A note on the term structure of risk aversion in utility-based pricing systems

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Abstract

We study utility-based pricing systems for options written on a non-traded asset in the presence of a correlated traded asset. We develop and analyse a concept of the term structure of risk aversion which unables us to consider options of different maturities in a way which is consistent with the present value calculations. In our framework European options of different maturities are priced relatively to a given portfolio rather then rather then realtively to the market porfolio

1 Introduction

In this paper we develop further the analysis conducted by Musiela and Zariphopoulou (2001) on pricing of a derivative with expiration T written on a non-traded asset Y in the presence of a correlated traded asset S and of a riskless bond B with maturity T.

The tradable asset's price is a log-normal diffusion satisfying

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dW_s^1, \\ \\ S_t = S > 0. \end{cases} \tag{1}$$

The level of the non-traded asset is given by

$$\begin{cases} dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s, \\ \\ Y_t = y \in \mathcal{R}. \end{cases} \tag{2}$$

The processes W_t^1 and W_t are standard Brownian motions defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$,where $\mathcal{F}_t$ is the augmented σ-algebra generated by $\left(W_s^1, W_s, s \leq t\right)$. The Brownian motions are correlated with correlation $\rho \in (-1, 1)$. Assumptions on the drift and diffusion coefficients $b(\cdot, \cdot)$ and $a(\cdot, \cdot)$, respectively, are such that the above equation has a unique strong solution.

The bond price with maturity T is given by

$$B_s = e^{-r(T-s)}, \quad t \leq s \leq T. \tag{3}$$

The derivative to be priced is of European type with payoff $g(Y_T)$, at expiration T. The *writer's indifference price* of $g(Y_T)$ is defined as the amount h such that the investor is indifferent between the following two scenarios: optimize his utility payoff without employing the derivative and optimize his utility payoff taking into account, on the one hand the liability $g(Y_T)$ at expiration T, and on the other, the *compensation* h at time of inscription t. It turns out (see Musiela and Zariphopoulou (2001)) that when the individual risk preferences are modelled via an exponential utility function

$$U(x) = -e^{\gamma x} \tag{4}$$

with the risk aversion parameter $\gamma > 0$ then

$$h = h(y,t) = e^{-r(T-t)} \frac{1}{\gamma(1-\rho^2)} \ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t = y\right)\right), \tag{5}$$

where $\widetilde{\mathbb{P}}$ is given by

$$\widetilde{\mathbb{P}}(A) = E\left(\exp\left(-\rho\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\rho^2\frac{(\mu-r)^2}{\sigma^2}T\right)I_A\right), A \in \mathcal{F}_T. \tag{6}$$

The concept of investor's indifference used in the price determination refers to the comparison of the two value functions expressed in the forward wealth units. The compensation $\widetilde{h}$ at time of inscription t, is also expressed in the forward units, and hence is called the *writer's forward indifference price.* The amount $\widetilde{h}$ is given by the following formula (see Musiela and Zariphopoulou (2001))

$$\widetilde{h}(y,t) = \frac{1}{\gamma(1-\rho^2)} \ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t = y\right)\right) \tag{7}$$

$$= \frac{1}{\gamma(1-\rho^2)} \ln\left(E\left(e^{-\rho\frac{\mu-r}{\sigma}(W_T-W_t)-\frac{1}{2}\rho^2\frac{(\mu-r)^2}{\sigma^2}(T-t)}e^{\gamma\left(1-\rho^2\right)g(Y_T)} \,|Y_t = y\right)\right).$$

It is desirable for a pricing mechanism to satisfy what one may call a projection property. Namely, the price at time s of a claim with maturity T should be the same as the price calculated in the following two stages. First the price of the same claim at time t, assuming $s \leq t \leq T$, and then considering the result as a new claim with maturity t its price at time s. It is well known that the discounted arbitrage free prices are martingales and hence are given by the conditional expectations of the claims, calculated under the appropriate measures, and as such are linear projection operators. The forward indifference price given by the above formula depends on the risk aversion which in principle may

depend on the option maturity. In the first instance we assume the same risk aversion for both maturities T and t.

Seen from the date $s \leq t$ the price $\widetilde{h}(Y_t, t)$ if viewed as a claim written on the non-traded asset Y can be priced again, giving after straightforward transformations

$$\frac{1}{\gamma(1-\rho^2)} \ln\left(E\left(e^{-\rho\frac{\mu-r}{\sigma}(W_t-W_s)-\frac{1}{2}\rho^2\frac{(\mu-r)^2}{\sigma^2}(t-s)}e^{\gamma\left(1-\rho^2\right)\widetilde{h}(Y_t,t)}\,|Y_s=y\right)\right)$$

$$=\frac{1}{\gamma(1-\rho^2)} \ln\left(E\left(e^{-\rho\frac{\mu-r}{\sigma}(W_T-W_s)-\frac{1}{2}\rho^2\frac{(\mu-r)^2}{\sigma^2}(T-s)}e^{\gamma\left(1-\rho^2\right)g(Y_T)}\,|Y_s=y\right)\right)$$

which we recognize as the forward indifference writer's price at time s for the settlement date T of the claim $g(Y_T)$. We conclude then that the projection property holds for the forward writer's indifference price. The spot price (5) is defined by

$$h(y,t) = e^{-r(T-t)}\widetilde{h}(y,t)$$

thanks to the presence of the bond B with maturity T.

In this paper we develop further the concept of pricing based on the relationship of indifference. Instead of considering a single payoff we consider a portfolio of options with different payoffs and maturities. This leads to certain complications which are primarily due to the nonlinearity of the pricing formula (5) with respect to the payoff. We begin with the analysis of the price dependence on the option maturity which leads to the introduction in the following section of the concept of the term structure of risk aversion. Next we propose a pricing mechanism based on the indifference concept which is relative to a given portfolio. This enables us to benefit from the diversification effect when pricing the unhedgeable component of risk.

2 Term structure of risk aversion

In order to analyze the case of options with different maturities we assume from now on that we trade the discount bonds of all maturities T. Their price process are given by

$$B(s,T) = e^{-r(T-s)}, \quad t \leq s \leq T, \quad 0 \leq T \leq T_{\max}.$$

Assume one intends to write an option with maturity T_1whose payoff $g(Y_T)$ is determined at time $T \leq T_1$. Clearly because no additional risk is involved and for all t in the interval $[T, T_1]$ the forward to time T_1 writer's price, given by the formula (7), reduces to $g(Y_T)$. For all t from the interval $[0, T]$ the forward price can be computed using the projection property and the formula (7) applied to the forward price at time T, i.e., $g(Y_T)$, giving the spot price

$$e^{-r(T_1-t)}\frac{1}{\gamma(T_1)(1-\rho^2)} \ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma(T_1)\left(1-\rho^2\right)g(Y_T)}\,|Y_t=y\right)\right),$$

where $\gamma(T_1)$ indicates the dependence of the risk aversion on the option maturity. On the other hand the value at time T of the payoff $g(Y_T)$ at time T_1thanks to the presence of bonds with all maturities must equal $e^{-r(T_1-T)}g(Y_T)$. This can also be priced as a claim associated with maturity T and therefore its price must equal

$$e^{-r(T-t)}\frac{1}{\gamma(T)(1-\rho^2)}\ln\left(E_{\widetilde{\mathbb{P}}}\left(e^{\gamma(T)\left(1-\rho^2\right)e^{-r(T_1-T)}g(Y_T)}|Y_t=y\right)\right).$$

For the pricing system to be consistent across all maturities the two prices must coincide and hence we must have

$$\gamma(T)=\gamma e^{-rT},\gamma>0.$$

The problem now is that all the prices are expressed in units of a fixed time $t=0$ and not in terms of the current time t indicating that the risk aversion parameter γ must depend not only on the option maturity T but also on the current time t. A simple way to resolve this dilemma is to express all the relevant quantities in the current units. In particular the present value of the liability $g(Y_T)$ at time T is obviously equal to $e^{-r(T-t)}g(Y_T)$. It is therefore tempting to try to reconcile our previous results through the appropriate modifications of the risk aversion parameters. Namely, it seems that all one needs to do is to replace the former γ with $\gamma e^{-r(T-t)}$. Such a transformation requires the risk aversion which also depends on t. Unfortunately, this cannot be directly deduced from the analysis curried out in Musiela and Zariphopoulou (2001). In fact in order to accommodate for it one needs to reformulate the Merton's problem. The idea is to maximize utilities with and without an option expressing them in the current units rather then in the forward units to the option maturity which is the approach taken in Musiela and Zariphopoulou (2001). Namely, we are interested in the classical Merton's problem and the writer's problem for the of discounted payoffs, i.e.,

$$V(x,t)=\sup_{\mathcal{Z}}E\left(-e^{-\gamma e^{-r(T-t)}X_T}|X_t=x\right) \tag{8}$$

$$u(x,y,t)=\sup_{\mathcal{Z}}E\left(-e^{-\gamma e^{-r(T-t)}(X_T-g(Y_T))}/X_t=x,Y_t=y\right). \tag{9}$$

In both cases the investor starts, at time t, with initial endowment x and follows a sef-financing strategy by investing at time s the amounts, say π_s^0 and π_s, $t\leq s\leq T$, in the bond $B(s,T)$ and the traded risky asset S_s, respectively. The strategy generates wealth

$$X_s=\pi_s^0+\pi_s,t\leq s\leq T, \tag{10}$$

which satisfies the controlled diffusion equation

$$\begin{cases} dX_s=rX_sds+(\mu-r)\pi_sds+\sigma\pi_sdW_s^1 \\ \\ X_t=x. \end{cases} \tag{11}$$

The supremum is taken over a set $\mathcal{Z}$ of admissible controls (also referred to later on as *policies*) which are $\mathcal{F}_s$-progressively measurable and satisfy the integrability condition $E\int_t^T \pi_s^2 ds < \infty$.

To solve for the value function of the first problem (8) we introduce the discounted with the savings account wealth process

$$\overline{X}_s = e^{-r(s-t)} X_s, \quad t \leq s.$$

Using (11), we deduce that $\overline{X}$ satisfies

$$\begin{cases} d\overline{X}_s = (\mu - r)\,\overline{\pi}_s ds + \sigma \overline{\pi}_s dW_s^1 \\ \\ \overline{X}_t = x, \end{cases} \tag{12}$$

where

$$\overline{\pi}_s = e^{-r(s-t)} \pi_s$$

is the discounted from time s to the current time t amount π_s invested in the traded risky asset S_s at time s. In terms of the discounted with the savings account wealth process problem (8) can be reformulated as follows

$$V(x,t) = \sup_{\mathcal{Z}} E\left(-e^{-\gamma \overline{X}_T} \left| \overline{X}_t = x \right.\right),$$

with $\overline{X}_s$ solving (12). Consequently, the first value function is given by

$$V(x,t) = -e^{-\gamma x} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \tag{13}$$

Recall that the value function derived in Musiela and Zariphopoulou (2001) of the classical Merton's problem expressed in the forward to time T units is given by

$$\widetilde{V}(x,t) = -e^{-\gamma e^{r(T-t)} x} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)}. \tag{14}$$

Note that the two value functions (13) and (14) coincide when one introduces the appropriate term structure into the risk aversion parameter, namely, when γ in Musiela and Zariphopoulou (2001) is replaced with $\gamma e^{-r(T-t)}$.

Now we can proceed with the writer's problem. The writer's value function can be written as follows

$$u(x,y,t) = \sup_{\mathcal{Z}} E\left(-e^{-\gamma \overline{X}_T} e^{\gamma e^{-r(T-t)} g(Y_T)} / \overline{X}_t = x, Y_t = y\right).$$

Moreover

$$u(x,y,t) = \overline{u}(x,y,1,t)$$

with

$$\overline{u}(x,y,z,t) = \sup_{\mathcal{Z}} E\left(-e^{-\gamma \overline{X}_T} e^{\gamma Z_T g(Y_T)} / \overline{X}_t = x, Y_t = y, Z_t = z\right),$$

where

$$Z_s = ze^{-r(s-t)}, \quad t \leq s$$

and hence also satisfies

$$\begin{cases} dZ_s = -rZ_s ds \\ \\ Z_t = z. \end{cases}$$

The function $\overline{u}$ solves the HJB equation

$$\begin{array}{l} \overline{u}_t + \max_{\overline{\pi}} \left(\frac{1}{2}\sigma^2 \overline{\pi}^2 \overline{u}_{xx} + \rho\sigma a\left(y,t\right) \overline{\pi}\overline{u}_{xy} + \mu\overline{\pi}\overline{u}_x \right) \\ + \frac{1}{2}a^2\left(y,t\right)\overline{u}_{yy} + b\left(y,t\right)\overline{u}_y - rz\overline{u}_z = 0 \end{array} \tag{15}$$

with the terminal condition

$$\overline{u}\left(x,y,z,T\right) = -e^{-\gamma x}e^{\gamma z g(y)}.$$

Working as in Musiela and Zariphopoulou (2001) we postulate a solution in the separable form, namely

$$\overline{u}\left(x,y,z,t\right) = -e^{-\gamma x}F\left(y,z,t\right)$$

and we get the following equation for F

$$\begin{array}{c} F_t + \frac{1}{2}a^2\left(y,t\right)F_{yy} + \left(b\left(y,t\right) - \rho\frac{\mu - r}{\sigma}a\left(y,t\right) \right) F_y \\ -rzF_z - \frac{1}{2}\rho^2 a^2\left(y,t\right)\frac{F_y^2}{F} = \frac{\left(\mu - r\right)^2}{2\sigma^2}F \end{array} \tag{16}$$

with

$$F\left(y,z,T\right) = e^{\gamma z g(y)}.$$

Looking for a solution in the form

$$F\left(y,z,t\right) = v\left(y,z,t\right)^{\delta}$$

with

$$\delta = \frac{1}{1-\rho^2}$$

yields that v must solve the linear parabolic PDE

$$\begin{cases} v_t + \frac{1}{2}a^2\left(y,t\right)v_{yy} + \left(b\left(y,t\right) - \rho\frac{\mu - r}{\sigma}a\left(y,t\right) \right) v_y - rzv_z = \frac{\left(\mu - r\right)^2\left(1-\rho^2\right)}{2\sigma^2}v, \\ \\ v\left(y,z,T\right) = e^{\gamma\left(1-\rho^2\right)z g(y)}. \end{cases} \tag{17}$$

From the Feynman-Kac formula we have, under the appropriate integrability conditions, that v admits the stochastic representation

$$v\left(y,z,t\right) = E_{\widetilde{\mathbb{P}}}\left(e^{\gamma\left(1-\rho^2\right)Z_T g(Y_T) - \frac{\left(\mu - r\right)^2}{2\sigma^2}\left(1-\rho^2\right)(T-t)} \left| Y_t = y, Z_t = z \right. \right), \tag{18}$$

where the measure $\widetilde{\mathbb{P}}$ is defined in (6). It follows that the writer's value function takes the form

$$\overline{u}(x,y,z,t)$$
$$= -e^{-\gamma x} e^{-\frac{(\mu-r)^2}{2\sigma^2}(T-t)} \left(E_{\widetilde{\mathbb{P}}} \left(e^{\gamma\left(1-\rho^2\right) z e^{-r(T-t)} g(Y_T)} \left| Y_t = y \right. \right) \right)^{\frac{1}{1-\rho^2}}$$

and consequently the writer's price is given by

$$h(y,t) = \frac{1}{\gamma\left(1-\rho^2\right)} \ln E_{\widetilde{\mathbb{P}}} \left(e^{\gamma\left(1-\rho^2\right) e^{-r(T-t)} g(Y_T)} \left| Y_t = y \right. \right). \tag{19}$$

Note that the writer's price (5) coincides with (19) when γ in Musiela and Zariphopoulou (2001) is replaced with $\gamma e^{-r(T-t)}$.

3 Reference

Musiela M. and T. Zariphopoulou, Indifference prices and related measures, Technical Report (2001).