Qualitative analysis of optimal investment strategies in log-normal markets*

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Abstract

We provide a concise study of the qualitative behavior of the optimal investment feedback policies and optimal weights, and of the local (absolute and relative) risk tolerance and risk aversion functions in a log-normal market model. We examine their spatial and temporal monotonicity, and their spatial concavity. We also examine their robustness with respect to the investor’s risk tolerance coefficient as well as their dependence on the market parameters. We establish new results and provide short alternative proofs to existing ones.

1 Introduction

We study qualitative properties of the optimal investment strategies and other related quantities of a risk averse investor who trades in a finite horizon and aims to maximize her expected utility from terminal wealth.

Trading takes place among a riskless bond and $N$ stocks whose prices are log-normally distributed. This optimal investment problem, introduced by Merton [32], is readily solved either by using duality and related martingale analysis, or PDE techniques applied to the associated Hamilton-Jacobi-Bellman equation. In the latter case, the first order conditions yield the so-called optimal feedback controls which, in turn, generate the optimal investment processes.

Despite the simplicity of both the market setting and the underlying stochastic optimization problem, as well as the popularity of the latter and its

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important extensions in the academic investment literature, relatively little is known about the qualitative behavior of the optimal strategies, except when the utility is homothetic. In this case, the optimal feedback portfolios are linear functions of the current wealth and independent of time, and thus their structure can be easily analyzed.

Our aim herein is to consider general utility functions and carry out a detailed qualitative study for the optimal feedback portfolio functions and optimal weights, as well as for the local risk aversion and risk tolerance coefficients. Some of the properties we analyze have been studied before, either in discrete or continuous time settings, but we use an alternative methodology which yields much shorter and direct proofs. Some other properties we consider are new.

A by-product of our work is the compilation of a concise study with old and new results for the optimal strategies and related quantities. We discuss the questions we consider next.

- **When are the optimal allocations and optimal weights increasing in the wealth argument?**

  For general risk preferences, this question was examined by Arrow [1] who showed that, in a single period problem with one risky stock, the optimal investment in the latter is increasing in wealth if and only if the investor’s utility exhibits decreasing absolute risk aversion (DARA), and as long as the risk premium is positive. He, also, showed that the fraction of wealth invested in the stock, known as the average propensity to invest, is decreasing in wealth if and only if the utility exhibits increasing relative risk aversion (IRRA). Since this seminal work, it has become common in the economic literature to assume that the utility exhibits DARA and IRRA; these properties are also known as the Arrow hypothesis. Similar results were later produced for discrete time models (see, among others, [24], [30], [34], [36], [40] and [41], as well as [11] and [16] and references therein).

  Borell [5] is, to our knowledge, the first who studied the above questions in a continuous time setting. The main result in [5] is that, in the log-normal model, if the investor’s risk aversion coefficient is decreasing in wealth then, this spatial monotonicity is inherited at all trading times to the absolute local risk aversion function and, in turn, to the optimal portfolios. He also showed analogous monotonicity results for the relative quantities, namely, if the investor’s relative risk aversion is decreasing (increasing) in wealth then, the local relative risk aversion function and optimal weights are also decreasing (increasing) functions at all times.

  Similar results were later obtained using a different methodology by Xia in [47].

  Herein, we also study this question and provide a much shorter proof, alternative to the ones in [5] and [47].

- **When are the optimal allocations convex (concave) in the wealth argument?**
The concavity of the optimal portfolios is related to the so-called marginal propensity to invest \((MPI)\), which is defined as the derivative of the optimal portfolio with respect to wealth. Then, \(MPI\) is decreasing (increasing) in wealth if and only if the optimal portfolio function is concave (convex) in wealth. Such properties have been analyzed in single-period models (see, among others, [12], and [11] and [16] and references therein).

We show that if the risk tolerance is a strictly concave (convex) function, then the same property is inherited to the optimal portfolio at all trading times.

- **When are the optimal allocations and optimal weights increasing (decreasing) with respect to time?**

Very little is known about this important property in continuous time settings. While it was not, naturally, an issue in single-period models, it has not received appropriate attention even in discrete time settings. To our knowledge, Gollier and Zeckhauser [15] are the only ones who have studied it in a multi-period model. They investigated the connection between horizon length/age of investors and curvature of their absolute risk aversion.

Herein, we prove similar results in the continuous-time setting. We show that if the absolute risk tolerance coefficient is a concave (convex) function of wealth, then the optimal feedback allocations and optimal weights are always increasing (decreasing) as time increases. The proof of this surprising finding is a mere direct consequence of the concavity (convexity) properties mentioned earlier. The temporal behavior of the optimal policies have also been examined in more extended continuous time model settings in [10] and [28]. Therein, however, the generality of the model did not allow for specific results as the one herein.

- **Are optimal allocations and optimal weights robust with respect to risk preferences?**

The question whether two individuals with ordered risk aversion coefficients invest in an analogous way at all trading times has been analyzed extensively in single- and multi-period settings (see, among others, [39] and [48]).

In continuous time models, such a result was established in [47] for log-normal prices. Herein, we provide a much shorter proof of this result. While the result holds for a slightly smaller class of solutions than the one in [47], it bypasses several lengthy penalization and approximation arguments, and is based on entirely different methodology. The arguments are short and direct and are, in turn, used to derive robust bounds for the optimal policies, and the local risk tolerance and risk aversion coefficients.

- **How do the optimal allocations, local risk tolerance and local risk aversion functions depend on the market parameters?**

It follows easily that all the above monotonicity, concavity and robustness results hold for both the optimal investments and the local risk tolerance and
risk aversion functions as these quantities are mere scalar multiples of each other (see (13) and (15)). This is not, however, the case for the dependence of these quantities on market coefficients, for the latter appear rather implicitly in the relevant formulae. We examine this dependence.

For the case of one stock, we find, for example, that while the optimal feedback portfolio function is always decreasing in the stock’s volatility, the analogous monotonicity of the local absolute risk tolerance depends exclusively on the spatial convexity (concavity) of the risk tolerance coefficient.

The aforementioned continuous time results in [5] and [47] were proved via martingale methods applied to the dual problem. Herein, we use throughout a different methodology which relies on properties of solutions of auxiliary linear and nonlinear partial differential equations. The non-linear equations are satisfied by the local risk tolerance function, the local risk aversion and the optimal feedback strategies. A key transformation, see (25), relates their solutions to the ones of the heat equation and its derivatives. Several properties are then established by looking at how analogous properties of the terminal data are propagated at earlier times. The robustness, the related bounds, and the dependence on market parameters are derived by applying appropriate comparison results.

One should note, however, that all these approaches have similar characteristics as they focus on the dual and not the primal problem. The PDE methodology used herein, however, allows for substantially more tractability, makes the comparison arguments more transparent and requires much shorter technical arguments. More importantly, it enables us to analyze the time behavior and the dependence of the optimal portfolios on market parameters.

A by-product of our methodology is the construction of alternative representations for the optimal portfolio and wealth processes via the underlying auxiliary harmonic functions. These stochastic representations are, naturally, equivalent to the ones derived by duality arguments but yield more direct and easier to handle formulae. Using these expressions, we readily derive the sensitivities of the optimal processes with respect to initial wealth.

The key transformation (25) is similar to a transformation first used in [35] (see (29) and (31) therein), where the so called time-monotone forward investment performance process is constructed in a portfolio model with Itô diffusions price processes. In [35] equations similar to (8) and (22) arise and, in turn, analogous stochastic representations to (26) and (28) (cf., respectively, (14) and (19), and (23) and (24) in [35]).

Despite the fact that the transformations and the equations look alike, there are fundamental differences in the two settings. Herein, we analyze the log-normal model and with terminal utility data. Therein, one looks at an Itô diffusion model with initial utility data and no terminal horizon. As a result, equations (8) and (22) for the Merton problem are well posed but the ones in [35] correspond to ill-posed problems which might not have solutions.

We stress that while the transformations we use in [35] enable us to deduce closed-form solutions in a general market model, they are not applicable to mod-
els of expected utility of terminal wealth beyond the log-normal setting. This is a direct consequence of the time monotonicity that both the value function herein and the forward investment process in [35] have.

The paper is organized as follows. In section 2 we introduce the investment model and review the classical results. In section 3 we derive the auxiliary equations and give some background results, and also provide alternative stochastic representations for the optimal processes. In section 4 we examine the monotonicity - spatial and temporal - and the concavity of the optimal portfolios. In section 5, we establish the robustness results and provide related bounds. In section 6, we examine the sensitivities of the optimal portfolios, and of the local risk tolerance and risk aversion functions with respect to market parameters, as well as the sensitivity of the optimal wealth and portfolio processes with respect to initial wealth.

2 The log-normal investment model

We brieﬂy recall the classical Merton problem ([32]), its value function and optimal policies. Trading takes place in $[0, T]$, with the horizon $T$ being arbitrary but ﬁxed.

The market environment consists of one riskless and $N$ risky securities. The risky securities are stocks and their prices are modelled as log-normal processes. Namely, for $i = 1, \ldots, N$, the price $S_i^t$, $0 \leq t \leq T$, of the $i^{th}$ risky asset satisfies

$$dS_i^t = S_i^t \left( \mu^i dt + \sum_{j=1}^{N} \sigma^{ij} dW_j^t \right),$$

with $S_0^i > 0$, for $i = 1, \ldots, N$. The process $W_t = (W_1^t, \ldots, W_N^t)$, $t \geq 0$, is a standard $N$-dimensional Brownian motion, deﬁned on a ﬁltered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, it is assumed that the underlying ﬁltration, $\mathcal{F}_t$, coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$.

The coefﬁcients $\mu^i$ and $\sigma^i = (\sigma^{1i}, \ldots, \sigma^{Ni})$, $i = 1, \ldots, N$, $t \geq 0$, are constants with values in $\mathbb{R}$ and $\mathbb{R}^N$, respectively. For brevity, we use $\sigma$ to denote the $N \times N$ matrix volatility $((\sigma^{ij})$, whose $i^{th}$ column represents the volatility $\sigma^i = (\sigma^{1i}, \ldots, \sigma^{Ni})$ of the $i^{th}$ risky asset. Alternatively, we write (1) as

$$dS_i^t = S_i^t \left( \mu^i dt + \sigma^i \cdot dW_t \right).$$

The riskless asset, the savings account, offers constant interest rate $r > 0$. We denote by $\mu$ the $N \times 1$ vector with coordinates $\mu^i$ and by $1$ the $N$-dimensional vector with every component equal to one.

We assume that the volatility matrix is invertible, and deﬁne the vector

$$\lambda = (\sigma^T)^{-1} (\mu - r 1).$$

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It is throughout assumed that all entries of \( \sigma^{-1}\lambda \) are positive. This assumption is imposed for mere convenience so that all the properties of the optimal portfolios and the local risk tolerance are aligned (see Remark 10).

Starting at \( t \in [0, T] \) with an initial endowment \( x > 0 \), the investor invests at any time \( s \in (t, T] \) in the riskless and risky assets. The present value of the amounts invested are denoted, respectively, by \( \pi_s^0 \) and \( \pi_s^i \), \( i = 1, \ldots, N \).

The present value of her investment is, then, given by

\[
\sum_{k=0}^{N} \pi_s^k \quad \text{with} \quad \pi_s^k = \sum_{i=1}^{N} \pi_s^i.
\]

We will refer to \( \pi_s^k \) as the discounted wealth generated by the strategy \( \pi_s^0, \pi_s^1, \ldots, \pi_s^N \). The investment strategies will play the role of control processes and are taken to be self-financing. We easily deduce that the discounted wealth satisfies

\[
dX_s = \sigma \pi_s \cdot (\lambda ds + dW_s),
\]

with initial wealth \( X_t = x \), and where the (column) vector, \( \pi_s = \left( \pi_s^i; \quad i = 1, \ldots, N \right) \).

The investment process \( \pi_s \) is admissible if \( \pi_s \in F_s \), \( E \left( \int_t^T |\pi_s|^2 ds \right) < \infty \) and the associated wealth remains non-negative, \( X_s^* \geq 0 \), \( 0 \leq t \leq s \leq T \). We denote the set of admissible strategies by \( A \).

The investor’s utility function at \( T \) is given by \( U : \mathbb{R}_+ - \mathbb{R} \), and it is assumed to be a strictly concave, strictly increasing and \( C^4 \) \((0, \infty)\) function, satisfying the usual Inada conditions

\[
\lim_{x \to 0} U'(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} U'(x) = 0.
\]

We recall the inverse of the marginal utility \( U' \), \( I : \mathbb{R}_+ - \mathbb{R}_+ \),

\[
I(x) = \left( U' \right)^{-1}(x)
\]

and assume that, for some \( \gamma > 0 \), it satisfies the polynomial growth condition

\[
I(x) \leq \gamma + x^{-\gamma}.
\]

The value function is defined as the maximal expected utility,

\[
u(x, t) = \sup_{\pi \in A} E_T \left( U \left( X_T^\pi \right) \right) \left| X_t^\pi = x \right) \],

where \( X_s^\pi, \quad s \in (t, T] \), solves (3).

The above stochastic optimization problem has been widely studied and completely solved. We provide the main results below without a proof (see, for example, [2] and [22]).

**Proposition 1** i) The value function \( u \in C^{4,1} \left( \mathbb{R}_+ \times [0, T] \right) \) is strictly increasing and strictly concave in the spatial variable, and solves the Hamilton-Jacobi-Bellman (HJB) equation,

\[
u_t - \frac{1}{2} \left| \lambda \right|^2 \frac{\partial^2}{\partial x^2} = 0,
\]

with \( u(x, T) = U(x) \), with \( \lambda \) as in (2).
ii) The optimal portfolio process is given, for \( s \in [t,T] \), by

\[
\pi_s^* = \pi^* (X^*_s, s) = (\pi^*_{1,s} (X^*_s, s), ..., \pi^*_{N,s} (X^*_s, s)),
\]

where the optimal feedback portfolio vector \( \pi^* : \mathbb{R}_+ \times [0,T] \rightarrow \mathbb{R}_+^N \) is given by

\[
\pi^*(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)} \sigma^{-1} \lambda,
\]

and \( X^*_s, s \in [t,T] \), is the optimal wealth process solving (3) with \( \pi^*_s \) given in (9) being used.

Associated with any utility function are the absolute risk tolerance and absolute risk aversion coefficients, denoted by the functions \( R(x) \) and \( A(x) \), given respectively, for \( x \geq 0 \), by

\[
R(x) = -\frac{U'(x)}{U''(x)} \quad \text{and} \quad A(x) = -\frac{xU''(x)}{U'(x)}.
\]

Standing assumptions for the risk tolerance coefficient is that \( R(0) = 0 \) and that \( R(x) \) is strictly increasing for \( x \geq 0 \), previously mentioned as the DARA hypothesis.

The relative risk tolerance and relative risk aversion coefficients, denoted by the functions \( \tilde{R}(x) \) and \( \tilde{A}(x) \), are the normalized by wealth analogues of (11),

\[
\tilde{R}(x) = -\frac{xU'(x)}{xU''(x)} \quad \text{and} \quad \tilde{A}(x) = -\frac{xU''(x)}{xU'(x)}.
\]

For intermediate trading times \( t \in [0,T) \), one then defines the associated local, or indirect, absolute and relative coefficients. The local absolute risk tolerance, \( r(x,t) \), and the local absolute risk aversion, \( \gamma (x,t) \), are defined on \( \mathbb{R}_+ \times [0,T) \) and given, respectively, by

\[
r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)} \quad \text{and} \quad \gamma(x,t) = -\frac{u_{xx}(x,t)}{u_x(x,t)},
\]

with \( u \) being the value function, while the corresponding local relative risk tolerance, \( \tilde{r}(x,t) \), and local relative risk aversion, \( \tilde{\gamma}(x,t) \), are given, respectively, by

\[
\tilde{r}(x,t) = \frac{r(x,t)}{x} \quad \text{and} \quad \tilde{\gamma}(x,t) = \frac{\gamma(x,t)}{x}.
\]

It is then immediate from (9), (12) and (13) that the optimal portfolio, \( \pi^*_s \), and optimal weight processes, denoted by \( \tilde{\pi}^*_s \), are given, for \( s \in [t,T] \), respectively, by

\[
\pi^*_s = r(X^*_s,s) \sigma^{-1} \lambda \quad \text{and} \quad \tilde{\pi}^*_s = \frac{r(X^*_s,s)}{X^*_s} \sigma^{-1} \lambda.
\]
3 Auxiliary equations, harmonic functions and feedback controls

We start our study considering two auxiliary partial differential equations which we use throughout. The first is a known fast-diffusion type equation that the local absolute risk tolerance and the optimal portfolios solve (cf. (16) and (18)). The second is the classical heat equation satisfied by a function, denoted by $H$, which is defined in (21) and is linked with the local risk tolerance function as in (25).

The function $H$ plays a pivotal role in the analysis herein, for the various derivatives of the local risk tolerance and the optimal feedback portfolio functionals reduce to rather simple expressions involving only its spatial derivatives. These derivatives also solve the heat equation, and this equation admits comparison principle. As a result, various complicated comparison results, which are needed for proving the desired monotonicity, robustness, and sensitivity properties, reduce to direct comparisons of solutions to the involved heat equations.

The function $H$ also helps us derive stochastic representations of the optimal wealth and portfolio processes. They are provided in (26) and (27), (28).

We note that (21) is a mere variation of the Fenchel-Legendre transformation which has been widely used for the analysis of the Merton problem. Moreover (see Proposition 7), the alternative stochastic representations (26) and (27) are also variations of known stochastic expressions derived from duality theory. However, the representations we use herein turn out to be quite convenient for the analysis and study of various complicated nonlinear expressions of interest, and this enables us to derive results we did not know before.

**Proposition 2** i) The local absolute risk tolerance function $r(x,t)$ (cf. (13)) satisfies, for $(x,t) \in \mathbb{R}_+ \times (0,T]$, the equation

$$r_t + \frac{1}{2} |\lambda|^2 r^{xx} = 0$$

with

$$r(x,T) = R(x).$$

ii) Each component, $\pi^{*,i}$, $i = 1, ..., N$, of the optimal feedback portfolio vector $\pi^*$ solves, for $(x,t) \in \mathbb{R}_+ \times (0,T]$,

$$\pi^{*,i}_t + \frac{1}{2} m_i^2 \left(\pi^{*,i}\right)^2 \pi^{*,i}_{xx} = 0$$

with

$$\pi^{*,i}(x,T) = k_i R(x).$$

where $k_i$ is the $i^{th}$ element of the vector $\sigma^{-1} \lambda$ and $m_i = k_i^{-1} |\lambda|$.

iii) For each $i = 1, ... N$, and $t \in [0,T]$,

$$\lim_{x \to 0} \pi^{*,i}(x,t) = \lim_{x \to 0} r(x,t) = 0.$$

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Corollary 3 The local absolute risk aversion function $\gamma(x,t)$ (cf. (13)) satisfies, for $(x,t) \in \mathbb{R}_+ \times (0,T]$, the porous medium equation

$$\gamma_t = \frac{1}{2} |\lambda|^2 \left( \frac{1}{\gamma} \right)_{xx}$$

with

$$\gamma(x,T) = A(x).$$

The derivation of (18) can be found, among others, in [18], [20] and [47]; see, also, the old note of Black [3]. Equation (16) follow trivially from (10) and (18).

Finally, assertion (20) is well known; it can be proved, for example, by sending $x \to 0$ in equalities (3.2) and (3.3) in [47].

Proposition 4 Let $I : \mathbb{R}_+ \to \mathbb{R}_+$ be given by (5) and assume that it satisfies the growth condition (6). Let, also, $H : \mathbb{R} \times [0,T] \to \mathbb{R}_+$ be defined by

$$u_x (H(x,t), t) = \exp \left( -x - \frac{1}{2} |\lambda|^2 (T-t) \right),$$

where $u(x,t)$ is the value function (cf. (7)). Then,

i) the function $H(x,t)$ solves the heat equation,

$$H_t + \frac{1}{2} |\lambda|^2 H_{xx} = 0$$

with terminal condition

$$H(x,T) = I(e^{-x}).$$

ii) For each $t \in [0,T]$, $H(x,t)$ is strictly increasing and of full range,

$$\lim_{x \to -\infty} H(x,t) = 0 \text{ and } \lim_{x \to \infty} H(x,t) = \infty.$$  

Proof. The fact that $H$ solves the heat equation follows directly from (8) and (21). The existence and uniqueness of solutions to (22) follows directly from (6) and (23) (see, for example, [46]).

The monotonicity assertion follows from the fact that the function $H_x(x,t)$ solves the heat equation with positive terminal condition, since we have that $H_x(x,T) = -e^{-x} I'(e^{-x}) > 0$, and standard comparison arguments (see, among others, [38] and [46]). To show (24), we use (21) and that the value function $u(x,t)$ satisfies, for $t \in [0,T]$ the Inada conditions (4) (see, for example, [22]).

Besides the comparison results for solutions of the heat equation, which we do not list, we will be using two other results, namely, the preservation of the log-convexity and log-concavity of its positive solutions. The log-convexity property is a mere consequence of Hölder’s inequality while the log-concavity is more involved. For the latter, we refer the reader to Theorem 1.3 in [8] or to [6] and, for the case of boundary data, to [23]. The one-dimensional case we consider was first proved in [43]. For the reader’s convenience, we state these two properties next and present their proofs in the Appendix.
Proposition 5 Let $h : \mathbb{R} \times [0, T] \to \mathbb{R}_+$ be the solution of the heat equation

$$ h_t + \frac{1}{2} |\lambda|^2 h_{xx} = 0, $$

with terminal data $h(x, T) = h_0(x)$, with $h_0 \in C^2(\mathbb{R})$ satisfying, for $x \in \mathbb{R}$, $h_0(x) > 0$ and the growth assumption $h_0(x) \leq \gamma + e^{\gamma x}$, $\gamma > 0$. Then, for each $t \in [0, T)$, the following assertions hold.

i) If $h_0(x)$ is a log-convex function, then $h(x, t)$ is also log-convex.

ii) If $h_0(x)$ is a log-concave function, then $h(x, t)$ is also log-concave.

As mentioned earlier, the next result is pivotal for the analysis herein and will be used repeatedly throughout. It follows directly from (13) and (21), and provides a useful representation of the local risk tolerance function via the spatial inverse of the auxiliary harmonic function and its first spatial derivative.

Proposition 6 The local absolute risk tolerance function $r(x, t)$ is given, for $(x, t) \in \mathbb{R}_+ \times (0, T]$, by

$$ r(x, t) = H_x \left( H^{(-1)}(x, t), t \right), $$

where $H(x, t)$ satisfies (22) and (23).

3.1 The optimal wealth, portfolio and portfolio weights processes

We provide representation results for the optimal wealth, portfolio and portfolio weights processes. As (26) and (28) show, these processes are represented as harmonic functionals of time and the current value of the Brownian motion that drives the stock price processes (1). On the other hand, the optimal portfolio weights processes are given via the solution to a Burger’s equation, the current time and current value of the Brownian motion.

The stochastic representations (26) and (28) are derived using arguments analogous to the ones first developed in [35] for the optimal forward wealth and portfolio processes in an Ito diffusion model. We note, however, that in [35] the optimal processes are provided via analogous harmonic functionals that are rescaled in stochastic time (see (23) and (24) therein). Such rescaling is not really meaningful for the Merton problem, for it corresponds to a mere artificial change of the trading horizon. For a recent application of (26) in both the forward and the classical expected utility frameworks, we refer the reader to [33].

For the reader’s convenience, we provide the key steps of the proof. For simplicity, we assume that the initial time is $t = 0$.

Proposition 7 i) The optimal wealth $X_t^*$, $t \in [0, T]$, starting at $x$ at time 0, and the associated optimal investment vector $\pi_t^*$ are given, respectively, by the processes

$$ X_t^* = H \left( H^{(-1)}(x, 0) + |\lambda|^2 t + \lambda \cdot W_t, t \right) $$

(26)
\[ \pi_t^* = H_x \left( H^{-1}(X_t^*, t), t \right) \sigma^{-1} \lambda \]  
\[ = H_x \left( H^{-1}(x, 0) + \lambda^2 t + \lambda \cdot W_t, t \right) \sigma^{-1} \lambda, \]  
(27)

where \( H : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^+ \) satisfies (22) and (23).

ii) The optimal portfolio weights vector \( \tilde{\pi}_t^* = \frac{\pi_t^*}{\pi_t^*} \) is given by

\[ \tilde{\pi}_t^* = H_x \left( H^{-1}(X_t^*, t), t \right) \sigma^{-1} \lambda \]  
\[ = \tilde{H} \left( H^{-1}(X_t^*, t), t \right) \sigma^{-1} \lambda = \tilde{H} \left( H^{-1}(x, 0) + \lambda^2 t + \lambda \cdot W_t, t \right) \sigma^{-1} \lambda \]  
(29)

where the function \( \tilde{H} : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}^+ \) satisfies \( \tilde{H} (x, t) = \frac{\partial}{\partial x} \log H (x, t) \) and solves the Burger’s equation

\[ \tilde{H}_t + \frac{1}{2} |\lambda|^2 \tilde{H}_{xx} + |\lambda|^2 \tilde{H} \tilde{H}_x = 0, \]  
(30)

with terminal condition

\[ \tilde{H} (x, T) = -e^{-x} \frac{I''(e^{-x})}{I( e^{-x})}. \]

**Proof.** Assertion (27) is derived from (15) and (25). In turn, (28) follows from (27) and (26). Therefore, it remains to show (26).

For this, define the auxiliary process \( N_t, t \geq 0, \)

\[ N_t = H^{-1}(x, 0) + \lambda^2 t + \lambda \cdot W_t \]  
(31)

and apply Ito’s formula to the process \( H (N_t, t) \). We then have that

\[ dH (N_t, t) = H_x (N_t, t) dN_t + \frac{1}{2} H_{xx} (N_t, t) d \langle N \rangle_t + H_t (N_t, t) dt \]

\[ = H_x (N_t, t) \left( |\lambda|^2 dt + \lambda \cdot dW_t \right) + \left( H_t (N_t, t) + \frac{1}{2} |\lambda|^2 H_{xx} (N_t, t) \right) dt \]

\[ = H_x (N_t, t) \left( |\lambda|^2 dt + \lambda \cdot dW_t \right), \]

where we used that \( H \) solves (22). We easily deduce that the process \( H (N_t, t) \) is the optimal wealth generated by \( \pi_t^* \) and we conclude.

To establish assertion (29) one uses the definition of \( \tilde{\pi}_t^*, (26) \) and (28) and that \( H (x, t) \) solves (22). \( \blacksquare \)

We remark that one could use standard duality results to derive (26) and (28). The construction is in reverse order, in that the wealth representation (26) is established first and then (27) follows from a direct application of Ito’s formula and (3). This is highlighted below.
Remark 8 Let $Z_T$ be the Radon-Nikodym derivative of the unique martingale measure, say $Q$, with respect to the historical measure $P$. Let also $\xi^x \in \mathbb{R}_+$ be the optimal Lagrange multiplier satisfying the budget constraint $E_P(Z_T I(\xi^x Z_T)) = x$, with $I$ as in (5). Then, we have that $\xi^x = \exp \left( -H^{(-1)}(x,0) - \frac{1}{2} |\lambda|^2 T \right)$. In turn, for $t \in [0, T]$, the optimal wealth process is given by

$$X_t^* = E_Q( I(\xi^x Z_T)) = H \left( H^{(-1)}(x,0) + |\lambda|^2 t + \lambda \cdot W_t,t \right),$$

with $H(x,t)$ satisfying (22) and (23).

4 Spatial and temporal properties of optimal allocations and optimal weights

We study the questions stated in the introduction on the monotonicity and concavity of the optimal allocations and optimal weights with respect to wealth, as well as their monotonicity with respect to time. Because these quantities are affine transformations of the local absolute and relative risk tolerances (cf. (15)), we only provide the related proofs for the latter. For completeness though, we state each result for all related quantities.

4.1 Monotonicity in wealth

We start with the monotonicity properties of the optimal allocations and optimal weights. We recall the standing assumption that the risk tolerance coefficient $R(x)$ is strictly increasing for $x \geq 0$. As mentioned earlier, these properties were first established for the log-normal model in [5] and later in [47]. Below, we provide alternative short proofs to these results.

**Proposition 9** Let the absolute risk tolerance coefficient $R(x)$, $x \geq 0$, be strictly increasing. Then, for each $t \in [0, T]$ and $x \geq 0$,

i) the optimal allocations $\pi^*(x,t)$, $x \geq 0$, $i = 1, ..., N$, are strictly increasing.

ii) the local absolute risk tolerance and risk aversion coefficients $r(x,t)$ and $\gamma(x,t)$ are, respectively, strictly increasing and strictly decreasing.

**Proof.** Differentiating (25) for $t = T$ yields,

$$R'(x) = \frac{H_{xx}(x,T)}{H_x(x,T)} \bigg|_{z=H^{(-1)}(x,T)}.$$

(32)

Using the monotonicity of $H(x,t)$ and (24), we see that $R(x)$ is strictly increasing if and only if the auxiliary function $H(x,T)$ is strictly concave, $H_{xx}(x,T) > 0$.

On the other hand, the function $H_{xx}(x,t)$ also solves the heat equation (22) with terminal condition $H_{xx}(x,T) = (I(e^{-r}))''$. Using (6), we easily deduce
that the terminal condition $H_{xx} (x, T)$ satisfies the appropriate conditions for existence and uniqueness of solutions. A straightforward application of the comparison principle then yields that $H_{xx} (x, t) > 0$, for $(x, t) \in \mathbb{R} \times [0, T)$. Using (25) once more, we have that

$$r_x (x, t) = \frac{H_{xx} (z, t)}{H_x (z, t)} \bigg|_{z=H^{-1}(x, t)},$$

and using that $H_x (x, t) > 0$ (see part (ii) in Lemma 4) we easily conclude. ■

**Remark 10** In section 2, it was assumed that all entries of the vector $\sigma^{-1} \lambda$ are positive. As the above proof shows, this was made for mere convenience, so that the signs of the risk tolerance and the optimal feedback portfolios are the same. The results in the above proposition and in all subsequent ones can be readily generalized if the elements of $\sigma^{-1} \lambda$ are of different signs.

We continue with the analogous monotonicity properties for the relative optimal weights. We note that similarly to [5], our proofs rely crucially on the preservation of log-concavity of solutions to the heat equation.

**Proposition 11** Let the relative risk tolerance coefficient $\tilde{R} (x) , x > 0$, be an increasing (decreasing) function. Then, for each $t \in [0, T)$ and $x > 0$,

i) the optimal weights $\tilde{\pi}^{*i} (x, t) , i = 1, ..., N$, are increasing (decreasing).

ii) the local relative risk tolerance and risk aversion coefficients $\tilde{r} (x, t)$ and $\tilde{\gamma} (x, t)$ are, respectively, increasing (decreasing) and decreasing (increasing) functions.

**Proof.** Differentiating (25) at $t = T$ yields,

$$\tilde{R}' (x) = \frac{\partial^2}{\partial x^2} \log H (z, T) \bigg|_{z=H^{-1}(x, T)}.$$

Therefore, $\tilde{R} (x)$ is an increasing function if and only if $H (x, T)$ is log-convex. Applying Proposition 5 for $h_0 (x) = \tilde{R}' (x)$ we deduce that, for each $t \in [0, T)$ and $x \geq 0$,

$$\frac{\partial^2}{\partial x^2} \log H (x, t) > 0.$$

Using (25) once more, we deduce that

$$\frac{\partial}{\partial x} (\tilde{r} (x, t)) = \frac{\partial^2}{\partial x^2} \log H (z, t) \bigg|_{z=H^{-1}(x, t)},$$

and using (24) we conclude.

The analogous results are readily derived when the relative risk tolerance is decreasing. In this case, one uses the preservation of the log-concavity property of the solution to the heat equation. ■
4.2 Concavity/convexity in wealth

We examine the spatial concavity/convexity properties of the optimal allocations and optimal weights. Whether the risk tolerance coefficient is concave or convex has been a topic of long-standing debate. We refer the reader to [15] for an extensive discussion as well as to [27] and [31]. As it is argued therein, there are arguments and results which support both assumptions. In particular, as established in [19], concave risk tolerance implies that the risk aversion is proper, standard and risk vulnerable (cf., respectively, [37], [25] and [14]). The empirical study in [17] also suggests that the risk tolerance is a concave function of wealth. For a portfolio problem with explicit solutions with a convex risk tolerance, we refer the reader to [9], see also [49].

Proposition 12 The following assertions hold:

i) Let the absolute risk tolerance coefficient $R(x)$, $x \geq 0$, be concave. Then, for each $t \in [0, T]$ and $x \geq 0$, the optimal allocations $\pi^{a,i}(x,t)$, $i = 1, \ldots, N$, are concave. Moreover, for each $t \in [0, T]$ and $x \geq 0$, the local absolute risk tolerance and risk aversion functions $r(x,t)$ and $\gamma(x,t)$ are, respectively, concave and convex.

ii) Let the absolute risk tolerance coefficient $R(x)$, $x \geq 0$, be convex. Then, for each $t \in [0, T]$ and $x \geq 0$, the optimal allocations $\pi^{a,i}(x,t)$, $i = 1, \ldots, N$, and the local absolute risk tolerance coefficient $r(x,t)$ are convex.

iii) Let the absolute risk aversion coefficient $A(x)$, $x \geq 0$, be concave. Then, for each $t \in [0, T]$ and $x \geq 0$, the optimal allocations $\pi^{a,i}(x,t)$, $i = 1, \ldots, N$, are convex functions. Moreover, for each $t \in [0, T]$ and $x \geq 0$, the local absolute risk tolerance and risk aversion coefficients $r(x,t)$ and $\gamma(x,t)$ are, respectively, convex and concave.

Proof. To establish (i), we differentiate (25) twice at $t = T$ to obtain

$$R''(x) = \frac{\partial^2}{\partial x^2} \left( \log H_x(z,T) \right) \bigg|_{z=H^{(-1)}(x,T)}.$$  

Therefore, $R(x)$ is concave if and only if $H_x(x,T)$ is log-concave. In turn, we have that the function $H_x$ solves the heat equation (22) with positive log-concave terminal data. Applying Proposition 5 for $h_0(x) = R''(x)$, we deduce that for each $t \in [0, T)$ and $x \geq 0$, the function $H_x(x,t)$ is also log-concave. Differentiating (25) twice yields

$$r_{xx}(x,t) = \frac{\partial^2}{\partial x^2} \left( \log H_x(z,t) \right) \bigg|_{z=H^{(-1)}(x,t)},$$

and using (24) we conclude. From (13) we have

$$\gamma_{xx}(x,t) = \frac{-r_{xx}(x,t) r^2(x,t) - 2(r_x(x,t))^2 r(x,t)}{r^4(x,t)},$$

and the convexity of $\gamma(x,t)$ follows.
Assertion (ii) follows along similar arguments but using, instead, the preservation of the log-convexity property of solutions to the heat equation that $H_x(x, t)$ solves.

For (iii), we first observe that if the risk aversion coefficient $A(x)$ is a concave function, then (33), for $t = T$, yields that $R(x)$ is convex and, in turn, the results of (ii) hold. 

Note that the convexity of $R(x)$ (resp. $A(x)$) does not determine whether $A(x)$ (resp. $R(x)$) is concave or convex.

4.3 Monotonicity in time

This result shows that it is exclusively the curvature of the absolute risk tolerance coefficient that determines the time monotonicity of the optimal feedback investment function.

The proof of the result is remarkably simple and just uses the fact that the local risk tolerance function solves the fast-diffusion equation (16).

**Proposition 13** The following assertions hold:

Let the absolute risk tolerance coefficient $R(x)$, $x \geq 0$, be concave (convex). Then, for each $x \geq 0$, the optimal allocations $\pi^{*,i}(x, t)$, $i = 1, ..., N$, are increasing (decreasing) functions of time. Moreover, the local absolute risk tolerance and risk aversion coefficients $r(x, t)$ and $\gamma(x, t)$ are, respectively, increasing and decreasing functions of time.

ii) Let the absolute risk aversion coefficient $A(x)$, $x \geq 0$, be concave. Then, for each $x \geq 0$, the optimal allocations $\pi^{*,i}(x, t)$, $i = 1, ..., N$, are decreasing functions of time. Moreover, the local absolute risk tolerance and risk aversion coefficients are, respectively, decreasing and increasing functions.

**Proof.** We only show (i) since the rest of the arguments follow easily. To this end, recall from Proposition 12 that if $R(x)$ is concave (convex) then, for each $t \in [0, T)$, the absolute local risk tolerance $r(x, t)$ is also concave (convex). From equation (16) we, then, observe than $r_t(x, t) \leq 0 \geq 0$, and the assertion follows.

For the case of concave terminal risk tolerance, the above result is quite surprising for it goes against the conventional wisdom that as investors get older they should decrease their allocations in the risky assets. This feature is central in the management of life-cycle and target-date funds. We refer the reader, among others, to [4], [7], [42], [44] and [45].

As pointed out earlier, there are studies supporting the use of utility functions with concave as well as convex risk tolerance functions.

The temporal behavior of the value function and the optimal policies have been examined in more extended model settings in [10] and [28]. Therein, however, the generality of the model did not allow for specific results as the one above. After this work was completed, the authors discovered that in [29] the case of convex risk tolerance and its effect on the time behavior of the optimal policy was established using duality and martingale techniques.
5 Robustness of the optimal allocations and optimal weights

In this section we investigate the robustness of the optimal allocations and optimal weights with respect to ordered terminal data. Specifically, we consider two investors whose absolute risk tolerance coefficients, say $R_1(x)$ and $R_2(x)$, satisfy, for $x \geq 0$,

$$R_1(x) \leq R_2(x),$$

and we examine whether the above inequality implies that, for all $(x, t) \in \mathbb{R}_+ \times [0, T)$ and each $i = 1, ..., N$, the optimal feedback functionals $\pi_{1,i}^*(x, t)$ and $\tilde{\pi}_{1,i}^*(x, t)$, $j = 1, 2$, are, respectively, similarly ordered,

$$\pi_{1,i}^*(x, t) \leq \pi_{2,i}^*(x, t) \quad \text{and} \quad \tilde{\pi}_{1,i}^*(x, t) \leq \tilde{\pi}_{2,i}^*(x, t).$$

It is immediate that one needs to check whether (34) is preserved at all previous times, namely, if for $(x, t) \in \mathbb{R}_+ \times [0, T)$,

$$r_1(x, t) \leq r_2(x, t).$$

To our knowledge, these robustness questions have been investigated only by Xia in [47] for the log-normal model. Therein, the author considered an approximating sequence of penalized versions of the fast-diffusion equation in (16) and used duality arguments to obtain comparison of their solutions. As he mentions (see Remark 4.3 in [47]), it is quite difficult to obtain comparison results directly from (16).

However, as we show below, such comparison results can be obtained. The key idea is to consider an auxiliary equation, specifically, the one satisfied by the square of the risk tolerance function (cf. (46)), and establish the comparison for this equation rather than for the original one. The comparison for the fast-diffusion equation (16) then follows using the positivity of the local absolute risk tolerance functions.

Our proof is substantially shorter and more direct than the one in [47]. We note though that the results hold for a slightly smaller class of absolute risk tolerance coefficients. Specifically, we consider risk tolerances $R(x)$ which satisfy, for $x \geq 0$,

$$R'(x) < \infty,$$

and, moreover, their square $R^2(x)$ is a semi-superharmonic function ($SSH$), namely, it satisfies the inequality

$$(R^2(x))'' \leq K,$$

for some $K \in \mathbb{R}$. We provide examples of such absolute risk tolerance coefficients in the sequel.

Before we present the main robustness result, we provide the following auxiliary properties. We remind the reader that $R(x)$ is taken to be strictly increasing ($DARA$ hypothesis). We note however that all arguments in Propositions
14 and 16 below go through even when $R(x)$ is not monotone, at the expense of slightly more complicated arguments. However, we do not consider this case because of modeling considerations.

**Proposition 14** Assume that the absolute risk tolerance coefficient $R(x)$ satisfies (37) and (38), for $x \geq 0$, and some $K \in \mathbb{R}$. Then, the following assertions hold:

i) The function $r^2(x,t)$ is also SSH, uniformly in $t$, namely, for $(x,t) \in \mathbb{R}_+ \times [0,T),$ 
   \[(r^2(x,t))_{xx} \leq K. \tag{39}\]

ii) The absolute risk tolerance coefficient $R(x)$ satisfies, for $x \geq 0$, and $C = \sqrt{|K|}/2$,
   \[R'(x) \leq C \quad \text{and} \quad R(x) \leq Cx. \tag{40}\]

In turn, for $(x,t) \in \mathbb{R}_+ \times [0,T),$ 
   \[r_x(x,t) \leq C \quad \text{and} \quad r(x,t) \leq Cx. \tag{41}\]

**Proof.** i) To show (39), we first use (25) for $t = T$ to obtain 
   \[\left( R^2(x) \right)' = 2 \left( H^{(-1)}(x,T) \right)' H_x \left( H^{(-1)}(x,T) \right) H_{xx} \left( H^{(-1)}(x,T), T \right) \]
   \[= 2H_{xx} \left( H^{(-1)}(x,T), T \right). \]

In turn,
   \[\left( R^2(x) \right)'' = 2 \left( H^{(-1)}(x,T) \right)' H_{xxx} \left( H^{(-1)}(x,T), T \right) \]
   and, thus,
   \[\left( R^2(x) \right)'' = 2 \frac{H_{xxx}(z,T)}{H_x(z,T)} \bigg|_{z = H^{(-1)}(x,T)}. \tag{42}\]

Using (24) and (38), we deduce that, for $x \geq 0$,
   \[H_{xxx}(x,T) \leq MH_x(x,T), \tag{43}\]
   with $M = \frac{1}{2}K$. Classical comparison results for the heat equation, which both $H_{xxx}(x,t)$ and $MH_x(x,t)$ satisfy, then imply that the above inequality is preserved for previous times,
   \[H_{xxx}(x,t) \leq MH_x(x,t), \]
   for $(x,t) \in \mathbb{R}_+ \times [0,T).$ On the other hand, using (25) once more yields 
   \[\left( r^2(x,t) \right)_{xx} = 2 \frac{H_{xxx}(z,t)}{H_x(z,t)} \bigg|_{z = H^{(-1)}(x,t)}, \]
   and (39) follows from (43).
The first part of (40) follows from a direct adaptation of Proposition 2.4 in [13]; for completeness we highlight the main steps. We only look at the case that \( K > 0 \).

To this end, from (38) we have that the function \( R^2 (x) - \frac{1}{2} K x^2 \) is concave. Therefore,

\[
R^2 (y) - \frac{1}{2} K y^2 \leq R^2 (x) - \frac{1}{2} K x^2 + \left( R^2 (x) - \frac{1}{2} K x^2 \right)' (y - x),
\]

for any \( y > 0 \). Taking \( y = x + h, x \geq 0 \), we deduce that for all \( h \in \mathbb{R} \),

\[
\frac{1}{2} Kh^2 + \left( R^2 (x) \right)' h + R^2 (x) \geq 0.
\]

Therefore, we must have \( \left( (R^2 (x))' \right)^2 - 2 K R^2 (x) \leq 0 \).

In turn, using that \( R (x) \) and \( R' (x) \) are nonnegative we easily conclude.

The second part of (40) follows directly from the first and the fact that \( R (0) = 0 \).

To show the first part of (41), we use that \( R' (x) \leq C \) (cf. (40)) and \( R' (x) = \frac{H_{xz}(x,t)}{H_x(x,t)} \) to deduce that \( \frac{H_{xz}(x,t)}{H_x(x,t)} \leq C \). Working as above we obtain that, for \((x, t) \in \mathbb{R}_+ \times (0, T)\), \( H_{xz} (x, t) \leq CH_x (x, t) \), and we easily conclude.

The second part of (41) follows along similar arguments. \( \blacksquare \)

Remark 15 In [47], the admissible class of risk tolerance functions satisfy \( R (x) \leq M (1 + x) \), for \( x \geq 0 \) and \( M > 0 \). This property allows for risk tolerances with \( \lim_{x \to 0} R' (x) = \infty \), which, however, are excluded herein due to (37). It is this property that makes our admissible class smaller than the one in [47]. Such a case is, for example, \( R (x) = \sqrt{x} \). This risk tolerance is SSH, for it satisfies \( (R^2 (x))'' = 0 \), but fails to satisfy (37) at \( x = 0 \).

We are now ready to present the main robustness result.

Proposition 16 Assume that the absolute risk tolerance coefficients \( R_1 (x) \) and \( R_2 (x) \) satisfy, for \( x \geq 0 \),

\[
R_1 (x) \leq R_2 (x), \tag{44}
\]

and inequalities (37) and (38). Then, for \((x, t) \in \mathbb{R}_+ \times [0, T)\):

i) the optimal feedback allocations \( \pi^{*,i}_j (x, t) \) and optimal weights, \( \tilde{\pi}^{*,i}_j (x, t) \), \( x > 0, i = 1, \ldots, N, \) and \( j = 1, 2 \), satisfy

\[
\pi^{*,i}_1 (x, t) \leq \pi^{*,i}_2 (x, t) \quad \text{and} \quad \tilde{\pi}^{*,i}_1 (x, t) \leq \tilde{\pi}^{*,i}_2 (x, t).
\]

ii) the local absolute and relative risk tolerance and risk aversion functions satisfy

\[
r_1 (x, t) \leq r_2 (x, t) \quad \text{and} \quad \gamma_1 (x, t) \geq \gamma_2 (x, t) \quad \text{and} \quad \check{r}_1 (x, t) \leq \check{r}_2 (x, t) \quad \text{and} \quad \check{\gamma}_1 (x, t) \geq \check{\gamma}_2 (x, t). \tag{45}
\]
Proof. We only show the first inequality in (45), since all other assertions follow trivially. We will provide the proof in a slightly more general way.

To this end, we consider two functions, say \( r_1 (x, t) \) and \( r_2 (x, t) \), with the following properties: they are \( C^2,1 (\mathbb{R}_+ \times [0, T]) \), positive with \( r_1 (0, t) = r_2 (0, t) = 0 \), and, for each \( t \in [0, T) \), strictly increasing. They are, also, respectively, a sub- and super-solution of (16) and satisfy at \( t = T \) the inequality (44). Moreover, for \( (x, t) \in \mathbb{R}_+ \times [0, T] \) and \( i = 1, 2 \), \( r_i (x, t) \) and \( r_i^2 (x, t) \) satisfy, respectively, (41) and (38).

We first observe that the fast-diffusion equation in (16) yields that the function \( F : \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R}_+ \) defined by

\[
F (x, t) = r^2 (x, t)
\]
solves the quasi-linear equation

\[
F_t + \frac{1}{2} |\lambda|^2 F F_{xx} - \frac{1}{4} |\lambda|^2 F_x^2 = 0 \tag{46}
\]

with \( F (x, T) = R^2 (x) \). To facilitate the exposition we will work with \( \tilde{F} : \mathbb{R}_+ \times [0, T] \rightarrow \mathbb{R}_+ \), defined by \( \tilde{F} (x, t) = F (x, T - t) \). Then,

\[
\tilde{F}_t - \frac{1}{2} |\lambda|^2 \tilde{F} \tilde{F}_{xx} + \frac{1}{4} |\lambda|^2 \tilde{F}_x^2 = 0 \tag{47}
\]

with \( \tilde{F} (x, 0) = R^2 (x) \). Therefore, the functions \( f (x, t) \) and \( g (x, t) \) given by

\[
f (x, t) = r_1^2 (x, T - t) \quad \text{and} \quad g (x, t) = r_2^2 (x, T - t)
\]
satisfy, respectively,

\[
f_t - \frac{1}{2} |\lambda|^2 f f_{xx} + \frac{1}{4} |\lambda|^2 f_x^2 \leq 0,
\]

\[
g_t - \frac{1}{2} |\lambda|^2 g g_{xx} + \frac{1}{4} |\lambda|^2 g_x^2 \geq 0,
\]

and

\[
f (0, t) = g (0, t) = 0 \quad \text{and} \quad f (x, 0) \leq g (x, 0).
\]

Moreover, by the SSH assumption, we have that for some \( K \in \mathbb{R} \), for all \( x \geq 0 \),

\[
f_{xx} (x, 0) \leq K \quad \text{and} \quad g_{xx} (x, 0) \leq K.
\]

For the rest of the proof, we assume without loss of generality that \( K > 0 \).

We are going to show that, for \( t \in (0, T) \),

\[
f (x, t) \leq g (x, t). \tag{48}
\]

We follow part of the proof of Theorem 3.1 in [13] for comparison of positive solutions to (47) which are also SSH. Because we work with smooth solutions
of (47), various arguments in [13] are simplified, in particular the ones involving "doubling" the space variables.

To this end, we first introduce the auxiliary functions

\[ f_m(x, t) = \left(1 - \frac{t}{m}\right) f(x, t) \quad \text{and} \quad g_m(x, t) = \left(1 - \frac{t}{m}\right) g(x, t), \]

where

\[ m = \min(T, m_0) \quad (49) \]

where the constant \( m_0 \) will be chosen in the sequel, see (59).

Then, for \((x, t) \in \mathbb{R}_+ \times [0, m]\),

\[ 0 \leq f_m(x, t) \leq f(x, t) \quad \text{and} \quad 0 \leq g_m(x, t) \leq g(x, t). \quad (50) \]

Moreover,

\[ f_m(x, 0) \leq g_m(x, 0) \quad \text{and} \quad f_m(x, m) = g_m(x, m) = 0, \quad (51) \]

and

\[ f_m(0, t) = g_m(0, t) = 0. \quad (52) \]

We then easily obtain that \( f_m \) and \( g_m \) satisfy, respectively,

\[ \left(1 - \frac{t}{m}\right) f_{m,t} + \frac{1}{m} f_m - \frac{1}{2} |\lambda|^2 f_m f_{m,xx} + \frac{1}{4} |\lambda|^2 f_{m,x}^2 \leq 0 \quad (53) \]

and

\[ \left(1 - \frac{t}{m}\right) g_{m,t} + \frac{1}{m} g_m - \frac{1}{2} |\lambda|^2 g_m g_{m,xx} + \frac{1}{4} |\lambda|^2 g_{m,x}^2 \geq 0. \quad (54) \]

As argued in [13], straightforward bootstrapping argumentation can be used to establish (48) for \((x, t) \in \mathbb{R}_+ \times (0, T]\) once it is shown that, for \((x, t) \in \mathbb{R}_+ \times [0, m]\),

\[ f_m(x, t) \leq g_m(x, t). \]

To this end, we consider the test function \( \varphi(x) = 1 + x^4 \), for \( x \geq 0 \). We are going to show that, for any \( \varepsilon > 0 \), we have

\[ f_m(x, t) \leq g_m(x, t) + \varepsilon \varphi(x). \]

We argue by contradiction, i.e. we assume that there exists \( \varepsilon > 0 \) such that

\[ \sup_{(x, t) \in \mathbb{R}_+ \times [0, m]} (f_m(x, t) - g_m(x, t) - \varepsilon \varphi(x)) > 0. \quad (55) \]

Let \((\hat{x}, \hat{t})\) be any point such that \( f_m(\hat{x}, \hat{t}) - g_m(\hat{x}, \hat{t}) - \varepsilon \varphi(\hat{x}) > 0 \). We easily get from (50) and (41) that \( h(\hat{x}, \hat{t}) \leq C\hat{x}^2 \), for \( h = f_m, g_m \), which together with the growth of \( \varphi(x) \) yields that \( \hat{x} < \infty \).

Next, we observe that the extremum in (55), denoted by \((\hat{x}, \hat{t})\) is an interior point in \((0, \infty) \times (0, m)\). Indeed, if \((\bar{x}, \bar{t})\) is such that \( \bar{t} = 0 \) or \( \bar{t} = m \) we get
a contradiction from (51), while if, for some \( \tilde{t} \in (0, m) \), \( (\tilde{x}, \tilde{t}) = (0, \tilde{t}) \), we contradict (52).

At the interior maximum \( (\tilde{x}, \tilde{t}) \) in (55) we, then, have

\[
f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) > \varepsilon (1 + \tilde{x}^4) \tag{56}
\]

\[
f_{m,t}(\tilde{x}, \tilde{t}) - g_{m,t}(\tilde{x}, \tilde{t}) = 0, \quad f_{m,x}(\tilde{x}, \tilde{t}) - g_{m,x}(\tilde{x}, \tilde{t}) = 4\varepsilon \tilde{x}^3 \tag{57}
\]

and

\[
f_{m,xx}(\tilde{x}, \tilde{t}) - g_{m,xx}(\tilde{x}, \tilde{t}) \leq 12\varepsilon \tilde{x}^2. \tag{58}
\]

From (53) and (54) we deduce

\[
\frac{1}{m} \left( f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) \right) \leq \frac{1}{2} |\lambda|^2 \left( f_m(\tilde{x}, \tilde{t}) f_{m,xx}(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) g_{m,xx}(\tilde{x}, \tilde{t}) \right)
- \frac{1}{4} |\lambda|^2 \left( f_{m,x}^2(\tilde{x}, \tilde{t}) - g_{m,x}^2(\tilde{x}, \tilde{t}) \right).
\]

In turn,

\[
\left( \frac{1}{m} - \frac{1}{2} |\lambda|^2 K \right) \left( f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) \right)
\leq \frac{1}{2} |\lambda|^2 \left( f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) \right)
- \frac{1}{4} |\lambda|^2 \left( f_{m,x}^2(\tilde{x}, \tilde{t}) - g_{m,x}^2(\tilde{x}, \tilde{t}) \right) \left( f_{m,x}(\tilde{x}, \tilde{t}) + g_{m,x}(\tilde{x}, \tilde{t}) \right).
\]

Because \( f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) > 0 \) (cf. (56)), the (38) property of \( f_m \), and (57) and (58) above yield

\[
\left( \frac{1}{m} - \frac{1}{2} |\lambda|^2 K \right) \left( f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) \right)
\leq 6\varepsilon |\lambda|^2 g_m(\tilde{x}, \tilde{t}) \tilde{x}^2 - \varepsilon |\lambda|^2 \tilde{x}^3 \left( f_{m,x}(\tilde{x}, \tilde{t}) + g_{m,x}(\tilde{x}, \tilde{t}) \right).
\]

Using that the functions \( r_1(x, t) \) and \( r_2(x, t) \), and in turn \( f_m(x, t) \) and \( g_m(x, t) \), are strictly increasing, the above inequality yields that at \( (\tilde{x}, \tilde{t}) \) we must have,

\[
\varepsilon \left( \frac{1}{m} - \frac{1}{2} |\lambda|^2 K \right) (1 + \tilde{x}^4) \leq 6\varepsilon |\lambda|^2 g_m(\tilde{x}, \tilde{t}) \tilde{x}^2.
\]

Finally, using that \( g_m(\tilde{x}, \tilde{t}) \leq \left( 1 - \frac{\kappa}{m} \right) \frac{K}{2} \tilde{x}^2 \leq \frac{K}{2} \tilde{x}^2 \), the above inequality gives \( \frac{1}{m} - \frac{1}{2} |\lambda|^2 K \leq 3 |\lambda|^2 K \) and, in turn, that

\[
\frac{1}{m} \leq \frac{7}{2} |\lambda|^2 K.
\]
We easily see that if we choose $m_0$ in (49) such that
\[ \min (T, m_0) < \frac{2}{7 |\lambda|^2 K}, \] (59)
we get a contradiction, and we easily conclude. ■

Next, we provide examples of utility functions whose absolute risk tolerance coefficient satisfies (37) and (38).

**Example 17** The popular power case, $U(x) = \frac{1}{\gamma} x^\gamma$, $\gamma \in (0, 1)$, has $R^2(x) = \left(\frac{1}{1-\gamma}\right)^2 x^2$, satisfies (38) (as equality) with $K = 2 \left(\frac{1}{1-\gamma}\right)^2$.

**Example 18** Consider a utility function whose inverse marginal $I(x)$ (cf. (5)) is of the form
\[ I(x) = \int_0^N x^{-y} \nu(dy), \] (60)
for some finite and positive Borel measure, and with compact support ($N < \infty$).

Then, (23) yields that the related harmonic function is given, at $t = T$, by $H(x, T) = \int_0^N e^{x y} \nu(dy)$. From (32) we deduce that, for $x \geq 0$,
\[ R'(x) = 2 \int_0^N y^2 e^{H'(x) T y} \nu(dy) \int_0^N y e^{H'(x) T y} \nu(dy) \leq 2N \]
and (37) follows. Moreover, (38) holds since (cf. (42))
\[ (R^2(x))'' = 2 \int_0^N y^3 e^{H'(x) T y} \nu(dy) \int_0^N y e^{H'(x) T y} \nu(dy) \leq 2N^2. \]

We finish this section with a corollary of the above results which yields time-independent upper and lower bounds for the optimal policies. These bounds are linear functions of wealth. As a result, we are able to relate policies under general risk preferences with the ones of power utility, say $U(x) = \frac{1}{\delta} x^\delta$, with the parameter $\delta$ depending only on the slope of the risk tolerance coefficient at $x = 0$, as described in (61) below. We remind the reader that $R'(0) < \infty$ (cf. (37)).

**Corollary 19** Let the absolute risk tolerance coefficient $R(x)$ be such that $R'(0) > 0$ and define $\delta < 1$ as
\[ \delta = 1 - \frac{1}{R'(0)}. \] (61)
Then, for all $t \in [0, T]$, the optimal allocations $\pi^*, i (x, t)$, $i = 1, \ldots, N$, satisfy:

---

1 These utilities as well as the ones for which the representation (60) holds for their marginals $U'(x)$ (but not for their inverses $I(x)$) have various desirable properties. A detailed analysis and comparative study of these two families can be found in [21].
i) If \( R(x) \) is a concave function,

\[
\pi^* (x,t) \leq \frac{1}{1 - \delta} x \sigma^{-1} \lambda. \tag{62}
\]

ii) If \( R(x) \) is a convex function,

\[
\pi^* (x,t) \geq \frac{1}{1 - \delta} x \sigma^{-1} \lambda.
\]

**Proof.** We only show (62). To this end, we first note that because \( R(x) \) is concave and \( R(0) = 0 \), we have that \( R(x) \leq R'(0) x \). Trivially, the function \( R'(0) x \) satisfies (38) with \( K = 2R'(0) \). Using the comparison result in Proposition 16, we deduce that for \( (x,t) \in \mathbb{R}_+ \times [0,T) \), \( r(x,t) \leq R'(0) x \). In turn, \( (10) \) yields that \( \pi^* (x,t) \leq R'(0) x \sigma^{-1} \lambda, \) \( i = 1, \ldots, N \), and we conclude. ■

**Corollary 20** Let the absolute risk tolerance \( R(x) \) be such that, for all \( \alpha > 0 \), \( \alpha R(x) \leq R(\alpha x) \) (resp. \( \alpha R(x) \geq R(\alpha x) \)). Then, for \( (x,t) \in \mathbb{R}_+ \times [0,T) \), we have \( \alpha r(x,t) \leq r(\alpha x,t) \) (resp. \( \alpha r(x,t) \geq r(\alpha x,t) \)).

**Proof.** Define the function \( \hat{r}(x,t) = \alpha^{-1} r(\alpha x,t) \). Direct differentiation yields that it satisfies (16) and \( \hat{r}(x,T) \geq r(x,T) \). Therefore, \( (x,t) \in \mathbb{R}_+ \times [0,T) \), \( \hat{r}(x,t) \geq r(x,t) \) and we conclude. ■

### 6 Sensitivity analysis

#### 6.1 Sensitivities with respect to market parameters

We study the dependence on the market coefficients of the local absolute risk tolerance and risk aversion functions \( r(x,t) \) and \( \gamma(x,t) \), as well as of the optimal feedback portfolios.

For reasons we discuss later on, we start with the case in which only one stock is traded. We denote by \( \sigma \) and \( \lambda \), respectively, its volatility coefficient and Sharpe ratio (cf. (2)), and assume that \( \lambda, \sigma > 0 \). We note that even in this simple case, it is not immediate how the market parameters affect \( r(x,t) \), \( \gamma(x,t) \) and \( \pi^* (x,t) \), for \( \lambda \) and \( \sigma \) appear in a convoluted way. For example, while the terminal condition \( r(x,T) \) is independent on \( \lambda \), for \( t \in [0,T) \), we have that \( r(x,t;\lambda) = - \frac{u_x(x,t;\lambda)}{u_{xx}(x,t;\lambda)} \), with both \( u_x(x,t;\lambda) \) and \( u_{xx}(x,t;\lambda) \) depending on \( \lambda \), as it can be seen from the HJB equation (8). To our knowledge, sensitivity analysis for these functions has not been carried out to date for general utility functions.

For the case of a single stock, the equations that \( r(x,t) \) and \( \pi^* (x,t) \) solve (cf. (16) and (18)), simplify to

\[
r_t + \frac{1}{2} \lambda^2 r^2 r_{xx} = 0 \quad \text{and} \quad \pi^*_t + \frac{1}{2} \sigma^2 (\pi^*)^2 \pi^*_{xx} = 0 \tag{63}
\]

with

\[
r(x,T) = - \frac{U''(x)}{U''''(x)} \quad \text{and} \quad \pi^*(x,T) = - \frac{\lambda}{\sigma} \frac{U''(x)}{U''''(x)}. \tag{64}
\]

\]
We start with the monotonicity of the local absolute risk tolerance and risk aversion functions with respect to $\lambda$ and $\sigma$. As we show below, their dependence on the market parameters depends exclusively on the curvature of the absolute risk tolerance coefficient $R(x)$. This result is quite surprising and in contradistinction with the analogous monotonicity properties of the optimal investment policy $\pi^*(x, t)$ which are always preserved independently of the curvature of the risk tolerance coefficient.

A standing assumption for the next two propositions is that inequalities (37) and (38) is satisfied and, therefore, the comparison results in Proposition 16 hold.

**Proposition 21**

i) Let the absolute risk tolerance coefficient $R(x), x \geq 0,$ be concave. Then, for $(x, t) \in \mathbb{R}_+ \times [0, T)$, the local absolute risk tolerance $r(x, t)$ is decreasing in $\lambda$ and increasing in $\sigma$. The reverse assertion holds for the local absolute risk aversion $\gamma(x, t)$.

ii) Let the absolute risk tolerance coefficient $R(x), x \geq 0,$ be convex. Then, for $(x, t) \in \mathbb{R}_+ \times [0, T)$, the local risk tolerance function $r(x, t)$ is increasing in $\lambda$ and decreasing in $\sigma$. The reverse assertion holds for the local absolute risk aversion $\gamma(x, t)$.

**ii) Let the absolute risk aversion coe¢ cient $A(x), x \geq 0,$ be concave. Then, for $(x, t) \in \mathbb{R}_+ \times [0, T)$, the local absolute risk aversion function $\gamma(x, t)$ is increasing in $\lambda$ and decreasing in $\sigma$. The reverse assertion holds for the local risk tolerance function $r(x, t)$.

**Proof.** We only show the first part of assertion (i) since the rest of the statements follow trivially. To this end, let $\lambda > \hat{\lambda}$, and denote by $r(x, t; \lambda)$ and $r(x, t; \hat{\lambda})$ the corresponding solutions to (63) and (64). We are going to show that, for $(x, t) \in \mathbb{R}_+ \times [0, T]$,

$$r(x, t; \lambda) \leq r(x, t; \hat{\lambda}).$$

(65)

As shown in Proposition 12, if $R(x)$ is concave, the local absolute risk tolerance function $r(x, t)$ is also concave, for each $t \in [0, T)$. Therefore,

$$r_t \left( x, t; \hat{\lambda} \right) + \frac{1}{2} \lambda^2 r(x, t; \hat{\lambda}) r_{xx} \left( x, t; \hat{\lambda} \right)$$

$$= r_t \left( x, t; \lambda \right) + \frac{1}{2} \hat{\lambda}^2 r(x, t; \hat{\lambda}) r_{xx} \left( x, t; \hat{\lambda} \right)$$

$$+ \frac{1}{2} \left( \lambda^2 - \hat{\lambda}^2 \right) r(x, t; \hat{\lambda}) r_{xx} \left( x, t; \hat{\lambda} \right) \leq 0,$$

where we used that $r(x, t; \hat{\lambda})$ solves (16) and $r_{xx} \left( x, t; \hat{\lambda} \right) \leq 0$. Therefore, $r(x, t; \hat{\lambda})$ is a super-solution to the equation that $r(x, t; \lambda)$ solves with $r(x, T; \lambda) = r(x, T; \hat{\lambda})$. Using the comparison result in Proposition 16, (65) follows. □
The next result examines the monotonicity of the optimal feedback portfolio \( \pi^*(x,t) \) in terms of \( \lambda \) and \( \sigma \). We show that independently of the convexity (concavity) of the absolute risk tolerance coefficient, and thus of the analogous properties of the terminal condition \( \pi^*(x,T) \), the optimal portfolio function is always increasing in \( \lambda \) and decreasing in \( \sigma \).

**Proposition 22** The optimal feedback portfolio function \( \pi^*(x,t) \) is always increasing in \( \lambda \) and decreasing in \( \sigma \).

**Proof.** Let \( \lambda > \hat{\lambda} \) and denote by \( \pi^*(x,t;\lambda) \) and \( \pi^*(x,t;\hat{\lambda}) \) the corresponding portfolio functions. Then, for fixed \( \sigma \), both functions solve equation (63) which is independent of the Sharpe ratio, while \( \pi^*(x,T;\lambda) \geq \pi^*(x,T;\hat{\lambda}) \). Using the robustness result of Proposition 16 we easily conclude.

The monotonicity with respect to \( \sigma \) is slightly more involved since \( \sigma \) appears in both the equation and the terminal condition. To this end, we define the auxiliary function \( \tilde{\pi}(x,t;\sigma) = \sigma \pi^*(x,t;\sigma) \), which solves
\[
\tilde{\pi}_t + \frac{1}{2} \tilde{\pi}^2 \tilde{\pi}_{xx} = 0 \quad \text{and} \quad \tilde{\pi}(x,T) = \lambda R(x).
\]
Therefore, \( \tilde{\pi}(x,t;\sigma) \) is independent of \( \sigma \), yielding \( \frac{\partial \pi(x,t;\sigma)}{\partial \sigma} = 0 \). In turn, we deduce that \( \frac{\partial \pi^*(x,t;\sigma)}{\partial \sigma} = -\frac{1}{\sigma} \pi^*(x,t;\sigma) \) and using the positivity of \( \pi^*(x,t;\sigma) \) we conclude.

For the investment model with more than one stock analogous results can be obtained even though the intuitive interpretation of the market parameter \( m_i \) below is limited.

**Proposition 23**

i) Let the absolute risk tolerance coefficient \( R(x) \) be concave (convex). Then, for \( (x,t) \in \mathbb{R}_+ \times [0,T) \), the local absolute risk tolerance \( r(x,t) \) is decreasing (increasing) in \( |\lambda|^2 \).

ii) Let the absolute risk aversion coefficient \( A(x) \) be concave. Then, for \( (x,t) \in \mathbb{R}_+ \times [0,T) \), the local risk aversion \( \gamma(x,t) \) is increasing in \( |\lambda|^2 \).

From (18) and (19) we have that each optimal feedback portfolio component \( \pi^*_{-i}(x,t) \) satisfies
\[
\pi^*_{-i} + \frac{1}{2} m_i^2 (\pi^*_{-i})^2 \pi^*_{-i,xx} = 0 \quad \text{and} \quad \pi^*_{-i}(x,T) = \frac{|\lambda|}{m_i} R(x).
\]

We then have the following result.

**Proposition 24** For each \( i = 1, ..., N \), the optimal feedback component \( \pi^*_{-i}(x,t) \) is increasing in \( |\lambda| \) and decreasing in \( m_i \), where \( m_i = k_i^{-1} |\lambda| \) with \( k_i \) being the \( i^{th} \) element of the vector \( \sigma^{-1} \lambda \).
6.2 Sensitivities of the optimal processes in terms of the initial wealth

We conclude with the sensitivities of the optimal wealth and portfolio processes with respect to the initial condition. They follow from direct differentiation in the stochastic representations (26) and (28). Analogous sensitivities were first obtained for the optimal forward processes in Propositions 5 and 6 in [35].

To our knowledge, with the exception of (66), which was derived for a much more general market setting in [26], the rest of the assertions are, in the context of the Merton model, new.

**Proposition 25** The following assertions hold:

i) Let \( X^*_t, \ t \in [0,T], \) be the optimal wealth process with \( X^*_0 = x. \) Then,

\[
\frac{\partial X^*_t}{\partial x} = \frac{r(X^*_t, t)}{r(x, 0)},
\]

and

\[
\frac{\partial^2 X^*_t}{\partial x^2} = \frac{r_x(X^*_t, t) - r_x(x, 0)}{r(x, 0)} \frac{\partial X^*_t}{\partial x} = \frac{r(X^*_t, t)}{r^2(x, 0)} (r_x(X^*_t, t) - r_x(x, 0)).
\]

ii) Let \( \pi^*_t, \ t \in [0,T], \) be the optimal portfolio process. Then,

\[
\frac{\partial \pi^*_t}{\partial x} = \frac{\partial X^*_t}{\partial x} r_x(X^*_t, t) \sigma^{-1} \lambda = \frac{r(X^*_t, t)}{r(x, 0)} r_x(X^*_t, t) \sigma^{-1} \lambda
\]

and

\[
\frac{\partial^2 \pi^*_t}{\partial x^2} = \left( \frac{\partial^2 X^*_t}{\partial x^2} r_x(X^*_t, t) + \left( \frac{\partial X^*_t}{\partial x} \right)^2 r_{xx}(X^*_t, t) \right) \sigma^{-1} \lambda
\]

\[
= \frac{r(X^*_t, t)}{r^2(x, 0)} (r_x(X^*_t, t) (r_x(X^*_t, t) - r_x(x, 0)) + r(X^*_t, t) r_{xx}(X^*_t, t)) \sigma^{-1} \lambda.
\]

**Corollary 26** Let \( H(x, t) \) be the solution of (22) and (23) and \( N_t \) as in (31). Then,

\[
\frac{\partial X^*_t}{\partial x} = \frac{H_x(N_t, t)}{H_x(N_0, 0)} \quad \text{and} \quad \frac{\partial^2 X^*_t}{\partial x^2} = \frac{H_x(N_t, t)}{H^2_x(N_0, 0)} \left( \frac{H_{xx}(N_t, t)}{H_x(N_t, t)} - \frac{H_{xx}(N_0, 0)}{H_x(N_0, 0)} \right).
\]

Moreover,

\[
\frac{\partial \pi^*_t}{\partial x} = \frac{H_{xx}(N_t, t)}{H_x(N_0, 0)} \sigma^{-1} \lambda
\]

and

\[
\frac{\partial^2 \pi^*_t}{\partial x^2} = \frac{H_x(N_t, t)}{H^2_x(N_0, 0)} \times \left( \frac{H_{xx}(N_t, t)}{H_x(N_t, t)} - \frac{H_{xx}(N_0, 0)}{H_x(N_0, 0)} \right) + \frac{H_{xxx}(N_t, t) H_x(N_t, t) - H^2_{xx}(N_t, t)}{H^2_x(N_t, t)} \sigma^{-1} \lambda.
\]
Appendix

Proof of Proposition 5.

For simplicity, we set $|\lambda|^2 = 1$.

i) We need to show that for $\alpha \in (0, 1)$ and $(x, t) \in \mathbb{R} \times [0, T]$,

$$h(\alpha x + (1 - \alpha)y, t) \leq h(x, t)^\alpha h(y, t)^{1-\alpha}.$$ 

The Feynman-Kac formula, the log-convexity of the terminal datum and Hölder’s inequality yield

$$h(\alpha x + (1 - \alpha)y, t) = E(h_0(\alpha (x + W_{T-t}) + (1 - \alpha) (y + W_{T-t})))$$

$$\leq E((h_0(x + W_{T-t}))^\alpha (h_0(y + W_{T-t}))^{1-\alpha})$$

$$\leq (E(h_0(x + W_{T-t})))^\alpha (E(h_0(y + W_{T-t})))^{1-\alpha}$$

$$= (h(x, t))^\alpha (h(y, t))^{1-\alpha}.$$ 

ii) We need to show that for $\alpha \in (0, 1)$ and $(x, t) \in \mathbb{R} \times [0, T]$,

$$h(\alpha x + (1 - \alpha)y, t) \geq h(x, t)^\alpha h(y, t)^{1-\alpha}.$$ 

The Prékopa-Leindler inequality yields that, if for $0 < \alpha < 1, z, z' \in \mathbb{R}$, the positive functions $f, m, n$ satisfy

$$f(\alpha z + (1 - \alpha)z') \geq (m(z))^{\alpha} (n(z'))^{1-\alpha},$$

then, for $z \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f(z) \, dz \geq \left( \int_{-\infty}^{\infty} m(z) \, dz \right)^\alpha \left( \int_{-\infty}^{\infty} n(z) \, dz \right)^{1-\alpha}.$$ 

The log-concavity of $h_0(x)$ yields that for $\alpha \in (0, 1), z, z' \in \mathbb{R}$,

$$h_0(\alpha z + (1 - \alpha)z') \geq (h_0(z))^{\alpha} (h_0(z'))^{1-\alpha}.$$ 

Next, fix $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$ and define the functions

$$f(z; x, y, t) = e^{-\frac{(\alpha z + (1 - \alpha)y - z)^2}{4(T-t)}} h_0(z)$$

$$m(z; x, t) = e^{-\frac{(z-y)^2}{4(T-t)}} h_0(z) \quad \text{and} \quad n(z; y, t) = e^{-\frac{(y-z)^2}{4(T-t)}} h_0(z).$$

We easily see, that

$$f(\alpha z + (1 - \alpha)z'; x, y, t) \geq (m(z; x, t))^{\alpha} (n(z'; y, t))^{1-\alpha}.$$ 

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Indeed, from the log-concavity of the functions $h_0(x)$ and $e^{-x^2}$ we have

$$f(\alpha z + (1 - \alpha)z'; x, y, t) = e^{-\frac{(\alpha z + (1 - \alpha)z')^2}{4(t - t')}} h_0(\alpha z + (1 - \alpha)z')$$

$$\geq e^{-\frac{(\alpha x + (1 - \alpha)y - (1 - \alpha)z')^2}{4(t - t')}} (h_0(z))^\alpha (h_0(z'))^{1-\alpha}$$

$$\geq \left( e^{-\frac{(x-z)^2}{4(t-T)}} h_0(z) \right)^\alpha \left( e^{-\frac{(y-z')^2}{4(t-T)}} h_0(z') \right)^{1-\alpha}.$$ 

Therefore,

$$\int_{-\infty}^{\infty} e^{-\frac{(\alpha z + (1 - \alpha)z)^2}{4(t - t')}} h_0(z) \, dz = \left( \int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4(t-T)}} h_0(z) \, dz \right)^\alpha \left( \int_{-\infty}^{\infty} e^{-\frac{(y-z')^2}{4(t-T)}} h_0(z') \, dz \right)^{1-\alpha},$$

and we easily conclude.

References


