T-s)} h_y\left(Y_s, s\right) dW_s^{\perp}. \tag{3.23}$$

Theorem 3.2. *The forward residual wealth $\widetilde{L}_s$ satisfies (3.21) and hence is a local martingale under the measure $\mathbb{P}^*$, constructed in (3.1). The preference-adjusted exponential residual risk process defined in (3.22), is a positive local martingale under the measure $\mathbb{P}$, satisfying (3.23). Both process $\widetilde{L}_s$ and $\widetilde{Z}_s$ are martingales under the measures $\mathbb{P}^*$ and $\mathbb{P}$, respectively, subject to the appropriate integrability conditions. The payoff $g(Y_T)$ admits under $\mathbb{P}$ the following representation in terms of its indifference price, the price of the traded assets S and B changes, and the unhedgeable risk component which is due to the presence of the second Brownian motion*

$$\begin{aligned} g(Y_T) = {} & \frac{h(Y_t,t)}{B_t} + \int_t^T \frac{\rho a(Y_s,s)\, h_y(Y_s,s)}{\sigma S_s B_s} dS_s \\ & - \int_t^T \frac{\rho a(Y_s,s)\, h_y(Y_s,s)}{\sigma B_s^2} dB_s \\ & + \sqrt{1-\rho^2} \int_t^T \frac{a(Y_s,s)\, h_y(Y_s,s)}{B_s} dW_s^{\perp} \\ & - \frac{1}{2}\gamma\left(1-\rho^2\right) \int_t^T \left(\frac{a(Y_s,s)\, h_y(Y_s,s)}{B_s} \right)^2 ds. \end{aligned}$$

The rest of this section is dedicated to alternative probabilistic representation of the indifference price. Motivated by the arbitrage-free representation of derivative prices in complete models as expectations of the discounted payoffs under the martingale measure, we aim for similar representations. The underlying idea relies upon expressing h as the value function of a stochastic optimization problem evaluated at the optimum.

The specification of the relevant stochastic control problem and the characterization of h as its solution with the optimality of the relevant policies is executed along well known arguments. The basic ingredients are the smoothness of h and the uniqueness of the viscosity solutions of (3.24). The proof of the latter follows arguments developed in Duffie and Zariphopoulou (1993). The regularity of h follows from its exact formula (3.11) and from the smoothing properties of the infinitesimal generators of diffusions. We choose not to present the rigorous arguments herein but rather to proceed with a heuristic discussion and state the rigorous results, together with the appropriate technical conditions in the sequel. To this end, we recall the quasi-linear equation satisfied by the indifference price, namely

$$\begin{cases} h_t + \frac{1}{2}a^2(y,t)\, h_{yy} + b(y,t)\, h_y + a(y,t)\, \phi(y,t,h_y) = rh \\ \qquad h(y,T) = g(y), \end{cases} \tag{3.24}$$

where the action functional ϕ is given by

$$\phi(y,t,h_y) = \frac{1}{2} e^{r(T-t)} \gamma \left(1-\rho^2\right) a(y,t)\, h_y^2 - \rho \frac{\mu - r}{\sigma} h_y.$$

Define the dual convex function ϕ^* of ϕ by

$$\phi^* (y, t, \alpha) = \max_z (\alpha z - \phi (y, t, z)),$$

where α will subsequently be identified as a control variable. Simple calculations give the following expression

$$\phi^* (y, t, \alpha) = \frac{1}{2} \left(\alpha + \rho \frac{\mu - r}{\sigma} \right)^2 \left(e^{r(T-t)} \gamma \left(1 - \rho^2\right) a (y, t) \right)^{-1}.$$

Moreover, because

$$\phi (y, t, z) = \max_\alpha (\alpha z - \phi^* (y, t, \alpha))$$

quasi-linear equation (3.24) can be written also in the form

$$h_t + \frac{1}{2} a^2 (y, t) h_{yy} + b (y, t) h_y + a (y, t) \max_\alpha (\alpha h_y - \phi^* (y, t, \alpha)) = rh,$$

or, alternatively, as

$$h_t + \max_\alpha \left(\frac{1}{2} a^2 (y, t) h_{yy} + (b (y, t) + a (y, t) \alpha) h_y \right.$$
$$\left. - \frac{1}{2} \left(\alpha + \rho \frac{\mu - r}{\sigma} \right)^2 \left(e^{r(T-t)} \gamma \left(1 - \rho^2\right) \right)^{-1} \right) = rh. \tag{3.25}$$

The above equation can be viewed as the HJB-equation of a stochastic control problem that we discuss below. To this end, we need to define a diffusion process with the infinitesimal generator of the form

$$\frac{\partial}{\partial t} + \frac{1}{2} a (y, t) \frac{\partial^2}{\partial y^2} + (b (y, t) + a (y, t) \alpha (y, t)) \frac{\partial}{\partial y}$$

assuming that the function $\alpha (y, t)$ is such that

$$E \exp \left(\int_0^T \alpha_s dW_s - \frac{1}{2} \int_0^T \alpha_s^2 dt \right) = 1, \tag{3.26}$$

where

$$\alpha_s = \alpha (Y_s, s).$$

Let $\mathbb{P}^\alpha$ be the probability measure given by

$$\mathbb{P}^\alpha (A) = E \left(\exp \left(\int_0^T \alpha_t dW_t - \frac{1}{2} \int_0^T \alpha_t^2 dt \right) I_A \right), \quad A \in \mathcal{F}_T. \tag{3.27}$$

It follows that under $\mathbb{P}^\alpha$, the process

$$W_t^\alpha = W_t - \int_0^t \alpha_s ds, \quad 0 \le t \le T,$$

is a Brownian motion and, hence, the semimartingale decomposition of the process Y under the measure $\mathbb{P}^{\alpha}$ is given by

$$dY_t = \left(b\left(Y_t,t\right) + a\left(Y_t,t\right)\alpha\left(Y_t,t\right)\right)dt + a\left(Y_t,t\right)dW_t^{\alpha}.$$

Clearly, the process Y, under the measure $\mathbb{P}^{\alpha}$, is the required diffusion. Consequently, the writer's price can be written in the familiar value function form

$$h\left(y,t\right) = e^{-r(T-t)}\sup_{\alpha} H^{\alpha}\left(y,t\right),$$

where

$$H^{\alpha} = E_{\mathbb{P}^{\alpha}}\left(g\left(Y_T\right) - \frac{1}{2\gamma\left(1-\rho^2\right)}\int_t^T \left(\alpha_s + \rho\frac{\mu-r}{\sigma}\right)^2 ds\left|Y_t = y\right.\right). \tag{3.28}$$

Note that, by changing the measure from $\mathbb{P}^{\alpha}$ to $\mathbb{P}$, we also obtain

$$H^{\alpha}\left(y,t\right) = E\left(\exp\left(\int_t^T \alpha\left(Y_s,s\right)dW_s - \frac{1}{2}\int_t^T \alpha^2\left(Y_s,s\right)ds\right)\right.$$
$$\left.\times\left(g\left(Y_T\right) - \frac{1}{2\gamma\left(1-\rho^2\right)}\int_t^T\left(\alpha\left(Y_s,s\right) + \rho\frac{\mu-r}{\sigma}\right)^2 ds\right)\left|Y_t = y\right.\right),$$

where the process Y is given by (2.2).

From the first order conditions in the HJB equation (3.25), we derive the optimal feedback law

$$\alpha_0\left(y,t\right) = e^{r(T-t)}\gamma\left(1-\rho^2\right)a\left(y,t\right)h_y\left(y,t\right) - \rho\frac{\mu-r}{\sigma}. \tag{3.29}$$

Applying the suboptimal control $\alpha_s = -\rho\frac{\mu-r}{\sigma}$ in (3.28), we also obtain an obvious lower bound for the price. Before proceeding any further, we summarize the main results below.

Theorem 3.3. *The indifference writer's price $h\left(y,t\right)$ of the claim $g\left(Y_T\right)$ written on the non-traded asset Y satisfies*

$$h\left(y,t\right) = e^{-r(T-t)}\sup_{\alpha} H^{\alpha}\left(y,t\right) = e^{-r(T-t)}H^{\alpha_0}\left(y,t\right)$$
$$= e^{-r(T-t)}E_{\mathbb{P}^{\alpha_0}}\left(g\left(Y_T\right)|Y_t = y\right)$$
$$-\frac{1}{2}\gamma\left(1-\rho^2\right)e^{-r(T-t)}E_{\mathbb{P}^{\alpha_0}}\left(\int_t^T e^{2r(T-s)}a^2\left(Y_s,s\right)h_y^2\left(Y_s,s\right)ds\left|Y_t = y\right.\right).$$

In the above formula $H^{\alpha}\left(y,t\right)$ is given by (3.28), the supremum is taken over a class of feedback laws $\alpha\left(y,t\right)$ satisfying (3.26), and the optimal feedback law α_0 is defined in (3.29). Moreover,

$$h\left(y,t\right) \geq e^{-r(T-t)}E_{\widetilde{\mathbb{P}}}\left(g\left(Y_T\right)|\,Y_t = y\right),$$

where the measure $\widetilde{\mathbb{P}}$ is defined in (2.31).

We conclude this section by identifying a *martingale measure* for the forward price process $\widetilde{H}_s = \widetilde{h}(Y_s, s)$, $t \leq s \leq T$ defined in (3.19). Recall that the function $\widetilde{h}$ is defined in (3.11) and the function w is the solution to (3.12) with μ replaced by $\mu - r$. Alternatively, $\widetilde{h}(y,s) = e^{r(T-s)}h(y,s)$, while the function h is defined in (2.32). Straightforward application of the Ito formula gives

$$d\widetilde{H}_s = \widetilde{h}_y(Y_s, s)\, a(Y_s, s)\left(dW_s - \alpha_1(Y_s, s)\, ds\right), \tag{3.30}$$

where

$$\alpha_1(y,t) = \frac{1}{2}\gamma\left(1-\rho^2\right) a(y,t)\, \widetilde{h}_y(y,t) - \rho\frac{\mu - r}{\sigma}. \tag{3.31}$$

Because the process

$$W_t - \int_0^t \alpha_1(Y_s, s)\, ds, \quad 0 \leq t \leq T,$$

is a Brownian motion under the measure $\mathbb{P}^{\alpha_1}$ defined in (3.27), we have the following result.

Theorem 3.4. *The forward price process $\widetilde{H}_s = \widetilde{h}(Y_s, s)$, $t \leq s \leq T$, defined in (3.19), satisfies (3.30) and hence is a local martingale under the measure $\mathbb{P}^{\alpha_1}$, and a martingale under the appropriate integrability conditions.*

Remark 3.5. The buyer's price case can be analyzed in the same way. Assumptions under which the α_0 and α_1 satisfy (3.26) are given in the following section. Note that the measures $\mathbb{P}^{\alpha_0}$ and $\mathbb{P}^{\alpha_1}$ depend on the option payoff. This is consistent with the non-linear character of the indifference based pricing method. Another natural property is the price dependence on the risk aversion γ.

4 Sensitivity analysis

In this section we analyze the sensitivity of the pricing function h to the changes of certain variables and parameters. In particular, we derive analytic expressions for the derivatives of h with respect to the variable y and the correlation ρ, the risk aversion γ and the Sharpe ratio $\frac{\mu-r}{\sigma}$. To ease the presentation, we recall that

$$h(y,t) = e^{-r(T-t)}\frac{1}{\gamma(1-\rho^2)}\log w(y,t), \tag{4.1}$$

where the function w is the unique solution to

$$\begin{cases} w_t + \frac{1}{2}a^2(y,t)\, w_{yy} + \left(b(y,t) - \rho\frac{\mu-r}{\sigma}a(y,t)\right) w_y = 0 \\ \\ w(y,T) = e^{\gamma\left(1-\rho^2\right)g(y)}. \end{cases} \tag{4.2}$$

Introducing the function $z(y,t) = w_y(y,t)$ and, differentiating the above equation and the terminal condition with respect to y, we observe that z satisfies

$$\left\{ \begin{array}{c} z_t + \frac{1}{2}a^2(y,t)\, z_{yy} + (a(y,t)\, a_y(y,t) + c(y,t))\, z_y + c_y(y,t)\, z = 0 \\ \\ z(y,T) = \gamma\left(1-\rho^2\right) g_y(y)\, e^{\gamma\left(1-\rho^2\right)g(y)}, \end{array} \right. \tag{4.3}$$

with $c(y,t)$ given by

$$c(y,t) = b(y,t) - \rho\frac{\mu - r}{\sigma} a(y,t). \tag{4.4}$$

For simplicity, we assume that the coefficients a, b and the payoff g are absolutely continuous functions of y, with derivatives on y bounded uniformly on s. Note that it is quite easy to weaken the assumption concerning the payoff. In fact the function $w(y,t)$ is C^∞ in y for $t < T$ for any bounded Borel measurable payoff $g(y)$. Function z will then satisfy equation (4.3) with the appropriately defined terminal condition.

In order to obtain a probabilistic interpretation of $z(y,t)$ we define the function

$$\alpha_2(y,t) = a_y(y,t) - \rho\frac{\mu - r}{\sigma}$$

and the associated measure $\mathbb{P}^{\alpha_2}$ defined along the lines (3.27). Under $\mathbb{P}^{\alpha_2}$, we have

$$z(y,t) = E_{\mathbb{P}^{\alpha_2}}\left(\gamma\left(1-\rho^2\right) g_y(Y_T)\, e^{\gamma\left(1-\rho^2\right)g(Y_T)} e^{\int_t^T c_y(Y_s,s)ds} | Y_t = y\right).$$

Moreover, the Radon Nikodym density of the measure $\mathbb{P}^{\alpha_2}$ with respect to the measure $\widetilde{\mathbb{P}}$ can be easily computed, namely

$$\begin{aligned} \frac{d\mathbb{P}^{\alpha_2}}{d\widetilde{\mathbb{P}}} = \frac{d\mathbb{P}^{\alpha_2}}{d\mathbb{P}}\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}} &= \exp\left(\int_0^T \alpha_2(Y_s,s)\, dW_s - \frac{1}{2}\int_0^T \alpha_2^2(Y_s,s)\right) ds \\ &\times \exp\left(\rho\frac{\mu - r}{\sigma} W_T + \frac{1}{2}\rho^2\left(\frac{\mu - r}{\sigma}\right)^2 T\right) \\ &= \exp\left(\int_0^T a_y(Y_s,s)\, d\widetilde{W}_s - \frac{1}{2}\int_0^T a_y^2(Y_s,s)\, ds\right), \end{aligned}$$

where the process

$$\widetilde{W}_s = W_s + \rho\frac{\mu - r}{\sigma}s, \;\; 0 \le s \le T$$

is a Brownian motion under the measure $\widetilde{\mathbb{P}}$. This yields the following result.

Theorem 4.1. *Assume that the coefficients a and b are absolutely continuous functions of y with spatial derivatives a_y and b_y, bounded uniformly in t. Moreover, assume also that the payoff function g and its derivative g_y are bounded.*

Then the derivative $h_y(y,t)$ of the writer's indifference price $h(y,t)$ given in (2.32) is bounded and is equal to

$$h_y(y,t) = e^{-r(T-t)} \frac{1}{\gamma(1-\rho^2)} \frac{w_y(y,t)}{w(y,t)},$$

where

$$w(y,t) = E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t = y\right)$$

and

$$\begin{aligned} w_y(y,t) = E_{\widetilde{\mathbb{P}}} \Big(\gamma\left(1-\rho^2\right) g_y(Y_T)\, & e^{\gamma\left(1-\rho^2\right)g(Y_T)+\int_t^T c_y(Y_s,s)ds} \\ & \times e^{\int_t^T a_y(Y_s,s)d\widetilde{W}_s - \frac{1}{2}\int_t^T a_y^2(Y_s,s)ds} \,|Y_t = y\Big), \end{aligned}$$

with c defined in (4.4).

In a number of previously stated results we have implicitly assumed that the appropriate integrability conditions related to the choices of the option payoffs and the non-traded asset dynamics are fulfilled. They all referred to the conditions under which the controls of the type (3.4) satisfies (3.26) and hence allow for the appropriate measure transformations. Moreover, they also imply martingale rather than local martingale properties of the previously studied processes. Now we formulate a set of self-evident sufficient conditions for the above implication to hold.

Theorem 4.2. *Assume that the coefficients a and b are absolutely continuous functions of y with derivatives a_y and b_y, bounded uniformly in t, and that*

$$E\exp\left(A\int_0^T a(Y_s,s)\,dW_s - \frac{1}{2}A^2\int_0^T a^2(Y_s,s)\,ds\right) = 1 \qquad (4.5)$$

for arbitrary constant $A > 0$. Assume also that the payoff function g and its derivative g_y are bounded. Then the controls α_i, $i = 0,1,2$, satisfy (3.26). Moreover, the process $\widetilde{Z}_s$, defined in (3.22), is a martingale under $\mathbb{P}$, while the process $\widetilde{L}_s$, given by (3.21), is a martingale under $\mathbb{P}^*$.

Computation of the derivative of the price with respect to the risk aversion parameter γ follows the same arguments. Using formula (4.1) we get

$$h_\gamma(y,t) = -\frac{1}{\gamma}h(y,t) + e^{-r(T-t)} \frac{1}{\gamma(1-\rho^2)} \frac{w_\gamma(y,t)}{w(y,t)},$$

where

$$w(y,t) = E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t = y\right).$$

Given that the measure $\widetilde{\mathbb{P}}$ does not depend on γ, one can directly differentiate the integrand under the expectation which leads to

$$w_\gamma(y,t) = E_{\widetilde{\mathbb{P}}}\left(\left(1-\rho^2\right) g(Y_T)\, e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t = y\right).$$

It is interesting to analyze the monotonicity of the price $h(y,t) = h^{(\gamma)}(y,t)$ on the risk aversion parameter γ. To this end one can try to show that $h_\gamma(y,t) \geq 0$. Instead, we assume $0 < \gamma_1 < \gamma_2$, and note that the non-linear term in (3.13) is always positive and hence

$$\begin{aligned}
0 &= rh^{(\gamma_1)} - \Bigg(h_t^{(\gamma_1)} + \frac{1}{2}a^2(y,t)\, h_{yy}^{(\gamma_1)} + c(y,t)\, h_y^{(\gamma_1)} \\
&\quad + \frac{1}{2}e^{r(T-t)}\gamma_1\left(1-\rho^2\right)a^2(y,t)\left(h_y^{(\gamma_1)}\right)^2\Bigg) \\
&\geq rh^{(\gamma_1)} - \Bigg(h_t^{(\gamma_1)} + \frac{1}{2}a^2(y,t)\, h_{yy}^{(\gamma_1)} + c(y,t)\, h_y^{(\gamma_1)} \\
&\quad + \frac{1}{2}e^{r(T-t)}\gamma_2\left(1-\rho^2\right)a^2(y,t)\left(h_y^{(\gamma_1)}\right)^2\Bigg)
\end{aligned}$$

which implies that $h^{(\gamma_1)}$ is a sub-solution of (3.13) with $\gamma = \gamma_2$. Moreover, the terminal condition $g(y)$ is independent of γ. Using comparison argument (see, for example, Ishii and Lions (1990)) we conclude that $h^{(\gamma_1)} \leq h^{(\gamma_2)}$ and hence the writer's price is increasing with respect to the risk aversion γ. Clearly, the buyer's price is decreasing with respect to γ. Finally, taking the limit as $\gamma \to 0$ we observe that

$$h^{(0)}(y,t) = e^{-r(T-t)}E_{\widetilde{\mathbb{P}}}\left(g(Y_T)\,|Y_t = y\right) \tag{4.6}$$

for both the writer's and buyer's prices.

Theorem 4.3. *The writer's (resp. buyer's) price of the bounded payoff* $g(Y_T)$ *is increasing (resp. decreasing) with respect to the risk aversion* γ*. The price corresponding to zero risk aversion are given by (4.6). The derivative* h_γ *of the writer's price is given by*

$$\begin{aligned}
h_\gamma(y,t) &= -\frac{1}{\gamma}e^{-r(T-t)}\frac{1}{\gamma\left(1-\rho^2\right)}\ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t = y\right)\right) \\
&\quad + \frac{1}{\gamma}e^{-r(T-t)}\frac{E_{\widetilde{\mathbb{P}}}\left(g(Y_T)\, e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t = y\right)}{E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t = y\right)}.
\end{aligned}$$

We continue with the calculation of the derivative with respect to the correlation ρ. Our method of using different probabilistic representations of solutions works again. In this instance, it is more convenient to represent the price with respect to the historical measure $\mathbb{P}$ rather than the already adjusted measure $\widetilde{\mathbb{P}}$ which depends on the correlation. To this end, we write the price in the following form

$$h = \frac{e^{-r(T-t)}}{\gamma\left(1-\rho^2\right)} \ln\left(E\left(e^{-\rho\frac{\mu-r}{\sigma}W_T-\frac{1}{2}\rho^2\frac{(\mu-r)^2}{\sigma^2}T} e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t=y\right)\right) \tag{4.7}$$

and then take the derivative with respect to ρ. After straightforward simplifications we get the following result presented again in terms of the measure $\widetilde{\mathbb{P}}$.

Theorem 4.4. *The derivative h_ρ of the writer's price of the bounded payoff $g\left(Y_T\right)$ with respect to the correlation is given by*

$$h_\rho\left(y,t\right) = \frac{2\rho}{1-\rho^2} h\left(y,t\right)$$

$$-e^{-r(T-t)}\frac{1}{\gamma\left(1-\rho^2\right)}\left(\frac{\mu-r}{\sigma}\frac{E_{\widetilde{\mathbb{P}}}\left(\widetilde{W}_T e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t=y\right)}{E_{\widetilde{\mathbb{P}}}\left(e^{\gamma(1-\rho^2)g(Y_T)}\,|Y_t=y\right)}\right.$$

$$\left.+2\gamma\rho\frac{E_{\widetilde{\mathbb{P}}}\left(g\left(Y_T\right)e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t=y\right)}{E_{\widetilde{\mathbb{P}}}\left(e^{\gamma(1-\rho^2)g(Y_T)}\,|Y_t=y\right)}\right).$$

Formula (4.7) can also be used to calculate the derivative of h with respect to the Sharpe ratio $\frac{\mu-r}{\sigma}$. The method is self-evident by now and therefore we only state the result.

Theorem 4.5. *The derivative h_λ of the writer's price of the bounded payoff $g\left(Y_T\right)$ with respect to the Sharpe ratio $\lambda=\frac{\mu-r}{\sigma}$ is given by*

$$h_\lambda\left(y,t\right) = -e^{-r(T-t)}\frac{\rho}{\gamma\left(1-\rho^2\right)}\frac{E_{\widetilde{\mathbb{P}}}\left(\widetilde{W}_T e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_t=y\right)}{E_{\widetilde{\mathbb{P}}}\left(e^{\gamma(1-\rho^2)g(Y_T)}\,|Y_t=y\right)}.$$

Remark 4.6. To analyze the dependence of the price on the notional principal note that the price at time t, when $Y_t=y$, of the payoff $cg\left(Y_T\right), c\geq 1$, is equal to $ch^{(c\gamma)}\left(y,t\right)$ which is greater than $ch^{(\gamma)}\left(y,t\right)$ because of monotonicity of the price with respect to the risk aversion γ. Hence, the price dependence on the notional principal is nonlinear. Namely, the writer's price of the claim $cg\left(Y_T\right), c\geq 1$, is greater than c times the price of the claim $g\left(Y_T\right)$, and the buyer's price is smaller.

5 Model specification

The integrability condition (4.5) imposed on the diffusion coefficient $a(y,t)$ of the non-traded asset Y appears to be very restrictive. For example, it rules out the lognormality assumption for the dynamics of Y. Note, however, that it is in fact directly related to the integrability condition imposed on the payoff $g(Y_T)$. In order to explain this point in greater detail, we now concentrate our attention on the case of time homogeneous diffusions representing the dynamics of Y. To this end, we assume that the functions a and b are independent of t. More precisely, we start with an interval

$$I = (l, r), \quad -\infty \leq l < r \leq \infty.$$

We assume that the coefficients a and b are known absolutely continuous functions from I into $\mathcal{R}$ with bounded derivatives and, that

$$a(y) > 0, \ \ \forall\ y \in I.$$

They are specified exogenously by a statistical analysis of the data or by some other arguments. Observe that the concept of correlation between the traded asset S and the non-traded asset Y is invariant with respect to the monotonic transformations of the data. In fact, one way to specify ρ would be to measure the historical correlation between the returns on the traded asset S and the increments of the following transformation of the levels of the non-traded asset Y

$$k(y) = \int_{y_0}^{y} \frac{1}{a(u)} du,$$

where $y_0 \in I$. It is obvious that the process $Z = k(Y)$ satisfies

$$dZ_s = dk(Y_s) = \left(\frac{b(Y_s)}{a(Y_s)} - a_y(Y_s) \right) ds + dW_s$$

$$= \widehat{b}(Z_s)\, ds + dW_s,$$

where

$$\widehat{b}(z) = \frac{b\left(k^{-1}(z)\right)}{a\left(k^{-1}(z)\right)} - a_y\left(k^{-1}(z)\right)$$

and here k^{-1} stands here for the inverse of the function k. In order to satisfy the previous assumptions, the new drift would need to be absolutely continuous with a bounded derivative which clearly is dependent on properties of original coefficients a and b. More importantly though, instead of pricing the payoff $g(Y_T)$ written on the original process Y, one could transform the data and consider the equivalent payoff $g\left(k^{-1}(Z_T)\right)$. If the function $g\left(k^{-1}(z)\right)$ and its derivative are bounded. one can apply the previous analysis to the process Z rather than to the original process Y. This is because the integrability condition (4.5) is satisfied in this case.

The above discussion points out to a natural candidate for the dynamics of the non-traded asset, namely a Gauss-Markov process for which $a(y) = a$ and $b(y) = b_0 + b_1 y$, where a, b_0 and b_1 are constants. Obviously this class contains a very important case of a mean reverting Ornstein-Uhlenbeck process frequently used in financial modelling. In fact under such assumptions one can price a larger class of payoffs including the standard call and put options.

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