# Remarks on hedging and numeraire consistency in an incomplete binomial model

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### Abstract

In this paper, we provide an explicit pricing algorithm in an incomplete market framework and we investigate the associated hedging startegies. We also explore the numeraire consistency of the emerging pricing scheme.

## 1    Introduction

The classical arbitrage free pricing methodology is based upon the idea of risk replication. One makes investment allocations in order to dynamically replicate future liability. Clearly, this approach breaks down when the liability cannot be hedged completely by taking positions in the market. In such situations, of the so called incomplete markets, the notion of value itself and the associated risk management methodology have to be redefined.

A successful method for the pricing of unhedgeable risks is based on utility indifference. The central idea is to replace the traditional replication argument by the constraint of optimal behavior as an investor. Optimality refers to the individual risk preferences captured in terms of a utility and the corresponding value function. The price represents the amount which makes the investor indifferent to the various investment opportunities.

This concept of value was introduced originally by Hodges and Neuberger (1989) in relation to analysis of the transactions costs impact on the price. Latter on it was also extended to deal with other contexts of incompleteness. Analysis of indifference price properties was based so far on two dual approaches. The first one aims at solving the so called primary expected utility problem. Typically this approach works well in a Markovian setting in which it is straightforward to deduce that the indifference price satisfies a quasi-linear equation. Probabilistic representations of solutions to such equations can be derived in some special cases (cf. Musiela and Zariphopoulou (2001)). The dual approach seems to be more general. It leads to a structural result providing the indifference price representation as a supremum over the set of martingale measures of the difference between the expected value of the payoff and a certain penalty

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function, both depending on the martingale measure (cf. Frittelli (2000), Rouge and El Karoui (2000)). Unfortunately, none of the above approaches provides a general algorithm to compute the indifference price or gives a clear intuition for it.

One may argue that in many respects success of the arbitrage free pricing methodology lies in its simplicity. The main idea, especially when presented in the context of a one-period binomial model, is both intuitively obvious and mathematically trivial (see, for example, Musiela and Rutkowski (1997)). Unfortunately, neither of the above methods provide clear intuition for the indifference price. The main objective of this paper is to build intuition for this concept of value. We work with the classical one-period binomial model to which incompleteness is introduced by adding two elementary outcomes to the probability space. In spirit, this corresponds to the situation in which one considers options written on a non-traded asset, like in Davis (1999), (2000) or Musiela and Zariphopoulou (2001), and uses another dependent asset for pricing and risk management. Alternatively, one can also see this as related to the models with constraints, like in Rouge and El Karoui (2000).

It turns out that the hedgeable and unhedgeable components of risk are priced by financial markets and insurance valuation methods, respectively. Namely, the total risk at the end of a time period is conditioned on the hedgeable one leaving the risk that cannot be hedged. Then, this risk is priced by certainty equivalent. In the second step, the remaining hedgeable risk is priced by arbitrage. This defines the total risk at the beginning of the period and the valuation algorithm may be repeated. The first step uses insurance valuation method and returns a modified payoff in line with the preferences. The second step prices this new payoff by arbitrage. Each time reference is made to the same measure which, on the one hand, does not alter the relevant conditional distribution for the insurance valuation method and, on the other, posses a martingale property for the financial markets valuation approach. It is the minimal relative entropy martingale measure. The details are presented in Section 2. Section 3 focuses on the indifference price dependence on the risk aversion. Hedging related issues are discussed in Section 4. Numeraire independence of the indifference price and the associated structure of consistent utilities is addressed in Section 5.

## 2  Price representation

Consider a one-period model of a market with one riskless asset and two risky assets, of which only one is traded. For simplicity, consider initially zero interest rate. The current values of the risky traded and non-traded assets are denoted by $S_0$ and $Y_0$, respectively. The traded asset value at the end of the period is denoted by $S_T$. We assume that $S_T \in \left\{ S^d, S^u \right\}$ with $0 < S^d < S^u$. The non-traded asset value $Y_T$ satisfies $Y_T \in \left\{ Y^d, Y^u \right\}, Y^d < Y^u$.

Let $\Omega = \left\{ \omega_1, \omega_2, \omega_3, \omega_4 \right\}$ be a probability space and let $\mathbb{P}$ be a probability measure on the $\sigma-$algebra $\mathcal{F} = 2^\Omega$ of all subsets of $\Omega$. For each $i = 1, 2, 3, 4$ let $p_i = \mathbb{P}\left\{ \omega_i \right\} > 0$. View $S_T$ and $Y_T$ as random variables, i.e., as functions from $\Omega$

into $\mathbb{R}$ defined by

$$
\begin{array}{llll}
S_T(\omega_1) = S^u, & Y_T(\omega_1) = Y^u & \quad S_T(\omega_3) = S^d, & Y_T(\omega_3) = Y^u \\
S_T(\omega_2) = S^u, & Y_T(\omega_2) = Y^d & \quad S_T(\omega_4) = S^d, & Y_T(\omega_4) = Y^d.
\end{array}
$$

Clearly, such a model of a market is incomplete. There are four elementary outcomes in the probability space and only two assets available for trading and hence there are risk that cannot be hedged. One way to handle such situations is to use the price concept based on utility indifference. Below we recall the familiar definition.

Consider a portfolio consisting of $\alpha$ shares of stock and the amount $\beta$ invested in the riskless asset. The current value $X_0 = x$ of this portfolio is equal to $\beta + \alpha S_0 = x$. Its wealth $X_T$ at the end of the period is given by

$$
X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0).
$$

Consider now a claim $G$ settling at the end of the period, at time $T$. Mathematically speaking, $G$ is just a random variable defined on $\Omega$ with values in $\mathbb{R}$, namely $G(\omega_i) = g_i$ for $i = 1, 2, 3, 4$. In pricing of $G$, we need to specify our risk preferences. We choose to work with exponential utility of the form

$$
-e^{-\gamma x}, x \in \mathbb{R}, \gamma > 0.
$$

Optimality of investments, which will ultimately yield the price of $G$, is examined via the value function

$$
V_G(x) = \sup_\alpha E_\mathbb{P}\left(-e^{-\gamma(X_T-G)}\right) = e^{-\gamma x}\sup_\alpha E_\mathbb{P}\left(-e^{-\gamma\alpha(S_T-S_0)+\gamma G}\right). \tag{1}
$$

**Definition 1** *The indifference price of the claim $G$ is defined as the amount $\nu$ for which the two value functions $V_G$ and $V_0$, defined in (1) and corresponding to the claims $G$ and $0$, respectively, coincide. Namely, $\nu$ is the amount which satisfies*

$$
V_0(x) = V_G(x + \nu) \tag{2}
$$

*for any initial wealth $x$.*

As already mentioned before, there is a general structural result which gives a probabilistic representation of $\nu$. In order to present it we begin with a characterization of all equivalent martingales measures. Recall that the measure $Q$ on $\Omega$ is an equivalent martingale measure if $Q\{\omega_i\} = q_i > 0, q_1 + q_2 + q_3 + q_4 = 1$ and

$$
E_Q S_T = S_0. \tag{3}
$$

Obviously, condition (3) holds if and only if

$$
S^u(q_1 + q_2) + S^d(q_3 + q_4) = S_0
$$

or equivalently if and only if

$$q_1 + q_2 = q = \frac{S_0 - S^d}{S^u - S^d}. \tag{4}$$

Consequently, the set $\mathcal{Q}_e$ of equivalent martingale measures coincides with the following set of vectors in $\mathbb{R}_+^4$

$$\mathcal{Q}_e = \{(q_1, q_2, q_3, q_4) : q_1 + q_2 + q_3 + q_4 = 1, q_1 + q_2 = q\}, \tag{5}$$

where $q$ is defined in (4).

Recall also that the relative to $\mathbb{P}$ entropy of $Q$ is defined by

$$H(Q \,|\mathbb{P}) = \sum_{i=1}^{4} q_i \log \frac{q_i}{p_i}. \tag{6}$$

Therefore the martingale measure $\mathbb{Q} \in \mathcal{Q}_e$ with the minimal relative to $\mathbb{P}$ entropy is given by the solution to the problem

$$H(\mathbb{Q} \,|\mathbb{P}) = \min_{Q \in \mathcal{Q}_e} H(Q \,|\mathbb{P}).$$

It is straightforward to see that $\mathbb{Q}\{\omega_i\} = q_i^0, i = 1, ..., 4$, where

$$q_1^0 = q\frac{p_1}{p_1 + p_2}, q_2^0 = q\frac{p_2}{p_1 + p_2}, q_3^0 = (1 - q)\frac{p_3}{p_3 + p_4}, q_4^0 = (1 - q)\frac{p_4}{p_3 + p_4}, \tag{7}$$

while $q$ is defined in (4). Consequently, the minimal level of the relative entropy is

$$H(\mathbb{Q} \,|\mathbb{P}) = q \log \frac{q}{p_1 + p_2} + (1 - q) \log \frac{1 - q}{p_3 + p_4} \tag{8}$$

Using the above notation we are now ready to present the general structural result (see for example Delbaen et al. 2002). Namely, the indifference price $\nu$ of the claim $G$ is given by the following formula

$$\nu = \sup_{Q \in \mathcal{Q}_e} \left( E_Q G - \frac{1}{\gamma} \left( H(Q \,|\mathbb{P}) - H(\mathbb{Q} \,|\mathbb{P}) \right) \right). \tag{9}$$

**Remark 2** *The above representation of the indifference price is valid in general incomplete models, where the discounted price process are only assumed to follow locally bounded semimartingales. This clearly demonstrates universality of the indifference price concept. Unfortunately, formula (9) cannot be used directly to compute $\nu$. Also, it does not provide a clear intuition for this new concept of value.*

Consider, for example, a claim of the form $G = g_1(S_T)$. Clearly in this case the indifference price must coincide with the arbitrage free price for there is no risk that cannot be hedged. Indeed one can construct a nested complete one-period binomial model. Namely, let $\Omega' = \{\omega_1', \omega_2'\}$, where $\omega_1' = \{\omega_1, \omega_2\}$

and $\omega_2' = \{\omega_3, \omega_4\}$, set $p' = p_1 + p_2$ and view $S_T$ as a random variable defined on $\Omega'$ with the values $S_T(\omega_1') = S^u$ and $S_T(\omega_2') = S^d$. Obviously, the arbitrage free price is equal to $qg(S^u) + (1-q)g(S^d)$, where $q$ is given in (4) and it coincides with the indifference price calculated using (9), characterization of the equivalent martingale measures (4) and the definition (6).

Consider now a claim of the form $G = g_2(Y_T)$ and assume for simplicity that the random variables $S_T$ and $Y_T$ are independent under the measure $\mathbb{P}$. Intuitively, in this case the presence of the traded asset should not affect the price. Indeed, working directly with the value function (1) and the definition (2) it is straightforward to deduce that

$$\nu = \frac{1}{\gamma} \log E_{\mathbb{P}} e^{\gamma g_2(Y_T)}.$$

Therefore, the indifference price coincides with one of the actuarial valuation principles, namely, certainty equivalent. Note however that this representation of the price does not follow from (9) trivially.

The situation gets even more complicated when one considers a claim of the form

$$G = g_1(S_T) + g_2(Y_T).$$

One could be tempted to price $G$ by first pricing $g_1(S_T)$ by arbitrage, next pricing $g_2(Y_T)$ by certainty equivalent, and adding the results. Intuitively this should work when $S_T$ and $Y_T$ are independent, however this also cannot work under strong dependence between the two variables, for example when $Y_T$ is a function of $S_T$.

To get a better intuition for what happens we will analyze further the value function (1). Elementary transformations lead to the following expression

$$V_G(x) = e^{-\gamma x} \sup_\alpha \left( -e^{-\gamma \alpha (S^u - S_0)} \left( e^{\gamma g_1} p_1 + e^{\gamma g_2} p_2 \right) \right.$$

$$\left. -e^{-\gamma \alpha (S^d - S_0)} \left( e^{\gamma g_3} p_3 + e^{\gamma g_4} p_4 \right) \right).$$

The optimal number of shares $\alpha_0$ which maximizes $V_G(x)$ can be calculated by simple differentiation, it is equal to

$$\alpha_0 = \frac{1}{\gamma (S^u - S^d)} \log \frac{(S^u - S_0)(e^{\gamma g_1} p_1 + e^{\gamma g_2} p_2)}{(S_0 - S^d)(e^{\gamma g_3} p_3 + e^{\gamma g_4} p_4)}. \tag{10}$$

Further straightforward but tedious calculations lead to the following representation of the value function

$$V_G(x) = -e^{-\gamma x} \frac{1}{q^q (1-q)^{1-q}} \left( e^{\gamma g_1} p_1 + e^{\gamma g_2} p_2 \right)^q \left( e^{\gamma g_3} p_3 + e^{\gamma g_4} p_4 \right)^{1-q}, \tag{11}$$

where

$$q = \frac{S_0 - S^d}{S^u - S^d}. \tag{12}$$

For $G = 0$ the value function takes the form

$$V_0(x) = -e^{-\gamma x} \frac{(p_1 + p_2)^q (p_3 + p_4)^{1-q}}{q^q (1-q)^{1-q}}. \tag{13}$$

It easily follows from the definition of the indifference price (2) and the representations (11) (13) of the relevant value functions that

$$\nu = \frac{1}{\gamma} \left( q \log \frac{e^{\gamma g_1} p_1 + e^{\gamma g_2} p_2}{p_1 + p_2} + (1-q) \log \frac{e^{\gamma g_3} p_3 + e^{\gamma g_4} p_4}{p_3 + p_4} \right). \tag{14}$$

As expected the pricing formula depends on both, the historical probabilities $p_i$ and the risk neutral probability $q$ associated with the nested complete one-period Cox-Ross-Rubinstein model. In the remaining part of this section we will derive an alternative to (9) and intuitively more meaningful probabilistic representation of the indifference price $\nu$.

From now on it will be convenient to think that the claim $G$ is in fact a function, say $g$, of $S_T$ and $Y_T$. Consistently with the previous notation $G = g(S_T, Y_T)$ and $g(S^u, Y^u) = g_1, g(S^u, Y^d) = g_2, g(S^d, Y^u) = g_3, g(S^d, Y^d) = g_4$. Note that the expressions involving the historical probabilities can be written in terms of the conditional expectations, namely

$$\frac{e^{\gamma g_1} p_1 + e^{\gamma g_2} p_2}{p_1 + p_2} = E_{\mathbb{P}} \left( e^{\gamma G} | S_T = S^u \right)$$

and

$$\frac{e^{\gamma g_3} p_3 + e^{\gamma g_4} p_4}{p_3 + p_4} = E_{\mathbb{P}} \left( e^{\gamma G} | S_T = S^d \right).$$

A simple way to represent the price $\nu$ with respect to one probability distribution is to identify an equivalent martingale measure $\mathbb{Q}^0$ with the same conditional distribution of the non-traded asset given the traded asset as for the historical measure $\mathbb{P}$. More specifically, take the martingale measure $\mathbb{Q}^0$ for which

$$\mathbb{Q}^0 \{ Y_T = Y^u | S_T = S^u \} = \frac{q_1^0}{q} = \frac{p_1}{p_1 + p_2} \quad \text{or} \quad q_1^0 = q \frac{p_1}{p_1 + p_2}$$

$$\mathbb{Q}^0 \{ Y_T = Y^d | S_T = S^u \} = \frac{q_2^0}{q} = \frac{p_2}{p_1 + p_2} \quad \text{or} \quad q_2^0 = q \frac{p_2}{p_1 + p_2}$$

$$\mathbb{Q}^0 \{ Y_T = Y^u | S_T = S^d \} = \frac{q_3^0}{1-q} = \frac{p_3}{p_3 + p_4} \quad \text{or} \quad q_3^0 = (1-q) \frac{p_3}{p_3 + p_4}$$

$$\mathbb{Q}^0 \{ Y_T = Y^d | S_T = S^d \} = \frac{q_4^0}{1-q} = \frac{p_4}{p_3 + p_4} \quad \text{or} \quad q_4^0 = (1-q) \frac{p_4}{p_3 + p_4}.$$

Obviously, as evidenced by (7) $\mathbb{Q}^0 = \mathbb{Q}$ and hence it coincides with the minimal relative entropy martingale measure. We summarize our results in the Proposition below

**Proposition 3** *The indifference price of the claim $G = g\left(S_T, Y_T\right)$ is given by the following expression*

$$\nu = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma G}\,|S_T\right)\right),$$

*where $\mathbb{Q}$ is the minimal relative entropy martingale measure, i.e.,*

$$\mathbb{Q}\left\{\omega_1\right\} = q\frac{p_1}{p_1 + p_2}, \mathbb{Q}\left\{\omega_2\right\} = q\frac{p_2}{p_1 + p_2},$$

$$\mathbb{Q}\left\{\omega_3\right\} = \left(1 - q\right)\frac{p_3}{p_3 + p_4}, \mathbb{Q}\left\{\omega_4\right\} = \left(1 - q\right)\frac{p_4}{p_3 + p_4},$$

*and*

$$q = \frac{S_0 - S^d}{S^u - S^d}.$$

**Remark 4** *Note that the pricing algorithm seems to be intuitively obvious. One should condition the total risk at the end of a time period on the hedgeable one in order to extract the risk that cannot be hedged. Then, price that risk by certainty equivalent, and in the second step, price the remaining hedgeable risk by arbitrage. It is also clear that one should not alter the conditional distribution from its historical values. Indeed, all relevant to the pricing market information has been already extracted.*

To illustrate further intuitively natural properties of the indifference price, consider the claim of the form $G = g_1\left(S_T\right) + g_2\left(Y_T\right)$. Simple calculations show that the price $\nu$ is equal to

$$\nu = E_{\mathbb{Q}}\left(g_1\left(S_T\right)\right) + E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma g_2(Y_T)}\,|S_T\right)\right),$$

meaning that the indifference price of such a claim is the sum of the arbitrage price of $g_1\left(S_T\right)$ and the arbitrage price of $\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma g_2(Y_T)}\,|S_T\right)$ which in turn is the certainty equivalent of $g_2\left(Y_T\right)$ calculated with respect to the conditional distribution of $Y_T$ given $S_T$ under the measure $\mathbb{Q}$, or under the measure $\mathbb{P}$ because they coincide. Clearly the unhegeable risk has been extracted optimally and priced by the certainty equivalent, and the remaining risk has been priced by arbitrage.

## 3    Dependence on the risk aversion

The nonlinearity in the pricing algorithm corresponds to the risk preferences allocated to the unhedgeable component of risk. In the general case of the claim $G = g\left(S_T, Y_T\right)$ we have

**Proposition 5** *The function*

$$\gamma \to \nu(\gamma) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma G}\,|S_T\right)\right)$$

*from $\mathbb{R}_+$ into $\mathbb{R}$ is increasing and continuous. Moreover,*

$$\nu(0+) = E_{\mathbb{Q}}G,$$

$$\nu(\infty-) = q\max(g_1, g_2) + (1-q)\max(g_3, g_4) = E_{\mathbb{Q}}\|G\|_{L^{\infty}_{\mathbb{Q}\{\cdot|S_T\}}},$$

$$\nu'(0+) = \frac{1}{2}E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(G\,|S_T\right)\right),$$

*and hence*

$$\nu(\gamma) = E_{\mathbb{Q}}G + \frac{1}{2}E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(G\,|S_T\right)\right)\gamma + o(\gamma).$$

*The indifference price is consistent with the no arbitrage principle, namely,*

$$\inf_{Q\in\mathcal{Q}_a} E_Q G \le E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma G}\,|S_T\right)\right) \le \sup_{Q\in\mathcal{Q}_a} E_Q G,$$

*where $\mathcal{Q}_a$ is the set of absolutely continuous with respect to $\mathbb{P}$ martingale measures.*

**Proof.** Let $0 < \gamma_1 < \gamma_2$, then by the Holder inequality we get

$$E_{\mathbb{Q}}\left(e^{\gamma_1 G}\,|S_T\right) \le \left(E_{\mathbb{Q}}\left(e^{\gamma_2 G}\,|S_T\right)\right)^{\frac{\gamma_1}{\gamma_2}}$$

and hence the indifference price $\nu(\gamma)$ is an increasing function of the risk aversion $\gamma$. Continuity is trivial, the limiting values at $0$ and the first order expansion can be either derived directly or using the formula

$$\gamma\nu'(\gamma) = E_{\mathbb{Q}}\frac{E_{\mathbb{Q}}\left(Ge^{\gamma G}\,|S_T\right)}{E_{\mathbb{Q}}\left(e^{\gamma G}\,|S_T\right)} - \nu(\gamma)$$

and its derivative. The limit at $\infty$ is a consequence of the formula below

$$\lim \frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma G}\,|S_T = S^u\right) = \max(g_1, g_2) = \|G\|_{L^{\infty}_{\mathbb{Q}\{\cdot|S_T=S^u\}}}$$

and of the analogous formula for the case of conditioning on the event $S_T = S^d$. To show consistency with the no arbitrage principle assume, without loss of generality, that $g_1 < g_2$ and $g_3 < g_4$ and note that

$$\inf_{Q\in\mathcal{Q}_a} E_Q G \le E_{\mathbb{Q}}G = \nu(0+) \le \nu(\infty-) = E_{\overline{Q}}G \le \sup_{Q\in\mathcal{Q}_a} E_Q G,$$

where $\overline{Q}$ is a martingale measure for which

$$\overline{Q}\{\omega_1\} = 0, \overline{Q}\{\omega_2\} = q, \overline{Q}\{\omega_3\} = 0, \overline{Q}\{\omega_4\} = 1 - q.$$

∎

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# 4 Hedging related issues

The second important aspect of our analysis is hedging. Note that the optimal number of shares (10) can be split into two parts, namely,

$$\alpha_0 = \alpha_0^0 + \alpha_0^1,$$

where

$$\alpha_0^0 = \frac{1}{\gamma\left(S^u - S^d\right)} \log \frac{\left(S^u - S_0\right)\left(p_1 + p_2\right)}{\left(S_0 - S^d\right)\left(p_3 + p_4\right)}$$

is the optimal number of shares corresponding to the case when $G = 0$, while

$$\alpha_0^1 = \frac{1}{\gamma\left(S^u - S^d\right)} \log \frac{\left(e^{\gamma g_1} p_1 + e^{\gamma g_2} p_2\right)\left(p_3 + p_4\right)}{\left(p_1 + p_2\right)\left(e^{\gamma g_3} p_3 + e^{\gamma g_4} p_4\right)}$$

$$= \frac{1}{\gamma\left(S^u - S^d\right)} \log \frac{E_{\mathbb{Q}}\left(e^{\gamma G}\left|S_T = S^u\right.\right)}{E_{\mathbb{Q}}\left(e^{\gamma G}\left|S_T = S^d\right.\right)} = \frac{\partial \nu}{\partial S_0}$$

represents the residual optimal number of shares due to the presence of the option. The residual optimal wealth generated due to the derivative contract is denoted by $L_t, t = 0, T$. Clearly $L_0 = \nu$, which is the option price, and at $T$ we have

$$L_T = \nu + \frac{\partial \nu}{\partial S_0}\left(S_T - S_0\right). \tag{15}$$

It follows that the process $L$ is a martingale under any martingale measure, and hence in particular under the minimal relative entropy martingale measure $\mathbb{Q}$. The amount

$$L_T - G \tag{16}$$

represents the surplus, i.e., the difference between the optimal level of the residual wealth and the option payoff. Obviously, when $G$ depends only on $S_T$, i.e., $G = g\left(S_T\right)$, then $L_T = G$. In general, the expected surplus, given by

$$E_{\mathbb{Q}}\left(L_T - G\right) = \nu - E_{\mathbb{Q}} G = \frac{1}{2} E_{\mathbb{Q}}\left(Var_{\mathbb{Q}}\left(G\left|S_T\right.\right)\right)\gamma + o\left(\gamma\right)$$

and calculated under the minimal relative entropy martingale measure $\mathbb{Q}$ is positive. The expected utility of the surplus, calculated under the original measure $\mathbb{P}$, is equal to $-1$ and hence is the same as the utility of 0 surplus. Indeed, because we have

$$\nu + \frac{\partial \nu}{\partial S_0}\left(S^u - S_0\right) = \frac{1}{\gamma} \log E_{\mathbb{P}}\left(e^{\gamma G}\left|S_T = S^u\right.\right),$$

$$\nu + \frac{\partial \nu}{\partial S_0}\left(S^d - S_0\right) = \frac{1}{\gamma} \log E_{\mathbb{P}}\left(e^{\gamma G}\left|S_T = S^d\right.\right),$$

straightforward transformations give

$$E_{\mathbb{P}}\left(-e^{-\gamma(L_T - G)}\right) = E_{\mathbb{P}}\left(-e^{-\gamma(L_T - G)}\left|S_T = S^u\right.\right)\mathbb{P}\left\{S_T = S^u\right\}$$

$$+E_{\mathbb{P}}\left(-e^{-\gamma(L_T-G)}\,\big|S_T=S^d\right)\mathbb{P}\left\{S_T=S^d\right\}=-1.$$

Below we summarize the results related to hedging

**Proposition 6** *The residual optimal wealth (15) is a martingale under any martingale measure. The expected surplus $E_{\mathbb{Q}}\left(L_T-G\right)$ is positive. The expected utility of the surplus is $E_{\mathbb{P}}\left(-e^{-\gamma(L_T-G)}\right)=-1$ and hence it coincides with the utility of $0$ surplus. Alternatively, the certainty equivalent value of the surplus $L_T-G$ is $\frac{1}{\gamma}\log E_{\mathbb{P}}e^{-\gamma(L_T-G)}=0$.*

**Remark 7** *It is straightforward to observe that all the above results remain valid in the case when $Y_T$ is an arbitrary, and not only two valued random variable, and the claim $G$ is bounded.*

# 5 Numeraire independence and consistent utilities

One of the fundamentally important properties of the arbitrage free price is its independence of a numeraire. The indifference price will also be numeraire independent provided the appropriate relationships between the relevant units are build into the utility/value function.

Consider first the relationship between the forward and spot units, that is, the case of a non-zero interest rate $r$ over the time period. It is straightforward to verify that working with the forward wealth process leads to the same value function (11), where the argument $x$ represents the forward value of the initial wealth. Consequently, the indifference price as defined in (2) is also expressed in the forward units. Therefore the spot indifference price of the claim $G$ is given by

$$\frac{1}{1+r}E_{\mathbb{Q}}\left(\frac{1}{\gamma_F}\log E_{\mathbb{Q}}\left(e^{\gamma_F G}\,|S_T\right)\right),$$

where $\gamma_F$ represents the risk aversion associated with the forward units. Alternatively, discounting the terminal wealth $X_T$ and the claim $G$ by the interest rate $r$, i.e., expressing both, the wealth process and the claim in the spot units leads to the situation which can be reduced to the zero interest rate case by changing the former claim $G$ into $\frac{G}{1+r}$ . Consequently, the spot indifference price of the claim $G$ is given by

$$\nu\left(\gamma,G\right)=E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\frac{G}{1+r}}\,|S_T\right)\right). \tag{17}$$

Note that the two prices coincide for all claims if

$$\gamma_F=\frac{\gamma}{1+r}. \tag{18}$$

This translates into specification (18) of the risk aversion parameter associated with the forward units, with the base risk aversion parameter $\gamma$ being associated

with the spot units. The measure $\mathbb{Q}$ is the minimal relative entropy martingale measure for the process $S_t^* = S_t/B_t, t = 0, T$ of the stock price $S_t$ discounted by the bond price $B_t$ with $B_0 = 1$ and $B_T = 1 + r$.

Suppose now that we have chosen the stock price $S$ as a numeraire. In this case, we look for a measure $\mathbb{Q}^S$ under which the process $B_t^* = B_t/S_t$ of the bond price discounted by the stock price is a martingale. Obviously all such measures are determined by the equality

$$E_{\mathbb{Q}^S} B_T^* = B_0^*,$$

or more explicitly by

$$\frac{1+r}{S^u} \left(q_1^S + q_2^S\right) + \frac{1+r}{S^d} \left(q_3^S + q_4^S\right) = \frac{1}{S_0},$$

where $\mathbb{Q}^S \{\omega_i\} = q_i^S, i = 1, 2, 3, 4$. Setting $q^S = q_1^S + q_2^S$, one finds that

$$q^S = \left(\frac{1}{S^d} - \frac{1}{(1+r)\,S_0}\right) \frac{S^u S^d}{S^u - S^d}.$$

We choose the martingale measure $\mathbb{Q}^S$ with the minimal relative to $\mathbb{P}$ entropy by setting

$$\mathbb{Q}^S \{\omega_1\} = q^S \frac{p_1}{p_1 + p_2}, \mathbb{Q}^S \{\omega_2\} = q^S \frac{p_2}{p_1 + p_2},$$

$$\mathbb{Q}^S \{\omega_3\} = \left(1 - q^S\right) \frac{p_3}{p_3 + p_4}, \mathbb{Q}^S \{\omega_4\} = \left(1 - q^S\right) \frac{p_4}{p_3 + p_4}.$$

Note that the wealth process $X$ expressed in terms of the numeraire $S$ refers to the number of shares of the stock held in the portfolio at times $0$ and $T$. Therefore the risk aversion parameter $\gamma_S$ corresponding to this unit must be adjusted accordingly in order to be consistent with the base risk aversion parameter $\gamma$ associated with the spot units. Clearly, if we set

$$\gamma_S = \frac{\gamma S_T}{1 + r} \tag{19}$$

then the pricing algorithm is still valid. Indeed, associated with this numeraire, the certainty equivalent of the payoff $G$ is equal to

$$\frac{1}{\gamma_S} \log E_{\mathbb{P}} \left(e^{\gamma_S \frac{G}{S_T}} | S_T\right). \tag{20}$$

Moreover, because the conditional distributions are the same for the measure $\mathbb{P}$ and $\mathbb{Q}^S$ the indifference price should be given by

$$S_0 E_{\mathbb{Q}^S} \left(\frac{1}{\gamma_S} \log E_{\mathbb{Q}^S} \left(e^{\gamma_S \frac{G}{S_T}} | S_T\right)\right). \tag{21}$$

But with $\gamma_S$ given by (19) we get that (21) equals

$$S_0 E_{\mathbb{Q}^S}\left(\frac{1+r}{\gamma S_T}\log E_{\mathbb{P}}\left(e^{\frac{\gamma S_T}{1+r}\frac{G}{S_T}}|S_T\right)\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\frac{G}{1+r}}|S_T\right)\right)$$

because

$$S_0 E_{\mathbb{Q}^S}\frac{G}{S_T} = E_{\mathbb{Q}}\frac{G}{1+r} \tag{22}$$

for any payoff $G$ dependent on $S_T$ only (see Chapter 1 in Musiela and Rutkowski (1997)).

Consider now a general case of an arbitrary numeraire $N$ and denote by $\gamma_N$ the associated with it risk aversion parameter. Certainty equivalent of the unhedgeable risk associated with $N$ is

$$\frac{1}{\gamma_N}\log E_{\mathbb{P}}\left(e^{\gamma_N\frac{G}{N_T}}|S_T\right).$$

Obviously, the base risk aversion parameter $\gamma$ is associated with the riskless bond $B$. We have the following result.

**Proposition 8** *Indifference prices are numeraire independent if and only if*

$$\gamma_N = \frac{\gamma N_T}{1+r}. \tag{23}$$

**Proof.** The *if* part repeats the above arguments for an arbitrary numeraire. Denote by $\mathbb{Q}^N$ the martingale measure associated with the numeraire $N$ and with the minimal relative to $\mathbb{P}$ entropy and assume that for all $G$

$$N_0 E_{\mathbb{Q}^N}\left(\frac{1}{\gamma_N}\log E_{\mathbb{Q}^N}\left(e^{\gamma_N\frac{G}{N_T}}|S_T\right)\right) = E_{\mathbb{Q}}\left(\frac{1}{\gamma}\log E_{\mathbb{Q}}\left(e^{\gamma\frac{G}{1+r}}|S_T\right)\right). \tag{24}$$

Using (22) with $S$ replaced by $N$ and the fact that the conditional distributions coincide we transform the left hand side of (24) into

$$E_{\mathbb{Q}}\left(\frac{N_T}{\gamma_N(1+r)}\log E_{\mathbb{Q}}\left(e^{\gamma_N\frac{G}{N_T}}|S_T\right)\right)$$

and the *only if* statement follows. $\blacksquare$

**Remark 9** *The pricing algorithm consists of two steps. In the first we calculate the certainty equivalent which itself depends on two things. Namely, specification of the probability distribution and the choice of units. The probability distribution choice is obvious. One should work with the conditional distribution of the total risk, given the hedgeable one, under the original measure $\mathbb{P}$. The choice of units is arbitrary but the risk aversion parameter must to be adjusted accordingly if we want the certainty equivalent to be numeraire independent. In other words, the concept of utility and risk preferences is formulated with respect to a base unit, say the spot. For any other unit, the utility may be defined in a consistent*

*way by converting accordingly the risk aversion parameter like in (18), (19) or generally in (23). The second step of the pricing algorithm is the risk-neutral valuation which is, of course, numeraire independent. Remarkably, the indifference price is calculated with respect to a martingale measure which has minimal relative to $\mathbb{P}$ entropy and hence it does no change the conditional distribution of the total risk, given the hedgeable one.*

**Proposition 10** *As a function of the claim $G$ the indifference price $\nu(\gamma, G)$ given by (17) satisfies the following properties*

$$\nu(\gamma, 0) = 0,$$

$$\nu(\gamma, G + c) = \nu(\gamma, G) + c,$$

$$G_1 \leq G_2 \Rightarrow \nu(\gamma, G_1) \leq \nu(\gamma, G_2),$$

$$\nu(\gamma, \alpha G_1 + (1 - \alpha) G_2) \leq \alpha \nu(\gamma, G_1) + (1 - \alpha) \nu(\gamma, G_2).$$

**Proof.** Translation invariance and monotonicity are obvious. Convexity follows using Holder inequality. ∎

# 6  Generalizations

Extension of the pricing algorithm to the multi-period discrete time models is intuitively obvious and its ingredients can already be found in Smith and McCradle (1998). Even if not all the relevant mathematical steps are trivial, they essentially only refer to the stochastic dynamic principle. One needs to identify the price at the beginning of a time period with the payoff at the end of the previous time period and then iterate the algorithm. Passage from the discrete to the continuous time models uses standard mathematical arguments. It appears that, in general, the indifference price is related to the concept of non-linear expectations introduced in Coquet et al. (2002). In a Markovian model set up the algorithm provides a new probabilistic representation of solutions to the quasi-linear parabolic equations which the indifference pricing function satisfies. All of these results require much more advanced mathematical arguments and will be dealt with in the separate papers.

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