On the optimal portfolio problem with partial information and related mean field games*†

Panagiotis E. Souganidis‡and Thaleia Zariphopoulou§

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Abstract

We study optimal portfolio choice models in markets with partial information about the stock's drift. We first solve the single agent problem for general utilities using a new approach that yields regularity of the value function and closed-form expressions for the optimal processes. We then consider a $N-$player game in which players interact through the law of peers' wealth and study its mean field limit. This leads to a mean field game with common noise in a reduced, complete information market with unbounded controls in both the drift and the volatility. For its solution, we derive a new master system, comprised by the master equation and an optimality condition for the candidate mean field equilibrium control. We analyze the cases of separable couplings and general utilities. Using insights from indifference valuation, we represent the value of the game as a compilation of the solution to the problem of a single player without competition and a function solving a non-local quasilinear pde in the space of measures. When the couplings depend only on the average of peers' wealth, we derive explicit solutions and various regularity results for representative cases.

Key words: Mean field games, master equation, unbounded controlled common noise, portfolio choice, partial information, relative performance, indifference valuation, arbitrage-free pricing.

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‡Department of Mathematics, University of Chicago; souganidis@uchicago.edu

§Departments of Mathematics and IROM, McCombs School of Business, The University of Texas at Austin, and the Oxford-Man Institute, University of Oxford; zariphop@math.utexas.edu

1 Introduction

The paper contributes to the areas of optimal portfolio management with partial information and to mean field games (MFG) in such markets with common noise and unbounded controls. In the first part of the work, we revisit the single agent (no competition) expected utility problem of terminal wealth. Under general utilities, we produce regularity and new representation formulae for the value function, and the optimal wealth and portfolio processes. The results fill a gap between existing explicit expressions for special (homothetic) utilities and abstract results derived with general duality arguments. In the second part of the paper, we introduce a $N-$player game and its continuous limit, where the agents interact because of relative performance concerns involving the law of peers' wealth. This leads to a mean field game with common noise in a reduced, complete information market with unbounded controls in both the drift and the volatility. For its solution, we derive a new master system, comprised by the master equation and an optimality condition for the candidate mean field equilibrium control.

We analyze the cases of separable couplings and general utilities. Using insights from indifference valuation, we view the value of the game as the writer's value of a claim given by the coupling at the optimum. This guides us to represent the value of the game as a compilation of the solution to the single player problem without competition and a function solving a non-local quasilinear pde in the space of measures. When the couplings depend only on the average of peers' wealth, we derive explicit solutions and various regularity results for representative cases.

The underlying market model consists of a riskless bond (of zero interest rate) and a stock of price S with unknown drift Θ, partially observed through an observations process Y. Classical results from filtering lead to a reduced, complete information market model with the stock and the observations process solving

$$\begin{cases} dS_s = b\left(Y_s, s\right) S_s ds + S_s dW_s \quad \text{in} \quad (0,T] \quad \text{and} \quad S_0 = S > 0, \\ dY_s = b\left(Y_s, s\right) ds + dW_s \quad \text{in} \quad (0,T] \quad \text{and} \quad Y_0 = y \in \mathbb{R}. \end{cases} \tag{1}$$

The drift process

$$b\left(Y_s, s\right) = \mathbb{E}\left[\,\Theta \middle|\, \mathcal{F}_s^Y\right],$$

is known as the best estimator $\hat{\Theta}$ given the observations from Y, and the innovation process W is a Brownian motion defined on $[0,T]$ as

$$W_s = Y_s - \int_0^s \hat{\Theta}_u \mathrm{d}u.$$

In this market, we first consider the maximal expected utility problem of a single agent in the absence of competition. The state controlled process $\mathcal{X}$ models her wealth of the (single) agent and satisfies

$$d\mathcal{X}_s = b\left(Y_s, s\right) \alpha_s ds + \alpha_s dW_s \quad \text{in} \quad (t,T] \quad \text{and} \quad \mathcal{X}_t = x \in \mathbb{R}, \tag{2}$$

with α being the control policy, representing the amount invested in stock, which is assumed to be measurable only with respect to Y; we denote the set of such controls as $\mathcal{A}$.

The value function is defined as

$$u\left(x,y,t\right)=\sup_{\alpha\in\mathcal{A}}\mathbb{E}\left[\left.J(\mathcal{X}_{T})\right|\mathcal{X}_{t}=x,Y_{t}=y\right]\quad\text{in }\mathbb{R}\times\mathbb{R}\times\left[0,T\right],\tag{3}$$

with J being the agent's utility at horizon T.

We establish that u is the unique smooth solution of the associated HJB equation,

$$u_{t}-\frac{\left(bu_{x}+u_{xy}\right)^{2}}{2u_{xx}}+\frac{1}{2}u_{yy}+bu_{y}=0\ \text{ in }\ \mathbb{R}\times\mathbb{R}\times\left[0,T\right),\tag{4}$$

with terminal condition

$$u(x,y,T)=J(x).\tag{5}$$

The optimal feedback control $a^{*}:\mathbb{R}\times\mathbb{R}\times[0,T]\rightarrow\mathbb{R}$ is given by

$$a^{*}=-\frac{bu_{x}+u_{xy}}{u_{xx}},\tag{6}$$

and satisfies an autonomous equation (see (96)).

We show that the optimal wealth and portfolio processes $\mathcal{X}^{*}$ and α^{*} are represented in the closed-form,

$$\mathcal{X}_{s}^{*}=H(H^{(-1)}(x,y,t)+L_{t,s},Y_{s},s)\ \text{ in }\ (t,T]\ \text{ and }\ \mathcal{X}_{t}^{*}=x,\tag{7}$$

and

$$\begin{aligned}\alpha_{s}^{*}&=c(Y_{s},s)H_{z}(H^{(-1)}(x,y,t)+L_{t,s},Y_{s},s)\\&\quad+H_{y}(H^{(-1)}(x,y,t)+L_{t,s},Y_{s},s)\ \text{ in }\ (t,T],\end{aligned}\tag{8}$$

where the inverse functions are with respect to the $x-$argument.

The process L is defined as

$$L_{t,s}=\int_{t}^{s}b(Y_{\rho},\rho)c(Y_{\rho},\rho)d\rho+\int_{t}^{s}c(Y_{\rho},\rho)dW_{\rho}\ \text{ in }\ (t,T]\ \text{ and }\ L_{t,t}=0,\tag{9}$$

and the auxiliary function H solves the linear pde

$$H_{t}+\frac{1}{2}c^{2}H_{zz}+cH_{zy}+\frac{1}{2}H_{yy}=0\ \text{ in }\mathbb{R}\times\mathbb{R}\times\left[0,T\right),\tag{10}$$

with terminal condition

$$H(z,y,T)=\left(J'\right)^{(-1)}\left(e^{-z}\right),\tag{11}$$

where $c : \mathbb{R} \times [0, T] \to \mathbb{R}$ is given by

$$c = b + k_y, \tag{12}$$

and k solves

$$k_t + \frac{1}{2} k_{yy} = \frac{1}{2} b^2 \quad \text{in} \quad \mathbb{R} \times [0, T] \text{ and } k(y, T) = 0. \tag{13}$$

It follows that (7) yields a closed-form expression for the value function, that is,

$$u(x, y, t) = \mathbb{E}\left[J\left(H(H^{(-1)}(x, y, t) + L_{t,T}, Y_T, T) \right) \Big| X_t = x, Y_t = y \right]. \tag{14}$$

To the best of our knowledge, formulae (7), (8) and (14), and the regularity results we produce herein are entirely new when the utility is arbitrary.

Optimal portfolio problems in high-dimensional complete markets, as the market model (1), have been extensively studied using duality techniques. While duality is applicable in general semimartingale models, it is usually difficult, if possible at all, to obtain regularity and closed-form solutions unless one works with homothetic utilities. In Markovian models, duality arguments show that the inverse marginal value function satisfies a linear pde, which follows directly from the HJB equation and Fenchel transform. On the other hand, as we comment further in Section 2, the emerging linear pde is not uniformly elliptic, and thus it is not a priori known if smooth solutions exist. To our knowledge, several questions in this direction remain open.

Herein, we develop an alternative approach that addresses these issues and, furthermore, produces tractable, closed-form solutions for general utilities beyond the homothetic ones. Our method is based on a different linearization which involves a pair of candidate solutions, and not just a single equation as in the classical duality theory. Still, one of the emerging equations is not uniformly elliptic, which we study by first looking at the driftless case ($b \equiv 0$) and a modified utility. Structurally, the linearization resembles the one introduced in Musiela and Zariphopoulou [51] for an one-dimensional ill-posed HJB equation arising in time-monotone forward utilities. However, here we deal with a two-dimensional well-posed linear pde which requires entirely different arguments.

The second part of the paper considers a $N-$player game with relative performance criteria in the common market (see (1)). This is an important area of research in financial economics and applied finance on optimal allocation/fund management problems with relative performance concerns, since fund management is always performed and evaluated in relation to a benchmark (index, returns of competitors, clustered financial targets, etc.). The prevailing way to classify these models is based on whether agents compete while investing in a same, common market (asset diversification) or, more generally, also include individual assets which are inaccessible to their competitors (asset specialization). For both categories, the existing applied papers primarily consider only two player games, single period models, and linear or quadratic criteria (see, among others, [1], [6], [7], [13], [14], [23], [24], [36], [37], [46], [53] and [56]).

The literature in continuous time is relatively recent. Espinoza and Touzi introduced in [25] a $N-$player asset diversification game for players with common exponential utility and linear competition functions. The work was extended in a two-player game by Anthropelos, Geng and Zariphopoulou [2] under forward criteria. Lacker and the second author [41] provided the first mean field game formulation under asset specialization but, still, under linear competition and exponential utilities. They, also, studied the special case of power utilities and geometric competition function, and non-negative wealth constraints. The MFG in [41] was defined probabilistically and, for both cases, static (random) equilibria were constructed when both common noise (common assets) and individual noise (specialized assets) were included. Stemming from [41], a substantial literature in MFG has been produced allowing, among others, for intermediate consumption, external habit formation, Ito-diffusion dynamics, relaxed controls, and mean-variance criteria (see, for example, [8], [12], [27], [28], [39], [40], [59] and others).

Despite the generality of the market models, all existing works considered only two pairs of utilities and couplings, specifically, either exponential utilities and linear couplings (in an infinite wealth domain) or power utilities and multiplicative couplings (in semi-finite wealth domain). The only work that incorporated both general utilities and general couplings on the mean of the peer aggregate wealth is by the authors in [55]. Subsequently, in the context of forward criteria, [59] allowed for general time-monotone forward processes and linear couplings.

Another modeling framework that has not been adequately studied, if at all, in MFG with relative performance is partial information. Frequently, the drift of the traded asset is not fully known in contrast to the volatility that is easier to estimate. Portfolio models under partial information (and in the absence of competition) have been well studied; see, among others, [11], [15], [21], [34] and [42]. Providing complete bibliography is beyond the scope herein). However, to our knowledge, with the exception of Deng, Su and Zhou [22] and Zhang and Huang [61] for rather special cases, and Huang and Sun [31] for a mean-variance game with finite players, the work in partial information in MFG with relative performance is rather limited. MFG with partial information have been considered by various authors; see, for example, Benoussan and Yam [9], Sen and Caines [52] and Shmaya and Zilotto [54]. In all these works, controls were allowed only in the drift and, furthermore, assumed to be in a compact admissible set. However, optimal portfolio models have, inherently, controls appearing in both the drift and the volatility, and, in addition, these controls are in general unbounded. Finally, all works on MFG with relative performance assume couplings that depend only on the mean of peer's aggregate wealth (or its geometric analogue when the utility is power/logarithmic). Herein, we depart from this assumption and allow for general dependence on the law of peers' wealth. We make all the above precise next.

We build on the $N-$player game and its MFG analogue developed in [55] and introduce their extensions within market models as in (1).

The value function of the $i^{th}-$player is defined as the terminal expected reward,

$$u^{N,i}\left(x_1,...,x_N,y,t\right)= \sup_{\pi_i\in\mathcal{A}}\mathbb{E}\Big[J\Big(X_T^{N,i},\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\delta_{X_T^{N,j}}\Big)|X_t=(x_1,...,x_N),Y_t=y\Big], \tag{15}$$

where $(x_1,...,x_N)\in\mathbb{R}^N$, $y\in\mathbb{R}$ and the payoff function J is assumed to be the common utility of all players. The controlled state processes $X^{N,i}$ follow (2) and the observations process Y is as in (1).

The above value functions are expected to satisfy their related Hamilton Jacobi Bellman (HJB) equations (see (108)), which in turn lead to a linear system for the candidate optimal feedback policies (see (109)). This system, however, is non-tractable due to the interlinked dependence of controls and value functions and their derivatives, for which even regularity results are lacking. This motivates us to consider the limiting case, as the number of players goes to infinity.

Passing formally to the limit, we derive a system of two equations satisfied by the value function $U:\mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T]\to\mathbb{R}$ and the optimal strategy $\pi^*:\mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T]\to\mathbb{R}$, where $\mathcal{P}_2$ is the space of probability measures on $\mathbb{R}$ with finite second moment.

The first equation is the master equation

$$\begin{aligned}&U_t+\frac{1}{2}\pi^{*2}U_{xx}+\pi^*(bU_x+U_{xy})+\pi^*\mathcal{L}_1U+\mathcal{L}_2U+b\mathcal{L}_3U+\mathcal{L}_4U\\&+\frac{1}{2}U_{yy}+bU_y=0\ \text{ in }\ \mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T)\,,\end{aligned} \tag{16}$$

with terminal condition

$$U(x,y,m,T)=J(x,m), \tag{17}$$

where, for $v:\mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T)\to\mathbb{R}$, we write

$$\mathcal{L}_1v(x,y,m.t)=\int\pi^*\left(z,y,m,t\right)v_{xm}\left(x,y,m,z,t\right)\mathrm{d}m\left(z\right), \tag{18}$$

$$\begin{aligned}&\mathcal{L}_2v(x,y,m,t)=\\&\frac{1}{2}\int\int\pi^*\left(z_1,y,m,t\right)\pi^*(z_2,y,m,t)v_{mm}\left(x,y,m,z_1,z_2,t\right)\mathrm{d}m\left(z_1\right)\mathrm{d}m\left(z_2\right)\\&+\frac{1}{2}\int\left(\pi^*(z,y,m,t)\right)^2v_{mz}\left(x,y,m,z,t\right)\mathrm{d}m\left(z\right),\end{aligned} \tag{19}$$

$$\mathcal{L}_3 v(x,y,m.t) = \int \pi^*(z,y,m,t) v_m(x,y,m,z,t)\, \mathrm{d}m(z), \tag{20}$$

and

$$\mathcal{L}_4 v(x,y,m.t) = \int \pi^*(z,y,m,t) v_{ym}(x,y,m,z,t)\, \mathrm{d}m(z). \tag{21}$$

The second equation of the master system is the optimality/compatibility condition

$$\pi^* U_{xx} + \mathcal{L}_1 U = -(bU_x + U_{xy}) \ \text{ in } \ \mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T]. \tag{22}$$

The master system is beyond the scope of the current MFG theory; see, Lasry and Lions [43, 44], Lions [45], Huang, Malhame and Caines [30], Carmona and Delarue [18] and Cardaliaguet, Delarue, Lasry and Lions [16]. Indeed, so far, the existing theory deals with bounded controls, and, more importantly, homogeneous in space and "uncontrolled" coefficients for the common noise. With all these provisions, in the classical MFG theory the system consisting of (16), (17) and (22) collapses to a single infinite dimensional pde.

We, in turn, focus on the popular case of separable payoffs, that is,

$$J(x,m) = J(x - C(m)), \tag{23}$$

where $C : \mathcal{P}_2 \to \mathbb{R}$ is, in general, a function of the law of peers' wealth.

We obtain the closed-form solution

$$U(x,y,m,t) = u(x - f(y,m,t), y, t), \tag{24}$$

to the master equation (16) and (17), where u is as in (3), that is, the value function of the single agent problem in the absence of competion.

Function $f : \mathbb{R}\times\mathcal{P}_2\times[0,T] \to \mathbb{R}$ solves the non-local terminal value problem

$$f_t + \mathcal{L}_2 f + \mathcal{L}_4 f + \frac{1}{2} f_{yy} = 0 \ \text{ in } \ \mathbb{R}\times\mathcal{P}_2\times[0,T) \ \text{ and } f(y,m,T) = C(m), \tag{25}$$

and the optimality condition (22) becomes

$$\pi^*(x,y,m,t) - \mathcal{L}_1 f(x,y,m,t) = f_y(y,m,t) + a^*(x - f(y,m,t), y, t), \tag{26}$$

where a^* is the optimal feedback control of the single player problem; see (6).

The intuition behind representation (24) comes from interpreting the representative agent as the writer of claim $C(m_T^*)$ (at the optimum).

We provide details about the connection of the mean field game and an indifference valuation problem of the representative agent in subsection 5.4.

If the couplings depend only on the first moment $\bar{m} = \int z \mathrm{d}m(z)$ of peers' wealth, that is, for $m \in \mathcal{P}_2$,

$$C(m) = C(\bar{m}),$$

we analyze in detail the cases of exponential utility and arbitrary couplings, and of general utilities and linear couplings.

For exponential utilities $J : \mathbb{R} \to \mathbb{R}$ like $J(x) = -e^{-x}$ and general couplings, the terminal value problem (25) becomes autonomous, that is,

$$f_t + \frac{1}{2}\pi^{*2} \int\int f_{\bar{m}\bar{m}}(y, \bar{m}, z_1, z_2, t)\mathrm{d}m(z_1)\mathrm{d}m(z_2)$$

$$+ \pi^* \int f_{my}(y, \bar{m}, z, t)dm(z) + \frac{1}{2}f_{yy}(y, \bar{m}, t) = 0 \text{ in } \mathbb{R}\times\mathbb{R}\times [0, T),$$

$$f(y, \bar{m}, T) = C(\bar{m}),$$

and (22) yields the wealth-independent mean field equilibrium control $\pi^* : \mathbb{R}\times\mathcal{P}_2\times [0, T] \to \mathbb{R}$, given by

$$\pi^*(y, m, t) = \frac{f_y(y, \bar{m}, t) + c(y, t)}{1 - f_{\bar{m}}(y, \bar{m}, t)}.$$

The mean field value is given, for $(x, y, \bar{m}, t) \in \mathbb{R} \times \mathbb{R}\times\mathcal{P}_2\times [0, T]$ and with k as in (13), by

$$U(x, y, \bar{m}, t) = -e^{-(x-f(y,\bar{m},t))+k(y,t)}$$

and the mean field equilibrium wealth process X^* can be written as

$$X_s^* = \mathcal{X}_s^{x-f(y,\bar{m},t),*} + \mathcal{E}\left(\int \mathcal{X}_s^{x-f(y,\bar{m},t),*}\mathrm{d}m(x), Y_s, s\right) \text{ in } (t, T] \text{ and } X_t^* = x,$$

where

$$\mathcal{E}(y, z, t) = f(y, G(y, z, t), t) \quad \text{and} \quad G(y, z, t) = (\cdot - f(y, \cdot, t))^{(-1)}(y, z, t).$$

For general utilities and linear couplings, that is, $C(\bar{m}) = \theta\bar{m}$ with $\theta \in (0, 1)$, we obtain

$$f(y, \bar{m}, t) = \theta\bar{m} \quad \text{and} \quad U(x, y, \bar{m}, t) = u(x - \theta\bar{m}, y, t).$$

The mean field equilibrium wealth and portfolio processes X^* and π^* are given by

$$X_s^* = \mathcal{X}_s^{x-\theta\bar{m},*} + \frac{\theta}{1-\theta}\int \mathcal{X}_s^{x-\theta\bar{m},*}\mathrm{d}m(x) \text{ in } (t, T] \text{ and } X_t^* = x,$$

and

$$\pi_s^* = \alpha_s^{x-\theta\bar{m},*} + \frac{\theta}{1-\theta}\int \alpha_s^{x-\theta\bar{m},*}\mathrm{d}m(x) \text{ in } (t, T],$$

where $\mathcal{X}^{x-\theta\bar{m},*}$ and $\alpha^{x-\theta\bar{m},*}$ are the corresponding optimal processes of the single agent problem, given by (7) and (8) and starting at $x - \theta\bar{m}$.

The above decomposition interprets the mean field equilibrium process $X^{x,*}$ as the sum of the optimal wealth $\mathcal{X}^{x-\theta\bar{m},*}$ of the single agent starting at $x - \theta\bar{m}$ plus its mean in terms of the initial distribution of x weighted by the factor $\frac{\theta}{1-\theta}$.

The mean field equilibrium policy $\pi_s^{x,*}$ has an analogous decomposition.

Organization of the paper. The paper is organized as follows. In section 2, we revisit the market model under partial information and provide the new, alternative solution approach and the closed-form solutions for general utilities for the single agent without competition problem. In section 3, we introduce the $N-$player game and its continuum limit, and derive the master system for general utilities and couplings. In section 4, we study the class of separable payoffs and analyze representative cases in section 5. We conclude in section 6.

2 The portfolio choice problem under partial information for single agent without competition

We revisit the optimal portfolio choice problem for single agent without competition in a finite horizon with general utilities, and with partial information for the stock's drift. We introduce an alternative solution approach, establish regularity for the value function, and produce closed-form expressions for the optimal wealth and portfolio processes. We also present representative examples which cover the so-called SAHARA utilities and utlities with completely monotonic inverse marginals (CMIM).

2.1 The market model and background results on filtering

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports a Brownian motion B with filtration $\mathbb{F}^B = \left\{\mathcal{F}_t^B;\ 0 \le t \le T\right\}$ and a random variable $\Theta : \Omega \to [\theta_1, \theta_2]$, with $0 < \theta_1 < \theta_2 < \infty$, such that

$$\begin{cases} \Theta \text{ is independent of } B \text{ under } \mathbb{P} \text{ and has prior probability} \\ \text{distribution } \nu(A) = \mathbb{P}\left[\Theta \in A\right] \ \text{ for } A \in \mathcal{B}\left(\mathbb{R}^+\right), \\ \text{such that, for each } y \in \mathbb{R}, \displaystyle\int_{\theta_1}^{\theta_2} e^{y\theta} \mathrm{d}\nu(\theta) < \infty. \end{cases} \tag{27}$$

We consiser the observations process

$$Y_t = \Theta t + B_t \ \text{ in } \ (0,T] \quad \text{and} \quad Y_0 = y \in \mathbb{R}, \tag{28}$$

and introduce the $\mathbb{P}$-augmentation $\mathbb{F}^Y = \left\{\mathcal{F}_t^Y;\ 0 \le t \le T\right\}$ of $\mathcal{F}^Y = \left\{\sigma\left(Y_s\right); 0 \leqslant s \leqslant t \le T\right\}$, and the $\mathbb{P}$-augmentation $\mathbb{G} = \left\{\mathcal{G}_t; 0 \le t \le T\right\}$ of the enlarged filtration generated by both Θ and B,

$$\mathcal{G}_t^{\Theta,B} = \left\{\sigma\left(\Theta, B_s\right); 0 \leqslant s \leqslant t \le T\right\} = \sigma\left(\Theta\right) \vee \mathcal{F}_t^B.$$

We assume a financial market with two securities. The first is a riskless bond taken to be the numeraire and offering zero interest rate. The second security is a risky stock whose (discounted by the numeraire) price S satisfies

$$dS_t = \Theta S_t dt + S_t dB_t = S_t dY_t \text{ in } (0,T] \text{ and } \ S_0 > 0. \tag{29}$$

The key ingredient is that one cannot observe directly either Θ or the Brownian motion B, but only the levels of process S. In other words, it is only the information $\mathbb{F}^Y$ generated by the observations process Y that is accessible to the agent.

In this market, the set of admissible strategies $\mathcal{A}$ consists of self-financing investment strategies π representing the (discounted) amount allocated in the stock and measurable with respect to the information generated by the observations process, that is,

$$\mathcal{A} := \left\{\pi : \ \pi_t \in \mathcal{F}_t^Y \quad \text{and} \quad \mathbb{E}\int_0^T \pi_t^2 dt < \infty\right\}. \tag{30}$$

Remark 1. *Sde* (29) *may have the more general form*

$$dS_s = \Theta S_s dt + \sigma S_s dB_s \ \text{ in } (0,T] \quad \text{and} \quad S_0 > 0,$$

where σ is a known (positive) coefficient, representing the volatility of the stock. To ease the presentation, we assume $\sigma \equiv 1$, as the general case can be directly incorporated by a simple rescaling.

Next, we review some classical results from filtering which yield a reduced complete information model; see, among others, [3], [11], [33] and [34].

Let $F : \mathbb{R}\times[0,T] \to \mathbb{R}^+$ and $b : \mathbb{R}\times[0,T] \to \mathbb{R}^+$ be defined as

$$F(y,t) = \int_{\theta_1}^{\theta_2} e^{y\theta - \frac{1}{2}\theta^2 t} \mathrm{d}\nu(\theta) \quad \text{and} \quad b = \frac{F_y}{F}. \tag{31}$$

It follows that F solves the (ill-posed) heat equation

$$F_t + \frac{1}{2}F_{yy} = 0 \ \text{ in } \ \mathbb{R}\times(0,T] \quad \text{and} \quad F(y,0) = \int_{\theta_1}^{\theta_2} e^{y\theta} d\nu(\theta), \tag{32}$$

is strictly convex in y and strictly decreasing in t, and satisfies, for some constants $c_1 = c_1(\nu(\theta)), c_2 = c_2(\nu(\theta)) > 0$ and all $y \in \mathbb{R}$,

$$c_1 e^{-c_1|y|} \le F(y,T) \le c_2 e^{c_2|y|}. \tag{33}$$

Finally, F is absolutely monotonic, which yields that

$$b_y = \frac{F_{yy}F - (F_y)^2}{F^2} > 0 \text{ in } \mathbb{R}\times[0,T]. \tag{34}$$

It follows that

$$0 < \theta_1 \le b \le \theta_2 \quad \text{and} \quad 0 < b_y \le \theta_2^2 - \theta_1^2 \ \text{ in } \ \mathbb{R}\times[0,T]. \tag{35}$$

The best estimator of Θ is given by

$$\hat{\Theta}_t = \mathbb{E}\left[\Theta | \mathcal{F}_t^Y\right] = \begin{cases} b(Y_t,t) & \text{if } \ t \in (0,T], \\ \int_{\theta_1}^{\theta_2} \theta\nu(d\theta) & \text{if } \ t = 0, \end{cases}$$

and the so-called innovations process W, defined for $t \in [0,T]$, by

$$W_t = Y_t - \int_0^t \hat{\Theta}_s ds = Y_t - \int_0^t b(Y_s, s) ds, \tag{36}$$

is a $\left(\mathbb{F}^Y, \mathbb{P}\right)$ standard Brownian motion.

In view of the above, the original sdes (29) and (28) can be now written as

$$dS_t = b\left(Y_t, t\right) S_t dt + S_t dW_t \text{ in } (0,T] \quad \text{and} \quad S_0 = S > 0, \tag{37}$$

and

$$dY_t = b\left(Y_t, t\right) dt + dW_t \text{ in } (0,T] \quad \text{and} \quad Y_0 = y \in \mathbb{R}. \tag{38}$$

The model consisting of (37) and (38) constitutes a complete information market model with a single local factor Y.

For each $\alpha \in \mathcal{A}$, the agent's wealth, denoted by $\mathcal{X}$, solves

$$d\mathcal{X}_s = b\left(Y_s, s\right) \alpha_s ds + \alpha_s dW_s \text{ in } (t,T] \text{ and } \mathcal{X}_t = x \in \mathbb{R}. \tag{39}$$

Both $\mathcal{X}$ and α are expressed in discounted by the bond units and are, thus, unitless quantities. This property is important for the proper definitions of the payoffs in the upcoming MFG as they are, in general, allowed to depend on wealths, their laws, means, quantiles and higher moments.

The value function $u : \mathbb{R} \times \mathbb{R} \times [0,T] \to \mathbb{R}$ is defined as

$$u(x,y,t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[J(\mathcal{X}_T) \middle| \mathcal{X}_t = x, Y_t = y \right]. \tag{40}$$

We introduce next the main assumptions for the utility function $J : \mathbb{R} \to \mathbb{R}$, the risk tolerance $r : \mathbb{R} \to \mathbb{R}^+$ and the inverse marginal utility $I : \mathbb{R}^+ \to \mathbb{R}$, defined respectively by

$$r = -\frac{J'}{J''} \text{ and } I = (J')^{(-1)}. \tag{41}$$

We assume that

$$\begin{gathered} J \text{ is a strictly concave and strictly increasing map in } \mathcal{C}^4(\mathbb{R}), \text{ and} \\ \text{there exist } A, B, c > 0 \text{ and } \gamma > 1 \text{ such that, for all } x \in \mathbb{R}, \\ B \le r(x) \le \sqrt{Ax^2 + B}, \quad |r'| \le A \text{ and } \lim_{z \to \infty} z^\gamma I(z) = c. \end{gathered} \tag{42}$$

The assumptions above are mild and are satisfied by a large class of utility functions like, for example, the exponential utility

$$J(x) = -C_1 e^{-Bx} + C_2 \text{ for some } C_1 > 0 \text{ and } C_2 \in \mathbb{R}, \tag{43}$$

for which the risk tolerance is constant $r(x) = B$, and by the so-called SAHARA utilities, introduced in [51] (see, also, [19] and [60]), that are represented via their risk tolerances in the parametric form

$$r(x) = \sqrt{Ax^2 + B} \text{ for } A, B > 0. \tag{44}$$

They, also, apply to the extended family of utilities with completely monotonic inverse marginals (CMIM), introduced in [51] and further examined in [48]. We further elaborate on these utilities in subsection 2.3.

We note that (41) implies that

$$-zI'(z) = r(I(z)) \quad \text{and} \quad zI'(z) + z^2 I''(z) = r(I(z))r'(I(z)). \tag{45}$$

It is not surprising that, since the market dynamics (37) and (38) compose a complete market model, (40) can be linearized. Such linearization is inherent in the duality approach and, in Markovian models, a linear pde can be easily derived from the HJB equation and Fenchel transform. However, for general utilities, this pde is two-dimensional and not uniformly elliptic and, thus, regularity results are not a priori known. To the best of our knowledge, this issue has not been addressed in the existing literature, except for the case of homothetic utilities where the HJB equation reduces to a simple one-dimensional pde.

Here, we circumvent this issue by first looking at the auxiliary problem (47) for which the related linear pde (48) is one-dimensional and, thus, regularity results are easy to obtain. We stress that this dimensionality reduction works not because of utility homotheticity. Rather it results from suitably modifying the utility and, furthermore, removing the drift in the wealth dynamics, which essentially results in the optimal wealth process being static in the $x-$argument, as manifested in (62).

A similar optimization problem was considered in [34] but in a different context within the duality approach employed therein. Here, problem (47) plays a different role. Firstly, it helps us establish regularity for the original problem under much milder and direct assumptions on the utility function. Secondly, the expressions (see (62) below) for the optimal processes guide us how to construct their general analogues, (86) and (92), whose form turns out to be quite useful in the MFG we examine later on.

2.2 An auxiliary expected utility problem

We consider a fictitious portfolio choice problem with modified terminal utility and dynamics. Specifically, let $W^{\mathbb{Q}}$ be a Brownian motion on probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and, for $t \in [0, T]$, consider the processes $\tilde{\mathcal{X}}$ and $\mathcal{Y}$ solving

$$\begin{aligned} d\tilde{\mathcal{X}}_s &= \tilde{\alpha}_s dW^{\mathbb{Q}} \text{ in } (t, T] \text{ and } \tilde{\mathcal{X}}_t = x \in \mathbb{R}, \\ d\mathcal{Y}_s &= dW^{\mathbb{Q}} \text{ in } (t, T] \text{ and } \mathcal{Y}_t = y \in \mathbb{R}. \end{aligned} \tag{46}$$

The value function $w : \mathbb{R} \times \mathbb{R} \times [0, T] \to \mathbb{R}$ is defined by

$$w(x, y, t) = \sup_{\tilde{\alpha} \in \tilde{\mathcal{A}}} \mathbb{E}_{\mathbb{Q}} \left[J\left(\tilde{\mathcal{X}}_T\right) F(\mathcal{Y}_T, T) \middle| \tilde{\mathcal{X}}_t = x, \mathcal{Y}_t = y \right], \tag{47}$$

with F as in (31) and $\tilde{\mathcal{A}}$ defined similarly to (30).

To find the value function $w(x,y,t)$ and construct the optimal processes $\tilde{\mathcal{X}}^*$ and $\tilde{\alpha}^*$, we first introduce a key auxiliary function and study its properties.

Proposition 2. *(i) There exists a unique smooth solution* $h:\mathbb{R}\times\mathbb{R}\times[0,T]\to\mathbb{R}$ *to the terminal value problem*

$$h_t+\frac{1}{2}h_{yy}=0 \text{ in } \mathbb{R}\times[0,T] \quad \text{and} \quad h(z,y,T)=I\left(\frac{e^{-z}}{F(y,T)}\right), \tag{48}$$

with I *and* F *as in (41) and (31) respectively, such that, for each* $(y,t)\in\mathbb{R}\times[0,T]$,

$$\begin{aligned}&\lim_{z\to-\infty}h(z,y,t)=-\infty \quad \text{and} \quad \lim_{z\to\infty}h(z,y,t)=\infty,\\ &h_y>0 \quad \text{and} \quad h_z>0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}\times[0,T],\end{aligned} \tag{49}$$

and, for each (y,t) *in* $\mathbb{R}\times[0,T]$*, the inverse in* z *function* $h^{(-1)}(\cdot,y,t):\mathbb{R}\to\mathbb{R}$ *exists and is strictly increasing.*

(ii) For all $(x,y,t)\in\mathbb{R}\times\mathbb{R}\times[0,T]$ *and the constants* A *and* B *in* (42),

$$h_y(z,y,t)\leq\theta_2\sqrt{Ah^2(z,y,t)+Be^{A(T-t)}}, \tag{50}$$

and

$$-Ah_z(z,y,t)\leq h_{yz}(z,y,t)\leq Ah_z(z,y,t). \tag{51}$$

Proof. (i) The smoothness and uniqueness results follow from (31), (33) and (42).

Since

$$h_y(z,y,T)=-\frac{F_y(y,T)}{F(y,T)}\frac{e^{-z}}{F(y,T)}I'\left(\frac{e^{-z}}{F(y,T)}\right)>0, \tag{52}$$

and

$$h_z(z,y,T)=-\frac{e^{-z}}{F(y,T)}I'\left(\frac{e^{-z}}{F(y,T)}\right)>0, \tag{53}$$

the maximum principle yields (49).

(ii) At $t=T$, (52) combined with (35), (42) and (45), and the terminal condition in (48) imply that

$$h_y(z,y,T)=b(y,T)r\left(I\left(\frac{e^{-z}}{F(y,T)}\right)\right)\leq\theta_2\sqrt{Ah^2(z,y,T)+B}.$$

Let $g:\mathbb{R}\times\mathbb{R}\times[0,T]\to\mathbb{R}^+$ and $C:\mathbb{R}\times\mathbb{R}\times[0,T]\to\mathbb{R}$ be defined by

$$g(z,y,t)=\theta_2\sqrt{Ah^2(z,y,t)+Be^{A(T-t)}}, \tag{54}$$

and

$$C(z,y,t)=\frac{ABe^{A(T-t)}}{2g^3}(g+h_y),$$

and note that

$$h_y(\cdot,\cdot,T) \leq g(\cdot,\cdot,T) \text{ in } \mathbb{R}\times\mathbb{R}, \tag{55}$$

and, in view of (54) and the first inequality in (49), $C>0$ in $\mathbb{R}\times\mathbb{R}\times[0,T]$.

We claim that $h_y - g$ solves

$$(h_y-g)_t + \frac{1}{2}(h_y-g)_{yy} + C(h_y-g) = 0 \text{ in } \mathbb{R}\times\mathbb{R}\times[0,T). \tag{56}$$

To ease the notation, we assume that $\theta_2 = 1$. Then direct calculations yield that

$$g_t + \frac{1}{2}g_{yy} = \frac{ABe^{A(T-t)}}{2g^3}\left(g^2 - (h_y)^2\right).$$

Using (55) and (56) and applying the maximum principle we find that

$$h_y \leq g \text{ in } \mathbb{R}\times\mathbb{R}\times[0,T],$$

and, thus, (50) holds.

Finally, using again the maximum principle, we note that to show (51), it suffices to establish it for $t=T$.

To this end, note that (31) and (52) give

$$h_{yz}(z,y,T) = b(y,T)h_{zz}(z,y,T),$$

and, thus, after using (45) and (53),

$$h_{yz}(z,y,T) = b(y,T)h_z(z,y,T)\,r'\left(I\left(\frac{e^{-z}}{F(y,T)}\right)\right).$$

The latter equality, (35) and the derivative estimate in (42) give

$$|h_{yz}(z,y,T)| = b(y,T)h_z(z,y,T)\left|r'\left(I\left(\frac{e^{-z}}{F(y,T)}\right)\right)\right| \leq Ah_z(z,y,T),$$

and we conclude.

□

Next, we construct the value function w and the optimal processes $\tilde{\mathcal{X}}^*$ and $\tilde{\alpha}^*$.

Proposition 3. *Assume that $\mathcal{Y}$ solves (46). Then:*

(i) The value function w defined in (47) *is the unique in $\mathcal{C}^{2,2,1}(\mathbb{R}\times\mathbb{R}\times[0,T])$ solution to the terminal value HJB equation*

$$\begin{aligned} &w_t + \max_{\alpha}\left(\frac{1}{2}\alpha^2 w_{xx} + \alpha w_{xy}\right) + \frac{1}{2}w_{yy} \\ &= w_t - \frac{(w_{xy})^2}{2w_{xx}} + \frac{1}{2}w_{yy} = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}\times[0,T), \\ &w(x,y,T) = J(x)\,F(y,T), \end{aligned} \tag{57}$$

and is given, for h as in (48), by

$$w(x,y,t) = \mathbb{E}_{\mathbb{Q}}\left[J(h\left(h^{(-1)}\left(x,y,t\right), \mathcal{Y}_T, T\right) F(\mathcal{Y}_T,T) \Big| \mathcal{Y}_t = y \right]. \tag{58}$$

Furthermore, for each $(z,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$,

$$w_x\left(h\left(z,y,t\right),y,t\right) = e^{-z}. \tag{59}$$

(ii) For each $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$, *the optimal feedback policy is given by*

$$\tilde{\alpha}^*(x,y,t) = -\frac{w_{xy}\left(x,y,t\right)}{w_{xx}\left(x,y,t\right)} = h_y\left(h^{(-1)}(x,y,t),y,t\right), \tag{60}$$

and satisfies, for each $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$, *the inequalities*

$$0 < \tilde{\alpha}^*(x,y,t) \le \theta_2\sqrt{Ax^2 + Be^{A(T-t)}} \quad \text{and} \quad \tilde{\alpha}^*_x(x,y,t) \le \theta_2 A. \tag{61}$$

(iii) The optimal processes $\tilde{\mathcal{X}}^*$ *and* $\tilde{\alpha}^*$ *are given, respectively, for each* $(x,y) \in \mathbb{R}\times\mathbb{R}$ *and* $0 \le t \le s \le T$, *by*

$$\tilde{\mathcal{X}}^*_s = h\left(h^{(-1)}\left(x,y,t\right), \mathcal{Y}_s, s\right) \quad \text{and} \quad \tilde{\alpha}^*_s = h_y\left(h^{(-1)}\left(x,y,t\right), \mathcal{Y}_s, s\right). \tag{62}$$

Proof. (i) A straightforward modification of the results in [58] shows that the value function (47) is the unique viscosity solution of (57) in the class of functions that are strictly concave and strictly increasing in x, for each $(y,t) \in \mathbb{R}\times[0,T]$.

We now construct a smooth solution $\hat{w}$ of (57) in the same class of functions which, by uniqueness, will coincide with w.

To this end, consider the terminal value problem

$$\begin{aligned} &K_t + \frac{1}{2}K_{yy} = 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}\times[0,T), \\ &K(z,y,T) = J\Big(I\big(\frac{e^{-z}}{F(y,T)}\big)\Big)F(y,T), \end{aligned} \tag{63}$$

which, in view of (31), (33) and (42), is well-posed and has a unique smooth solution.

Using the terminal condition in (48), we rewrite

$$K(z,y,T) = J\left(h(z,y,T)\right)F(y,T), \tag{64}$$

and find

$$K_z(z,y,T) = h_z(z,y,T)J'\left(I\left(\frac{e^{-z}}{F(y,T)}\right)\right)F(y,T) = e^{-z}h_z(z,y,T).$$

In addition, for each fixed $z \in \mathbb{R}$, functions $K_z(z,y,t)$ and $e^{-z}h_z(z,y,t)$ solve in $\mathbb{R}\times[0,T)$ the same heat equation (see (48) and (63)) and, thus, by uniqueness, for each $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$,

$$K_z(z,y,t) = e^{-z}h_z(z,y,t). \tag{65}$$

Next, we introduce the smooth function $\hat{w} : \mathbb{R}\times\mathbb{R}\times[0,T] \to \mathbb{R}$ given by

$$\hat{w}(x,y,t) = K\Big(h^{(-1)}(x,y,t),y,t\Big), \tag{66}$$

with $h^{(-1)}$ as in Proposition 2.

We show that, for each $(y,t) \in \mathbb{R}\times[0,T]$, $\hat{w}$ is strictly increasing and strictly concave in x, and, in addition, it satisfies (57).

To ease the notation, we write

$$p(x,y,t) = h^{(-1)}(x,y,t). \tag{67}$$

Then, for each $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$,

$$p_t = -\frac{h_t(p,y,t)}{h_z(p,y,t)}, \quad p_x = \frac{1}{h_z(p,y,t)} \tag{68}$$

$$p_{xx} = -\frac{h_{zz}(p,y,t)}{\left(h_z(p,y,t)\right)^3}, \quad p_y = -\frac{h_y(p,y,t)}{h_z(p,y,t)} \tag{69}$$

$$\begin{aligned} p_{yy} = -\frac{1}{h_z\,(p,y,t)} &\left(\left(\frac{h_y(p,y,t)}{h_z(p,y,t)}\right)^2 h_{zz}(p,y,t)\right. \\ &\left. -2\frac{h_y(p,y,t)}{h_z(p,y,t)}h_{zy}(p,y,t) + h_{yy}(p,y,t)\right) \end{aligned} \tag{70}$$

and

$$p_{xy} = -\frac{1}{\left(h_z\,(p,y,t)\right)^2}\left(-\frac{h_y(p,y,t)}{h_z(p,y,t)}h_{zz}(p,y,t) + h_{zy}(p,y,t)\right), \tag{71}$$

It follows from (65) and (66) that

$$\hat{w}_t = p_t K_z(p,y,t) + K_t(p,y,t),$$

and

$$\hat{w}_x = p_x K_z(p,y,t) = \frac{K_z(p,y,t)}{h_z(p,y,t)} = e^{-p}. \tag{72}$$

Furthermore,

$$\hat{w}_{xx} = -p_x\hat{w}_x,\ \tilde{w}_{xy} = -p_y\tilde{w}_x \ \text{ and } \ \tilde{w}_y = p_y K_z(p,y,t) + K_y(p,y,t), \tag{73}$$

and

$$\tilde{w}_{yy} = p_{yy}K_z(p,y,t) + \left(p_y\right)^2 K_{zz}(p,y,t) + 2p_y K_{zy}(p,y,t) + K_{yy}(p,y,t).$$

Next, we use that K solves (63) and show that $\hat{w}$ satisfies (57). Indeed,

$$\begin{aligned}
&\hat{w}_t - \frac{1}{2}\frac{(\hat{w}_{xy})^2}{\hat{w}_{xx}} + \frac{1}{2}\hat{w}_{yy} \\
&= \left(p_t + \frac{1}{2}p_{yy}\right) K_z(p,y,t) + \frac{1}{2}\frac{(p_y)^2}{p_x}\hat{w}_x + \frac{1}{2}(p_y)^2 K_{zz}(p,y,t) + p_y K_{zy}(p,y,t).
\end{aligned}$$

In addition, from (65) we deduce that

$$K_{zz} = -e^{-z}h_z + e^{-z}h_{zz} \quad \text{and} \quad K_{zy} = e^{-z}h_{zy}.$$

Combining the above and using (72), we find, after using (48), that

$$\hat{w}_t - \frac{1}{2}\frac{(\hat{w}_{xy})^2}{\hat{w}_{xx}} + \frac{1}{2}\hat{w}_{yy} = -\big(h_t + \frac{1}{2}h_{yy}\big)h_z = 0.$$

Finally, we deduce from (72) that $\hat{w}_x(x,y,t) > 0$. Then (73) together with the monotonicity of $h^{(-1)}$ yield $\hat{w}_{xx}(x,y,t) < 0$. We easily conclude that $\hat{w} \equiv w$ in $\mathbb{R} \times \mathbb{R} \times [0,T]$.

To show (58), we first observe that (63) yields a probabilistic representation for K, that is,

$$\begin{aligned}
K(z,y,t) &= \mathbb{E}_{\mathbb{Q}}\left[J\left(I\left(\frac{e^{-z}}{F(\mathcal{Y}_T,T)} \right) \right) F(\mathcal{Y}_T,T) \middle| \mathcal{Y}_t = y \right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[J\left(h\left(z,\mathcal{Y}_T,T\right)\right) F(\mathcal{Y}_T,T) \middle| \mathcal{Y}_t = y \right],
\end{aligned}$$

where the last equality follows from the terminal condition in (48). Then, (58) and (59) follow respectively from (66), using that $\hat{w} \equiv w$ and (72).

(ii) The first order conditions in (57) imply the first equality in (60). For the second equality, we use (67), (68), (69) and (72) to obtain

$$\tilde{\alpha}^*(x,y,t) = -\frac{p_y(x,y,t)}{p_x(x,y,t)} = h_y(p,y,t) = h_y(h^{(-1)}(x,y,t), y, t). \tag{74}$$

The first inequality in (61) follows from (50). For the second, we observe that

$$\tilde{\alpha}^*_x(x,y,t) = \frac{h_{yz}(h^{(-1)}(x,y,t), y, t)}{h_z(h^{(-1)}(x,y,t), y, t)},$$

and use (51).

(iii) Using the feedback policy (74), the state controlled sde in (46) becomes

$$d\tilde{\mathcal{X}}^*_s = \tilde{\alpha}^*(\tilde{\mathcal{X}}^*_s, \mathcal{Y}_s, s) dW^{\mathbb{Q}}_s \text{ in } (t,T] \text{ and } \tilde{\mathcal{X}}^*_t = x. \tag{75}$$

We claim that, for each fixed $(y,t) \in \mathbb{R} \times [0,T]$, (75) has a unique strong solution given by

$$\tilde{\mathcal{X}}^*_s = h\left(h^{(-1)}(x,y,t), \mathcal{Y}_s, s\right) \text{ in } (t,T] \text{ and } \tilde{\mathcal{X}}^*_t = x. \tag{76}$$

We begin by verifying that $\tilde{\mathcal{X}}^*$ satisfies (75). Indeed, applying Ito's rule to (76) and using (48) gives

$$dh\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_s,s\right)=h_y\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_s,s\right)dW_s^{\mathbb{Q}}.$$

From (61), we have that, for $s\in[t,T]$,

$$\left(h_y\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_s,s\right)\right)^2\leq Ax^2+Be^{A(T-t)},$$

and, thus,

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\int_t^T\left(h_y\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_s,s\right)\right)^2\mathrm{d}s\\&\qquad\leq A\big(h^{(-1)}\left(x,y,t\right)\big)^2(T-t)+\int_t^T Be^{A(T-s))}\mathrm{d}s.\end{aligned}$$

Moreover, (74) evaluated at (76) yields

$$\tilde{\alpha}^*(\tilde{\mathcal{X}}_s^*,\mathcal{Y}_s,s)=\tilde{\alpha}^*(h\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_s,s\right),\mathcal{Y}_s,s)=h_y\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_s,s\right),$$

and both equalities in (62) follow.

Lastly, we show that for each fixed $(y,t)\in\mathbb{R}\times[0,T]$, (75) has a unique solution. Arguing by contradiction, we assume that there exist two solutions, say $\tilde{\mathcal{X}}_1$ and $\tilde{\mathcal{X}}_2$, with $\tilde{\mathcal{X}}_{1,t}=\tilde{\mathcal{X}}_{2,t}=x$. Then, (75) and (61) yield, for $s\in[t,T]$, that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left(\tilde{\mathcal{X}}_{1,s}-\tilde{\mathcal{X}}_{2,s}\right)^2&=\mathbb{E}_{\mathbb{Q}}\int_t^s\left(\tilde{\alpha}^*(\tilde{\mathcal{X}}_{1,\rho},\mathcal{Y}_\rho,\rho)-\tilde{\alpha}^*(\tilde{\mathcal{X}}_{2,\rho},\mathcal{Y}_\rho,\rho)\right)^2d\rho\\&\leq\theta_2A\mathbb{E}_{\mathbb{Q}}\int_t^s\left(\tilde{\mathcal{X}}_{1,\rho}-\tilde{\mathcal{X}}_{2,\rho}\right)^2d\rho,\end{aligned}\tag{77}$$

and uniqueness follows from Grönwall's inequality. □

Remark 4. *Alternatively (58) may be derived from (47) and (62), using that at $t=T$, we have that $\tilde{\mathcal{X}}_T^*=h\left(h^{(-1)}\left(x,y,t\right),\mathcal{Y}_T,T\right)$.*

We now revert the analysis to the original problem (40), starting with the specification and regularity of its solution.

Proposition 5. *The value function u defined in (40) is the unique in $\mathcal{C}^{2,2,1}\left(\mathbb{R}\times\mathbb{R}\times[0,T]\right)$, strictly concave and strictly increasing in x, solution to the terminal value HJB equation*

$$\begin{cases}u_t+\max_\alpha\left(\frac{1}{2}\alpha^2u_{xx}+\alpha\left(bu_x+u_{xy}\right)\right)+\frac{1}{2}u_{yy}+bu_y=\\ u_t-\dfrac{\left(bu_x+u_{xy}\right)^2}{2u_{xx}}+\frac{1}{2}u_{yy}+bu_y=0\quad in\quad\mathbb{R}\times\mathbb{R}\times[0,T),\\ u(x,y,T)=J(x),\end{cases}\tag{78}$$

and is given by

$$u(x,y,t)=\frac{w(x,y,t)}{F(y,t)}, \tag{79}$$

with w as in Proposition 3 and F as in (31)*.*

Proof. A straightforward adaptation of the results in [58] yield that the value function (40) is the unique viscosity solution in the class of strictly increasing and strictly concave in x solutions, for each $(y,t)\in\mathbb{R}\times[0,T]$, solutions.

Let $\hat{u}=\dfrac{w}{F}$. We show that $\hat{u}\equiv u$ on $\mathbb{R}\times\mathbb{R}\times[0,T]$.

For this, it suffices to establish that $\hat{u}\in\mathcal{C}^{2,2,1}(\mathbb{R}\times\mathbb{R}\times[0,T])$ is strictly concave and strictly increasing in x for each $(y,t)\in\mathbb{R}\times[0,T]$, and that it satisfies (78). The first three properties follow easily from the analogous properties of w, so we only show that $\hat{u}$ solves (78).

Direct calculations yield, after substituting the above in (57) and using that w solves (57) and (31), that

$$w_t-\frac{1}{2}\frac{(w_{xy})^2}{w_{xx}}+\frac{1}{2}w_{yy}=\big(\hat{u}_t-\frac{1}{2}\frac{(b\hat{u}_{xy}+\hat{u}_x)^2}{\hat{u}_{xx}}+\frac{1}{2}\hat{u}_{yy}+b(y,t)\hat{u}_y\big)F=0,$$

Since $F>0$ in $\mathbb{R}\times[0,T]$, we easily conclude.

□

Next, we introduce two auxiliary functions, H and k, that will be used in the construction of the optimal processes X^* and α^*.

Proposition 6. *Let k solve the terminal value problem* (13)*. Then $H:\mathbb{R}\times\mathbb{R}\times[0,T]\to\mathbb{R}$ given by*

$$H(z,y,t)=(u_x)^{(-1)}\left(e^{-z+k(y,t)},y,t\right), \tag{80}$$

where (-1) *denotes the inverse in the $x-$argument, is a* $\mathcal{C}^{2,2,1}(\mathbb{R}\times\mathbb{R}\times[0,T])$ *solution to the terminal value problem* (10) *and* (11) *where $c:\mathbb{R}\times[0,T]\to\mathbb{R}$ given by* (12) *solves the terminal value problem*

$$c_t+\frac{1}{2}c_{yy}=0 \quad in \quad \mathbb{R}\times[0,T) \quad and \quad c(y,T)=b(y,T). \tag{81}$$

Looking at (59) and (80), it is clear that H resembles the auxiliary function h. We stress, however, that contrary to (59), the equation satisfied by H is not uniformly elliptic, and, therefore, it is not a priori known whether a smooth solution exists.

Proof. It follows easily from the assumptions that k is well defined, since for all $(y,t)\in\mathbb{R}\times[0,T]$,

$$-\theta_2(T-t)<k(y,t)<-\theta_1(T-t).$$

Furthermore, (79) yields that $u_x > 0$ in $\mathbb{R} \times \mathbb{R}\times [0,T]$ and, therefore, for each $(y,t) \in \mathbb{R}\times [0,T]$, the map $x \to u_x(x,y,t)$ is invertible, and, thus, H is well defined.

Rewriting (80) as

$$u_x(H(z,y,t)y,t) = e^{-z+k(y,t)}, \tag{82}$$

and using (59) and (79), we obtain that, for each $(z,y,t) \in \mathbb{R}\times\mathbb{R} \times [0,T]$,

$$H(z,y,t) = h(z - n(y,t), y, t), \tag{83}$$

with

$$n = k + \ln F. \tag{84}$$

Then, (83) yields

$$H_t = -n_t h_z + h_t,\ H_z = h_z,\ H_{zz} = h_z,\ H_{xy} = -n_y h_{zz} + h_{zy},\ H_y = -n_y h_z + h_y,$$

and

$$H_{yy} = -n_{yy} h_z + n_y^2 h_{zz} - 2 n_y h_{zy} + h_{yy},$$

where h and its derivatives are evaluated at $(z - n(y,t), y, t)$.

Moreover, (84) gives

$$\frac{F_y}{F} + k_y - n_y = 0 \quad \text{in } \ \mathbb{R}\times [0,T].$$

Using (31), (32) and (80), we deduce that

$$n_t + \frac{1}{2} n_{yy} = \frac{1}{F}\big(F_t + \frac{1}{2}F_{yy}\big) + \big(k_t + \frac{1}{2}k_{yy} - \Big(\frac{F_y}{F}\Big)^2\big) = k_t + \frac{1}{2}k_{yy} - \frac{1}{2}b^2 = 0.$$

Recalling (12) we obtain, after some routine calculations, that

$$\begin{aligned} &H_t + \frac{1}{2}c^2 H_{zz} + cH_{zy} + \frac{1}{2}H_{yy} \\ &= \big(h_t + \frac{1}{2}h_{yy}\big) - \big(n_t + \frac{1}{2}n_{yy}\big)h_z \\ &+ \frac{1}{2}\big(\frac{F_y}{F} + k_y - n_y\big)^2 h_{zz} + \big(\frac{F_y}{F} + k_y - n_y\big)h_{zy} = 0. \end{aligned}$$

Finally, to show (81) we use (12) and (13) to obtain

$$c_t + \frac{1}{2}c_{yy} = b_t + \frac{1}{2}b_{yy} + b b_y.$$

On the other hand, (31) yields

$$F_{yt} = b_t F + b F_t \ \text{ and } \ F_{yyy} = b_{yy} F + 2 b_y F_y + b F_{yy},$$

which together with (32) imply

$$b_t F + b F_t + \frac{1}{2}\left(b_{yy} F + 2 b_y F_y + b F_{yy}\right) = \big(b_t + \frac{1}{2}b_{yy} + b_y \frac{F_y}{F}\big)F = 0.$$

The claim follows since $F > 0$ in $\mathbb{R}\times [0,T]$.

$\square$

We are now ready to state the main results which provide closed-form expressions for the value function u, the optimal wealth process $\mathcal{X}^\star$ and the associated optimal control process $\alpha^\star$. As mentioned earlier, these results are, to the best of our knowledge, new and of independent interest.

Theorem 7. *The value function u in (40) is represented, for $(x,y,t) \in \mathbb{R} \times \mathbb{R} \times [0,T]$, as*

$$u(x,y,t) = \mathbb{E}\left[J\left(H(H^{(-1)}(x,y,t) + L_{t,T}, Y_T, T)\right) \Big| X_t = x, Y_t = y \right], \tag{85}$$

with the process L defined by (9) *with H and c solving respectively* (10), (11) *and* (81).

The optimal wealth process $\mathcal{X}^*$ is as

$$\mathcal{X}^*_s = H(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s) \ \text{ in } (t,T] \ \text{ and } \ \mathcal{X}^*_t = x, \tag{86}$$

and the optimal feedback control α^* is given by

$$\alpha^* = -\frac{bu_x + u_{xy}}{u_{xx}} \quad \text{in } \ \mathbb{R} \times \mathbb{R} \times [0,T], \tag{87}$$

and satisfies, with $\tilde{\alpha}^*$ as in (60),

$$\alpha^* = \tilde{\alpha}^* \quad \text{in } \ \mathbb{R} \times \mathbb{R} \times [0,T]. \tag{88}$$

Moreover, if A and B are as in (42), *then, for each $(x,y,t) \in \mathbb{R} \times \mathbb{R} \times [0,T]$,*

$$0 < \alpha^*(x,y,t) \le \theta_2 \sqrt{Ax^2 + Be^{A(T-t)}} \ \text{ and } \ \alpha^*_x(x,y,t) \le \theta_2 A. \tag{89}$$

The optimal control process α^* is given by

$$\alpha^*_s = \alpha^*(\mathcal{X}^*_s, Y_s, s), \tag{90}$$

with $\mathcal{X}^*$ solving the sde

$$d\mathcal{X}^*_s = b(Y_s,s)\alpha^*(\mathcal{X}^*_s, Y_s, s)ds + \alpha^*(\mathcal{X}^*_s, Y_s, s)dW_s \ \text{ in } \ (t,T] \quad \text{and} \quad \mathcal{X}^*_t = x, \tag{91}$$

and is represented, for $s \in [t,T]$, as

$$\begin{aligned} \alpha^*_s &= c(Y_s,s)H_z(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s) \\ &\qquad + H_y(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s) \\ &= \alpha^*\left(H\left(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s\right), Y_s, s\right). \end{aligned} \tag{92}$$

Proof. We only establish the claims for $\mathcal{X}^*$ and α^*, since (85) follows easily.

The first order conditions in (78) yield (87), while (79) gives

$$-\frac{bu_x + u_{xy}}{u_{xx}} = -\frac{w_{xy}}{w_{xx}},$$

and (88) follows, while (89) comes directly from (61).

Next, we note that (90) and (91) will follow once we show that the candidate control process is admissible and that (91) has a unique strong solution.

To this end, observe that (82) gives

$$H_z u_{xx}(H,y,t) = -u_x(H,y,t),$$

and

$$H_y u_{xx}(H,y,t) + u_{xy}(H,y,t) = k_y\,(y,t)\,u_x(H,y,t),$$

and, in turn, (12) and (87) imply

$$\alpha^*(H,y,t) = c(y,t)H_z(z,y,t) + H_y(z,y,t). \tag{93}$$

Using (93), (9) and (38), we write (91) as

$$\begin{aligned} d\mathcal{X}_s^* &= H_z\left(H^{(-1)}(\mathcal{X}_s^*, Y_s, s), Y_s, s\right)(b(Y_s,s)c(Y_s,s)ds + c(Y_s,s)dW_s) \\ &\quad + H_y\left(H^{(-1)}(\mathcal{X}_s^*, Y_s, s), Y_s, s\right)(b(Y_s,s)ds + dW_s) \\ &= H_z(H^{(-1)}(\mathcal{X}_s^*, \mathcal{Y}_s, s), \mathcal{Y}_s, s)\mathrm{d}L_{t,s} + H_y(H^{(-1)}(\mathcal{X}_s^*, \mathcal{Y}_s, s), \mathcal{Y}_s, s)\mathrm{d}\mathcal{Y}_s. \end{aligned} \tag{94}$$

To identify the unique, for each fixed (y,t) in $\mathbb{R}\times[0,T]$, solution to (94), we introduce the process $\hat{\mathcal{X}}$ given by

$$\hat{\mathcal{X}}_s = H(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s) \quad \text{for} \quad s \in [t,T].$$

Using (10) we find

$$\begin{aligned} &d\hat{\mathcal{X}}_s = d\left(H(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s)\right) \\ &= \left(H_z\left(H^{(-1)}(\hat{\mathcal{X}}_s, Y_s, s), Y_s, s\right)\right)dL_{t,s} + H_y\left(H^{(-1)}(\hat{\mathcal{X}}_s, Y_s, s), Y_s, s\right)dY_s \\ &+ \left(H_t(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s) + \frac{1}{2}c^2(Y_s,s)H_{zz}(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s)\right)\mathrm{d}s \\ &+ \left(c(Y_s,s)H_{zy}(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s) + \frac{1}{2}H_{yy}(H^{(-1)}(x,y,t) + L_{t,s}, Y_s, s)\right)\mathrm{d}s \\ &= H_z\left(H^{(-1)}(\hat{\mathcal{X}}_s, Y_s, s), Y_s, s\right)\mathrm{d}L_{t,s} + H_y\left(H^{(-1)}(\hat{\mathcal{X}}_s, Y_s, s), Y_s, s\right)dY_s, \end{aligned}$$

and, thus,

$$d\tilde{\mathcal{X}}_s^* = d\hat{\mathcal{X}}_s^* \text{ in } (t,T] \text{ and } \tilde{\mathcal{X}}_t^* = \hat{\mathcal{X}}_t^* = x.$$

The first claim of the theorem is a direct consequence of (86) evaluated at T and (40), while the last claim follows from (86), (90) and (93).

It remains to show that the Itô's integrals appearing in the expansions above are well defined, and that (94) has a unique solution for each $(y,t) \in \mathbb{R}\times[0,T]$.

These follow from (88) and arguments similar, albeit more tedious, to the ones used in the proof of Proposition 2.

□

Remark 8. *We note that (59) and (80), rewritten below as*

$$w_x(h(z,y,t),y,t) = e^{-z} \quad \textit{and} \quad u_x(H(z,y,t)y,t) = e^{-z+k(y,t)} \tag{95}$$

for problems (47) and (40), relate the marginals w_x and u_x to their corresponding linear equations (48) and (10). In the existing literature of portfolio choice in complete markets, linearization has been carried out directly for the inverse marginal $(u_x)^{(-1)}$ leading to a single linear pde. Herein, however, we use a different linearization approach, by seeking a pair of auxiliary functions (H,k) solving respective linear equations. The pair $(h,0)$ is easily identified as the analogue of (H,k) when $b(y,t) \equiv 0$ and the modified utility $J(x)F(y,T)$. Of course, the two approaches are equivalent but working with transformations (95) allow us to produce regularity results and produce closed-form solutions for general utilities.

We conclude with the derivation of an autonomous equation satisfied by the optimal feedback controls for problems (47) and (40). The equation generalizes the fast-diffusion equation that the risk tolerance function in the log-normal case satisfies; see, among others, [10], [32], [35] and [47].

Proposition 9. *Let r be the risk tolerance (41). The optimal feedback control functions $\tilde{a}^*$ and α^* for problems (47) and (40) satisfy the terminal value problem*

$$\begin{cases} R_t + \frac{1}{2}R^2 R_{xx} + R R_{xy} + \frac{1}{2}R_{yy} = 0 \quad \textit{in} \ \ \mathbb{R}\times\mathbb{R}\times[0,T)\,, \\ R(x,y,T) = b(y,T)r(x). \end{cases} \tag{96}$$

Proof. Given (88), it suffices to establish (96) for $\tilde{a}^*$.

Using (60) and differentiating $p = h^{(-1)}$ we find

$$\tilde{\alpha}_t^* = p_t h_{zy} + h_{yt}, \quad \tilde{\alpha}_x^* = p_x h_{zy},$$

$$\tilde{\alpha}_{xx}^* = p_{xx} h_{zy} + (p_x)^2 h_{yzz}, \quad \tilde{\alpha}_y^* = p_y h_{zy} + h_{yy},$$
$$\tilde{\alpha}_{xy}^* = p_{xy} h_{zy} + p_x p_y h_{zzy} + p_x h_{zyy},$$

and

$$\tilde{\alpha}_{yy}^* = p_{yy} h_{zy} + (p_y)^2 h_{zzy} + 2p_y h_{zyy} + h_{yyy},$$

where functions $\tilde{\alpha}_x^*, p$ and their derivatives are evaluated at (x,y,t), and h_y and its derivatives at $(p(x,y,t),y,t)$.

Grouping terms yields

$$\begin{aligned}
&\tilde{\alpha}_t^* + \frac{1}{2}(\tilde{\alpha}^*)^2 \tilde{\alpha}_{xx}^* + \tilde{\alpha}^* \tilde{\alpha}_{xy}^* + \frac{1}{2}\tilde{\alpha}_{yy}^* \\
&= \big(p_t + \frac{1}{2}(h_y)^2 p_{xx} + h_y p_{xy} + \frac{1}{2}p_{yy}\big) h_{zy} + \frac{1}{2}(h_y p_x + p_y)^2 h_{zzy} \\
&= \big(p_t + \frac{1}{2}(h_y)^2 p_{xx} + h_y p_{xy} + \frac{1}{2}p_{yy}\big) h_{zy},
\end{aligned}$$

where we used that h_y satisfies (48) and that $h_y p_x + p_y = 0$ (cf. (68) and (69)). Using (69), (70) and (71), we obtain

$$p_t + \frac{1}{2}(h_y)^2 p_{xx} + h_y p_{xy} + \frac{1}{2} p_{yy} = 0,$$

and we conclude. □

2.3 Examples

We discuss three representative examples of utilities. We start with the well studied exponential one, which we review for completeness and also to provide the formulae to be used in the upcoming MFG examples. Then, we turn to SAHARA utilities and, finally, to their CMIM extension.

2.4 Exponential utility

To ease the presentation, since the arguments are similar, we only consider the utiity $J : \mathbb{R} \to \mathbb{R}$ given by

$$J(x) = -e^{-x}. \tag{97}$$

Since, for $z > 0$, $I(z) = -\ln z$, (11) yields $H(z, y, T) = z$. Then, (10) implies that $H(z, y, t) = z$ for each $(z, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$, and, in turn, (86) and (92) give

$$\mathcal{X}_s^* = x + L_{t,s} \quad \text{and} \quad \alpha_s^* = c(Y_s, s), \tag{98}$$

with L as in (9) and c solving (81).

Furthermore, for $(x, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$ and k as in (13),

$$u(x, y, t) = \mathbb{E}\left[-\exp\left(-\left(x + L_{t,T}\right)\right)\right] = -e^{-x+k(y,t)}.$$

2.5 Utilities with asymptotically linear risk tolerance

We consider utility functions J characterized by their risk tolerance function of the parametric form (44). Such functions were introduced in [51] and the case of known drift, $b(y, t) \equiv b$, was studied in [60] and [19], where the acronym SAHARA (symmetric asymptotic hyperbolic risk aversion) utilities was introduced.

To ease the presentation, we work only with the special case that the risk tolerance is

$$r(x) = \sqrt{x^2 + 1},$$

which, from (11) and (41), corresponds, for $(z, y) \in \mathbb{R} \times \mathbb{R}$, to

$$H(z, y, T) = \sinh z. \tag{99}$$

It follows from (10)that, for $(z, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$,

$$H(y,t) = \frac{1}{2}e^{z}l_1(y,t) - \frac{1}{2}e^{-z}l_2(y,t), \tag{100}$$

where $l_1, l_2 : \mathbb{R} \times [0,T] \to \mathbb{R}$ are smooth solutions to

$$l_{1,t} + \frac{1}{2}l_{1,yy} + cl_{1,y} + \frac{1}{2}c^2 l_1 = 0 \text{ in } \mathbb{R}\times [0,T) \text{ and } l_1(y,T) = 1,$$

and

$$l_{2,t} + \frac{1}{2}l_{2,yy} - cl_{2,y} + \frac{1}{2}c^2 l_2 = 0 \text{ in } \mathbb{R}\times [0,T) \text{ and } l_2(y,T) = 1.$$

Since, for $i = 1, 2$, $l_i > 0$ in $\mathbb{R}\times [0,T]$, for $(z, y, t) \in \mathbb{R} \times \mathbb{R} \times [0, T]$, we have

$$H_z(z,y,t) = \frac{1}{2}e^{z}l_1(y,t) + \frac{1}{2}e^{-z}l_2(y,t) > 0, \tag{101}$$

and, thus, $H^{(-1)}$, which is inverse in the $z-$argument, exists and is given, for $(x, y, t) \in \mathbb{R} \times \mathbb{R}\times [0, T]$, by

$$H^{(-1)}(x,y,t) = \ln \frac{2x + \sqrt{4x^2 + 2l_2(y,t)}}{2l_1(y,t)}. \tag{102}$$

Following (86), (92), (100), (101) and (102), the optimal processes $\mathcal{X}^*$ and α^* are given by

$$\mathcal{X}^*_s = \frac{1}{2}e^{H^{(-1)}(x,y,t)+L_{t,s}}l_1(Y_s,s) - \frac{1}{2}e^{-\left(H^{(-1)}(x,y,t)+L_{t,s}\right)}l_2(Y_s,s),$$

and

$$\alpha^*_s = \frac{1}{2}e^{H^{(-1)}(x,y,t)+L_{t,s}}l_1(Y_s,s) + \frac{1}{2}e^{-\left(H^{(-1)}(x,y,t)+L_{t,s}\right)}l_2(Y_s,s).$$

2.6 Utilities with completely monotonic inverse marginals

A large class of utilities, which substantially extend the SAHARA ones, are utilities with completely monotonic (CM) inverse marginals (CMIM).

Classical results for the representation of CM functions yield that the inverse marginals of CMIM utilities are of the form

$$I(x) = \int x^{-\rho} d\mu\left(\rho\right), \tag{103}$$

for a suitable measure so that $I'(x) < 0$, $\lim_{x\to 0} I(x) = \infty$ and $\lim_{x\to\infty} I(x) = 0$.

CMIM utilities were introduced in [51] in the context of time-monotone forward utilities where a detailed study of the measure μ was carried out; see, also, [26] for related turnpike-type problems and [48] for their role in abstract semimartingale market models.

From (103) and (82), we deduce that CMIM utilities can be equivalently described by the related function $H(z,y,T)$, which takes, for $z\in\mathbb{R}$, the form

$$H(z,y,T)=\int e^{z\rho}\mathrm{d}\mu\left(\rho\right). \tag{104}$$

For example, (99) corresponds to measure $\mu\left(\rho\right)=\frac{1}{2}\left(\delta_1-\delta_{-1}\right)$.

With terminal condition (104), the solution to (10) is represented as

$$H(z,y,t)=\int e^{z\rho}m_\rho(y,t)d\mu\left(\rho\right), \tag{105}$$

where for each $\rho\in\operatorname{supp}\left(\mu\right)$, $l_\rho:\mathbb{R}\times[0,T]\to\mathbb{R}^+$ solves the terminal value problem

$$l_{\rho,t}+\frac{1}{2}l_{\rho,yy}+(\rho c)l_{\rho,y}+\frac{1}{2}(\rho c)^2l_\rho=0 \text{ in } \mathbb{R}\times[0,T) \text{ and } l_\rho(y,T)=1.$$

The optimal processes $\mathcal{X}^*$ and α^* are given, for $0\le t\le s\le T$, by

$$\mathcal{X}_s^*=\int e^{\rho\left(H^{(-1)}(x,y,t)+L_{s,t}\right)}l_\rho(Y_s,s)d\mu\left(\rho\right),$$

and

$$\alpha_s^*=\int \rho e^{\rho\left(H^{(-1)}(x,y,t)+L_{s,t}\right)}l_\rho(Y_s,s)d\mu\left(\rho\right).$$

Naturally, one needs to specify proper conditions on both the measure μ and the market dynamics so that the above quantities are well-defined. We leave these questions for future research.

Remark 10. *The form of the auxiliary function H in (99) and (104) allows us to guess solutions of the form (100) and (105), respectively. These candidate solutions are smooth as they solve individual one-dimensional linear pdes. This together with the uniqueness of viscosity solutions for equation (10) yield the result. However, outside the class of CMIM utilities, smooth solutions are not a priori guaranteed as the problem is two-dimensional and not uniformly elliptic.*

3 The N-player and the mean field game in the reduced complete information market

We consider a game of $N-$players in the market as in section 2, consisting of a stock with price S solving (37) with the local factor process Y satisfying (38), and a bond offering zero interest rate. The latter is taken to be the numeraire and all state and controlled processes below are expressed in discounted by it units.

The wealth process $X^{N,i}$ of the $i^{th}-$player, for $i=1,...,N$, evolves according to the sde

$$dX_s^{N,i}=b(Y_s,s)\pi_s^{N,i}ds+\pi_s^{N,i}dW_s \text{ in } (t,T] \text{ and } X_t^{N,i}=x_i\in\mathbb{R}, \tag{106}$$

with the drift coefficient b as in (31) and the Brownian motion W as in (36).

The control policies $\pi^{N,i}$, which are self-financing and satisfy $\pi^{N,i} \in \mathcal{A}$ as in (30), model the (discounted) amount invested in the stock.

To ease the presentation, in what follows we will be occasionally using the abbreviated notation $z^{1:N} = (z_1, ..., z_N)$. Moreover, we will not be repeating, unless important, the fact that $i = 1, \ldots, N$.

The value function of the $i^{th}-$player, for $(x_1, ..., x_N, y, t) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R} \times [0, T]$, is

$$
\begin{aligned}
&u^i(x_1, ..., x_N, y, t) \\
&= \sup_{\pi_i \in \mathcal{A}} \mathbb{E}\Big[J\Big(X_T^{N,i}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \delta_{X_T^{N,j}}\Big) | X_t^{N,1:N} = x_{1:N}, Y_t = y\Big].
\end{aligned}
\tag{107}
$$

The utility function denoted, by a slight abuse of notation, as J is common across players and depends on both their individual terminal wealth $X_T^{N,i}$ and the law of their peers' wealth at T. As mentioned above, since all involved processes are unitless, there is no issue with unit consistency in nonlinear payoffs as above, thus allowing for couplings that depend on higher moments, quantiles, distances among returns, and others; see [50] for the role of unit consistency,

We recall that a control process $\left(\pi^{N,1,*}, ..., \pi^{N,N,*}\right)$ is a Nash equilibrium of this game, if, for each $i = 1, ..., N$, and all $\pi^{N,i} \in \mathcal{A}$,

$$
\begin{aligned}
&\mathbb{E}\Big[J\Big(X_T^{N,i}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \delta_{X_T^{N,j,*}}\Big) | X_t^{N,1:j-1,*} = x_{1:i-1}, X_t^{N,i} = x_i, \\
&\qquad X_t^{N,j+1:N,*} = x_{i+1:N}, Y_t = y\Big] \\
&\leq \mathbb{E}\Big[J\Big(X_T^{N,i,*}, \frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \delta_{X_T^{N,j,*}}\Big) | X_t^{N,1:N,*} = x_{1:N}, Y_t = y\Big],
\end{aligned}
$$

where $X^{N,j,*}$ denotes the solution to (106) with control process $\pi^{N,j,*}$.

Next, we (formally) assume that there exist Nash equilibrium control processes $\left(\pi^{N,i,*}, ..., \pi^{N,N,*}\right)$ in the feedback form

$$
\pi_s^{N,i,*} = \pi^{N,i,*}\left(X_s^{N,1,*}, ..., X_s^{N,N,*}, Y_s, s\right),
$$

for some functions $\pi^{N,i,*} : \mathbb{R}^N \times \mathbb{R} \times [0, T] \to \mathbb{R}$ and processes $X^{N,i,*}$ solving (106) with controls $\pi^{N,i,*}$. Then, the value functions $\left(u^{N,1}, \ldots, u^{N,N}\right)$ are expected to satisfy the fully coupled system of terminal value Hamilton-Jacobi-

Bellman (HJB) equations

$$
\begin{aligned}
&u_t^{N,i} + \max_{\pi^{N,i}} \Big(\frac{1}{2} \left(\pi^{N,i}\right)^2 u^{N,i}_{x_i x_i} \\
&\quad + \pi^{N,i} \Big(b u^{N,i}_{x_i} + \sum_{j=1,\ j\neq i}^{N} \pi^{N,j,*} u^{N,i}_{x_i x_j} \Big) + \pi^{N,i} u^{N,i}_{x_i y} \Big) \\
&\quad + \frac{1}{2} \sum_{j=1,\ j\neq i}^{N} \sum_{k=1, k\neq i}^{N} \pi^{N,j,*} \pi^{N,k,*} u^{N,i}_{x_j x_k} + \sum_{j=1,\ j\neq i}^{N} \pi^{N,j,*} u^{N,i}_{x_j y} \\
&\quad + b \sum_{j=1,\ j\neq i}^{N} \pi^{N,j,*} u^{N,i}_{x_j} + \frac{1}{2} u^{N,i}_{yy} + b u^{N,i}_{y} = 0 \quad \text{in} \ \ \mathbb{R}^N \times \mathbb{R} \times [0,T], \\
&u^{N,i} \Big(x_1, ..., x_N, y, T \Big) = J \Big(x_i, \frac{1}{N-1} \sum_{j=1, j\neq i}^{N} \delta_{x_j} \Big).
\end{aligned} \tag{108}
$$

If, in addition, the maximum in (108) is well-defined in each respective HJB equation, we deduce - still formally - that the optimal feedback control functions $\left(\pi^{N,1,*}, ..., \pi^{N,N,*}\right)$ must satisfy the linear $N \times N$ system

$$
\pi^{N,i,*} u^{N,i}_{x_i x_i} + \sum_{j=1,\ j\neq i}^{N} \pi^{N,j,*} u^{N,i}_{x_i x_j} = -b u^{N,i}_{x_i} - u^{N,i}_{x_i y}, \quad i = 1, ..., N. \tag{109}
$$

This system, however, is not tractable due to the interlinked dependence of the feedback controls $\pi^{N,i,*\prime} s$ and the coefficients $u^{N,i}_{x_i}, u^{N,i}_{x_i x_i}, u^{N,i}_{x_i x_j}$ and $u^{N,i}_{x_i y}$, $i, j = 1, ..., N$. We actually note that it is not even clear if the individual value functions $u^{N,i\prime} s$ are smooth enough for the latter derivatives to exist. Similar difficulties were, also, encountered in the simpler system derived in the Black and Scholes market model ($b(y,t) \equiv b$) in [55].

Motivated by the intractability of the finite player game, we consider next a related mean field game.

3.1 The mean field game and its master system

The representative agent's state X solves, for $\pi \in \mathcal{A}$, the continuum analogue of sde (106), that is,

$$
dX_s = b(Y_s, s)\pi_s ds + \pi_s dW_s \ \text{in} \ (t,T] \ \text{ and } \ X_t = x, \tag{110}
$$

with b as in (31), and W and Y as in (36) and (38), respectively.

We emphasize that the derivation below is formal since we do not have estimates that would allow us to give a rigorous proof. To this end, for large number of players, we assume that the optimal feedback controls $\pi^{N,i,*}\left(x_1, ..., x_N, y, t\right)$ and the value functions $u^{N,i}\left(x_1, ..., x_N, y, t\right)$ satisfy, for each $i = 1, ..., N$,

$$
\pi^{N,i,*}\left(x_1, ..., x_N, y, t\right) \ \simeq \ \pi^{N,*}\left(x_i, y, m^{N,i}, t\right), \tag{111}
$$

and

$$u^{N,i}\left(x_1,...,x_N,y,t\right)\simeq u^N\left(x_i,y,m^{N,i},t\right), \tag{112}$$

where $m^{N,i}$ is the empirical measure of the rest of the players,

$$m^{N,i}=\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\delta_{x_j}. \tag{113}$$

We assume that, for each i, as $N\to\infty$, $m^{N,i}$ converge weakly to a common measure $m\in\mathcal{P}_2$ and, furthermore, that there exist $\pi^*,U:\mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T]\to\mathbb{R}$ such that, as $N\to\infty$,

$$\begin{aligned}\pi^{N,i,*}\left(x_i,y,m^{N,i},t\right)&\to\ \pi^*\left(x,y,m,t\right),\\ u^{N,i}\left(x_i,y,m^{N,i},t\right)&\to\ U\left(x,y,m,t\right).\end{aligned} \tag{114}$$

Then, following similar approximations for each term in (108), we find that the formal limit, as $N\to\infty$, of (108) takes the form (16) and the candidate mean field equilibrium optimal control $\pi^*\left(x,y,m,t\right)$ is expected to satisfy (22).

In summary, we expect the pair $\left(U\left(x,y,m,t\right),\pi^*\left(x,y,m,t\right)\right)$ to satisfy the system of equations (16) and (22).

For the convenience of the reader we present a formal derivation of the optimality condition (22). The master equation follows similarly.

It follows from (112) and (113) that, for $i\in\{1,\dots,N\}$ and $j\in\{1,\dots,N\}\setminus\{i\}$,

$$u^{N,i}_{x_ix_j}\simeq\frac{1}{N-1}u^N_{xm}(x_i,y,m^{N,i},x_j,t).$$

Inserting the above in (109) and using (111) we find

$$\begin{aligned}&\pi^{N,*}u^N_{xx}(x_i,y,m^{N,i},t)\\&+\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\pi^{N,*}(x_j,y,m^{N,i},t)u^N_{xm}(x_i,y,m^{N,i},x_j,t)\\&=-bu^N_x(x_i,y,m^{N,i},t)-u^N_{xy}(x_i,y,m^{N,i},t).\end{aligned}$$

After sending $N\to\infty$, the above expression yields formally (22).

3.1.1 A simplified master equation

The master equation (16) can be simplified to a similar equation corresponding to the case $b(y,t)\equiv 0$ and a modified terminal utility. Indeed, in analogy to (79), we set

$$W(x,y,m,t)=U\left(x,y,m,t\right)F(y,t),$$

with F solving (32). Then, direct calculations in (16) yield the master equation

$$\begin{aligned}
&W_t + \frac{1}{2}\pi^{*2} W_{xx} + \pi^* W_{xy} + \pi^* \mathcal{L}_1 W + \mathcal{L}_2 W + +\mathcal{L}_4 W \\
&\qquad\qquad + \frac{1}{2} W_{yy} = 0 \text{ in } \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2 \times [0,T), \\
&W(x,y,m,T) = J(x,m)F(y,T),
\end{aligned} \tag{115}$$

with $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_4$ defined in (18), (19) and (21) respectively.

Herein, we choose to work with the original version (16) for two reasons. Firstly, (16) provides a natural object to use in order to recover its analogue when $b(y,t) \equiv b > 0$, by sending ν in (27) to the Dirac function δ_b. Secondly, it is expected that the results for non-zero drift can be used to study MFG in complete markets with multi-dimensional local factors (local volatility, predictable means, and others) and general utilities, which arise in many applications but have not been studied so far.

4 A separable class of MFG with partial information

We study the master system (16) and (22) for payoffs of the separable form, that is,

$$J(x,m) = J\left(x - C\left(m\right)\right), \tag{116}$$

with J being the utility of the representative agent and $C > 0$ the coupling function on the law of peers' wealth. We allow for general dependence of C on m, thus extending all works so far which considered only couplings of form $C\left(m\right) = C(\bar{m})$. We examine the latter special case in subsection 5.2 and subsection 5.3. We note that the minus sign above is immaterial and is only used to allow for comparisons with existing works in portfolio choice with relative performance. Indeed, one may allow for homophilous interactions, in which players benefit from actions of their peers; see [29] for an example with exponential utilities.

4.1 Solution of the master system

We construct a solution pair U and π^* to system (16) and (22).

Arguing formally, we show that the value of the game can be represented as the value function u of the single agent problem (40) (no competition) with its argument translated by a function f that solves a non-local quasilinear infinite-dimensional pde whose coefficients depend on the mean field equilibrium feedback controls. We provide rigorous arguments when additional, but still general enough, modeling assumptions are introduced.

We recall that the value function u in (3) satisfies the terminal HJB equation (4) and (5) and that the optimal feedback control α^* is given by (6).

Assume next that there exists a smooth solution $f \in \mathbb{R} \times \mathcal{P}_2 \times [0,T] \to \mathbb{R}$ to

$$f_t + \mathcal{L}_2 f + \mathcal{L}_4 f + \frac{1}{2} f_{yy} = 0 \ \text{ in } \ \mathbb{R}\times\mathcal{P}_2\times[0,T]\,, \qquad f(y,m,T) = C(m), \tag{117}$$

with $\mathcal{L}_2$ and $\mathcal{L}_4$ as in (19) and (21) respectively, where $\pi^* : \mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T] \to \mathbb{R}$ satisfies, for each $(x,y,m,t) \in \mathbb{R}\times\mathcal{P}_2\times[0,T]$,

$$\pi^*(x,y,m,t) - \int \pi^*(z,y,m,t) f_m(x,y,z,t) dm(z) = f_y(x,y,m,t) + a^*(x - (y,m,t)\,, y, t). \tag{118}$$

The regularity of f is important for what follows. We note, however, that it is not known whether (117) has a smooth solution. Later we give an interpretation to this equation and provide a formula for the possible solution.

Next, we provide the main result of this section, representing the value U of the game in terms of the value function of the problem of no competition and function f above. We defer the interpretation of the solution to subsection 5.1 where we connect U to the value of the writer of claim $C(m_T^*)$ (at the optimum,) u being the value function of the "plain" investor and f being the dynamic price of $C(m_T^*)$.

Proposition 11. *Let u and a^* be as in* (4), (5) *and* (6)*, and assume that* (117) *admits a smooth solution with π^* satisfying* (118)*. Then, a solution to* (16) *is given, for $(x,y,m,t) \in \mathbb{R}\times\mathcal{P}_2\times[0,T]$, by*

$$U(x,y,m,t) = u(x - f(y,m,t), y, t), \tag{119}$$

with π^* satisfying (118).

Proof. Assuming that U and F satisfy (119) and (117), we first show (118).

Differentiating (119), where for simplicity we omit the dependence of U, U_t, U_x, $U_y, U_{xy}, U_{xx}, U_{yy}$ on (x,y,m,t), of U_m, U_{mx}, U_{my} on (x,y,m,z,t), of U_{mm} on (x,y,m,z_1,z_2,t), of f, f_y, f_{yy} on (y,m,t), of f_m on (y,m,z,t), of f_{mm} on (y,m,z_1,z_2,t) and of u and its derivatives on $(x - f(y,m,t), y, t)$, yields

$$U_t = -f_t u_x + u_t,\ U_x = u_x,\ U_{xx} = u_{xx},\ U_y = -f_y u_x + u_y,$$

$$U_{yy} = -f_{yy} u_x + (f_y)^2 u_{xx} - 2 f_y u_{xy} + u_{yy},\ U_{xy} = -f_y u_{xx} + u_{xy},$$

$$U_m = -f_m u_x,\ U_{xm} = -f_m u_{xx},\ U_{ym} = -f_{ym} u_x + f_m f_y u_{xx} - f_m u_{xy},$$

$$U_{mm} = f_m f_m u_{xx} + f_{mm} u_x,$$

Inserting the above in the optimality condition (22) we find (118).

Next we show that U defined in (119) indeed satisfies the master equation (16), as long as (117) and (118) hold, and all involved partial derivatives exist.

Inserting the above derivatives in (16) and grouping terms we find

$$-u_x A + u_t + \frac{1}{2} u_{xx} B + (b u_x + u_{xy}) C + \frac{1}{2} u_{yy} + b u_y = 0, \tag{120}$$

where u is evaluated at $(x - f(y, m, t), y, t)$ and the auxiliary quantities $A, B, C : \mathbb{R} \times \mathcal{P}_2 \times [0, T] \to \mathbb{R}$ are given by

$$\begin{aligned}
&A(y, m, t) = f_t(y, m, t) + \\
&\frac{1}{2} \int \int \pi^*(z_1, y, m, t) \pi^*(z_2, y, m, t) f_{mm}(y, m, z_1, z_2, t) dm(z_1) dm(z_2) \\
&+ \frac{1}{2} \int (\pi^*(z, y, m, t))^2 f_{mz}(y, m, z, t) dm(z) \\
&+ \int \pi^*(z, y, m, t) f_{my}(y, m, z, t) dm(z) + \frac{1}{2} f_{yy}(y, m, t),
\end{aligned}$$

$$\begin{aligned}
&B(x, y, m, t) = (\pi^*(x, y, m, t))^2 \\
&- 2\pi^*(x, y, m, t) \int \pi^*(z, y, m, t) f_m(x, y, m, z, t) \, dm(z) \\
&- 2\pi^*(x, y, m, t) f_y(y, m, t) + \Big(\int \pi^*(z, y, m, t) f_m(y, m, z, t) dm(z) \Big)^2 \\
&+ 2 f_y(m, y, t) \int \pi^*(z, y, m, t) f_m(y, m, z, t) dm(z) + (f_y(y, m, t))^2
\end{aligned}$$

and

$$\begin{aligned}
&C(x, y, m, t) = \pi^*(x, y, m, t) \\
&- \pi^*(x, y, m, t) \int \pi^*(z, y, m, t) f_m(x, y, m, z, t) \, dm(z) - f_y(y, m, t).
\end{aligned}$$

It follows that $B = C^2$ which, together with (118), yields

$$\frac{1}{2} u_{xx} B + (b u_x + u_{xy}) C = -\frac{(b u_x + u_{xy})^2}{2 u_{xx}}. \tag{121}$$

Combining the above and using the HJB equation (78) satisfied by u, we get that

$$u_t + \frac{1}{2} u_{xx} B + b u_x + u_{xy} + \frac{1}{2} u_{yy} + b u_y = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R} \times [0, T].$$

Thus, (120) reduces to

$$u_x A = 0 \quad \text{in } \mathbb{R} \times \mathbb{R} \times [0, T],$$

and the claim follows, since $u_x > 0$ in $\mathbb{R} \times \mathbb{R} \times [0, T]$.

□

5 Representative examples

We analyze two examples. The first considers exponential utilities and general couplings that are functions of the law of peers' wealth. In such settings, the mean field equilibrium control becomes independent of x and (117) reduces to the autonomous (125). The value of the game is given by (126). If, in addition, the coupling depends only on the mean of peers' wealth, equation (125) reduces further to (130), for which we provide a complete analysis, establishing existence, uniqueness and regularity. We, also, produce in closed-form the mean field equilibrium optimal processes in (141) and (144).

The second example considers general utilities and couplings that are linear functions of peers' wealth. In such settings, the value of the game is given by (149) and the mean field equilibrium processes are represented as (153) and (154). The optimal processes admit an intuitively pleasing decomposition in terms of the corresponding optimal processes in the singe agent problem with modified initial condition, and an additional component that involves averages with regards to the initial distribution of peers' wealth.

We recall (110), which at a mean field equilibrium control π^*, becomes

$$dX_s^* = b(Y_s, s)\pi_s^* ds + \pi_s^* dW_s \quad \text{and} \quad X_t^* = x, \tag{122}$$

and we denote by $\bar{X}^*$ its conditional on the common noise average,

$$\bar{X}_s^* = \int z dm_s^*(z) = \mathbb{E}\left[X_s^* \middle| \mathcal{F}_s^W\right] \text{ in } [t,T] \text{ and } \; X_t^* = x, \; \bar{X}_t^* = \bar{m}, \tag{123}$$

with x, $\bar{m} \in \mathbb{R}$, and $m \in \mathcal{P}_2$ being the initial distribution of the players and m^* the conditional on $\mathcal{F}^W$ law of X^*.

5.1 Exponential utility and general couplings on the law of peers' wealth

As in subsection 2.4, here we consider the utility function $J(x) = -e^{-x}$ and recall that the value function and the optimal policy of the single player problem are given, for each $(x,y,t) \in \mathbb{R} \times \mathbb{R} \times [0,T]$, by

$$u(x,y,t) = -e^{-x+k(y,t)} \quad \text{and} \quad a^*(x,y,t) = c(y,t),$$

with the functions k and c as in (13) and (12).

Then, from (118), we deduce that

$$\pi^*(x,y,m,t) - \int \pi^*(z,y,m,t) f_m(y,m,z,t) dm(z) = f_y(y,m,t) + c(y,t),$$

which yields that the candidate optimal feedback control is independent of x, that is,

$$\pi^*(x,y,m,t) = \pi^*(y,m,t),$$

and, thus,

$$\pi^*(y,m,t) = \frac{f_y\,(y,m,t) + c(y,t)}{1 - \int f_m(y,m,z,t)dm(z)}. \tag{124}$$

Then (117) simplifies to

$$f_t + \mathcal{L}_2 f + \mathcal{L}_4 f + \frac{1}{2} f_{yy} = 0 \ \text{ in } \ \mathbb{R}\times\mathcal{P}_2\times[0,T]\,.$$

Therefore, using (124), we obtain the autonomous terminal value problem

$$\begin{aligned} &f_t + \frac{1}{2}a^2\hat{\mathcal{L}}_2(f) + a\hat{\mathcal{L}}_4(f) + \frac{1}{2}f_{yy} = 0 \ \text{ in } \ \mathbb{R}\times\mathcal{P}_2\times[0,T]\,, \\ &f(y,m,T) = C(m), \end{aligned} \tag{125}$$

where

$$a(y,m,t) = \frac{f_y(y,m,t) + c(y,t)}{1 - \int f_m(y,m,z,t)dm(z)},$$

$$\hat{\mathcal{L}}_2(f)(y,m,t) = \int\int f_{mm}(y,m,z_1,z_2,t)dm(z_1)dm(z_2) + \int f_{mz}(y,m,z,t)dm(z),$$

and

$$\hat{\mathcal{L}}_4(f)(y,m,t) = \int f_{ym}(y,m,z,t)dm(z).$$

The value of the game is given, for each $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$, by

$$U(x,y,m,t) = -e^{-(x-f(y,m,t))+k(y,t)}. \tag{126}$$

5.2 Exponential utility and general couplings on the mean of peer's wealth

We consider coupling functions $C:\mathcal{P}_2\to\mathbb{R}$ that depend only on the mean of peers' wealth, that is,

$$C(m) = C\,(\bar{m})\,. \tag{127}$$

The special case $C\,(\bar{m}) = \theta\bar{m}$, $\bar{m}\in\mathbb{R}$, with $\theta\in(0,1)$, is the only one so far analyzed in the literature under exponential utility. The general case $C(\bar{m})$ was recently solved in [55] for Black and Scholes markets and, herein, we extend these results for the single factor model (37) and (38).

As far as the coupling is concerned, we assume that $C:\mathbb{R}\to\mathbb{R}$ satisfies (127) with

$$\begin{gathered} C\in\mathcal{C}^2(\mathbb{R}),\ C(0)=0, \\ \text{and, for some constants } k_1,k_2,K,L>0 \text{ and all } z\in\mathbb{R}, \\ k_1 < 1 - C'\,(z) < k_2 \ \text{ and } \ (z - C\,(z))^{(-1)} \le Ke^{Lz^2}. \end{gathered} \tag{128}$$

We note that there is no assumption of monotonicity on C, which is one of the key assumptions in many references in the MFG literature.

In the sequel we provide a complete analysis of the value of the game, the mean field equilibrium feedback control, and the optimal processes.

We start with (125) and find an $x-$independent smooth solution $f:\mathbb{R}\times\mathcal{P}_2\times[0,T]\to\mathbb{R}$ to (117).

To this end, we use c given by (12) and introduce the auxiliary smooth function

$$q(y,t)=\begin{cases} -\frac{1}{2}\int_0^t c_y(0,s)\,ds+\int_0^y c(\rho,t)d\rho \;\text{ in }\; (0,\infty)\times[0,T], \\ -\frac{1}{2}\int_0^t c_y(0,s)\,ds \text{ if } \;(y,t)\in\{0\}\times[0,T], \\ -\frac{1}{2}\int_0^t c_y(0,s)\,ds-\int_y^0 c(\rho,t)d\rho \;\text{ in }\; (-\infty,0)\times[0,T]. \end{cases} \tag{129}$$

Proposition 12. *Assume* (127) *and* (128)*. Then,* (125) *reduces to*

$$\begin{aligned} &f_t+\frac{1}{2}\left(\frac{f_y+c}{1-f_{\bar m}}\right)^2 f_{\bar m\bar m}+\frac{f_y+c}{1-f_{\bar m}}f_{y\bar m}+\frac{1}{2}f_{yy}=0 \;\; \textit{in} \;\; \mathbb{R}\times\mathbb{R}\times[0,T), \\ &f(y,\bar m,T)=C(\bar m), \end{aligned} \tag{130}$$

which has a unique smooth solution given, for $(y,\bar m,t)\in\mathbb{R}\times\mathbb{R}\times[0,T]$*, by*

$$f(y,\bar m,t)=\bar m-(g(y,\cdot,t))^{(-1)}(y,\bar m,t)-q(y,t), \tag{131}$$

where g solves, for q as in (129) and each $\bar m\in\mathbb{R}$,

$$\begin{cases} g_t+\frac{1}{2}g_{yy}=0 \;\; \textit{in} \;\; \mathbb{R}\times[0,T), \\ g(y,\bar m,T)=(\cdot-C(\cdot)-q(y,T),y,T)^{(-1)}(\bar m,y,T). \end{cases} \tag{132}$$

Furthermore,

$$1-f_{\bar m}>0 \;\textit{in}\; \mathbb{R}\times\mathbb{R}\times[0,T]. \tag{133}$$

Proof. We show first that (130) can be reduced to (134) below, which corresponds to the case $c(y,t)=0$ in $\mathbb{R}\times[0,T]$. Then, we construct a unique smooth solution to (134) and define $\hat f$ as in (135). In turn, we show that $\hat f$ solves (130) with $\hat f(y,\bar m,T)=C(\bar m)$, and that it is smooth. Thus, by the uniqueness of viscosity solutions, it coincides with f.

Let $q(y,t)$ be as in (129), and consider the solution $h=h(y,\bar m,t)$ to

$$\begin{aligned} &h_t+\frac{1}{2}\left(\frac{h_y}{1-h_{\bar m}}\right)^2 h_{\bar m\bar m}+\frac{h_y}{1-h_{\bar m}}h_{y\bar m}+\frac{1}{2}h_{yy}=0 \;\;\text{in}\;\; \mathbb{R}\times\mathbb{R}\times[0,T], \\ &h(y,\bar m,T)=C(\bar m)+q(y,T). \end{aligned} \tag{134}$$

We assume for now that (134) has a unique smooth solution, a fact that we will verify at the end of the proof, and we introduce $\hat f:\mathbb{R}\times\mathbb{R}\times[0,T]\to\mathbb{R}$ given by

$$\hat f(y,\bar m,t)=h(y,\bar m,t)-q(y,t) \;\;\text{in}\;\; \mathbb{R}\times\mathbb{R}\times[0,T], \tag{135}$$

and claim that it solves (130).

Since $\hat{f}(y,\bar{m},T)=C(\bar{m})$ is obvious, we show that it solves the pde in (130).

We first recall that, since $c\,(y,t)$ solves (81), we have

$$\int_0^y c_t(\rho,t)d\rho = -\frac{1}{2}c_y(y,t)+\frac{1}{2}c_y(0,t) \ \text{ in } \ (0,\infty)\times[0,T]\,,$$

and, similarly,

$$\int_y^0 c_t(\rho,t)d\rho = -\frac{1}{2}c_y(y,t)+\frac{1}{2}c_y(0,t) \ \text{ in } \ (-\infty,0)\times[0,T]\,.$$

Then, if $y\geq 0$,

$$\hat{f}_t = h_t+\frac{1}{2}c_y,\ \hat{f}_{\bar{m}} = h_{\bar{m}},\ \hat{f}_y = h_y - c,\ \hat{f}_{\bar{m}\bar{m}} = h_{\bar{m}\bar{m}}.$$

Inserting the above derivatives in (134) yields that $\hat{f}$ solves (130) in $[0,\infty)\times\mathbb{R}\times[0,T)$. A similar calculation yields the claim in $(-\infty,0]\times\mathbb{R}\times[0,T)$.

To prove that h is the unique smooth solution to the terminal value problem (134), we define, for each $(y,t)\in\mathbb{R}\times[0,T]$, the map $g(y,\cdot,t):\mathbb{R}\to\mathbb{R}$ by

$$g(y,\bar{m},t)=(\cdot - h\,(y,\cdot,t))^{(-1)}\,(y,\bar{m},t) \quad \text{in} \ \ \mathbb{R}\times\mathbb{R}\times[0,T]\,,$$

and claim that, for each $\bar{m}\in\mathbb{R}$, $g(\cdot,\bar{m},\cdot)$ is a solution to

$$\begin{aligned} &g_t+\frac{1}{2}g_{yy}=0 \ \text{ in } \mathbb{R}\times[0,T)\,,\\ &g\,(y,\bar{m},T)=(\cdot - F(\cdot)-p(y,T),y,T)^{(-1)}\,(y,\bar{m},T)\,. \end{aligned} \tag{136}$$

Since the terminal condition follows trivially, we only need to show that g satisfies the equation in (136).

Let $N:\mathbb{R}\times\mathbb{R}\times[0,T]\to\mathbb{R}$ be defined as

$$N(y,\bar{m},t)=\bar{m}-h(y,\bar{m},t). \tag{137}$$

Then, N solves the terminal value problem

$$\begin{aligned} &N_t+\frac{1}{2}\left(\frac{N_y}{N_{\bar{m}}}\right)^2 N_{\bar{m}\bar{m}}-\frac{N_y}{N_{\bar{m}}}N_{y\bar{m}}+\frac{1}{2}N_{\bar{m}\bar{m}}=0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}\times[0,T)\,,\\ &N(y,\bar{m},T)=\bar{m}-F(\bar{m})+c(y,T). \end{aligned} \tag{138}$$

The definitions of h and g imply that $N\,(y,g(y,\bar{m},t),t)=\bar{m}$. In turn, evaluating equation (138) at $g(y,\bar{m},t)$ and using the above yields

$$-\frac{g_t}{g_{\bar{m}}}+\frac{1}{2}\left((g_y)^2 N_{\bar{m}\bar{m}}+2g_yN_{\bar{m}y}+N_{yy}\right)=-\frac{1}{g_{\bar{m}}}\left(g_t+\frac{1}{2}g_{yy}\right)=0.$$

To conclude, we consider the terminal value problem (136) with $q(y,T)$ as in (129) for $t=T$. Then, g is smooth and, furthermore, the maximum principle yields $g_{\bar{m}} > 0$ in $\mathbb{R}\times\mathbb{R}\times[0,T]$. Thus, g is invertible in $\bar{m}$, and its inverse $g(y,\cdot,t)^{(-1)}(y,\bar{m},t) = N(y,\bar{m},t)$ is well defined and smooth. In turn, (137) yields that h is well defined and smooth, and (131) follows.

The rest of the proof follows.

□

Next, we construct the mean field equilibrium processes, X^* and π^*.

Proposition 13. *Let f as in Proposition 12, k as in (13) and introduce the map $G:\mathbb{R}\times\mathbb{R}\times[0,T]$ given by*

$$G(y,\bar{m},t) = (\cdot - f(y,\cdot,t))^{(-1)}(y,\bar{m},t). \tag{139}$$

The mean field value is given by

$$U(x,y,m,t) = -e^{-(x-f(y,\bar{m},t))+k(y,t)} \quad \text{in } \mathbb{R}\times\mathbb{R}\times[0,T]. \tag{140}$$

The mean field equilibrium process, X^*, is represented, for $0\le t\le s\le T$, as

$$X_s^* = x - f(y,\bar{m},t) + L_{t,s} + f(Y_s, G(Y_s, x - f(y,\bar{m},t) + L_{t,s}, s), s), \tag{141}$$

with the process L defined in (9). Alternatively, for $0\le t\le s\le T$,

$$\begin{aligned} X_s^* &= \mathcal{X}_s^{x-f(y,\bar{m},t),*} + f\left(Y_s, G(Y_s, \mathcal{X}_s^{x-f(y,\bar{m},t),*}, s\right), s) \\ &= \mathcal{X}_s^{x-f(\bar{m},y,t),*} + \mathcal{E}\left(Y_s, \mathcal{X}_s^{x-f(\bar{m},y,t),*}, s\right), \end{aligned} \tag{142}$$

where, for $(y,\bar{m},t)\in\mathbb{R}\times\mathbb{R}\times[0,T]$,

$$\mathcal{E}(y,\bar{m},t) = f(y, G(y,\bar{m},t), t), \tag{143}$$

and $\mathcal{X}^{x-f(y,\bar{m},y,t),*}$ is the optimal wealth in the single agent problem (40), starting at $x-f(y,\bar{m},t)$; see (86).

Moreover,

$$\bar{X}_s^* = G(Y_s, x - f(y,\bar{m},t) + L_{t,s}, s) \quad \text{in } \mathbb{R}\times\mathbb{R}\times[t,T],$$

and the mean field equilibrium control process, π^*, is given, $0\le t\le s\le T$, by

$$\pi_s^* = \frac{f_y\left(Y_s, \bar{X}_s^*, s\right) + c(Y_s,s)}{1 - f_{\bar{m}}\left(Y_s, \bar{X}_s^*, s\right)}. \tag{144}$$

Proof. From (124) we have

$$\pi_s^* = \pi^*\left(Y_s, \bar{X}_s^*, s\right) = \frac{f_y\left(Y_s, \bar{X}_s^*, s\right) + c(Y_s,s)}{1 - f_{\bar{m}}\left(Y_s, \bar{X}_s^*, s\right)},$$

with $\bar{X}_t^* = \bar{m}$, which is well defined in view of (133). Using the above and (122) yields, for $0 \le t \le s \le T$,

$$dX_s^* = \frac{f_y\left(Y_s, \bar{X}_s^*, s\right) + c(Y_s, s)}{1 - f_{\bar{m}}\left(Y_s, \bar{X}_s^*, s\right)}\left(b(Y_s, s)ds + dW_s\right),$$

with $X_t^* = x$ and, thus,

$$d\bar{X}_s^* = \frac{f_y\left(Y_s, \bar{X}_s^*, s\right) + c(Y_s, s)}{1 - f_{\bar{m}}\left(Y_s, \bar{X}_s^*, s\right)}\left(b(Y_s, s)ds + dW_s\right) \text{ in } (t, T) \quad \bar{X}_t^* = \bar{m}. \tag{145}$$

Applying Itô's formula to $f\left(Y_s, \bar{X}_s^*, , s\right)$ and using (130) gives

$$\mathrm{d}f\left(Y_s, \bar{X}_s^*, s\right) = f_{\bar{m}}\left(Y_s, \bar{X}_s^*, s\right) d\bar{X}_s^* + f_y\left(Y_s, \bar{X}_s^*, s\right) dY_s.$$

Using (145) and regrouping terms we obtain

$$\mathrm{d}f\left(Y_s, \bar{X}_s^*, s\right) = b(Y_s, s)\Big(\frac{f_y\left(\bar{X}_s^*, Y_s, s\right) + f_{\bar{m}}\left(\bar{X}_s^*, Y_s, s\right)c(Y_s, s)}{1 - f_{\bar{m}}\left(\bar{X}_s^*, Y_s, s\right)}\Big)ds$$

$$+\frac{f_y\left(\bar{X}_s^*, Y_s, s\right) + f_{\bar{m}}\left(\bar{X}_s^*, Y_s, s\right)c(Y_s, s)}{1 - f_{\bar{m}}\left(\bar{X}_s^*, Y_s, s\right)}dW_s.$$

Therefore, with the process L as in (9),

$$d(X_s^{x,*} - f\left(Y_s, \bar{X}_s^*, s\right)) = b(Y_s, s)c(Y_s, s)ds + c\left(Y, s\right)dW_s = dL_{t,s}.$$

It follows that

$$X_s^* - f\left(Y_s, \bar{X}_s^*, s\right) = x - f\left(y, \bar{m}, t\right) + L_{t,s},$$

and, in turn,

$$\bar{X}_s^* = G(Y_s, \bar{m} - f\left(y, \bar{m}, t\right) + L_{t,s}, s).$$

To show (142), we use (143) and recall (98).

□

5.3 General utility and couplings linear in the mean of peers' wealth

We consider general utilities J satisfying (42) and linear couplings, that is, for each $m \in \mathcal{P}_2$ and $\theta \in (0, 1)$,

$$C(m) = \theta\bar{m}. \tag{146}$$

First, we observe that

$$f\left(y, \bar{m}, t\right) = \theta\bar{m}, \tag{147}$$

is a smooth solution to (117) and, since $1 - f_{\bar{m}}(y, m, t) = 1 - \theta > 0$ in $\mathbb{R} \times \mathbb{R} \times [0, T]$, it is the unique smooth solution.

Since the optimality condition (118) reduces to

$$\pi^*(x,y,m,t) - \theta \int \pi^*(z,y,m,t)dm(z) = \alpha^*(x-\theta\bar{m},y,t),$$

with $\alpha^*(x,y,t)$ as in (87), it follows that

$$\int \pi^*(z,y,m,t)dm(z) = \frac{1}{1-\theta}\int \alpha^*(z-\theta\bar{m},y,t)dm(z),$$

and, thus, for each $(x,y,m,t) \in \mathbb{R}\times\mathbb{R}\times\mathcal{P}_2\times[0,T]$,

$$\pi^*\left(x,y,m,t\right) = \alpha^*\left(x-\theta\bar{m},y,t\right) + \frac{\theta}{1-\theta}\int \alpha^*\left(z-\theta\bar{m},y,t\right)dm\left(z\right). \tag{148}$$

Then, with u as in (40) and using (119) and (147), we obtain

$$U(x,y,m,t) = u\left(x-\theta\bar{m},y,t\right). \tag{149}$$

To construct the optimal mean field and portfolio processes, we note that it follows from (122) and (148), that X^* solves

$$dX_s^* = b(Y_s,s)\left(\alpha^*(X_s^*-\theta\bar{X}_s^*,Y_s,s) + \frac{\theta}{1-\theta}\int \alpha^*(z-\theta\bar{X}_s^*,Y_s,s)dm_s^*(z)\right)ds$$

$$+\left(\alpha^*(X_s^*-\theta\bar{X}_s^*,Y_s,s) + \frac{\theta}{1-\theta}\int \alpha^*(z-\theta\bar{X}_s^*,Y_s,s)dm_s^*(z)\right)dW_s,$$

with $X_t^* = x$ and $\bar{X}^*$ as in (123). Therefore,

$$d\bar{X}_s^* = \frac{1}{1-\theta}b(Y_s,s)\left(\int \alpha^*(z-\theta\bar{X}_s^*,Y_s,s)dm_s^*(z)\right)ds$$

$$+\frac{\theta}{1-\theta}\left(\int \alpha^*(z-\theta\bar{X}_s^*,Y_s,s)dm_s^*(z)\right)dW_s,$$

with $\bar{X}_t^* = \bar{m}$.

It follows that $X_t^* - \theta\bar{X}_t^* = x - \theta\bar{m}$ and, for $0 \le t \le s \le T$,

$$\begin{aligned} &\mathrm{d}\left(X_s^* - \theta\bar{X}_s^*\right) \\ &= b(Y_s,s)\alpha^*(X_s^*-\theta\bar{X}_s^*,Y_s,s)ds + \alpha^*(X_s^*-\theta\bar{X}_s^*,Y_s,s)dW_s. \end{aligned}$$

Working as in the proof in Theorem 7, we obtain that the sde above, which is autonomous in the argument $X^* - \theta\bar{X}^*$, has a unique strong solution, given by

$$X_s^* - \theta\bar{X}_s^* = H\left(H^{(-1)}\left(x-\theta\bar{m},y,t\right) + L_{t,s},Y_s,s\right), \tag{150}$$

with the process L as in (9) and the function H solving (10).

Next, averaging with regards to m^* and using the separability of the initial condition $x - \theta\bar{m}$ from the common noise in the process $X^* - \theta\bar{X}^*$ above, we obtain, with $m_t^* = m$, that

$$\bar{X}_s^* = \frac{1}{1-\theta} \int H\left(H^{(-1)}\left(x - \theta\bar{m}, y, t\right) + L_{t,s}, Y_s, s\right) dm(x). \tag{151}$$

The representation (86) for the optimal wealth in the single agent optimization problem (40) implies that, for $s \in [t, T]$,

$$\int H\left(H^{(-1)}\left(x - \theta\bar{m}, y, t\right) + L_{t,s}, Y_s, s\right) dm(x) = \int \mathcal{X}_s^{x-\theta\bar{m},*} dm(x). \tag{152}$$

Combining (150), (151) and (152), we obtain that the mean field equilibrium wealth process is given by

$$\begin{aligned} X_s^* &= H\left(H^{(-1)}\left(x - \theta\bar{m}, y, t\right) + L_{t,s}, Y_s, s\right) \\ &+ \frac{\theta}{1-\theta} \int H\left(H^{(-1)}\left(x - \theta\bar{m}, y, t\right) + L_{t,s}, Y_s, s\right) dm(x). \end{aligned}$$

In other words, reinstating the appropriate initial conditions we have, for $0 \leq t \leq s \leq T$, the decomposition

$$X_s^{x,*} = \mathcal{X}_s^{x-\theta\bar{m},*} + \theta\bar{X}_s^{\bar{m},*} = \mathcal{X}_s^{x-\theta\bar{m},*} + \frac{\theta}{1-\theta} \int \mathcal{X}_s^{x-\theta\bar{m},*} dm(x). \tag{153}$$

To construct the associated mean field equilibrium control process, we observe that, for $0 \leq t \leq s \leq T$,

$$\begin{aligned} \pi_s^* &= \pi^*\left(X_s^*, Y_s, m_s^*, s\right) \\ &= \alpha^*\left(X_s^* - \theta\bar{X}_s^*, Y_s, s\right) + \frac{\theta}{1-\theta} \int \alpha^*\left(z - \theta\bar{X}_s^*, Y_s, s\right) dm_s^*(z). \end{aligned} \tag{154}$$

From (150) and the second equality in (92), we obtain

$$\begin{aligned} \alpha^*\left(X_s^* - \theta\bar{X}_s^*, Y_s, s\right) &= \alpha^*\left(H\left(H^{(-1)}\left(x - \theta\bar{m}, y, t\right) + L_{t,s}, Y_s, s\right), Y_s, s\right) \\ &= \alpha^*\left(\mathcal{X}_s^{x-\theta\bar{m},*}, Y_s, s,\right) = \alpha_s^{x-\theta\bar{m},*}, \end{aligned}$$

the optimal process of the single agent problem (40) starting at $(x - \theta\bar{m}, y, t)$.

Thus,

$$\int \alpha^*\left(z - \theta\bar{X}_s^*, Y_s, s\right) dm_s^*(z) = \int \alpha^*\left(\mathcal{X}_s^{x-\theta\bar{m},*}, Y_s, s\right) dm(x) = \int \alpha_s^{x-\theta\bar{m},*} dm(x),$$

and conclude that the mean field equilibrium policy is represented as

$$\pi_s^{x,*} = \alpha_s^{x-\theta\bar{m},*} + \frac{\theta}{1-\theta} \int \alpha_s^{x-\theta\bar{m},*} dm(x). \tag{155}$$

We can interpret the above optimal processes as follows. At time t, the representative agent starts at x, and splits it to $x_1 = x - \theta\bar{m}$ and to $x_2 = \theta\bar{m}$. Using x_1, she follows the optimal policy $\alpha^{x_1,*}$ as in (40), which generates the process

$$X_{1,s}^{x_1,*} = H\left(H^{(-1)}(x_1, y, t) + L_{t,s}, Y_s, s\right).$$

With the remaining x_2, she follows policy $\frac{\theta}{1-\theta}\int \alpha_s^{x-\theta\bar{m},*}dm(x)$, obtaining

$$X_{2,s}^{x_2,*} = \frac{\theta}{1-\theta}\int \alpha^*\left(H\left(H^{(-1)}(x_1, y, t) + L_{t,s}, Y_s, s\right), Y_s, s\right)dm(x)$$
$$= \frac{\theta}{1-\theta}\int \mathcal{X}_s^{x-\theta\bar{m},*}dm(x).$$

We note that $\frac{\theta}{1-\theta}\int \alpha_s^{x-\theta\bar{m},*}dm(x)$ is a feasible policy but not optimal.

5.4 Separable payoffs and connection with indifference valuation and arbitrage-free pricing

We conclude by relating and interpreting the results for separable payoffs using elements from indifference valuation and arbitrage-free pricing. For the reader's convenience, we recall the notion of indifference price which, to avoid cumbersome notation, we present somewhat informally (we refer the reader to [17] and [49] for a general overview of the area).

Let Z_T be a given claim, represented as an $\mathcal{F}_T^Y-$measurable random variable and introduced at t. Then, its so-called writer's value function is defined as

$$w^Z(x, y, t) = \sup_{\mathcal{A}} \mathbb{E}_{\mathbb{P}}\left[\, J(\mathcal{X}_T - Z_T)|\, \mathcal{X}_t = x, Y_t = y, \mathcal{F}_t^Y\right],$$

with $\mathcal{X}$ and Y solving (39) and (38).

At time t, the indifference price of Z_T, denoted by $p_t(Z_T)$, is defined as the spatial input $p_t(Z_T) \in \mathcal{F}_t^Y$ such that, a.s. in $\mathbb{R}\times\mathbb{R}\times[0,T]$,

$$w^0(x, y, t) = w^Z(x + p_t(Z_T), y, t).$$

In other words, $p_t(Z_T)$ makes the agent "indifferent" between (i) investing optimally in the underlying market without the claim ($Z_T \equiv 0$) and (ii) receiving compensation $p_t(Z_T)$ at initiation time t, investing optimally in the same market and rendering Z_T at expiry time T.

We are also interested in the indifference price process $p_s(Z_T)$, $0 \le t \le s \le T$, and the sde satisfied by it, with $p_T(Z_T) = Z_T$.

In the general case of incomplete markets, calculating the indifference price process is a taunting task as the process $p_s(Z_T)$ is, in general, path-dependent and incorporates in a complicated way both the market dynamics and the writer's wealth.

If, however, the market is complete, as the one herein, and under mild integrability assumptions on Z_T, it can be shown that its indifference price process coincides with its so-called arbitrage-free price, which is given, for $0 \leq t \leq s \leq T$, by the martingale

$$p_s(Z_T) = \mathbb{E}_{\mathbb{Q}}[\, Z_T | \, \mathcal{F}_s^Y]. \tag{156}$$

The measure $\mathbb{Q}$ is the unique, absolutely continuous to the (physical) measure $\mathbb{P}$, that makes the (discounted) stock price a martingale[1]. It then follows, from the martingale representation theorem and the definition of $\mathbb{Q}$, that we can obtain the decomposition

$$dp_s(Z_T) = h_s dW_s^{\mathbb{Q}} = bh_s ds + h_s dW_s \text{ for } s \in (t,T) \text{ and } p_T(Z_T) = Z_T,$$

with W as in (36) and some process h, known as the "indifference hedge".

Furthermore, it can be shown that the above sde also coincides with the actual under $\mathbb{P}$ dynamics that generate claim Z_T, that is, we have $p_s(Z_T) = Z_s$, or equivalently, if the process Z solves

$$dZ_s = b(Y_s, s)h_s ds + h_s dW_s, \; Z_t = z, \quad 0 \leq t \leq s \leq T,$$

then

$$p_s(Z_T) = Z_s \text{ in } (t,T) \text{ and } p_t(Z_T) = z.$$

We may now draw analogies with the quantities we obtain in the MFG problem for the separable case (116). Viewing, at the optimum, the random variable $C\,(m_T^*)$ as a claim, the value function u in (40) as w^0 and inspecting equality (119), we write, with some abuse of notation,

$$\begin{aligned} u(x{-}f(y,m,t),y,t) &= \mathbb{E}\left[J(\mathcal{X}_T^{x-f(y,m,t),*}) \Big| \, \mathcal{X}_t^* = x - f(y,m,t), Y_t = y \right] \\ &= \mathbb{E}\left[J(X_T^{x,*} - C(m_T^*) \big| \, X_t^* = x, Y_t = y, m_t^* = m \right] = U(x,y,m,t). \end{aligned} \tag{157}$$

In other words, we can think of the mean field game value U as the value of the representative agent who starts with wealth x and "writes" claim $C(m_T^*)$.

In turn, we interpret the mean field equilibrium optimal processes X^* and π^* as follows. At time t, the representative agent splits the initial x into $x - f(y,m,t)$ and $f(y,m,t)$. With initial wealth $x - f\,(y,m,t)$, she implements the optimal policy $\alpha^{x-f(y,m,t),*}$, given by (92) and rewritten below for convenience,

$$\begin{aligned} \alpha_s^{x-f(y,m,t),*} &= c(Y_s,s)H_z(H^{(-1)}(x - f(y,m,t),y,t) + L_{t,s}, Y_s, s) \\ &\quad + H_y(H^{(-1)}(x - f(y,m,t),y,t) + L_{t,s}, Y_s, s). \end{aligned}$$

This generates process $\mathcal{X}^{x-f(y,m,t),*}$ and, in particular, the random variable $\mathcal{X}_T^{x-f(y,m,t),*}$ at terminal horizon T.

[1]The reader familiar with arbitrage-free pricing may see the connection between the pricing measure $\mathbb{Q}$ and the one appearing in the dynamics (46) of the auxiliary problem (47).

In parallel, using the initial wealth $f(y,m,t)$, the representative agent follows strategy C^* which generates $C(m_T^*)$ at T. In other words, the "claim" $C(m_T^*)$ admits the decomposition

$$C(m_T^*) = f(y,m,t) + \int_t^T b(Y_s,s)C_s^* ds + \int_t^T C_s^* dW_s.$$

Therefore, the net wealth generated at T, starting at x, following policies $\alpha^{x-f(y,m,t),*}$ and C^*, and fulfilling the liability $C(m_T^*)$ at T is given by

$$X_T^{x,*} - C(m_T^*)$$

$$= (x - f(y,m,t)) + f(y,m,t) + \int_t^T b(Y_s,s)\left(\alpha_s^{x-f(y,m,t),*} + C_s^*\right) ds$$

$$+ \int_t^T (\alpha_s^{x-f(y,m,t),*} + C_s^*) dW_s - C(m_T^*) = \mathcal{X}_T^{x-f(y,m,t),*},$$

and this is precisely what the left hand side in (157) expresses.

One may erroneously think that the above arguments trivialize the MFG problem, in the sense that one could "guess" (157) from starts and bypass the entire analysis. This is, of course, not correct since, contrary to the traditional indifference/arbitrage-free valuation problem, the claim in consideration $C(m_T^*)$ is not a priori given. Rather, $C(m_T^*)$ is being created by the dynamic interaction of the agents in $[t,T]$ and has initial value $f(y,m,t)$ that needs to be found by solving equation (117) and (118), or the autonomous equation (130) in subsection 5.2.

Finally, we comment that a Feynma-Kac-type representation of the function f solving (117) is consistent with the arbitrage-free price of the claim $C(m_T^*)$ in analogy to (156).

6 Conclusions

We revisited the classical optimal portfolio choice problem with partial information and studied related mean field games of relative performance, with couplings depending on the law of peers' wealth. For the former problem, we introduced an alternative solution approach and produced regularity and closed-form represenations for the value function and the optimal processes, for general utilities.

For the MFG games, we allowed for general dependence of the couplings on the law of peers' wealth, and produced a master system, comprised by the master equation and an optimality/compatibility condition of the mean field equilibrium feedback policy.

For both problems, we studied representative cases and interpreted the corresponding solutions.

There are several extensions of this work. Firstly, the methodology developed herein may be extended to complete market models with general factors, beyond reduced models generated from partial observations. Such models incorporate, for example, local volatility, predictable returns, and others, and have been studied by various authors for homothetic utilities; see, among others, [38], [57] and [58] for homothetic utilities.

Secondly, intermediate consumption and multi-assets may be, also, incorporated. It is expected that for separable payoffs, analogous connections with the single agent problem will hold, but the form of equation (117) is not obvious.

Thirdly, the closed-form representations for the optimal wealth and portfolio processes may be useful in studying the effects of collective investment behavior on asset returns. For example, one may study the so-called crowding risk ([4], [5], [20]) which arises when similar investment activity across many agents ends up diluting value and performance.

In a different direction, one may study MFG with relative performance defined in a semi-finite domain, for example in $[0, \infty)$. In the absence of competition, this case corresponds to non-negative wealth constraints. Generally speaking, there are two distinct ways to incorporate competition, multiplicatively and additively. The former was firstly modeled in [41] for the case of power utilities, where the effects of peer performance was formulated as the geometric average of the players' wealth. For such couplings, the non-negativity constraint is implicitly satisfied throughout, and the homotheticity of preferences allowed for explicit solutions that led to random (contant in time) equilibria. This specific case, power utilities and multiplicative-type couplings, was subsequently studied by many authors, with representative references being listed in the bibliography. However, the case of general utility and/or multiplicative couplings remains open. If, on the other hand, the competition is additive, as the case herein, one still expects a solution to be as in (24), that is,

$$U(x,t) = u(x - f(y,m,t), y, t) \tag{158}$$

where u is the solution of the single agent problem in $[0, \infty) \times [0, T]$ and $f > 0$ solving an equation like (25). However, the nonnegativity of the wealth argument in the right-hand-side above will result in $U(x,t)$ defined in a time-evolving (admissibility) domain of the form $x \geq f(y, m, t)$, $y \in \mathbb{R}$, $m \in \mathcal{P}_2$, and $t \in [0, T]$. If the equation satisfied by f has a unique, smooth enough solution, the admissibility domain can be fully characterized and representation (158) is expected to hold.

In both semi-definite domain cases, with multiplicative or additive couplings, it would be interesting to study the emerging indifference valuation problems which, however, are expected to be non-standard. Indeed, it is not clear what kind of "claim" multiplicative couplings would result to, at mean field equilibrium. For the additive case, the difficulties are entirely different, as questions related to "super-hedging", wealth range and the dynamic price of the coupling at the optimum may be challenging even in the complete market herein. The authors are currently working on both these directions.

Finally, incorporating idiosyncratic noise remains a challenging question, as the linearization steps will not hold. The authors are currently working on the exponential case and general couplings, allowing for individual stocks in addition to the common one.

References

[1] Agarwal, V., Daniel N.D. and N.Y. Naik, Flows, performance and managerial incentives in hedge funds. In: EFA 2003, annual conference, paper No. 501, 2003.

[2] Anthropelos, M., Geng, T. and T. Zariphopoulou, Competition in fund management under forward performance criteria, SIAM Journal on Financial Mathematics, 13(4), 2022, 1271-1320.

[3] Bain, A. and D. Crisan, Fundamentals of stochastic filtering, Stochastic Modeling and Applied Probability Series, Springer Verlag, 2009.

[4] Baltas, N., The impact of crowding in alternative risk premia investing, Financial Analysts Journal, 75(3), 89-104, 2019.

[5] Barroso, P., Edelen, R.M. and P. Karehnke, Crowding and tail risk in momentum returns, Journal of Financial and Quantitative Analysis, 57(4), 2021.

[6] Basak, S. and D. Makarov, Strategic asset allocation in money management, Journal of Finance, 69(1), 179-217, 2014.

[7] Basak, S. and D. Makarov, Competition among managers: equilibrium policies, cost-benefit implications and financial innovation, SSRN Electronic Journal, 2010.

[8] Bauerle, N. and T. Goll, Nash equilibria for relative investors with (non)linear price impact, arXiv:2303,18161.

[9] Bensoussan, A. and S. Yam, Mean field approach to stochastic control with partial information, ESAIM, Control, Optimization and Calculus of Variations, 27, 2021.

[10] Black, F. Individual investment and consumption under uncertainty. Portfolio Insurance: A guide to dynamic hedging, D.L. Luskin (Ed.), New York, John Wiley and Sons, 1988, 207-225; first version in 1968.

[11] Bjork, T., Davis, M.H.A. and C. Landen, Optimal investment under partial information, Mathematical Methods of Operations Research, 71(2), 2010, 371-399..

[12] Bo, J.L. and T.Q. Li, Approximation of Nash equilibrium of optimal consumption in stochastic growth model with jumps, Acta Mathematica Sinica, 38(9), 1621-1642, 2022.

[13] Boyle, P., Garlappi, L. Uppal, R. and T. Wang, Keynes meet Markowitz: the trade-off between familiarity and diversification, Management Science, 58(2), 253-272, 2012.

[14] Brown, S.J., Goetzmann, W. N. and J. Park, Careers and survival: competition and risk in the hedge fund and CTA industry, Journal of Finance, 56(5), 1829-1886, 2001.

[15] Brendl, S., Portfolio selection under incomplete information, Stochastic Processes and their Applications, 116(5), 2006, 701-723.

[16] Cardaliaguet, P., Delarue, F., Lasry, J.M. and P.-L. Lions, The master equation and the convergence problem for mean field games, Ann. Math. Stud. 201, 2019.

[17] Carmona, R. (Ed.), Indifference pricing: Theory and Applications, Princeton University Press, 2009.

[18] Carmona, R. and F. Delarue, Probabilistic theory of mean field games with applications I-II, Springer, Berlin, 2018.

[19] Chen, A., Pelsser, A. and M. Vellekoop, Modeling non-monotone risk aversion using SAHARA utility functions, Journal of Economic Theory, 146, 2074-2092, 2011.

[20] Chincarini, L., Lazo-Paz, R. and F. Moneta, Crowed spaces and anomalies, Journal of Banking and Finance, 182, 2026.

[21] Danilova, A., Monoyios, M. and A. Ng, Optimal investment with inside information and parameter uncertainty, Mathematics, Finance and Economics, 3, 13-38, 2010.

[22] Deng, C., Su, X. and C. Zhou, Relative wealth concerns and partial information and heterogeneous priors, SIAM Journal on Financial Mathematics, 15(2), 2024.

[23] De Marzo, P. and R. Kaniel, Contracting in peer networks, Journal of Finance, 78(5), 2725-2778, 2023.

[24] De Marzo, P., Kaniel, R. and I. Kremer, Relative wealth concerns and financial bubbles, Review of Financial Studies, 21(1), 19-50, 2008.

[25] Espinosa, G.-E. and N. Touzi, Optimal invstement under relative performance concerns, Mathematical Finance, 15(2), 221-257, 2015.

[26] Geng, T. and T. Zariphopoulou, Temporal and spatial turnpikes in Ito-diffusion markets under forward performance criteria, Numerical Algebra, Control and Optimization, 15(1), 243-272, 2025.

[27] Guo, X., Hu, R. and T. Zariphopoulou, Entropy regularization for mean field games with learning, Mathematics of Operations Research, published on line, 3239-3260, 2022.

[28] Fu, G. and C. Zhou, Mean field portfolio games, Finance and Stochastics, 27, 189-231, 2022.

[29] Hu, R. and T. Zariphopoulou, N-player and mean field games in Ito diffusion markets with competitive or homophilous interaction, Stochastic Analysis, Filtering and Stochastic Optimization, Springer-Verlag, (Eds. G. Yin and T. Zariphopoulou), 209-237, 2022.

[30] Huang, M., Malhame, R.P. and P.E. Caines, Large population stochastic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, Communications in Information systems, 6(3), 221-252, 2006.

[31] Huang, Y.J. and L.H. Sun, Partial information in a mean-variance portfolio selection game, Mathematical Finance, 36(2), 309-329, 2026.

[32] Huang, C.F. and T. Zariphopoulou, Turnpike behavior of long-term investments, Finance and Stochastics, 3(1), 1-20, 1999.

[33] Kallianpur, G., Stochastic filtering theory, Stochastic modeling and Applied Probability, Springer-Verlag, 1980.

[34] Karatzas, I. and X. Zhao, Bayesian adaptive portflio optimization, Option pricing, interest rates and risk management, Handbooks in Mathematical Finance, Jouini, E., Cvitanic, J. and M. Musiela Eds. Cambridge University Press, 632-669, 2001.

[35] Kallblad, S. and T. Zariphopoulou, On the Black's equation for the risk tolerance function, arXiv:1705.07472.

[36] Kempf, A. and S. Ruenzi, Tournaments in mutual fund families, Review of Financial Studies, 21(2), 1013-1036, 2008.

[37] Kempf, A., S. Ruenzi and T. Thiele, Employment risk, compensation incentives and managerial risk taking: evidence form the mutual fund industry, Journal of Financial Econometrics, 92(1), 92-108, 2009.

[38] Kim, T.S. and E. Omberg, Dynamic nonmyopic portfolio behavior, Review of Financial Studies, 9(10), 1996, 141-161.

[39] Kraft, H., Meyer-Wehmann A. and FT. Seifried, Dynamic asset allocation with relative wealth concerns in incomplete markets, Journal of Economics Dynamics and Control, 14(2), 263-281, 2020.

[40] Lacker, D. and A. Soret, Many player games for optimal consumption and investment under relative performance concerns, Mathematics, Finance and Economics, 14, 263-281, 2020.

[41] Lacker, D. and T. Zariphopoulou, Mean field and N-player games under relative performance concerns, Mathematical Finance, 29(4), 1003-1038, 2019.

[42] Lakner, P., Utility maximization under partial information, Stochastic Processes and their Applications 56, 1995, 247-249.

[43] Lasry, J.-M. and P.-L. Lions, Jeaux a champ moyen, I. Le cas stationnaire, C.R. Math. Acad. Sci. Paris, 343(9), 619-625, 2006.

[44] Lasry, J.-M. and P.-L. Lions, Jeaux a champ moyen, II. Horizon fini et controle optimal, C.R. Math. Acad. Sci. Paris, 343(10), 679-684, 2006.

[45] Lions, P.-L., Courses at College de France.

[46] Mitton, T. and K. Vorkink, Equilibrium undiversification and the preference for skewness, Review of Financial Studies, 20(4), 1255-1288, 2007.

[47] Monin, P. and T. Zariphopoulou, On the optimal wealth process in a lognormal market: applications to risk management, Journal of Financial Engineering, 1(2), 2014.

[48] Mostovyi, O., Sibru, M. and T Zariphopoulou, On the analyticity of the value function in optimal investment, Pure and Applied Functional Analysis, 9(3), 2024, 825-822.

[49] Musiela M. and T. Zariphopoulou, An example of indifference prices under exponential preferences, Finance and Stochastics, 8, 2004, 229-239.

[50] Musiela, M. and T. Zariphopoulou, Remarks on hedging and numeraire consistency in an incomplete binomial model, Technical Report, 2002.

[51] Musiela, M. and T. Zariphopoulou, Portfolio choice under space-time monotone performance criteria, SIAM Journal in Financial Mathematics, 1, 2010, 266-288.

[52] Sen, N. and P.E. Caines, Mean field games with partial information, SIAM Journal on Control and Optimization, 57(3), 2064-2091, 2019.

[53] Sirri, E.R. and P. Tuffano, Costly search and mutual fund flows, Journal of Finance, 53(5), 1589-1622, 1998.

[54] Shmaya, E. and B. Zilotto, Bayesian learning in mean field games, SIAM Journal on Control and Optimization, 63(3), 1783-1797, 2025.

[55] Souganidis, P. and T. Zariphopoulou, Mean-field games with unbounded controlled common noise in portfolio management with relative performance criteria, Mathematics and Financial Economics, published on line, 2024, 219-234.

[56] Uppal, R. and T. Wang, Model misspecification and underdiversification, Journal of Finance, 58(3), 2465-2486, 2003.

[57] Wachter, J., Asset allocation, Annual Review of Financial Economics, 2(1), 2010, 175-206.

[58] Zariphopoulou, T., Optimal investment and consumption with nonlinear stock dynamics, Mathematical Methods of Operations Research, 50, 1999, 271-296.

[59] Zariphopoulou, T., Mean-field and n-player games in Ito diffusion markets under forward performance criteria, Probability, Uncertainty and Quantitative Risk, 9(2), 2024, 123-148.

[60] T. Zariphopoulou and T. Zhou, Investment performance measurement under asymptotically linear local risk tolerance,Handbook of Numerical Analysis (P. G. Ciarlet), 15, 2009, 227-253.

[61] Zhang, A. and P. Huang, Optimal portfolio with relative perfromance and partial infromation: a mean field approach, Asian Journal of Control, Special Issue, 2023.