Closed Form Option Valuation with Smiles

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Current Version: August 26, 1999

File Reference: closed4.tex

Abstract

Assuming that the underlying local volatility is a function of stock price and time, we develop an approach for generating closed form solutions for option values for a certain class of volatility functions. The class is the set of volatility functions which solve the same partial differential equation as derivative security values in the Black Scholes model. We illustrate our results with three examples.

We are grateful for comments from Claudio Albanese, Anlong Li, Dilip Madan, Eric Reiner and the participants of the Columbia Conference on Applied Probability and Global Derivatives 99. They are not responsible for any errors. The third author acknowledges partial support from a Romnes fellowship and the Graduate School of the University of Wisconsin, Madison.

I Introduction

Notwithstanding the recent award of the Nobel prize in economics to Professors Merton and Scholes, option prices commonly violate the formulas developed in Merton[14] and Black-Scholes[2]. In particular, Rubinstein[18] has documented a pronounced dependence of implied volatility on the strike of the option, which has been consistently downward sloping since the crash of 1987. While the source of this volatility smile is still the subject of much debate, the intransigence of this smile has lead to a resurgence of interest in option pricing formulas which are capable of simultaneously explaining a cross-section of option prices.

While the volatility smile can be explained using a price process displaying stochastic volatility and/or jumps (see eg. Bates[1] for references), the hedging arguments underlying these option pricing models usually require continuous trading in one or more options. Unfortunately, the width of the bid-ask spreads in many options markets generally renders frequent option trading economically unviable. For such markets, it therefore seems prudent to first explore the class of option pricing models which permit replication using dynamic trading in stocks alone. In the diffusion class of such models, this class is characterized by a volatility function which depends on time, the stock price, and the stock price path (see Hobson and Rogers[11] for some path-dependent volatility specifications). Further restricting volatility dependence to just time and the stock price[1] has the computational advantage of reducing the dimensionality of the problem. This reduction can also yield closed form solutions providing further computational advantages.

Although these advantages of price-dependent volatility are well known, there are a limited number of volatility specifications which yield closed form solutions for options prices. It is well known that the constant, square root, and proportional volatility models can all be embedded within the constant elasticity of variance (CEV) model pioneered in Cox[6] (also see Schroeder[19] and Linetsky and Davidoff [13]). Goldenberg[10] generalizes the CEV model and studies various deterministic time and scale changes of the stock price process which map it to more tractable processes such as the square root process or standard Brownian motion. Similarly, Bouchouev and Isakov[4] and Li[12] write the stock price process as various functions of the standard Brownian motion and time.

A general approach for generating option pricing formulas for price-dependent volatility is in fact implicit

[1] Note that the volatility follows a stochastic process in this case. However increments in the volatility are perfectly correlated locally with increments in the stock price.

in the original Black Scholes paper. By regarding the stock as an option on the assets of the firm, options on the stock become de facto compound options. In this model, the volatility of the stock depends on the firm value. If the function relating the stock price to the firm value can be explicitly inverted[2], then the stock volatility can be expressed as an explicit function of the stock price.

This paper takes this economically motivated approach in order to generate closed form solutions for option prices with smiles. Our only departure is that we take the fundamental driver to be standard Brownian motion rather than firm value. The primary reason for this departure is analytical convenience, since more results are known for standard Brownian motion than geometric Brownian motion. When standard Brownian motion is taken as the fundamental driver, it becomes important to determine the conditions under which a future stock price depends only on the contemporaneous level of the driving standard Brownian motion, and thus does not depend on the Brownian path.

The purpose of this paper is to characterize the *entire* class of volatility functions which permit the stock price to be transformed into standard Brownian motion by scale changes alone[3]. Since many results are known for standard Brownian motion (see eg. Borodin and Salminen [3]), these results can be used to generate closed form solutions for option prices with smiles. We find that the class of volatility functions permitting this transformation is characterized by a fully nonlinear partial differential equation (p.d.e.), which we are nonetheless able to solve analytically. As a result, we explicitly obtain many new volatility functions which yield closed form solutions for option prices. We illustrate our results by obtaining several new option pricing formulas, which all involve only normal distribution or density functions.

The outline for this paper is as follows. Section II presents the nonlinear p.d.e. governing volatility, while Section III appends appropriate boundary conditions and characterizes the unique solution. Section IV shows how this solution can be used to generate option prices. Section V illustrates with three examples and the final section summarizes the paper and discusses extensions.

[2]This explicit invertibility requirement is not met in the Geske model[9].

[3]In a binomial framework, this is equivalent to assumption 5 in Rubinstein[18] p. 788 that all paths ending at the same node have the same risk-neutral probability and to the *computationally simple* criterion in Nelson and Ramaswamy[17].

II The Nonlinear p.d.e. for Volatility

In this section, we lay out our economic model and present necessary and sufficient conditions on the volatility function which permit the driving standard Brownian motion to be expressible as a scale change of the stock price process. Assuming these conditions hold, we also present a formula for this scale change. We show that the necessary and sufficient condition on the volatility function is that it satisfies a certain nonlinear p.d.e. The next section describes a simple technique for generating solutions to this p.d.e.

Our economic model assumes frictionless markets, no arbitrage, and that the underlying stock price process is a one dimensional diffusion starting from a positive value. We assume a proportional risk-neutral drift of $r - q$, where $r \geq 0$ is the constant risk-free rate and $q \geq 0$ is the constant dividend yield. We also assume that the absolute volatility rate is a positive $C^{2,1}$ function $a(S,t)$ of the stock price $S \in (0, \infty)$ and time $t \in (0,T)$, where T is some distant horizon exceeding the longest maturity of the option to be priced. Since this process will in general[4] have positive probability that the stock price hits zero, we absorb the stock price at the origin in this event. Let τ_o denote the first hitting time of the origin and let $\tau \equiv T \wedge \tau_o$ be a bounded stopping time. Let $\{S_t : t \in [0,\tau]\}$ denote the risk-neutral stock price process:

$$dS_t = (r-q)S_t dt + a(S_t,t)dW_t, \quad t \in [0,\tau], \tag{1}$$

where $\{W_t, t \in [0,\tau]\}$ is a standard Brownian motion (SBM) under the risk-neutral measure Q. To ensure that the forward price process is a martingale[5], we assume that for each t, $\lim_{S \uparrow \infty} a(S,t) = O(S)$.

The standard approach to option valuation is to find a function relating the option price to the stock price. In order to obtain closed form solutions, this approach requires that one know the transition density of the risk-neutral process (1). For realistic volatility functions, it is in general difficult to determine this density. In this paper, we take an SBM as the fundamental driver of all stochastic processes of interest. This is similar to the Black Scholes idea of taking a geometric Brownian motion describing the firm value as the fundamental driver. Of course in both cases, the transition density is very well known. The potential disadvantage of this approach is that the processes of interest, such as volatility and option prices, are

[4]Linetsky and Davidov[13] give a complete description of boundary classification behavior for one dimensional time-homogeneous diffusion processes with constant proportional drift. In our notation, they show that that if $a(S)$ grows as S^p as $S \downarrow 0$, then for $p \geq 1$, the origin is a natural boundary, for $p \in [\frac{1}{2}, 1)$, the origin is an exit boundary, while for $p < \frac{1}{2}$, the origin is a regular boundary point. In the first case, the origin is inaccessible, in the intermediate case, the process must absorb, and in the final case, a variety of boundary behavior is possible, but we impose absorption.

[5]See Linetsky and Davidov[13] for a complete description of possible behavior as $S \uparrow \infty$.

expressed in terms of the unobservable SBM or firm value. This disadvantage is overcome in our case if the SBM can be expressed in terms of the contemporaneous stock price. Appendix 1 derives the following necessary condition on the volatility function $a(S,t)$ which permits this representation:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2} + (r-q)S\frac{\partial a}{\partial S}(S,t) + \frac{\partial a}{\partial t}(S,t) = (r-q)a(S,t), \quad S>0, t\in[0,T]. \tag{2}$$

Note that (2) is the standard Black Scholes/Merton(BSM) p.d.e. describing the value of a claim $a(S,t)$ with dividend yield q, when the underlying price process has a diffusion coefficient $a(S,t)$. This fully nonlinear p.d.e. is a necessary condition on the volatility function $a(S,t)$ in order that the SBM is simply a scale change of the stock price. Equations (69) and (74) of the appendix imply that the scale change is $W_t = w(S_t,t;S_0)$ where:

$$w(S,t;S_0) = \int_{S_0}^{S}\frac{1}{a(Z,t)}dZ + \int_0^t\left[\frac{1}{2}\frac{\partial a(S,s)}{\partial S}\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,s)}\right]ds. \tag{3}$$

The converse is proved in Appendix 2: given a local volatility function $a(S,t)$ satisfying the nonlinear p.d.e. (2), the process $B_t \equiv \int_{S_0}^{S_t}\frac{1}{a(Z,t)}dZ + \int_0^t\left[\frac{1}{2}\frac{\partial a(S,s)}{\partial S}\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,s)}\right]ds$ is the SBM W.

III A Solution Class for the P.D.E.

This section presents a simple way to generate volatility functions which solve our fundamental non-linear p.d.e. (2). The next section shows how this technique can also be used to generate closed form option pricing formulas.

If the SBM W_t is regarded as an increasing function of the stock price, then one can conversely regard the stock price as the value of a derivative security written on W_t, whose value increases with this underlying. Taking this approach, let $s(w,t), w\geq L(t), t\in[0,T]$ be the spatial inverse of $w(S,t)$, i.e.

$$S_t = s(W_t,t), \quad t\in[0,T]. \tag{4}$$

For obvious reasons, we refer to $s(\cdot,\cdot)$ as the *stock pricing function.* The lower bound $L(t)$ of the domain of this function is an absorbing boundary for the SBM. It can be negative infinity, or it can be any known function of time.

By Itô's lemma, the stock price process can be written as:

$$dS_t = \left[\frac{\partial s}{\partial t}(W_t,t) + \frac{1}{2}\frac{\partial^2 s}{\partial w^2}(W_t,t)\right]dt + \frac{\partial s}{\partial w}(W_t,t)dW_t, \quad t\in[0,T]. \tag{5}$$

Equating coefficients on dt in (1) and (5) and using (4) gives a simple linear p.d.e. for the stock pricing function $s(w,t)$:

$$\frac{\partial s}{\partial t}(w,t) + \frac{1}{2}\frac{\partial^2 s}{\partial w^2}(w,t) = (r-q)s(w,t), \quad w \geq L(t), t \in [0,T]. \tag{6}$$

Equating coefficients of dW_t in (1) and (5) and using (4) gives a link between the *absolute volatility function* $a(S,t)$ and the stock pricing function $s(w,t)$:

$$a(s(w,t),t) = \frac{\partial s}{\partial w}(w,t), \quad w \geq L(t), t \in [0,T]. \tag{7}$$

Thus, if we can identify a solution $s(w,t)$ to the linear p.d.e. (6), then we obtain a corresponding solution $a(S,t)$ to the nonlinear p.d.e. (2).

To identify a subset of plausible solutions to (6), consider the terminal and growth conditions:

$$\lim_{w \downarrow L(t)} s(w,t) = 0, \quad t \in [0,T], \tag{8}$$

$$\lim_{w \uparrow \infty} s(w,t) = O(e^w), \quad t \in [0,T], \tag{9}$$

and:

$$\lim_{t \uparrow T} s(w,t) = \phi(w), \quad w \geq L(T), \tag{10}$$

where $\phi(w)$ is a positive increasing function also satisfying (8) and (9). Since $\phi(w)$ is positive and increasing, a minor extension of the classical results of Widder[20] implies that $s(w,t)$ is also positive and increasing in w. The lower boundary condition (8) ensures that the stock price absorbs at zero when SBM hits $L(t) < 0$, while the the upper boundary condition (9) ensures that the solution is bounded on bounded domains. The Feynman-Kac theorem can be used to find the continuous solution to the boundary value problem (BVP) consisting of the p.d.e. (6) and the boundary and terminal conditions (8),(9), and (10):

$$s(w,t) = e^{-(r-q)(T-t)} E^Q_{w,t}[\phi(W^a_T)], \quad w \geq L(t), t \in [0,T], \tag{11}$$

where $\{W^a_u; u \in [t,T]\}$ is an SBM absorbing at the lower barrier $L(t) < 0$ and starting at $w > L(t)$ at time t. Since $s(w,t)$ is increasing in $w \geq L(t)$, its spatial inverse $w(S,t)$ exists and is increasing in $S > 0$.

By (7), the corresponding boundary and terminal conditions for the volatility function are:

$$\lim_{S \downarrow 0} a(S,t) = 0, \quad t \in [0,T], \tag{12}$$

$$\lim_{S\uparrow\infty} a(S,t) = O\left(e^{w(S,t)}\right), \quad t \in [0,T], \tag{13}$$

and:

$$\lim_{t\uparrow T} a(S,t) = \phi'(w(S,T)), \quad S > 0. \tag{14}$$

By (4) and (7), a solution class for the nonlinear p.d.e. (2) subject to (12) to (14) is:

$$a(S,t) = \frac{\partial s}{\partial w}(w(S,t),t) \quad S > 0, t \in [0,T]. \tag{15}$$

Appendix 2 proves that the boundary value problem consisting of the nonlinear p.d.e. (2) and the boundary conditions (12) to (14) has a unique solution. It follows that (15) is this unique solution.

IV Option Pricing

This section interprets the SBM W_t and the absolute volatility $a(S_t,t)$ as prices of certain derivative securities. It also shows how to generate closed form formulas for transition densities and for option prices. The next section illustrates our results with three examples.

Before pricing options, it is worth noting that the SBM W_t is the forward price of a claim with the payoff $\phi^{-1}(S_T)$ at its maturity T, where $\phi^{-1}(\cdot)$ denotes the inverse of $\phi(w)$. The deferral of this exotic's premium payment to maturity induces zero drift under Q, while the terminal payoff induces unit volatility. Thus, $w(S_t,t)$ is a standard pricing function relating the time t forward price of the Brownian exotic paying $\phi^{-1}(S_T)$ at T to the time t spot price and time.

The local absolute volatility $a(S_t,t)$ can also be interpreted as the price process for an exotic equity derivative. To determine the payoff, note from (14) that the terminal absolute volatility $a(S,T)$ is the following function of the terminal spot price S_T:

$$a(S_T,T) = \phi'(W_T) = \frac{1}{\frac{\partial \phi^{-1}}{\partial S}(S_T)} = \lim_{h\downarrow 0} \frac{h}{\phi^{-1}(S_T+h) - \phi^{-1}(S_T)}.$$

From (2), it is clear that the claim whose value is the local absolute volatility also has a constant proportional payout of q.

Since we have been able to value these exotic derivatives, it should not be surprising that we can also value calls of some intermediate maturity $M \in [0,T)$. To relate the call value to the contemporaneous spot price, we first determine the function $\gamma(w,t)$ relating the call value to the price of the SBM W_t and time t. By Itô's lemma, this pricing function solves:

$$\frac{1}{2}\frac{\partial^2\gamma}{\partial w^2}(w,t) + \frac{\partial\gamma}{\partial t}(w,t) = r\gamma(w,t), \quad w > L(t), t \in (0,M), \tag{16}$$

subject to the boundary conditions:

$$\lim_{w\downarrow L(t)} \gamma(w,t) = 0, \quad \lim_{w\uparrow\infty} \gamma(w,t) = s(w,t)e^{-q(M-t)} - Ke^{-r(M-t)}, \quad t \in [0,M], \tag{17}$$

where $K > 0$ is the call strike, and subject to the terminal condition:

$$\gamma(W,M) = [s(w,M) - K]^+, \quad w > L(M). \tag{18}$$

By the Feynman-Kac representation theorem, the continuous solution to this BVP is:

$$\gamma(w,t) = e^{-r(M-t)}E^Q_{W,t}[s(W^a_M, M) - K]^+, \qquad w > L(t), t \in [0, M \wedge \tau), \tag{19}$$

where recall $\{W_u, u \in [t,M]\}$ is an SBM absorbing at the lower barrier $\{L(u), u \in (t,T)\}$ and starting at $w \geq L(t)$ at time t.

Equation (19) relates the call value to the SBM W_t and time t. To instead relate the call value to the stock price and time, we assume that $L(t) = L$, i.e. that the boundary along which the Brownian derivative vanishes is independent of time. In this case, the probability density function for absorbing SBM is known and if we let $c(S,t) = \gamma(w(S,t),t)$ in (19), then:

$$c(S,t) = e^{-r(M-t)}\int_L^\infty \frac{[s(z,M)-K]^+}{\sqrt{2\pi(M-t)}}\left\{\exp\left[-\frac{1}{2}\left(\frac{z-w(S,t)}{\sqrt{M-t}}\right)^2\right] - \exp\left[-\frac{1}{2}\left(\frac{z+w(S,t)-2L}{\sqrt{M-t}}\right)^2\right]\right\}dz, \tag{20}$$

for $S \geq 0, t \in [0, M \wedge \tau)$. This solution will be an explicit function of S and t if $s(z,M)$ and $w(S,t)$ can both be written explicitly in terms of their arguments.

Differentiating (20) twice w.r.t. the strike price K gives the risk-neutral probability that the stock price is at K at time M, given that the stock is at S at time t. Alternatively, the change of variables $S = s(w,t)$ for the absorbing SBM transition density expresses this probability as:

$$q(Z,M;S,t) = \frac{\frac{\partial w}{\partial S}(Z,M)}{\sqrt{2\pi(M-t)}}\left\{\exp\left\{-\frac{1}{2}\left[\frac{w(Z,M)-w(S,t)}{\sqrt{M-t}}\right]^2\right\} - \exp\left\{-\frac{1}{2}\left[\frac{w(Z,M)+w(S,t)}{\sqrt{M-t}}\right]^2\right\}\right\}.$$

Alternatively, since $\frac{\partial w}{\partial S}(Z,M) = \frac{1}{\frac{\partial s}{\partial w}} = \frac{1}{a(Z,M)}$, from (7):

$$q(Z,M;S,t) = \frac{1}{a(Z,M)\sqrt{2\pi(M-t)}}\left\{\exp\left\{-\frac{1}{2}\left[\frac{w(Z,M)-w(S,t)}{\sqrt{M-t}}\right]^2\right\} - \exp\left\{-\frac{1}{2}\left[\frac{w(Z,M)+w(S,t)}{\sqrt{M-t}}\right]^2\right\}\right\} \tag{21}$$

for $S > 0, t \in [0, M \wedge \tau)$. The density will be positive only if $w(S,M)$ is increasing in S and it will be explicit only if $w(S,M)$ is explicit in S.

An alternative derivation of the call pricing function $C(S,t), S \geq 0, t \in [0, M \wedge \tau)$ is obtained by integrating its payoff $(S-K)^+$ against this density, and discounting at the riskfree rate:

$$\begin{aligned} C(S,t) &= e^{-r(M-t)} \int_0^\infty (Z-K)^+ \frac{1}{a(Z,M)\sqrt{2\pi(M-t)}} \left\{\exp\left\{-\frac{1}{2}\left[\frac{w(Z,M)-w(S,t)}{\sqrt{M-t}}\right]^2\right\}\right. \\ &\qquad \left. - \exp\left\{-\frac{1}{2}\left[\frac{w(Z,M)+w(S,t)}{\sqrt{M-t}}\right]^2\right\}\right\} dZ. \end{aligned} \tag{22}$$

V Examples

This section first points out the Black-Scholes[2] proportional absolute volatility model and the Cox-Ross[7] constant absolute volatility model both fit into our framework. It then derives three new examples of volatility functions which yield closed form solutions for option prices. In general, the examples illustrate that our technique generates complicated, although realistic, volatility functions. Nonetheless, the option pricing formulas are all fairly simple and only require evaluating normal distribution and density functions.

The Black Scholes model can be put into our framework by considering an exponential stock payoff function $\phi(w) = \beta e^{\sigma w}, w \in \Re$. The stock pricing function then becomes $s(w,t) = \beta e^{(r-q-\frac{\sigma^2}{2})(T-t)+\sigma w}, w \in \Re, t \in [0,T]$. Evaluating this function at $w = W_t$ then yields a geometric Brownian motion. This process can never hit zero in contradiction to the reality that firms do go bankrupt. In contrast, Cox and Ross[7] modelled the stock price as having constant absolute volatility, which allows bankruptcy and induces hyperbolic "lognormal" volatility. Similarly, the three examples which follow all allow bankruptcy and all have the required property that $L(t) = L$.

V-A Stock Price is Hyperbolic Sine

Instead of taking the stock payoff as exponential, suppose more generally that it is given by the difference of two exponential functions. Specifically, suppose $\phi(w)$ satisfies:

$$\phi(w) = \begin{cases} \beta \sinh[\alpha(w - L)] & \text{if } w > L; \\ 0 & \text{if } w < L, \end{cases} \tag{23}$$

where recall $\sinh(x) \equiv \frac{e^x - e^{-x}}{2}$. Figure 1 graphs this payoff function and its inverse.

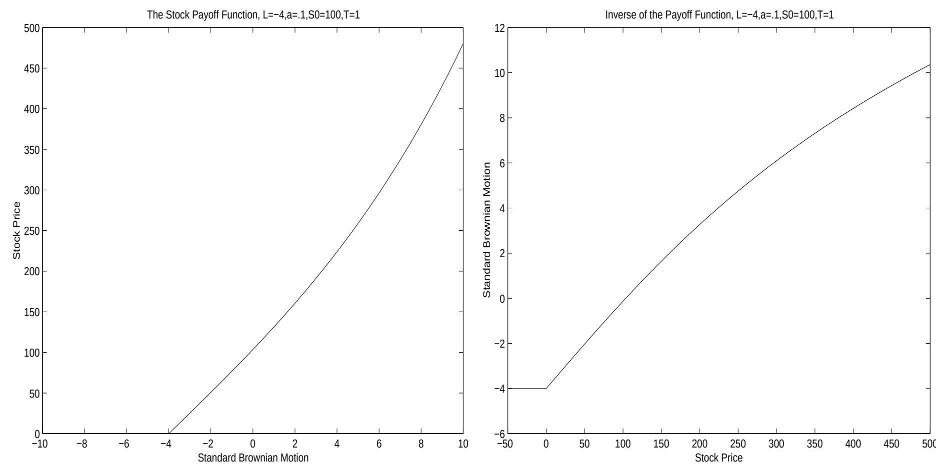

Figure 1: The Stock Payoff Function and Its Inverse

The solution of (6) subject to (8), (9), and (23) is:

$$s(w,t) = \beta e^{-\mu(T-t)} \sinh[\alpha(w - L)], \quad t \in [0,T], w > L, \tag{24}$$

where:

$$\mu \equiv r - q - \alpha^2/2. \tag{25}$$

Setting $s(0,0) = S_0$ relates the scaling constant to the initial stock price:

$$\beta = S_0 e^{\mu T} \text{csch}(-\alpha L), \tag{26}$$

where $\text{csch}(x) \equiv \frac{1}{\sinh(x)}$. Hence from (4):

$$S_t = S_0 e^{\mu t} \text{csch}(-\alpha L) \sinh[\alpha(W_t - L)], \qquad t \in [0,T]. \tag{27}$$

Figure 2 graphs this stock pricing function against the driving SBM and time. For each time, the stock price is an increasing convex function of the SBM, with greater slope and convexity than in the Black Scholes model with lognormal volatility α.

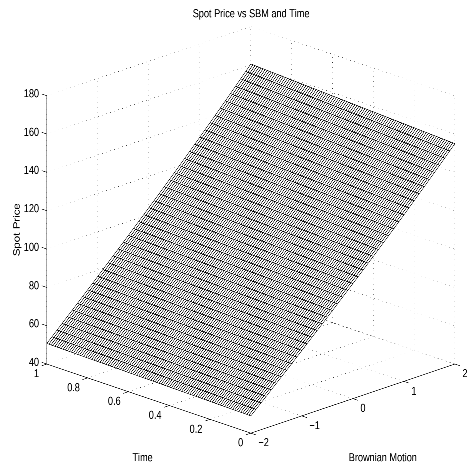

Figure 2: The Stock Pricing Function

To express the local volatility in terms of the stock price, solve (24) for w:

$$w(S,t) = L + \frac{\sinh^{-1}}{\alpha}\left(\frac{S}{\beta}e^{\mu(T-t)}\right), \qquad S > 0, t \in [0,T). \tag{28}$$

Differentiating w.r.t. S:

$$\frac{\partial w}{\partial S}(S,t) = \frac{1}{\alpha\sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}}}. \tag{29}$$

From (7), the local volatility is just the reciprocal:

$$a(S,t) = \alpha\sqrt{S^2 + \beta^2 e^{-2\mu(T-t)}}. \tag{30}$$

Dividing by the stock price gives the "lognormal" local volatility surface:

$$\sigma(S,t) \equiv \frac{a(S,t)}{S} = \alpha\sqrt{1 + \left(\frac{\beta}{Se^{\mu(T-t)}}\right)^2}. \qquad S > 0, t \in [0,\tau), \tag{31}$$

where β is given in (26). Figure 3 graphs this local volatility surface.

To understand the behavior of this local volatility function, note that the stock pricing function is proportional to $\sinh[\alpha(w-L)]$, which behaves linearly in w for w near L and exponentially in αw for w large. Thus, the volatility smile is approximately hyperbolic in S ("normal volatility") for S near zero, while it is asymptoting to the constant α ("lognormal volatility") for S high. As S increases from 0 to ∞, the volatility smile slopes downward in a convex fashion.

Just as α controls the asymptotic height of the volatility smile, the parameter L controls the "at-the-money" volatility. To see this, note that substituting (26) in (31) and evaluating the volatility function along

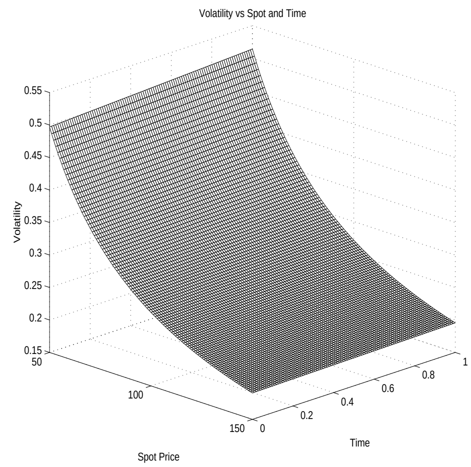

Figure 3: The Local Volatility Surface

$S = S_0 e^{\mu t}$ yields a stationary "at-the-money" volatility of:

$$\sigma(S_0 e^{\mu t}, t) = \alpha\sqrt{1 + \left(\frac{\beta}{S_0 e^{\mu T}}\right)^2} = \alpha\sqrt{1 + \operatorname{csch}^2(-\alpha L)}, \quad t \in [0, T],$$

from (26). Thus, as $L \downarrow -\infty$, at-the-money volatility approaches α. Conversely, as $L \uparrow 0$, volatility approaches infinity[6].

To get the risk-neutral density function, evaluate the Brownian pricing function w at $(S, t) = (Z, M)$:

$$w(Z, M) = L + \frac{\sinh^{-1}}{\alpha}\left(\frac{Z}{\beta} e^{\mu(T-M)}\right). \tag{32}$$

Similarly, evaluating the delta of the Brownian derivative at $(S, t) = (Z, M)$ yields:

$$\frac{\partial w}{\partial S}(Z, M) = \frac{1}{\alpha\sqrt{Z^2 + \beta^2 e^{-2\mu(T-M)}}}. \tag{33}$$

Substituting (28), (32), and (33) in (21) implies that the risk-neutral stock pricing density is:

$$\begin{aligned} q(Z, M; S, t) \quad = \frac{1}{\sqrt{2\pi(M-t)}} & \left\{ \exp\left\{ -\frac{1}{2}\left[\frac{\sinh^{-1}\left(\frac{Z}{\beta}e^{\mu(T-M)}\right) - \sinh^{-1}\left(\frac{S}{\beta}e^{\mu(T-t)}\right)}{\alpha\sqrt{M-t}}\right]^2 \right\} \right. \\ & \left. - \exp\left\{ -\frac{1}{2}\left[\frac{\sinh^{-1}\left(\frac{Z}{\beta}e^{\mu(T-M)}\right) + \sinh^{-1}\left(\frac{S}{\beta}e^{\mu(T-t)}\right)}{\alpha\sqrt{M-t}}\right]^2 \right\} \right\} \frac{1}{\alpha\sqrt{Z^2 + \beta^2 e^{-2\mu(T-M)}}}, \end{aligned} \tag{34}$$

for $S > 0, t \in [0, M \wedge \tau)$. Figure 4 graphs this density (termed the arcsinhnormal) against the future spot price and time. The downward sloping volatility surface graphed in Figure 3 cancels much of the positive skewness of the lognormal density leading to a close approximation of a Gaussian density.

[6] An explanation for the latter result stems from shareholders' willingness to gamble as bankruptcy becomes more likely.

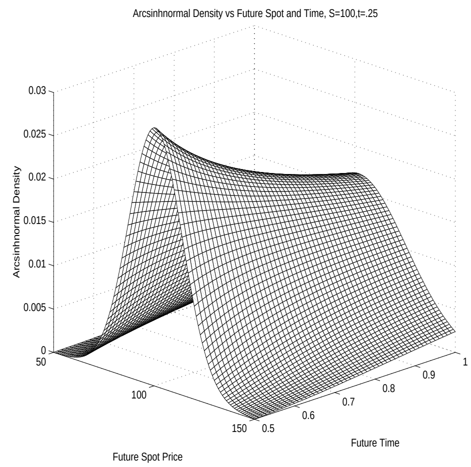

Figure 4: The Arcsinhnormal Probability Density Function

Integrating the call's payoff against this density yields the following pricing formula:

$$\begin{aligned}
C(S,t) &= \frac{e^{-q(M-t)}}{2}\left(S+\sqrt{S^2+\beta^2 e^{-2\mu(T-t)}}\right)[N(d_+ + \alpha\sqrt{M-t}) + N(d_- - \alpha\sqrt{M-t})] \\
&- \frac{\beta^2 e^{-q(M-t)}}{2e^{2\mu(T-t)}}\frac{1}{S+\sqrt{S^2+\beta^2 e^{-2\mu(T-t)}}}[N(d_+ - \alpha\sqrt{M-t}) + N(d_- + \alpha\sqrt{M-t})] \\
&- Ke^{-r(M-t)}[N(d_+) - N(d_-)],
\end{aligned}$$

where:

$$d_\pm \equiv \frac{\pm\sinh^{-1}\left(\frac{S}{\beta}e^{\mu(T-t)}\right) - \sinh^{-1}\left(\frac{K}{\beta}e^{\mu(T-t)}\right)}{\alpha\sqrt{M-t}}.$$

Figure 5 graphs the call value and time values of this model against the corresponding values in the Black-Scholes model with the same at-the-money implied volatility. The negative skewness apparent in the volatility surface and arcsinhnormal density function is manifested in higher out-of-the-money put prices and lower out-of-the-money call prices. Figure 6 plots the arcsinhnormal call value against the current stock price and time.

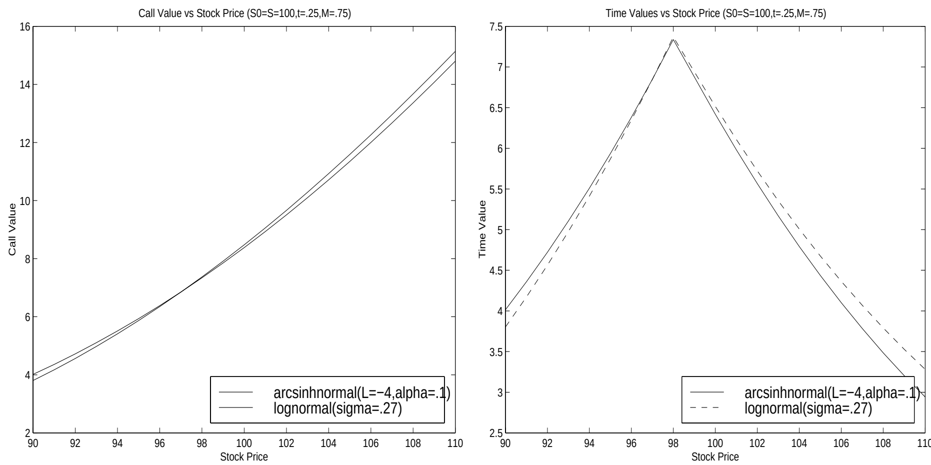

Figure 5: The Arcsinhnormal Call Value and Time Value vs. Black-Scholes

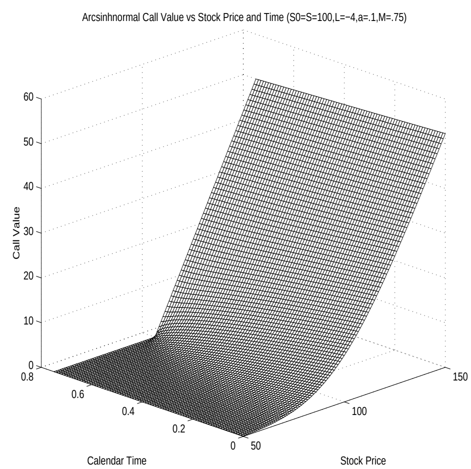

Figure 6: The Arcsinhnormal Call Valuation Function

V-B Stock Price is Depressed Cubic

A depressed cubic is a cubic polynomial with the constant and squared term suppressed. For example, the first two terms in the power series expansion of $\sinh(x)$ are given by the depressed cubic $x + \frac{x^3}{6}$. Historically, the first polynomial to be explicitly inverted after the quadratic is the depressed cubic, by Del Ferro around 1500[7].

Suppose that the final payoff $\phi(w)$ is described by the following depressed cubic:

$$\phi(w) = \begin{cases} \beta\left[(w-L)^3 + 3(\gamma - T)(w-L)\right] e^{(r-q)T} & \text{if } w > L; \\ 0 & \text{if } w < L, \end{cases} \tag{35}$$

for $\beta > 0$, $\gamma > T$. Figure 7 graphs this payoff function.

[7]However, the first publication of the solution was by Cardano in 1545, who extended the result to general cubics and to quartics (see [16]).

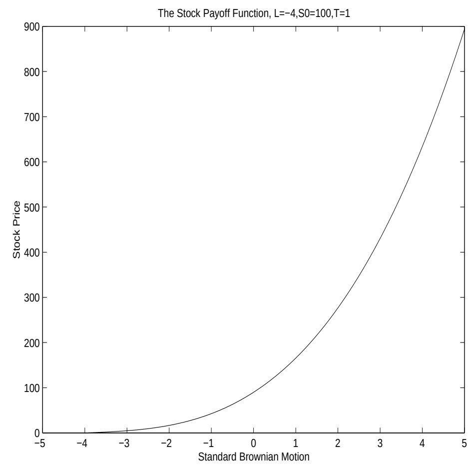

Figure 7: The Stock Payoff Function for the Depressed Cubic (L=-4, $\gamma - T = 1$)

The solution of (6) subject to (8), (9), and (35) is:

$$s(w,t) = \beta\left[(w-L)^3 + 3(\gamma - t)(w-L)\right]e^{(r-q)t}, \quad w > L, t \in [0,T]. \tag{36}$$

Setting $s(0,0) = S_0$ expresses the scaling constant in terms of the initial stock price:

$$\beta = -\frac{S_0}{L^3 + 3\gamma L}, \tag{37}$$

where recall $L < 0$. Figure 8 graphs this stock pricing function against the driving SBM and time. For each time, the stock price is an increasing convex function of the SBM.

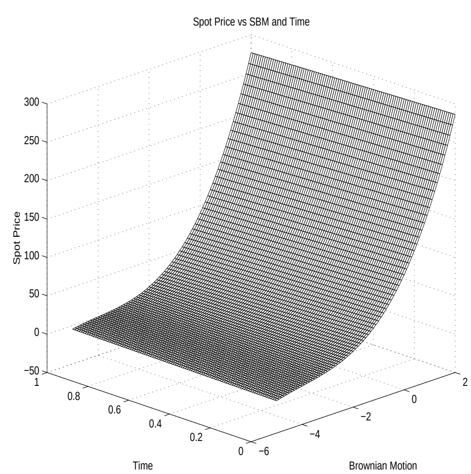

Figure 8: The Stock Pricing Function (L=-4, $\gamma - T = 1$)

To express the local volatility in terms of the stock price, we need to invert the depressed cubic (36).

Using Cardano's formula (see [16], pgs.8-10):

$$w(S,t) = L + \triangle(S,t), \qquad S \geq 0, t \in [0,T], \tag{38}$$

where:

$$\triangle(S,t) \equiv \rho_+^{1/3}(S,t) - \rho_-^{1/3}(S,t), \qquad S \geq 0, t \in [0,T], \tag{39}$$

and:

$$\rho_\pm(S,t) \equiv \pm\frac{S}{2\beta e^{(r-q)t}} + \sqrt{\left(\frac{S}{2\beta e^{(r-q)t}}\right)^2 + (\gamma - t)^3}. \tag{40}$$

Differentiating (36) w.r.t. w implies that:

$$s_w(w,t) = 3\beta\left[(w-L)^2 + \gamma - t\right] e^{(r-q)t}, \quad w \geq L, t \in [0,T]. \tag{41}$$

From (7), substituting (38) in (41) determines the absolute volatility:

$$a(S,t) = 3\beta[\triangle^2(S,t) + \gamma - t]e^{(r-q)t}, \qquad S > 0, t \in [0,\tau), \tag{42}$$

where β is given in (37). Dividing by the stock price yields the local volatility surface:

$$\sigma(S,t) \equiv \frac{a(S,t)}{S} = \frac{3}{S}\beta[\triangle^2(S,t) + \gamma - t]e^{(r-q)t}, \qquad S > 0, t \in [0,\tau). \tag{43}$$

Figure 9 graphs this local volatility surface.

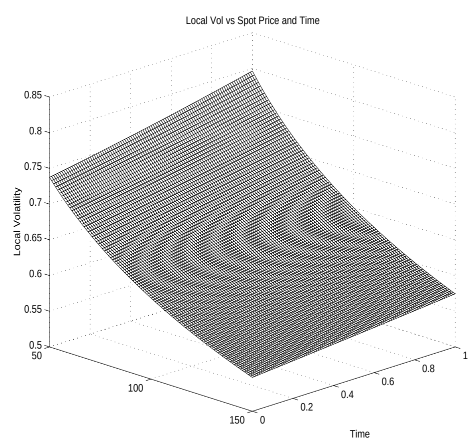

Figure 9: The Local Volatility Surface

To understand the behavior of this local volatility function, note that the stock pricing function again behaves linearly in w for w near L and is dominated by the w^3 term for w large. Thus, the volatility smile is approximately hyperbolic in S ("normal volatility") for S near zero, while it is asymptoting to zero like $\frac{1}{S^{1/3}}$ for S high. Thus, as S increases from 0 to ∞, the volatility smile again slopes downward in a convex fashion. The more negative is L, the smaller is β from (37), and the smaller is volatility from (43). Thus, the overall level of the volatility is governed by the likelihood of bankruptcy. The function $\triangle(S,t)$ in (43) has little time dependence and so the other free parameter γ controls the shape of the volatility function in the time variable. For interest rates above dividend yields, $\sigma(S,t)$ is increasing in t and the higher is γ, the lower is the slope in the time variable.

Substituting (38) in (21) gives the risk-neutral density:

$$q(Z,M;S,t) = \frac{1}{a(Z,M)\sqrt{2\pi(M-t)}}\left\{\exp\left\{-\frac{1}{2}\left[\frac{\triangle(Z,M)-\triangle(S,t)}{\sqrt{M-t}}\right]^2\right\} - \exp\left\{-\frac{1}{2}\left[\frac{\triangle(Z,M)+\triangle(S,t)}{\sqrt{M-t}}\right]^2\right\}\right\} \tag{44}$$

where $a(\cdot,\cdot)$ is given in (42) and $\triangle(\cdot,\cdot)$ is defined in (39). Figure 10 graphs this density (termed the depressed cube root density) against the future spot price and time. The downward sloping volatility surface graphed in Figure 9 again cancels much of the positive skewness of the lognormal density leading to a close approximation of a Gaussian density.

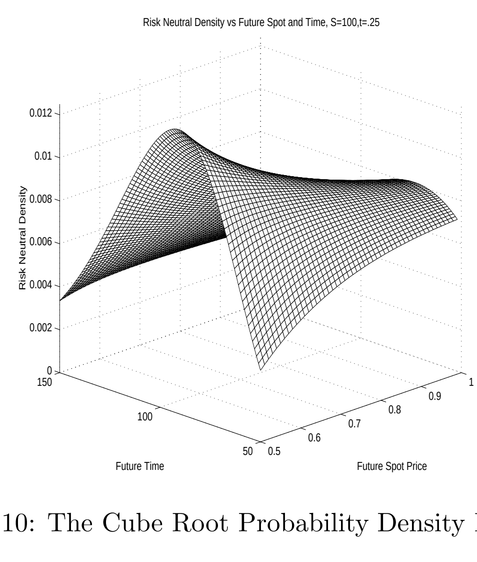

Figure 10: The Cube Root Probability Density Function

Integrating the call's payoff against this density yields the following valuation formula:

$$\begin{aligned} C(S,t) &= \beta e^{rt-qM}\sqrt{M-t}\left\{\left[3\gamma-2t-M+[\triangle(S,t)-\triangle(K,M)]^2+3\triangle(S,t)\triangle(K,M)\right]N'(d_+)\right. \\ &\quad \left. -\left[3\gamma-2t-M+[\triangle(S,t)+\triangle(K,M)]^2-3\triangle(S,t)\triangle(K,M)\right]N'(d_-)\right\} \end{aligned}$$

$$+Se^{-q(M-t)}[N(d_+) + N(d_-)] - Ke^{-r(M-t)}[N(d_+) - N(d_-)],$$

where:

$$d_\pm \equiv \frac{\pm\triangle(S,t) - \triangle(K,M)}{\sqrt{M-t}}.$$

Figure 11 graphs the call value and time values of this model against the corresponding values in the Black-Scholes model with the same at-the-money implied volatility. The negative skewness apparent in the volatility surface and density function is once again manifested in higher out-of-the-money put prices and lower out-of-the-money call prices. Figure 12 plots the call value against the current stock price and time.

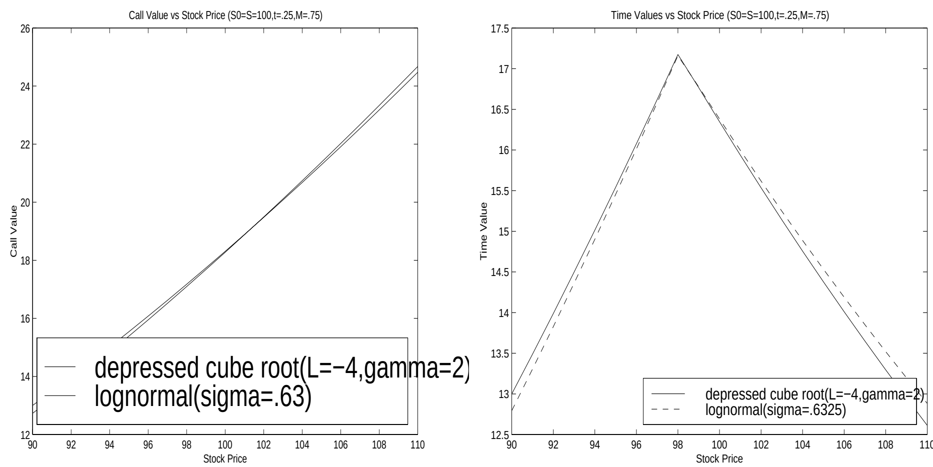

Figure 11: The Depressed Cubic Call Value and Time Value vs. Black-Scholes

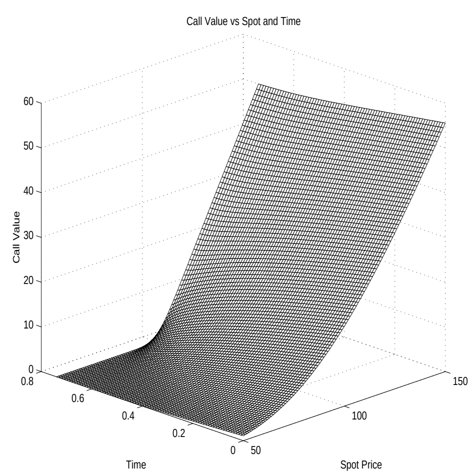

Figure 12: The Call Valuation Function in the Depressed Cubic Model

V-C Stock Price is Depressed Cubic of Hyperbolic Sine

Our final example synthesizes the previous two. Suppose that the final payoff $\phi(w)$ is described by the following depressed cubic:

$$\phi(w) = \begin{cases} \beta\left[\sinh^3[\alpha(w-L)] + 3\gamma\sinh[\alpha(w-L)]\right] & \text{if } w > L; \\ 0 & \text{if } w < L, \end{cases} \tag{45}$$

for $\beta > 0$, $\gamma > 0$. Figure 13 graphs this payoff function. Note that the payoff $b\{\sinh[\alpha(w-L)]\}^+$ used in

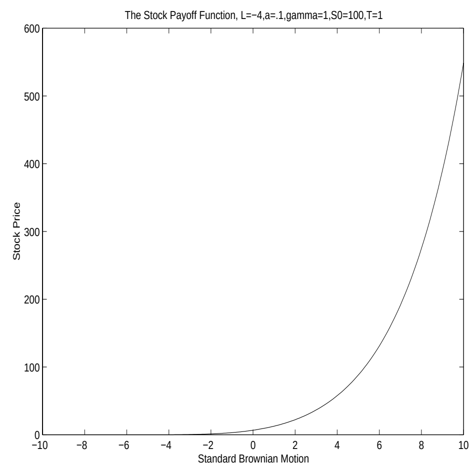

Figure 13: The Stock Payoff Function for the Depressed Cubic in Hyperbolic Sine(L=-4, $\alpha = .1$, $\gamma = 1$)

the first example can be obtained from (45) by setting $\gamma = \frac{b}{3\beta}$ and letting $\beta \downarrow 0$. Less obviously, the payoff in the second example can also be obtained from (45) by applying the linear operator $\frac{1}{3\gamma+6}\mathcal{D}_\alpha^3 + \mathcal{D}_\alpha$ where $\mathcal{D}_\alpha$ denotes differentiation w.r.t. α and letting $\alpha \downarrow 0$.

The solution of (6) subject to (8), (9), and (45) is:

$$s(w,t) = \beta e^{-\mu(T-t)}\left\{\sinh^3[\alpha(w-L)] + 3p(t)\sinh[\alpha(w-L)]\right\}, \quad t \in [0,T], w > L, \tag{46}$$

where $\mu \equiv r - q - \frac{9}{2}\alpha^2$ and $p(t) \equiv \frac{1-(1-4\gamma)e^{-4\alpha^2(T-t)}}{4}$. Setting $s(0,0) = S_0$ expresses the scaling constant in terms of the initial stock price:

$$\beta = \frac{S_0 e^{\mu T}}{\sinh^3[-\alpha L] + 3p(0)\sinh[-\alpha L]}, \tag{47}$$

where recall $L < 0$. Figure 14 graphs this stock pricing function against the driving SBM and time. For each time, the stock price is an increasing convex function of the SBM.

To express the local volatility in terms of the stock price, we need to solve the depressed cubic (46) in

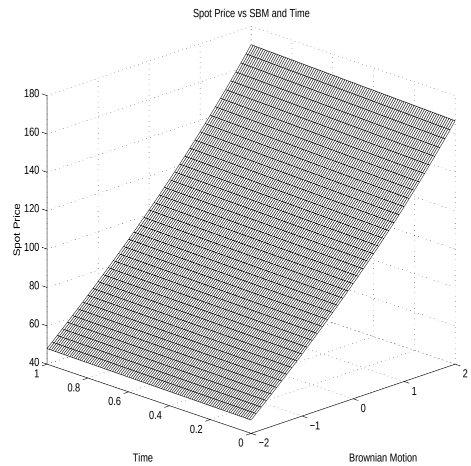

Figure 14: The Stock Pricing Function (L=-4, $\alpha = .1$, $\gamma = 1$)

$\sinh[\alpha(w - L)]$ for w. Again, using Cardano's formula:

$$w(S,t) = L + \frac{\sinh^{-1}[\triangle(S,t)]}{\alpha} \qquad S \geq 0, t \in (0,T), \tag{48}$$

where again:

$$\triangle(S,t) \equiv \rho_+^{1/3}(S,t) - \rho_-^{1/3}(S,t) \tag{49}$$

but now:

$$\rho_\pm(S,t) \equiv \pm\frac{Se^{\mu(T-t)}}{2\beta} + \sqrt{\left(\frac{Se^{\mu(T-t)}}{2\beta}\right)^2 + p^3(t)}. \tag{50}$$

Differentiating w.r.t. S implies that:

$$\frac{\partial w}{\partial S}(S,t) = \frac{e^{\mu(T-t)}}{3\alpha\beta\sqrt{1+\triangle^2(S,t)}}\frac{\rho_+^{1/3}(S,t)+\rho_-^{1/3}(S,t)}{\rho_+(S,t)+\rho_-(S,t)}, \quad S > 0, t \in [0,\tau]. \tag{51}$$

From (7), reciprocating gives the absolute volatility:

$$a(S,t) = \frac{3\alpha\beta\sqrt{1+\triangle^2(S,t)}}{e^{\mu(T-t)}}\frac{\rho_+(S,t)+\rho_-(S,t)}{\rho_+^{1/3}(S,t)+\rho_-^{1/3}(S,t)}, \quad S > 0, t \in [0,\tau), \tag{52}$$

where β is given in (47) and $\triangle(S,t)$ is given in (49). Dividing by the stock price yields the local volatility surface:

$$\sigma(S,t) \equiv \frac{a(S,t)}{S} = \frac{3\alpha\beta\sqrt{1+\triangle^2(S,t)}}{Se^{\mu(T-t)}}\frac{\rho_+(S,t)+\rho_-(S,t)}{\rho_+^{1/3}(S,t)+\rho_-^{1/3}(S,t)}, \quad S > 0, t \in [0,\tau). \tag{53}$$

Figure 15 graphs this local volatility surface.

To understand the behavior of this local volatility function, note that the stock pricing function again behaves linearly in w for w near L and thus is approximately hyperbolic in S ("normal volatility") for S

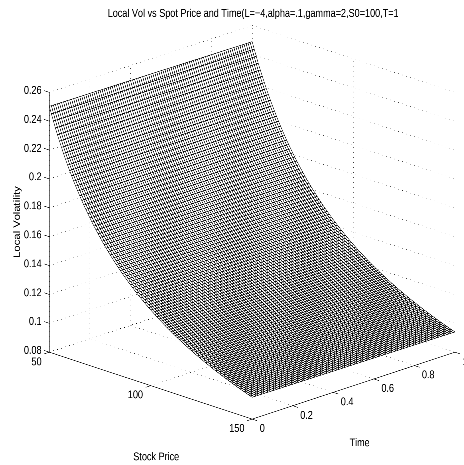

Figure 15: The Local Volatility Surface

near zero. The behavior for large stock prices is quite complicated, but graphical analysis indicates that the volatility reaches a minimum and slopes upwards for sufficiently large S. In contrast to the first two models, this third model has three free parameters α, L, and γ, allowing a wide variety of shapes.

Substituting (48) in (21) gives the risk-neutral density:

$$q(Z,M;S,t) = \frac{1}{a(Z,M)\sqrt{2\pi(M-t)}} \cdot \left\{ \exp\left\{ -\frac{1}{2}\left[\frac{\sinh^{-1}[\triangle(Z,M)] - \sinh^{-1}[\triangle(S,t)]}{\alpha\sqrt{M-t}}\right]^2 \right\} - \exp\left\{ -\frac{1}{2}\left[\frac{\sinh^{-1}[\triangle(Z,M)] + \sinh^{-1}[\triangle(S,t)]}{\alpha\sqrt{M-t}}\right]^2 \right\} \right\}, \quad (54)$$

where $a(\cdot,\cdot)$ is given in (42) and $\triangle(\cdot,\cdot)$ is defined in (49). Figure 16 graphs this density (termed the depressed cube root arcsinh density) against the future spot price and time. The downward sloping volatility surface graphed in Figure 15 again cancels much of the positive skewness of the lognormal density leading to a close approximation of a Gaussian density.

Integrating the call's payoff against this density yields the following valuation formula:

$$C(S,t) =$$

$$\beta \left\{ \frac{e^{-\mu(T-t)-q(M-t)}}{8} \left[\triangle(S,t) + \sqrt{\triangle^2(S,t)+1}\right]^3 \left[N(d_+ + 3\alpha\sqrt{M-t}) + N(d_- - 3\alpha\sqrt{M-t})\right] \right.$$

$$-3[1-4p(M)]\frac{e^{-(r-\frac{\alpha^2}{2})(M-t)-\mu(T-M)}}{8}\left[\triangle(S,t) + \sqrt{\triangle^2(S,t)+1}\right]\left[N(d_+ + \alpha\sqrt{M-t}) + N(d_- - \alpha\sqrt{M-t})\right]$$

$$+3[1-4p(M)]\frac{e^{-(r-\frac{\alpha^2}{2})(M-t)-\mu(T-M)}}{8}\frac{1}{\triangle(S,t) + \sqrt{\triangle^2(S,t)+1}}\left[N(d_+ - \alpha\sqrt{M-t}) + N(d_- + \alpha\sqrt{M-t})\right]$$

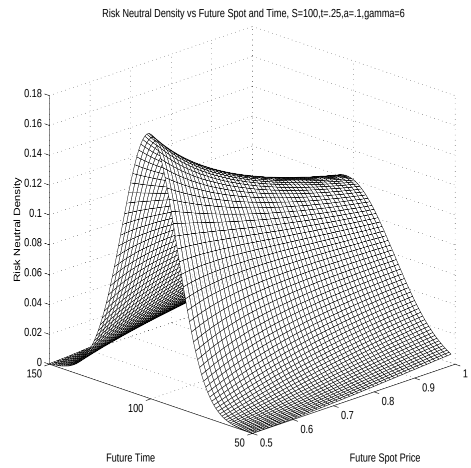

Figure 16: The Cube Root ArcSinh Probability Density Function

$$-\frac{e^{-\mu(T-t)-q(M-t)}}{8}\frac{1}{\left[\triangle(S,t)+\sqrt{\triangle^2(S,t)+1}\right]^3}\left[N(d_+-3\alpha\sqrt{M-t})+N(d_-+3\alpha\sqrt{M-t})\right]\Bigg\}$$

$$-Ke^{-r(M-t)}[N(d_+)-N(d_-)],$$

where $d_\pm \equiv \frac{\pm\sinh^{-1}[\triangle(S,t)]-\sinh^{-1}[\triangle(K,M)]}{\alpha\sqrt{M-t}}$. Figure 17 graphs the call value and time values of this model against the corresponding values in the Black-Scholes model with the same at-the-money implied volatility. The negative skewness apparent in the volatility surface and density function is once again manifested in higher out-of-the-money put prices and lower out-of-the-money call prices.

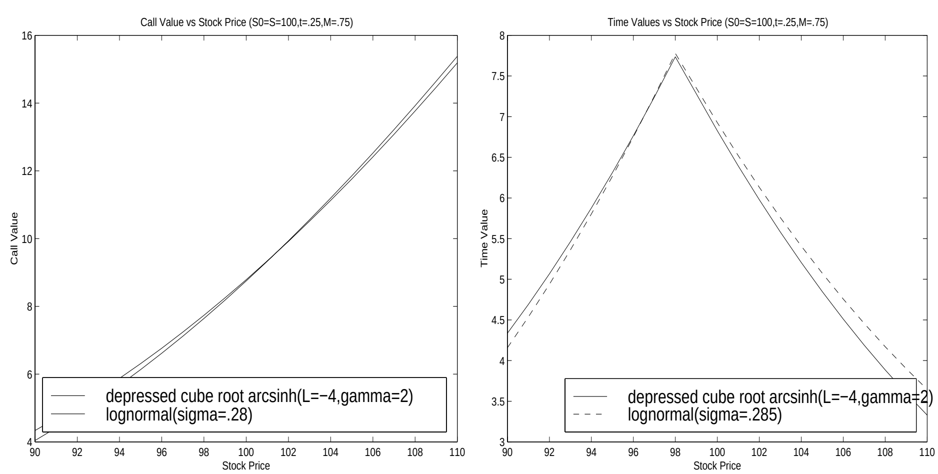

Figure 17: The Depressed Cubic Arcsinh Call Value and Time Value vs. Black-Scholes

Figure 18 plots the call value against the current stock price and time.

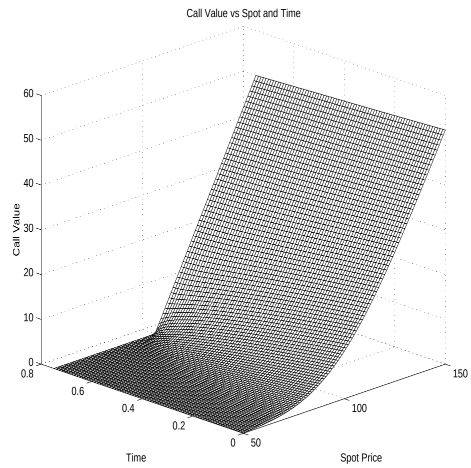

Figure 18: The Call Valuation Function in the Depressed Cubic in Hyperbolic Sine Model

VI Summary and Extensions

Assuming that the underlying local volatility is a function of stock price and time, we developed an approach for generating closed form solutions for option values when the volatility solves a certain nonlinear partial differential equation. The approach involves specifying a stock payoff function which has the key property that the stock pricing function can be explicitly calculated and inverted. We illustrated our results with three examples.

All three examples can be generalized. The first one involving the difference of two exponentials can be generalized to a payoff of the form:

$$\phi_1(w) = \frac{P_4(e^{\alpha(w-L)})}{e^{n\alpha(w-L)}}, \tag{55}$$

where P_4 is a polynomial of at most degree 4, α is a nonnegative constant, L is a negative constant, and $n = 0, 1, 2, 3,$ or 4. The coefficients of the polynomial should be chosen so that the payoff and the stock pricing function are monotonic. To invert the stock pricing function, the quartic root formula would be used. Similarly, the second example, which involved the depressed cubic, can be generalized to a payoff of the form:

$$\phi_2(w) = \frac{P_4(w-L)}{(w-L)^n}, \tag{56}$$

where again the payoff and stock pricing function should be monotonic. Galois theory can be used to generate polynomials in $e^{\alpha(w-L)}$ or in $(w - L)$, which are of higher order than quartic and whose roots may be computed. The third example involves composing two explicitly invertible maps, and nothing prevents composing more complex functions such as ϕ_1 and ϕ_2 or having multiple compositions.

Further extensions to this analysis would include developing pricing formulas for *path-dependent* options such as American or barrier options. For example, to extend our first illustration to down barrier options, note from (25) that if $\alpha = \sqrt{2(r-q)}$, then $\mu = 0$ and so the stock price process in (27) becomes:

$$S_t = S_0 \text{csch}(-\alpha L) \sinh[\alpha(W_t - L)], \qquad t \in [0, T]. \tag{57}$$

Since S_t hits some lower barrier H when the SBM hits a constant barrier, the joint density for the minimum stock price and the final stock price is easily determined analytically. To deal with up barrier options, the transition density for SBM absorbing at both lower and upper barriers would be used. To analytically approximate American options, the randomization technique discussed in Carr[5] could be used.

For stock or option payoff functions which are intractable analytically, the present analysis has implications for numerical analysis. If the volatility function satisfies the nonlinear p.d.e. (2), then a change of independent variables simplifies the standard valuation p.d.e. so that the underlying has unit diffusion and zero drift. European option values can therefore be obtained by quadrature, which can be applied recursively for Bermudan or compound options. American values can be obtained by the standard finite difference scheme for the heat equation (with a time and state dependent potential term). Alternatively, Monte Carlo simulation can be enhanced by simulating (absorbing) SBM rather than a complicated stock price process.

A generalization of the present analysis would change the clock as well as the scale. A deterministic time change allows the driver to be an Ornstein Uhlenbeck process, since it is a deterministic time and scale change of SBM. A deterministic time change would also bring the time-dependent Black Scholes analysis into the framework. The generalization of the representation formula $W_t = w(S_t, t; S_0)$, appearing in (64) of Appendix 1, is:

$$W_{t'} = w(S_t, t), \tag{58}$$

where:

$$t' = \theta(t), \tag{59}$$

with $\theta(t)$ an increasing differentiable function satisfying $\theta(0) = 0$ and $\theta'(t) > 0$ for $t \in [0, T]$.

Then the generalization of (3) is readily obtained:

$$w(S,t) = \sqrt{\theta'(t)} \int_{S_0}^{S} \frac{1}{a(Z,t)} dZ + \int_0^t \sqrt{\theta'(s)} \left[\frac{1}{2} \frac{\partial a(S,s)}{\partial S} \bigg|_{S=S_0} - \frac{(r-\delta)S_0}{a(S_0,s)} \right] ds, \quad S > 0, t \in [0,T). \tag{60}$$

The generalization of the nonlinear p.d.e. (2) for the absolute volatility function $a(S,t)$ is:

$$\frac{a^2(S,t)}{2} \frac{\partial^2 a(S,t)}{\partial S^2} + (r-q)S \frac{\partial a}{\partial S}(S,t) + \frac{\partial a}{\partial t}(S,t) = \left[r - q + \frac{\theta''(t)}{2\theta'(t)} \right] a(S,t), \quad S > 0, t \in (0,T). \tag{61}$$

The generalization of the p.d.e. (6) for the stock pricing function $s(w,t)$ is:

$$\frac{\partial s}{\partial t}(w,t) + \frac{\theta'(t)}{2} \frac{\partial^2 s}{\partial w^2}(w,t) = (r-\delta)s(w,t), \quad w \geq L(t), t \in [0,T], \tag{62}$$

while the link (7) between the absolute volatility $a(S,t)$ and the stock pricing function $s(x,t)$ generalizes to:

$$a(s(w,t),t) = \frac{\partial s}{\partial w}(w,t) \sqrt{\theta'(t)}, \quad w \geq L(t), t \in [0,T]. \tag{63}$$

A powerful alternative to a time change is a change of probability measure. To retain the path-independence property, one would quanto the Brownian derivative security into a second derivative security written on the stock. It can be shown that for *any* pair of time and space-dependent drift and volatility functions, a change of scale and measure can be developed which reduces the drift to zero and the absolute volatility to unity. Since the rate used to discount the stock payoff function will in general be state-dependent, the generalization of the linear p.d.e. (6) for the stock pricing function will in general be a nonlinear reaction diffusion equation.

In this paper, all three examples took the stock payoff to be an odd function of absorbing Brownian motion, which was equivalent on the relevant domain to the same function of standard Brownian motion. Drivers other than standard Brownian motion could also be considered. For example, one could consider the stock as an even function (eg. square, fourth power, or hyperbolic cosine) of reflecting Brownian motion, or more generally of a Bessel process. Other functions of multi-dimensional SBM can also be considered instead of the radius. Alternatively, the stock price process could depend on the driver's path statistics such as the extrema or local time. Stochastic time changes of standard Brownian motion can make the driver a pure jump Levy process. In the interests of brevity, these extensions are best left for future research.

Appendix 1: Necessity of Fundamental PDE for Path Independence

This appendix proves that the necessity of the PDE (2) in order that the standard Brownian motion be a function of the stock price and time. Let $w(S,t)$ denote a $C^{2,1}$ function defined on the domain $S>0$ and $t\in[0,T]$ which for each t, maps the stock price S to the SBM W, i.e.:

$$W_t = w(S_t, t). \tag{64}$$

Since W_t is SBM, a necessary condition on $w(S,t)$ is that $W_0 = 0$, which implies:

$$w(S_0, 0) = 0. \tag{65}$$

We further require that for each t, w is monotonically increasing in S and that the map be onto a region $[L(t),\infty)$, where $L(t)\in[-\infty,0)$ is some lower bound on the realizations of W_t. Applying Itô's lemma to (64) yields:

$$dW_t = \left[\frac{\partial w}{\partial t}(S_t,t) + (r-q)S_t\frac{\partial w}{\partial S}(S_t,t) + \frac{a^2(S_t,t)}{2}\frac{\partial^2 w}{\partial S^2}(S_t,t)\right]dt + a(S_t,t)\frac{\partial w}{\partial S}(S_t,t)dW_t. \tag{66}$$

Since W_t is SBM, a necessary condition on w and a is that the diffusion coefficient in (66) be equal to one:

$$\frac{\partial w}{\partial S}(S,t)a(S,t) = 1, \tag{67}$$

or equivalently:

$$\frac{\partial w}{\partial S}(S,t) = \frac{1}{a(S,t)}. \tag{68}$$

Integrating yields:

$$w(S,t;S_0) = \int_{S_0}^{S}\frac{1}{a(Z,t)}dZ + w(S_0,t), \tag{69}$$

where for $b>a$, $\int_b^a h(x)dx \equiv -\int_a^b h(x)dx$ and recall $w(S_0,0)=0$.

Differentiating (68) w.r.t. S gives:

$$\frac{\partial^2 w}{\partial S^2}(S,t) = -\frac{1}{a^2(S,t)}\frac{\partial a(S,t)}{\partial S}, \tag{70}$$

and differentiating (69) w.r.t. t gives:

$$\frac{\partial w}{\partial t}(S,t) = -\int_{S_0}^{S}\frac{1}{a^2(Z,t)}\frac{\partial a}{\partial t}(Z,t)dZ + \frac{\partial w}{\partial t}(S_0,t). \tag{71}$$

Since W_t is an SBM, a further necessary condition on w and a is that the drift coefficient in (66) be zero:

$$\left[\frac{\partial w}{\partial t}(S,t) + (r-q)S\frac{\partial w}{\partial S}(S,t) + \frac{a^2(S,t)}{2}\frac{\partial^2 w}{\partial S^2}(S,t)\right] = 0. \tag{72}$$

Substituting (71), (68), and (70) in (72) and simplifying gives:

$$-\int_{S_0}^{S}\frac{1}{a^2(Z,t)}\frac{\partial a}{\partial t}(Z,t)dZ + \frac{\partial w}{\partial t}(S_0,t) + (r-q)S\frac{1}{a(S,t)} - \frac{1}{2}\frac{\partial a(S,t)}{\partial S} = 0, \tag{73}$$

Since this equation holds for all S, in particular it holds at $S = S_0$, which in turn yields:

$$\frac{\partial w}{\partial t}(S_0,t) = \frac{1}{2}\frac{\partial a(S,t)}{\partial S}\bigg|_{S=S_0} - (r-q)S_0\frac{1}{a(S_0,t)}.$$

Integrating w.r.t. time and imposing (65):

$$w(S_0,t) = \int_0^t \left[\frac{1}{2}\frac{\partial a(S,s)}{\partial S}\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,s)}\right] ds. \tag{74}$$

Note from (72), (69) and (74) that $W_t = w(S_t,t)$ is the forward price at t of a claim with final payment:

$$w(S_T,T) = \int_{S_0}^{S_T}\frac{1}{a(Z,T)}dZ + \int_0^T \left[\frac{1}{2}\frac{\partial a(S,t)}{\partial S}\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,t)}\right] dt.$$

at T.

Differentiating (73) w.r.t. S gives:

$$-\frac{1}{a^2(S,t)}\frac{\partial a}{\partial t}(S,t) + (r-q)\frac{1}{a(S,t)} - (r-q)S\frac{1}{a^2(S,t)}\frac{\partial a}{\partial S}(S,t) - \frac{1}{2}\frac{\partial^2 a(S,t)}{\partial S^2} = 0.$$

Multiplying by $-a^2(S,t)$ produces the desired fundamental p.d.e. (2) for the absolute volatility:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2} + (r-q)S\frac{\partial a}{\partial S}(S,t) + \frac{\partial a}{\partial t}(S,t) = (r-q)a(S,t), \quad S > 0, t \in [0,T]. \tag{75}$$

Appendix 2: Sufficiency of Fundamental PDE for Path Independence

This appendix proves that if a positive function $a(S,t)$ satisfies the nonlinear PDE (2), then there exists a function $w(S,t)$ increasing in S, such that the process:

$$B_t \equiv w(S_t,t), \qquad t \in [0,\tau], \tag{76}$$

is the driving SBM, i.e. $B_t = W_t$ for all $t \in [0,\tau]$. This function is given by:

$$w(S,t) = \int_{S_0}^{S} \frac{1}{a(Z,t)} dZ + \int_0^t \left[\frac{1}{2} \frac{\partial}{\partial S}[a(S,s)]\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,s)} \right] ds, \quad S>0, t \in [0,T']. \tag{77}$$

To prove this result, Itô's lemma applied to (76) implies that for $t \in [0,\tau]$:

$$dB_t = \left[\frac{\sigma^2(S_t,t)}{2} \frac{\partial^2 w}{\partial S^2}(S_t,t) + (r-q)S_t \frac{\partial w}{\partial S}(S_t,t) + \frac{\partial w}{\partial t}(S_t,t) \right] dt + \frac{\partial w}{\partial S}(S_t,t)a(S_t,t)dW_t. \tag{78}$$

For B to be SBM, we need to show that its drift is zero:

$$\frac{a^2(S,t)}{2} \frac{\partial^2 w}{\partial S^2}(S,t) + (r-q)S \frac{\partial w}{\partial S}(S,t) + \frac{\partial w}{\partial t}(S,t) = 0, \quad S>0, t \in [0,\tau], \tag{79}$$

that its volatility is unity:

$$\frac{\partial w}{\partial S}(S,t)a(S,t) = 1, \quad S>0, t \in [0,\tau], \tag{80}$$

and that its initial level is zero:

$$B_0 = s(S_0,0) = 0. \tag{81}$$

Differentiating (77) w.r.t. S:

$$\frac{\partial w}{\partial S}(S,t) = \frac{1}{a(S,t)}, \quad S>0, t \in [0,\tau], \tag{82}$$

so (80) holds. Furthermore, since $a(S,t)$ is assumed positive, the function w is increasing in S. Evaluating (77) at $(S,t) = (S_0,0)$ implies (81) also holds. To show that (79) also holds, divide (2) by $a^2(S,t)$:

$$\frac{1}{2} \frac{\partial^2 a}{\partial S^2}(S,t) + \frac{(r-q)S}{a^2(S,t)} \frac{\partial a}{\partial S}(S,t) - \frac{r-q}{a(S,t)} + \frac{1}{a^2(S,t)} \frac{\partial a}{\partial t}(S,t) = 0, \quad S \geq 0, t \in [0,\tau). \tag{83}$$

Integrating both sides w.r.t. S implies:

$$\frac{1}{2} \frac{\partial a}{\partial Z}(Z,t)\bigg|_{Z=S_0}^{Z=S} - \frac{(r-q)Z}{a(Z,t)}\bigg|_{Z=S_0}^{Z=S} + \int_{S_0}^{S} \frac{1}{a^2(Z,t)} \frac{\partial a}{\partial t}(Z,t)dZ = 0, \quad S \geq 0, t \in [0,\tau), \tag{84}$$

where the constant of integration has been set to zero. Multiplying through by -1 implies that for $S \geq 0, t \in [0, \tau)$:

$$-\frac{a^2(S,t)}{2} \frac{\frac{\partial a}{\partial S}(S,t)}{a^2(S,t)} + \frac{(r-q)S}{a(S,t)} + \frac{1}{2}\frac{\partial}{\partial S}[a(S,t)]\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,t)} - \int_{S_0}^{S} \frac{1}{a^2(Z,t)} \frac{\partial a}{\partial t}(Z,t)dZ = 0. \tag{85}$$

Differentiating (82) w.r.t. S:

$$\frac{\partial^2 w}{\partial S^2}(S,t) = -\frac{\frac{\partial a}{\partial S}(S,t)}{a^2(S,t)}, \quad S > 0, t \in [0,\tau], \tag{86}$$

Differentiating (77) w.r.t. t:

$$\frac{\partial w}{\partial t}(S,t) = \frac{1}{2}\frac{\partial a}{\partial S}(S,t)\bigg|_{S=S_0} - \frac{(r-q)S_0}{a(S_0,t)} - \int_{S_0}^{S} \frac{1}{a^2(Z,t)} \frac{\partial a}{\partial t}(Z,t)dZ, \quad S > 0, t \in [0,\tau]. \tag{87}$$

Substituting (82),(86), and (87) in (85) gives (79).

Appendix 3: Uniqueness of Solutions for Fundamental PDE

In this appendix, we provide a uniqueness result for the solutions of (75) governing the absolute volatility.

Theorem: Let $a : [0, +\infty) \times [0, T] \to \Re$ be a solution of:

$$\frac{a^2(S,t)}{2}\frac{\partial^2 a(S,t)}{\partial S^2} + (r-q)S\frac{\partial a}{\partial S}(S,t) + \frac{\partial a}{\partial t}(S,t) = (r-q)a(S,t), \tag{88}$$

with:

$$a(0,t) = 0, \;\; t \in [0,T] \quad \text{and} \quad a(S,T) = g(S), \tag{89}$$

with $g \in C^2(0,+\infty)$, $g(0) = 0$ and $g(x) \sim 0(x)$ for x large. Then a is a unique solution of (88) and (89) in the class of functions satisfying $a(S,t) \sim 0(S)$ for S large and $|(a^2(S,t))_{ss}| \leq C$ for $(S,t) \in [0,+\infty) \times [0,T]$ and some given constant C.

Proof: First, we observe that if we let $\widetilde{a}(x,t) \equiv e^{(r-q)(T-t)}a(S,t)$ and $x \equiv Se^{(r-q)(T-t)}$, then $\widetilde{a}(x,t)$ solves:

$$\frac{\widetilde{a}^2(x,t)}{2}\frac{\partial^2 \widetilde{a}}{\partial x^2}(x,t) + \frac{\partial \widetilde{a}}{\partial t}(x,t) = 0, \tag{90}$$

and also satisfies the conditions (89).

In view of the above observation, it suffices to establish the uniqueness result for the solutions of (90) instead of (88). To this end, we define $F : [0,+\infty) \times [0,T] \to [0,+\infty)$ to be

$$F(x,t) = \widetilde{a}^2(x,t). \tag{91}$$

Direct calculations yield that F solves the quasi-linear equation:

$$F_t(x,t) + \frac{1}{2}F(x,t)F_{xx}(x,t) = F_x^2(x,t) \tag{92}$$

$$F(0,t) = 0, \;\; t \in [0,T], \text{ and } F(x,T) = g^2(x). \tag{93}$$

From the assumptions on $\widetilde{a}$, we get that $F(x,t) \sim 0(x^2)$ for x large and that $F(x,t)_{xx} \leq C$ for $(x,t) \in [0,+\infty) \times [0,T)$. Using a variation of the results of Fukuda, Ishii, and Tsutsumi[8], we get that (92) has a unique solution. Therefore, if $a_1(x,t)$ and $a_2(x,t)$ are two solutions of (88), satisfying also (89), the above uniqueness result yields that

$$a_1^2(x,t) = a_2^2(x,t). \tag{94}$$

Next, we look at the difference $G(x,t) = a_1(x,t) - a_2(x,t)$. Differentiation and use of (88) yield:

$$\begin{aligned} G_t(x,t) &= (a_1(x,t))_t - (a_2(x,t))_t \\ &= -\frac{1}{2}a_1^2(x,t)(a_1(x,t))_{xx} + \frac{1}{2}a_2^2(x,t)(a_2(x,t))_{xx} \\ &= -\frac{1}{2}a_1^2(x,t)G_{xx}(x,t) - \frac{1}{2}(a_2(x,t))_{xx}(a_1^2(x,t) - a_2^2(x,t)) \\ &= -\frac{1}{2}a_1^2(x,t)G_{xx}(x,t). \end{aligned}$$

Therefore, G solves:

$$G_t(x,t) + \frac{1}{2}a_1^2(x,t)G_{xx}(x,t) = 0 \tag{95}$$

with:

$$G(0,t) = 0 \quad \text{and} \quad G(x,T) = 0.$$

Working as above for $\widehat{G}(x,t) = a_2(x,t) - a_1(x,t)$ yields that $\widehat{G}$ solves:

$$\widehat{G}_t(x,t) = \frac{1}{2}a_2^2(x,t)\widehat{G}_{xx}(x,t) \tag{96}$$

which, in view of (94), coincides with (95). Moreover, $\widehat{G}(0,T) = 0$ and $\widehat{G}(x,T) = 0$. We can easily verify that equation (95) (or (96)) admits a comparison principle and we readily conclude that $a_1(x,t) = a_2(x,t)$ for $(x,t) \in [0,+\infty) \times [0,T]$.

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