Investments and forward utilities

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Abstract

This paper proposes a new approach for portfolio allocation. The novel concept of forward dynamic utility is introduced. General classes of such utilities are constructed by combining the local variational utility input with the market dynamics represented by multidimensional Ito process. Explicit closed form expressions for optimal allocations are obtained. They depend exclusively on the optimal wealth level and the utility-generated measure of risk.

1 Introduction

The aim of this paper is to introduce and analyze the notion of forward dynamic utility and to expose the role it plays in the investment context. Intuitively, a dynamic utility should represent, possibly changing over time, individual preferences of an agent. Information based on which the agent will adjust his preferences will be revealed over time and will be represented by the filtration $(\mathcal{F}_t, t \geq 0)$ defined of the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$. As such the dynamic utility we consider herein closely resembles the classical notion of recursive utility of consumption.

In contrast to this classical literature, our agent is not going to optimize utility of consumption. Instead, he will face an investment problem to which he will apply utility based measurement. Naturally, he will want to track his utility over time and, to this aim, represents by $x \in \mathbb{R}$ the aggregate amount invested. Consequently, his utility, denoted by $U(x,t)$, becomes a function of time and wealth, t and x. In particular, $U(x,t)$ is an $\mathcal{F}_t$- adapted process. As a function of x the utility $U(x,t)$, for each $t \geq 0$, is assumed to be increasing and concave.

Also in contrast to the classical literature, there is no pre-specified trading horizon at the end of which the utility datum is assigned. Rather, the agent

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starts with *today's* specification of his utility, $U(x,0)$, and then builds the process $U(x,t)$ for $t>0$ in relation to the information flow given by $(\mathcal{F}_t, t\geq 0)$. We call $t=0$ the forward normalization point. It turns out that his utility choice for time $t>0$ will be constrained by $(\mathcal{F}_t, t\geq 0)$. This, together with the choice of normalization point, distinguishes the forward dynamic utility from the recursive utility for which the aggregator can be specified exogenously and the value function is recovered backwards in time. The adjective *forward* refers to the fact that the utility is constructed forward in time starting from zero.

Let us first consider the case in which the agent specifies, at time $t=0$, her dynamic utility for all future times $t>0$, i.e., when the filtration $(\mathcal{F}_t,\ t\geq 0)$ is trivial and, thus, the utility $U(x,t)$ is a deterministic function. To fix ideas, consider the case when the variable x represents wealth in the discounted to time $t=0$ units. The discounting numeraire is the classical savings account. In order to capture the agent's *impatience* we assume that, as a function of time t, the utility $U(x,t)$ is decreasing, for each x. The rate of decay represents the degree of impatience that the agent expresses. As it will be seen latter on, depending on the context in which such a utility will be applied, structural properties of the rate of decay will emerge.

The idea of using utility to define preference for advancing future satisfaction (impatience according to Irving Fisher (1913) terminology) is well known in economic literature (see, for example, Koopmans (1960), Koopmans, Diamond and Williamson (1964)). It assumes that an alternative with higher utility is always preferred over one with lower utility, and indifference exists between alternatives of equal utility. Additionally, the degree of impatience must also be calibrated to the opportunities the agent enjoys. If the opportunities are not great and the impatience high, the agent will take no action because he will have a sense of loosing utility independently on how hard he tries to exploit the opportunities given to him. Consequently, we argue that the degree of his impatience cannot be higher than what he can achieve in terms of the expected utility, if he optimally exploits the opportunities. Otherwise he should take no action or search for a bigger set of opportunities. To summarize, the agent will dynamically adjust his preferences consistently with the filtration $(\mathcal{F}_t, t\geq 0)$ and his impatience will be compensated for by the opportunities given to him, provided he exploits them in full relatively to his preferences. An asset manager will have to specify his dynamic utility in a way that allows him to implement, for example, a passive enhanced strategy or a total return strategy. Otherwise he will not utilize the investment opportunity presented to him. Clearly his utility cannot be specified in isolation to the investment universe he can participate in.

Given the filtration and the opportunity set an agent may know exactly what she is going to do, i.e., she knows her optimal behavior. However, she may not know what dynamic utility this behavior corresponds to. In this situation her degree of impatience and hence her dynamic utility is implicit to her optimal behavior. For example, a hedger may favour self-financing strategies that eliminate maximal 'amount' of risk in a transaction. This corresponds to the dynamic utility for which the optimal strategy consists of holding a riskless bond.

Herein we provide an explicit characterization of the agent's forward dynamic utility process and the optimal portfolios. The construction of the dynamic utility is general and not based on specific assumptions on dimensionality and/or nature of the asset returns. The key idea is to build and combine stochastic and variational inputs that come, respectively, from the market environment and the investor's dynamic risk attitude.

The market input incorporates i) the returns of assets available in the market, ii) an index of relative performance (benchmark) or, alternatively, a numeraire choice, iii) feasibility and trading constraints, iv) the investor's view of the market away from its equilibrium and v) a subordinated investment time (scaling of calendar time). The market input is represented in (1) as well as via three processes (Y, Z, A) introduced, respectively, in (8), (9) and (7).

The variational utility input provides a differential constraint on the three utility traits of the investor, namely, his risk aversion, his preference to higher than lower wealth and his impatience. It is modelled as the solution, $u(x,t)$, of a fully nonlinear pde (6) with initial data given by $U(x,0)$. Note that due to the forward in time construction of utility, the relevant pde is posed inversely in time.

A quantity that plays pivotal role in finding the optimal utility volume and constructing the optimal portfolios is the local risk tolerance, $r(x,t) = -u_x(x,t)/u_{xx}(x,t)$. Using the utility nonlinear pde and the definition of $r(x,t)$ yields a *transport equation* for $u(x,t)$ with slopes of characteristics equal to (half of) the risk tolerance. Further calculations yield that $r(x,t)$ solves an autonomous, inverse in time *fast diffusion equation* (FDE) with conductivity function $r^2(x,t)$ (see (17)). Finally, using the (FDE) and the definition of local risk aversion, $\gamma(x,t) = r(x,t)^{-1}$ we derive an, inverse in time, *porous medium equation* (PME) ((20). This equation has exponent $m = -1$ ((PME) nonlinearity given by $F(\gamma) = \gamma^{-1}$). [1].

Combining the stochastic and variational input we construct the forward dynamic utility process. Along with this construction, the associated optimal portfolio processes are determined. Further stochastic analysis arguments yield a 2-dimensional system of SDE that describes the stochastic evolution of the (benchmarked) optimal wealth and the (benchmarked-subordinated) risk tolerance processes. Using appropriate measure and time changes, we, in turn, recover the so-called canonical representation of the wealth-risk tolerance SDE system. Its solution effectively produces an efficient frontier associated with optimal behavior under forward utility criteria.

The dynamics of the canonical wealth-risk tolerance SDE system are local functions of its coordinates. Using analytic arguments we provide closed-form solutions for the two state processes. Their functional representation uses the solution of a heat equation with a drift term. The coefficient of the latter is given in terms of the spatial gradient of the local risk tolerance at a reference point. The quantity that enters in this explicit construction of the efficient frontier

[1] For a concise treatment of the Porous Medium Equation and the Fast Diffusion Equation we refer the reader to the books of J.-L. Vasquez (2006a, 2006b).

is the so-called budget capacity that expresses the current available wealth in terms of the aggregate (up to present time) risk aversion (cf. (27)).

The paper is organized as follows. In Section 1 we describe the investment universe and introduce the concept of forward dynamic utility. In Section 3 we construct a class of such utilities as well as the associated optimal portfolios. In Section 4 we analyze the stochastic evolution of the optimal wealth and the risk tolerance, and we build the efficient frontier in the canonical market configuration. In Section 5 we provide examples of utility inputs and their associated risk tolerance functions.

Remark: *This is a preliminary and incomplete version, written in an informal fashion.*

Various technical assumptions have been implicitly introduced and many technical arguments have not been addressed. For example, the involved forward utility processes are taken to be martingales but situations might arise in which only characterization in terms of local martingales can be obtained. No explicit assumptions have been made with regards to the wealth domain and/or feasibility constraints. Little is said about existence, uniqueness and regularity of the solutions of the emerging PDE. Note that with the exception of the utility transport equation, they are all formulated inversely in time, a fact that poses horrendous difficulties in analyzing them. Additional challenges arise in the study of the (FDE) *that governs the risk tolerance as well as in the* (PME) *solved by the risk aversion. These issues are not addressed in this version. However, we provide examples that indicate that meaningful solutions to these PDE not only exist but, more importantly, yield a rich class of forward utilities that arise in well known and frequently used cases.*

2 Investment universe

The agent is allowed to invest in a financial market in which risky and riskless assets are traded. We choose to represent this universe by a standard multi-dimensional generalization of the Black-Scholes model as presented in Musiela and Rutkowski (2005). Namely, for $i = 1, ..., k$, the price of the i^{th} risky asset is modelled as an Ito process

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right)$$

or equivalently,

$$\begin{cases} dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t \right) \\ \\ S_0^i > 0. \end{cases} \tag{1}$$

Herein $W = \left(W^1, ..., W^d \right)$ is a standard $d-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ while $\cdot$ stands for the inner product.

The coefficients μ_t^i and σ_t^i follow bounded progressively measurable processes on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. A special, but particularly important, case is obtained by postulating that, for every i, the return μ^i on the asset i is constant and the volatility coefficient σ^i is represented by a fixed vector in $\mathbb{R}^d$.

For brevity, we write $\sigma = \sigma_t$ to denote the volatility matrix - that is, the time dependent random matrix $\left(\sigma_t^{ji};\ j = 1, ..., d, i = 1, ..., k\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. The riskless asset, the savings account, has the price process B satisfying

$$\begin{cases} dB_t = r_t B_t dt \\ \\ \quad B_0 = 1, \end{cases}$$

for a bounded, nonnegative, progressively measurable interest rate process r.

To ensure the absence of arbitrage opportunities, we postulate the existence of an $\mathcal{F}_t-$ progressively measurable process $\lambda \in \mathbb{R}^d$ such that the equality

$$\mu_t^i - r_t = \sum_{j=1}^d \sigma_t^{ji} \lambda_t^j = \sigma_t^i \cdot \lambda_t$$

is satisfied simultaneously for all $i = 1, ..., k$. Using vector and matrix notation, the above becomes

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t, \tag{2}$$

where σ^T stands for the matrix transpose, $\mathbf{1}$ denotes the $d-$dimensional vector with every component equal to one, and μ_t is the vector with components μ_t^i. The process λ is often referred to as a market price of risk. Note that it is not uniquely determined, in general.

The agent uses the opportunity to invest in order to satisfy his impatience, risk aversion and his preference to higher -than lower- wealth levels. Starting, at $t = 0$, with an initial endowment x at time zero, he invests at future times $t > 0$ in all available assets and follows a self-financing strategy. The present value of the amounts invested in the riskless and the i^{th} risky asset are denoted, respectively, by π_t^0 and π^i. Therefore, the present value of his investment is given by

$$X_t = \sum_{i=0}^k \pi_t^i.$$

It is straightforward to see that the process $(X_t, t \geq 0)$ satisfies

$$dX_t = \sum_{i=1}^k \pi_t^i \left(\mu_t^i - r_t\right) dt + \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot dW_t$$

$$= \sum_{i=1}^k \pi_t^i \sigma_t^i \cdot (\lambda_t dt + dW_t) = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t), \tag{3}$$

where π_t is the column vector $\pi_t = \left(\pi_t^i; i = 1, ..., k\right)$ and $X_t = x$. We will occasionally use super script π in the wealth process.

Remark 1 *Note that the wealth generated by the investment strategy is expressed in the discounted to time zero units. Therefore, the utility $U(x, 0)$ of wealth x at time $t = 0$ represents the utility of x today and for any $t > 0$ the number $U(x, t)$ represents the utility of the same wealth at any future date.*

We are now ready to define a *forward dynamic utility* in the context of the above investment opportunities. We denote the utility datum by $u_0(x)$.

Definition 2 *Let $U(x, t)$ be an $\mathcal{F}_t-$adapted process. We say that $U(x, t)$ is a forward dynamic utility if:*

- *as a function of x it is increasing and concave for each $t \geq 0$,*
- *it satisfies $U(x, 0) = u_0(x)$,*
- *for all $T \geq t$ and each self-financing strategy, represented by π, the associated discounted wealth X^π satisfies*

$$E_{\mathbb{P}}\left(U\left(X_T^\pi, T\right) | \mathcal{F}_t\right) \leq U\left(X_t^\pi, t\right), \tag{4}$$

- *for all $T \geq t$, there exists a self financing strategy, represented by π^*, for which the associated discounted wealth X^* satisfies*

$$E_{\mathbb{P}}\left(U\left(X_T^*, T\right) | \mathcal{F}_t\right) = U\left(X_t^*, t\right). \tag{5}$$

Forward utilities were first introduced by the authors in Musiela and Zariphopoulou (2003, 2005), see also, Musiela and Zariphopoulou (2005) and Musiela and Zariphopoulou (2006a).

As a simple example, consider the self-financing strategy to buy and hold a bond. The associated discounted wealth is constant and equal to the initial endowment x. Consequently the dynamic utility process satisfies, for $t \leq T$,

$$E\left(U(x, T) | \mathcal{F}_t\right) \leq U(x, t)$$

and hence it is a supermartingale. Clearly, the supermartingale property of the utility process $U\left(X_t^\pi, t\right)$ holds for any π.

3 A class of forward dynamic utilities

In this section we construct a class of forward utility processes. As it was mentioned in the Introduction, the construction is based on compiling the stochastic market input with the variational utility input. We start with the latter.

The agent chooses utility function $u_0(x)$ representing his preferences for *today*, i.e., for time $t = 0$. Then the utility input, $u(x, t)$, is constructed by solving the following fully nonlinear partial differential equation equation

$$\begin{cases} u_t u_{xx} = \frac{1}{2} u_x^2 \\ \\ u(x, 0) = u_0(x). \end{cases} \tag{6}$$

For the specification of the market input, we first consider two auxiliary $\mathcal{F}_t-$progressively measurable processes $\delta \in \mathbb{R}^d$ and $\phi \in \mathbb{R}^d$. We then define the so-called *market input processes* A, Y and Z. They will, respectively, play the role of a *subordinator,* a *benchmark* and a process that captures the investor's *market view away from equilibrium* and/or quantifies his *feasibility constraints.*

The subordinator process A is absolutely continuous and solves

$$\begin{cases} dA = \left|\sigma\sigma^{+}\left(\lambda+\phi\right)-\delta\right|^{2} dt \\ \\ A_0 = 0, \end{cases} \tag{7}$$

where σ^{+} stands for Moore-Penrose pseudo-inverse of σ.

The benchmark process Y is taken to satisfy

$$\begin{cases} dY = Y\delta \cdot \left(\lambda dt + dW\right) \\ \\ Y_0 = 1. \end{cases} \tag{8}$$

The process Z is an exponential martingale, namely,

$$\begin{cases} dZ = Z\phi \cdot dW \\ \\ Z_0 = 1. \end{cases} \tag{9}$$

In order to construct the forward dynamic utility we inject into the variational input, $u\left(x,t\right)$, the above market relevant information.

A quantity that will play instrumental role in the specification of the optimal utility volume and the optimal portfolio is the *local risk tolerance,* defined by

$$r\left(x,t\right) = -\frac{u_x\left(x,t\right)}{u_{xx}\left(x,t\right)} \tag{10}$$

with u solving (6).

We will be also using the *benchmarked-subordinated risk tolerance* process, defined as

$$R = r\left(\frac{X}{Y}, A\right) \tag{11}$$

with X, A and Y solving (3), (7) and (8).

We are now ready to prove the main Theorem.

Theorem 3 *Assume that $\sigma\sigma^{+}\delta = \delta$. Let A, Y and Z be defined as in (7), (8) and (9), while the function u is given by (6). Then,*

i) the process $U\left(x,t\right)$ defined by

$$U\left(x,t\right) = u\left(\frac{x}{Y_t}, A_t\right) Z_t \tag{12}$$

is a forward dynamic utility and

ii) the optimal portfolio vector is given by

$$\frac{1}{Y}\pi^{*}=\sigma^{+}\left(\left(\frac{X^{*}}{Y}-R^{*}\right)\delta+R^{*}\left(\lambda+\phi\right)\right) \tag{13}$$

with

$$R^{*}=r\left(\frac{X^{*}}{Y},A\right). \tag{14}$$

Herein, X^{*} is the optimal wealth process, given in (3) with π^{*} being used.

Proof. Let π an admissible policy and X the wealth process given in (3).

Apply Ito's formula to

$$U\left(x,t\right)=u\left(\frac{x}{Y_{t}},A_{t}\right)Z_{t}$$

to obtain

$$dU=du\left(\frac{X}{Y},A\right)Z$$

$$=\left(du\left(\frac{X}{Y},A\right)\right)Z+u\left(\frac{X}{Y},A\right)dZ+d\left\langle u\left(\frac{X}{Y},A\right),Z\right\rangle.$$

Moreover,

$$du\left(\frac{X}{Y},A\right)=u_{x}\left(\frac{X}{Y},A\right)d\left(\frac{X}{Y}\right)+u_{t}\left(\frac{X}{Y},A\right)dA+\frac{1}{2}u_{xx}\left(\frac{X}{Y},A\right)d\left\langle\frac{X}{Y}\right\rangle$$

and

$$d\left(\frac{X}{Y}\right)=\left(\frac{1}{Y}\sigma\pi-\frac{X}{Y}\delta\right)\cdot\left(\left(\lambda-\delta\right)dt+dW\right). \tag{15}$$

Consequently,

$$d\left\langle u\left(\frac{X}{Y},A\right),Z\right\rangle=u_{x}\left(\frac{X}{Y},A\right)d\left\langle\frac{X}{Y},Z\right\rangle$$

$$=u_{x}\left(\frac{X}{Y},A\right)\left(\frac{1}{Y}\sigma\pi-\frac{X}{Y}\delta\right)\cdot Z\phi dt,$$

$$u\left(\frac{X}{Y},A\right)dZ=u\left(\frac{X}{Y},A\right)Z\phi\cdot dW=U\phi\cdot dW$$

and

$$\left(du\left(\frac{X}{Y},A\right)\right)Z=u_{x}\left(\frac{X}{Y},A\right)Z\left(\frac{1}{Y}\sigma\pi-\frac{X}{Y}\delta\right)\cdot\left(\left(\lambda-\delta\right)dt+dW\right)$$

$$+u_{t}\left(\frac{X}{Y},A\right)Z\left|\sigma\sigma^{+}\left(\lambda+\phi\right)-\delta\right|^{2}dt+\frac{1}{2}u_{xx}\left(\frac{X}{Y},A\right)Z\left|\frac{1}{Y}\sigma\pi-\frac{X}{Y}\delta\right|^{2}dt.$$

Therefore, dropping the arguments in u_x, u_t and u_{xx} we can, in turn, write

$$dU = \left(u_x \frac{Z}{Y} \sigma \pi - u_x \frac{XZ}{Y} \delta - U\phi \right) \cdot dW$$

$$+ u_x Z \left(\frac{1}{Y} \sigma \pi - \frac{X}{Y} \delta \right) \cdot (\lambda - \delta)\, dt + u_t Z \left| \sigma \sigma^{+} (\lambda + \phi) - \delta \right|^2 dt$$

$$+ \frac{1}{2} u_{xx} Z \left| \frac{1}{Y} \sigma \pi - \frac{X}{Y} \delta \right|^2 dt + u_x Z \left(\frac{1}{Y} \sigma \pi - \frac{X}{Y} \delta \right) \cdot \phi dt.$$

Using the assumption $\sigma \sigma^{+} \delta = \delta$, the function r and the process R, defined respectively in (10) and (11), yields

$$dU = \left(u_x \frac{Z}{Y} \sigma \pi - u_x \frac{XZ}{Y} \delta - U\phi \right) \cdot dW$$

$$+ \frac{1}{2} u_{xx} Z \left| \frac{1}{Y} \sigma \pi - \left(\left(\frac{X}{Y} - R \right) \delta + R \sigma \sigma^{+} (\lambda + \phi) \right) \right|^2 dt$$

$$+ \left(u_t - \frac{1}{2} u_{xx} \left(\frac{u_x}{u_{xx}} \right)^2 \right) \left| \sigma \sigma^{+} (\lambda + \phi) - \delta \right|^2 dt.$$

Finally, using the utility equation (6) we get that

$$dU = \left(u_x \frac{Z}{Y} \sigma \pi - u_x \frac{XZ}{Y} \delta - U\phi \right) \cdot dW$$

$$+ \frac{1}{2} u_{xx} Z \left| \frac{1}{Y} \sigma \pi - \left(\left(\frac{X}{Y} - R \right) \delta + R \sigma \sigma^{+} (\lambda + \phi) \right) \right|^2 dt.$$

Now, we observe that the process U is a supermartingale for each π and it is a martingale for π^* which satisfies (13) with X^* representing the wealth process corresponding to π^* and R^* as in (14). ■

4 Risk tolerance and optimal portfolio processes

i) Three related PDE

We begin this section with the derivation of a partial differential equation that the risk tolerance function, given in (10), satisfies. We set the initial condition

$$r(x, 0) = r_0(x) = -\frac{u_0'(x)}{u_0''(x)} \tag{16}$$

where we recall that $u_0(x)$ represents the utility datum set at present time, $t = 0$.

Proposition 4 *The risk tolerance function $r(x,t)$, defined in (10), satisfies the following partial differential equation*

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ \\ r(x,0) = r_0(x). \end{cases} \tag{17}$$

Proof. Recall that

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}.$$

Hence

$$u_{tx} = u_x - \frac{1}{2} u_x \left(\frac{u_x u_{xxx}}{u_x^2} \right)$$

and

$$u_{txx} = u_{xx} - \frac{1}{2} u_{xx} \left(\frac{u_x u_{xxx}}{u_x^2} \right) - \frac{1}{2} u_x \left(\frac{u_x u_{xxx}}{u_x^2} \right)_x.$$

Moreover,

$$r_x = -1 + \frac{u_x u_{xxx}}{u_x^2}$$

and

$$r_{xx} = \left(\frac{u_x u_{xxx}}{u_x^2} \right)_x.$$

Consequently

$$r_t + \frac{1}{2} r^2 r_{xx} = -\frac{u_{tx}}{u_{xx}} + \frac{u_x u_{txx}}{u_{xx}^2} + \frac{1}{2} \left(\frac{u_x}{u_{xx}} \right)^2 \left(\frac{u_x u_{xxx}}{u_x^2} \right)_x = 0$$

and the statement follows. ■

The above equation belongs to the class of *fast diffusion equations* (FDE). It has conductivity coefficient equal to the square of the solution itself. The analysis of (FDE) is rather difficult, especially because (17) is also posed inversely in time. However, one can construct a rich class of examples that represent well known classes of utilities. A detailed analysis is carried out in Musiela and Zariphopoulou (2006b).

Example 1: The risk tolerance is given by

$$r(x,t;\alpha,\beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}.$$

Special cases of the above risk tolerance are the following:

i) For $\alpha = 0$,

$$r(x,t;0,\beta) = \sqrt{b}$$

yielding the exponential utility

$$u(x,t) = -e^{-\frac{x}{\sqrt{b}}+t}.$$

ii) For $\alpha = 1, \beta = 0$,

$$r(x,t;1,0) = \sqrt{x^2} = |x|$$

yielding, for $x > 0$, the logarithmic utility

$$u(x,t) = \log x - t.$$

iii) For $\alpha > 1, \beta = 0$,

$$r(x,t;\alpha,0) = \sqrt{\alpha x^2} = \sqrt{\alpha}\,|x|$$

yielding, for $x > 0$, the power utility

$$u(x,t) = \frac{1}{\gamma} x^{\gamma} e^{-\frac{1}{2}\frac{\gamma}{1-\gamma}t} \quad \text{and} \quad \gamma = \frac{\sqrt{a}-1}{\sqrt{\alpha}}.$$

Other utilities can be recovered for different ranges of the above parameters.

Example 2: The risk tolerance is given by

$$r(x,t;\alpha,\beta) = m(x;\alpha)\, n(t;\beta)$$

with

$$m(x;\alpha) = \varphi\left(\Phi^{-1}(x;\alpha)\right) \qquad \text{and} \qquad n(t;\beta) = \sqrt{t+\beta}$$

where

$$\Phi(x;\alpha) = \int_{\alpha}^{x} e^{z^2/2} dz \qquad \text{and} \qquad \varphi(x) = \frac{d\Phi(x;\alpha)}{dx}.$$

The associated variational utility is given by

$$u(x,t) = \Phi\left(\Phi^{-1}(x;\alpha) - \sqrt{t+\beta}\right).$$

Observe that the exponential, logarithmic and power utility are also special cases of multiplicative risk tolerance, corresponding to the choices $m(x;\alpha) = \alpha$, $m(x;\alpha) = \alpha x$ and $m(x;\alpha) = x$, with time factor $n(t;\beta) = 1$.

Using (17) and (6) we may derive an alternative, first order partial differential equation for the utility input**.** After routine differentiation, we deduce that u solves the *transport equation*

$$\begin{cases} u_t + \frac{1}{2} r(x,t)\, u_x = 0 \\ u(x,0) = u_0(x). \end{cases} \tag{18}$$

In this transport equation, the slope of the *characteristics curves* is given by

$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t).$$

Example 3: The risk tolerance of Example 1 produces a utility

$$u(x(t), t) = u_0(x)$$

along the curves

$$\frac{dx(t)}{dt} = \sqrt{\alpha x(t)^2 + \beta e^{-\alpha t}}.$$

Finally, a third partial differential equation may be derived that describes the movement of the local *risk aversion coefficient*, defined as

$$\gamma(x,t) = \frac{1}{r(x,t)}. \tag{19}$$

Differentiating the fast diffusion equation (17), we deduce that γ solves the *porous medium equation (PME)*

$$\begin{cases} \gamma_t = F(\gamma)_{xx} \\ \gamma(x,0) = r_0(x)^{-1} \end{cases} \tag{20}$$

where the (PME) nonlinearity is given by

$$F(\gamma) = \frac{1}{\gamma}.$$

ii) The wealth-risk tolerance SDE system, original market configuration

We are now ready to derive the SDE that governs the *benchmarked optimal wealth* and the *benchmarked-subordinated risk tolerance process.*

To this end, we recall that the dynamics of the optimal portfolio π^* defined in (13) depend on the dynamics of the processes $\frac{X^*}{Y}$ and R^*. The dynamics of $\frac{X^*}{Y}$ can be recovered from (15) by replacing π with π^*.

Proposition 5 *The processes $\frac{X^*}{Y}$ and R^* satisfy*

$$d\left(\frac{X^*}{Y}\right) = R^*\left(\sigma\sigma^+(\lambda+\phi) - \delta\right)\cdot\left((\lambda-\delta)\,dt + dW\right), \tag{21}$$

and

$$dR^* = r_x\left(\frac{X^*}{Y}, A\right) d\left(\frac{X^*}{Y}\right) \tag{22}$$

where the process A and Y are given, respectively, in (7) and (8).

Proof. Using the dynamics of X^* and Y we deduce

$$d\left(\frac{X^*}{Y}\right) = \left(\frac{1}{Y}\sigma\pi^* - \frac{X^*}{Y}\delta\right)\cdot\left((\lambda-\delta)\,dt + dW\right)$$

$$= \left(\sigma \sigma^{+} \left(\left(\frac{X^*}{Y} - R^* \right) \delta + R^* (\lambda + \phi) \right) - \frac{X^*}{Y} \delta \right) \cdot ((\lambda - \delta)\, dt + dW)$$

$$= R^* \left(\sigma \sigma^{+} (\lambda + \phi) - \delta \right) \cdot ((\lambda - \delta)\, dt + dW).$$

Moreover

$$dR^* = dr \left(\frac{X^*}{Y}, A \right) = r_x \left(\frac{X^*}{Y}, A \right) d \left(\frac{X^*}{Y} \right)$$

$$+ r_t \left(\frac{X^*}{Y}, A \right) dA + \frac{1}{2} r_{xx} \left(\frac{X^*}{Y}, A \right) d \left\langle \frac{X^*}{Y} \right\rangle$$

$$= r_x \left(\frac{X^*}{Y}, A \right) d \left(\frac{X^*}{Y} \right)$$

$$+ \left(r_t \left(\frac{X^*}{Y}, A \right) dA + \frac{1}{2} r_{xx} \left(\frac{X^*}{Y}, A \right) (R^*)^2 \right) dA$$

$$= r_x \left(\frac{X^*}{Y}, A \right) d \left(\frac{X^*}{Y} \right)$$

$$+ \left(r_t \left(\frac{X^*}{Y}, A \right) dA + \frac{1}{2} \left(r \left(\frac{X^*}{Y}, A \right) \right)^2 r_{xx} \left(\frac{X^*}{Y}, A \right) \right) dA$$

Using the fast diffusion equation (17) eliminates the last term above and, in turn,

$$dR^* = r_x \left(\frac{X^*}{Y}, A \right) d \left(\frac{X^*}{Y} \right).$$

■

iii) The SDE wealth-tolerance system in canonical market configuration

In order to further analyze the solutions of (21) and (22), we are first going to introduce a measure and time transformation. To this end, we let $\mathbb{P}^*$ be the probability measure defined, for $T > 0$, by

$$\mathbb{P}^* (A) = E_{\mathbb{P}} \left(\exp \left(\int_0^T (\lambda - \delta) \cdot dW - \int_0^T |\lambda - \delta|^2 \, dt \right) I_A \right), \quad A \in \mathcal{F}_T$$

with $\mathbb{P}$ being the original historical measure.

Clearly, the process

$$W_t^* = W_t - \int_0^t (\lambda - \delta)\, ds, t > 0,$$

is a Brownian motion under $\mathbb{P}^*$. Note that the measure $\mathbb{P}^*$ is a *martingale measure* for the benchmarked wealth, $\frac{X}{Y}$, generated by the self-financing strategy π, as follows trivially from (15).

Now, define

$$M_t = \int_0^t \left(\sigma\sigma^+ (\lambda + \phi) - \delta\right) \cdot dW^*$$

and observe that

$$\langle M \rangle = A.$$

Consequently, the process

$$w = M_{A^{(-1)}}, \tag{23}$$

where $A^{(-1)}$ stands for the inverse of A, is a one dimensional Brownian motion.

Next, we define the processes

$$\begin{cases} x_1 = \left(\frac{X^*}{Y}\right)_{A^{(-1)}} \\ \\ x_2 = R^*_{A^{(-1)}}. \end{cases} \tag{24}$$

Then, it follows directly from (21) and (22) that (x_1, x_2) solve the *system of SDE*

$$\begin{cases} dx_1(t) = x_2(t)\, dw(t) \\ \\ dx_2(t) = r_x(x_1(t), t)\, x_2(t)\, dw(t) \end{cases} \tag{25}$$

with initial conditions

$$x_1(0) = \frac{x}{y} \qquad \text{and} \qquad x_2(0) = r_x\left(\frac{x}{y}, 0\right), \tag{26}$$

w as in (23).

iv) Analytic solutions in canonical market configuration

The above system can be solved *analytically* as it is shown next.

To facilitate the exposition, we introduce the variable

$$z = \frac{x}{y}.$$

We will be also using a reference point (see (28) below) denoted by $\hat{z}$.

Theorem 6 *Let (25) be the SDE system for the process $x_1(t)$ and $x_2(t)$. Define the budget capacity function $h(z,t)$ by*

$$z = \int_{\hat{z}}^{h(z,t)} \frac{1}{r(\rho, t)} d\rho \tag{27}$$

with r being the local risk tolerance. Let $h_0(z) = h(z, 0)$, i.e.

$$z = \int_{\hat{z}}^{h_0(z)} \frac{1}{r_0(\rho)} d\rho. \tag{28}$$

Then h solves the linear partial differential equation

$$\begin{cases} h_t + \frac{1}{2} h_{zz} - k(t)\, h_z = 0 \\ \\ h(z,0) = h_0(z) \end{cases} \tag{29}$$

where

$$k(t) = -\frac{1}{2} r_x(\hat{z}, t). \tag{30}$$

Let also z be the process

$$z_t = h_0^{-1}(z) + \int_0^t k(s)\, ds + w_t. \tag{31}$$

Then the solution of (25) is given by

$$\begin{cases} x_1(t) = h(z_t, t) \\ \\ x_2(t) = h_x(z_t, t). \end{cases} \tag{32}$$

Proof. Differentiating (27) with respect to spatial argument z yields

$$r(h(z,t), t) = h_z(z,t). \tag{33}$$

Differentiating, also, with respect to t gives

$$\frac{h_t(z,t)}{r(h(z,t),t)} - \int_{\hat{z}}^{h(z,t)} \frac{r_t(\rho,t)}{r^2(\rho,t)} d\rho = 0$$

and, in turn,

$$\frac{h_t(z,t)}{r(h(z,t),t)} + \frac{1}{2} \int_{\hat{z}}^{h(z,t)} r_{xx}(\rho,t)\, d\rho = 0$$

where we used (17). Further calculations yield

$$\frac{h_t(z,t)}{r(h(z,t),t)} + \frac{1}{2} \left(r_x(h(z,t),t) - r_x(\hat{z},t) \right) = 0$$

and, subsequently,

$$h_t(z,t) + \frac{1}{2} r_x(h(z,t),t)\, h_z(z,t) - k(t)\, h_z(z,t) = 0$$

where we used (33) and (30).

Differentiating (33) we obtain

$$r_x(h(z,t),t)\, h_z(z,t) = h_{zz}(z,t) \tag{34}$$

and combining it with the last equation yields (29).

The initial condition $h_0(z)$ follows trivially.

Next consider the process z_t (cf. (31)) and observe that its Ito differential is

$$dz_t = k(t)\, dt + dw_t.$$

Therefore, if we set

$$x_1(t) \equiv h(z_t, t) \tag{35}$$

we deduce

$$\begin{aligned} dx_1(t) &= dh(z_t, t) \\ &= \left(h_t(z_t, t) + \frac{1}{2} h_{zz}(z_t, t) + k(t)\, h_z(z_t, t) \right) dt + h_z(z_t, t)\, dw_t \end{aligned}$$

and, in turn,

$$dx_1(t) = h_z(z_t, t)\, dw_t$$

where we used equation (29).

Next, using (33), we obtain

$$dx_1(t) = r(h(z_t, t), t)\, dw_t = r(x_1(t), t)\, dw_t$$

and the first equation in (32) is established.

For the initial condition $x_1(0)$ we easily get

$$x_1(0) = h(z_0, 0) = h_0(z_0) = h_0\left(h_0^{-1}\left(\frac{x}{y} \right) \right) = \frac{x}{y}.$$

Next, we set

$$x_2(t) \equiv h_z(z_t, t) \tag{36}$$

and following similar to the above argumentation yields

$$dx_2(t) = \left(h_{tz}(z_t, t) + \frac{1}{2} h_{zzz}(z_t, t) + k(t)\, h_{zz}(z_t, t) \right) + h_{zz}(z_t, t)\, dw_t.$$

Observe that because h solves the linear pde (29), the above drift vanishes.

Therefore,

$$\begin{aligned} dx_2(t) &= h_{zz}(z_t, t)\, dw_t \\ &= r_x(h(z_t, t), t)\, h_z(z_t, t)\, dw_t \end{aligned}$$

where we used (34). Using (35) and (36), we deduce

$$dx_2(t) = r_x(x_1(t), t)\, x_2(t)\, dw_t$$

and the second equation in (32) is established.

To conclude the proof, we observe that

$$x_2(0) = h_z(z_0, 0) = h_z\left(h_0^{-1}(z), 0 \right).$$

From (28), we have

$$\frac{1}{\left(h_0^{-1}(z) \right)'} = r_0(z).$$

On the other hand,

$$h_z\left(h_0^{-1}(z),0\right)=\frac{1}{\left(h_0^{-1}(z)\right)'}.$$

and, therefore,

$$r_0\left(\frac{x}{y}\right)=\frac{1}{\left(h_0^{-1}\left(\frac{x}{y}\right)\right)'}.$$

Combining the above, we deduce,

$$x_2(0)=\frac{1}{\left(h_0^{-1}\left(\frac{x}{y}\right)\right)'}=r_0\left(\frac{x}{y}\right)$$

and we conclude. ■

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