Representation of forward performance criteria with random endowment via FBSDE and its application to forward optimized certainty equivalent*

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Abstract

We extend the notion of forward performance criteria to settings with random endowment in incomplete markets. Building on these results, we introduce and develop the novel concept of *forward optimized certainty equivalent (forward OCE)*, which offers a genuinely dynamic valuation mechanism that accommodates progressively adaptive market model updates, stochastic risk preferences, and incoming claims with arbitrary maturities.

In parallel, we develop a new methodology to analyze the emerging stochastic optimization problems by directly studying the candidate optimal control processes for both the primal and dual problems. Specifically, we derive two new systems of forward-backward stochastic differential equations (FBSDEs) and establish necessary and sufficient conditions for optimality, and various equivalences between the two problems. This new approach is general and complements the existing one for forward performance criteria with random endowment based on backward stochastic partial differential equations (backward SPDEs) for the related value functions. We, also, consider representative examples for both forward performance criteria with random endowment and for forward OCE. Furthermore, for the case of exponential criteria, we investigate the connection between forward OCE and forward entropic risk measures.

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1 Introduction

This work contributes to the theory of forward performance criteria in incomplete markets. It studies forward performance processes in the presence of random endowment and, building on this, introduces the novel concept of *forward optimized certainty equivalent (forward OCE)*. In parallel, it develops a new methodological approach to study the emerging forward stochastic optimization problems through new, interesting on their own right, *forward-backward stochastic differential equations (FBSDEs)* satisfied by the optimal control processes of the primal and dual problems.

Random endowments are a central object of study in expected utility maximization and, furthermore, play an important role in indifference valuation/certainty equivalent, where they model the payoffs (or liabilities) to be priced and hedged. The aim herein is twofold: to develop a general framework to incorporate them within the broader class of forward performance processes and analyze the related optimal control processes, and to investigate how they can be used to generalize the widely-used notion of optimized certainty equivalent.

The motivation for the plan of study herein stems from various shortcomings of the classical (backward) setting, as highlighted next and further explained later in the paper. Recalling the standard paradigm in the random endowment literature, one pre-specifies at initial time $t = 0$ a quadruple consisting of i) the (longest) horizon $[0, T]$ within which random endowments will arrive, ii) the utility function U at the end of this horizon, iii) the underlying market model $\mathcal{M}_{[0,T]}$ and iv) the upcoming random endowments. In other words, these modeling ingredients are chosen statically, once and for all, at initiation. However, more flexibility is frequently needed as new random endowments might arrive at times not known before or arrive at random times (see [49]), the market model might be also revised (see [2]) and, furthermore, the risk preferences themselves could be modified (see [48]). Indeed, let us consider the following simple representative case: for simplicity, it is assumed that there is a random endowment, given by a random variable P_T specified at $t = 0$. However, at some future time τ, with $0 < \tau < T$, the utility maximizer learns that an additional payoff P_{T_1} is expected at time $T_1 < T$. One now sees that the solution of the problem in $[0, \tau]$ has not considered this updated information, yielding *a posteriori* time-inconsistent solutions. The situation becomes even more complex if this new random endowment actually arrives at time $T_1 > T$, for the utility U was pre-defined only at T, and not beyond this time. Therefore, a modified utility maximization problem in $[T, T_1]$ needs first to be defined in order to accommodate the new random endowment. Additional considerations arise if at an intermediate time, say $\tau' < T$, the market model is updated to $\mathcal{M}'_{[\tau',T]}$, which will also yield *a posteriori* time-inconsistent solutions, even if the assumptions on the random endowments remain the same.

Naturally, these limitations also have undesirable consequences on indifference prices. Indeed, the expected utility maximization framework does not allow to price in a time-consistent manner claims arriving at times not known at initiation, and especially when these new claims mature at instances beyond the pre-specified horizon, in which case the underlying model is not even well defined. It, also, fails to price, in a time-consistent manner, claims when the market model is being dynamically revised, or when the risk preferences themselves evolve stochastically. These limitations were one of the main motivations for the third author and M. Musiela to develop the theory of forward performance criteria in the early 2000s (see, among others, [41, 42, 43, 44, 45]).

Since then, the effort in the forward approach has mainly focused on, from the one hand, developing a probabilistic characterization of forward performance processes and, from the other, studying forward indifference prices for the special class of exponential criteria, see, for example, the stochastic partial differential equation (SPDE) approach initiated in [45], and also studied in [21, 22, 23, 24]. When the forward performance process is homothetic, like the exponential case, a new class of ergodic backward stochastic differential equations (BSDEs) have been proposed to characterize this class (see [17, 40]). When the forward performance process is time-monotonic, Widder's theorem has been applied for

their characterization ([6, 46]). In addition, the corresponding discrete-time theory has been recently explored extensively ([1, 2, 39, 48]). The applications of forward performance processes have extended to various domains, such as relative performance criteria ([4]), general semimartingale models ([12]), insurance ([16]), duality theory ([19, 52]), behavioral finance ([27]), regime-switching models ([30]), intertemporal consumption ([33]), model uncertainty ([34]), mean-field games ([50]), and maturity-independent risk measures ([53]).

The only studies of the forward performance process with random endowment were the ones for exponential criteria in the context of exponential forward indifference prices (see, among others, [38, 41, 42]). The most general result can be found in Chong et al. [17] who built on the work of forward entropic risk measures, initially developed in [53], and proposed a BSDE, coupled with an ergodic BSDE, representation of exponential forward indifference prices. However, how this BSDE/ergodic BSDE approach may be extended beyond the exponential case remains an open problem.

1.1 Main contributions

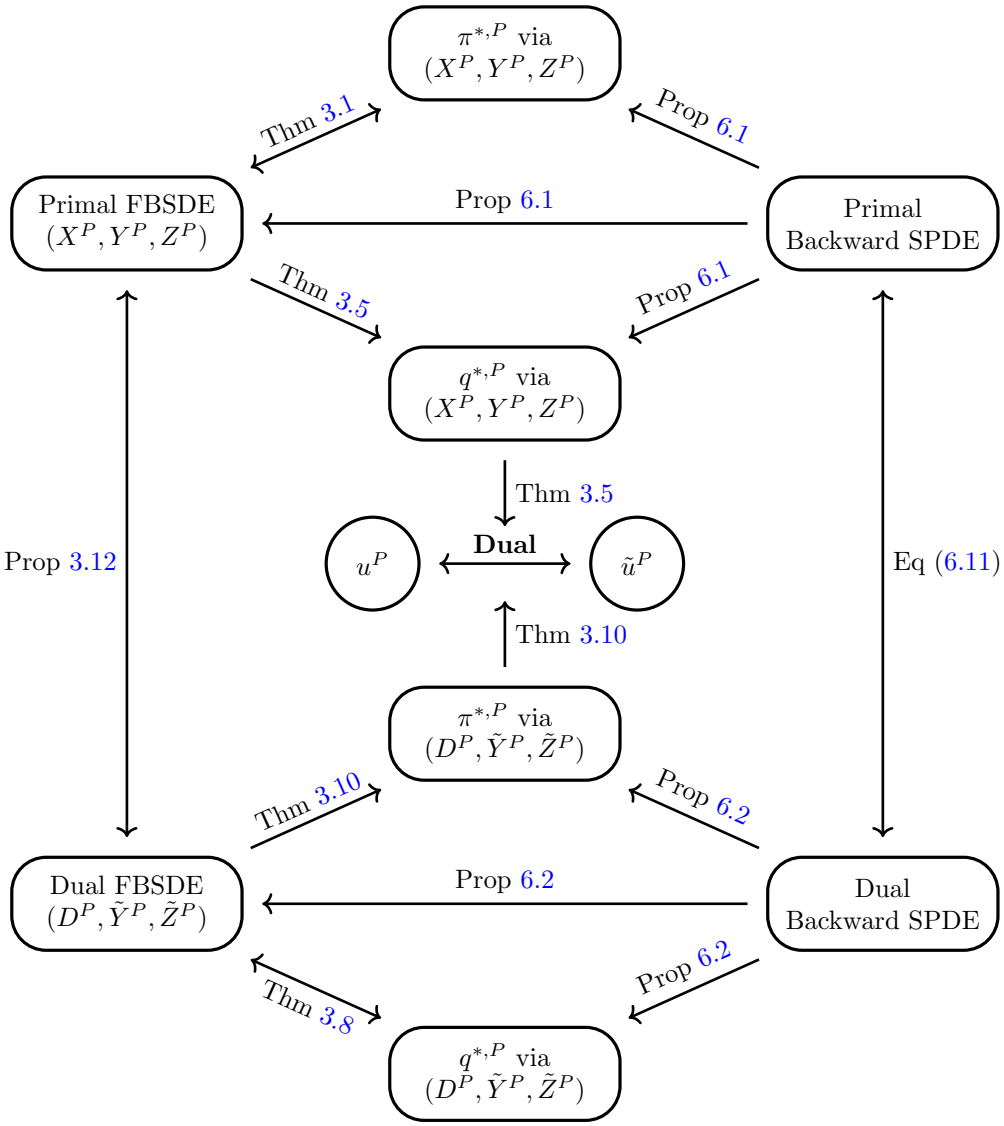
FBSDE for forward performance criteria with random endowment

Herein, we depart from both the specific class of exponential forward performance processes and the SPDE approach for general forward performance processes without random endowment. We take an entirely different approach by working directly with the optimal policies and the corresponding optimal state price density processes via the solutions of two FBSDEs, one referred to as the *primal FBSDE (3.3)* and the other as the *dual FBSDE (3.16)*. These two FBSDEs form a convex dual relationship akin to the primal and the dual problems of the forward performance process with random endowment. We demonstrate that both FBSDEs offer necessary and sufficient conditions for the forward performance process with random endowment and its convex dual.

We introduce the new methodology, establish necessary and sufficient optimality conditions via the new FBSDEs and explore equivalences between the primal and dual problems. To solve the associated FBSDEs, we employ the recently developed decoupling field method ([25, 26]), which enables us to address general forward performance criteria under specific conditions in a single stochastic factor model. Additionally, the exponential case allows for the decoupling of the forward and backward equations of the FBSDE, while the complete markets scenario provides explicit solutions.

The derivation of the primal and dual FBSDEs draws inspiration from Horst et al. [28], who considered the FBSDE characterization of utility maximization with random endowment but only from the primal perspective. However, our work differs from [28] as the static utility functions therein are being replaced by general forward performance processes. Their stochastic nature combined with their inherent martingale optimality leads to simpler and more interpretable terms in the obtained FBSDEs. Furthermore, we explore the convex dual of the primal FBSDE, presenting a novel approach to studying the convex duality in the forward setting. The results herein also provide a new perspective for forward performance processes in the absence of random endowment, yielding the self-generation property, first studied in [52] for the exponential case.

For the reader's convenience, we next provide a roadmap of the FBSDE approach and the underlying equivalences and dual relationships. For completeness, the roadmap also includes information about the formal derivation of the corresponding primal and dual backward SPDEs discussed in Section 6.



Glossary of notation.

P	random endowment
u^P	value function of the primal problem
$\tilde{u}^P$	value function of the dual problem
$\pi^{*,P}$	optimal control of the primal problem
$q^{*,P}$	optimal control of the dual problem

Figure 1: The FBSDE approach road map of main results.

Forward optimized certainty equivalent

Building on the results on forward performance processes with random endowment, we introduce the novel concept of the *forward optimized certainty equivalent (forward OCE)*. Its static counterpart, introduced by Ben-Tal and Teboulle in [8] and [9] (see also (5.1) here), is a decision-making criterion based on expected utility theory and represents the outcome of an optimal fund allocation where an investor can choose to allocate a portion of the money from the random endowment to spend. However, due to the static nature of the utility function, extending the existing OCE notion to a dynamic setting while maintaining time consistency poses conceptual challenges. Recent studies by Backhoff-Veraguas et al. [10] and [11] began with the convex dual representation of OCE and generalized it to a dynamic version by introducing an additional variable to ensure time consistency. While [10] and [11] generalize the static case of [8] and [9], the dynamic OCE therein is directly tied down to the pre-chosen horizon T even though they are dynamically time-consistent within $[0, T]$. This is a direct consequence of the fact that the underlying utility function is by nature tied down to T. Similar horizon dependence is also observed in the classical indifference prices.

One then poses the question if there is a way to construct an OCE-type valuation mechanism that is *horizon invariant*, or *maturity independent*, since we typically align the horizon with the longest maturity. We prefer to use the terminology maturity independent to align it with the studies in [53] and [17].

We stress that we do not expect that the OCE of a given claim would not depend on the maturity of it, an obviously wrong result. Rather we seek a valuation mechanism, like the classical conditional expectation, the exponential forward indifference pricing or the forward entropic risk measure, that does not depend *per se* on a specific horizon or maturity. This is what we develop herein within the OCE framework.

The key idea of the forward OCE notion: We present a novel perspective on the static OCE by interpreting it as the convex dual of the value function for a utility maximization problem with the random endowment within an auxiliary financial market whose driving noise is orthogonal to the random endowment. We, then, replace the static utility function with the forward performance process within this auxiliary financial market and, in turn, define a forward OCE as the convex dual of the value function for the forward performance process with the random endowment. This forward OCE represents the result of an optimal dynamic allocation of funds, where an investor faces the choice of saving a portion of their current wealth alongside the random endowment to maximize the forward performance process. The auxiliary variable introduced in [10] and [11] has a natural interpretation as a stochastic deflator, converting nominal values into real values. The forward OCE mirrors the optimal balance between maximizing the forward performance process from the saving alongside the random endowment and reducing spending due to this saving choice.

Unlike the framework proposed in [10] and [11], the newly defined forward OCE is both time consistent and maturity independent. Furthermore, in the forward OCE framework, we introduce an auxiliary financial market where the underlying asset can be utilized to partially hedge the risk associated with the random endowment. In contrast, in the dynamic OCE framework presented in [10] and [11], the driving noise in this auxiliary market is assumed to be orthogonal to the random endowment, resulting in null hedging effects. Finally, we apply the new definition and the related results to the exponential case and we demonstrate that the forward OCE aligns with the negative of the forward entropic risk measure proposed in [17].

The paper is organized as follows. In section 2, we introduce the market and recall the notion of forward performance criteria and their convex dual. In section 3, we present the primal and dual problems in the presence of random endowment, and the main results on the new FBSDEs, while in section 4, we solve the FBSDEs for the complete market case, the exponential forward performance case, and the more general Markovian case, respectively. In section 5, we introduce the notion of forward OCE and investigate the connection of exponential forward OCE with forward entropic risk

measure. In section 6, we discuss the backward SPDE approach and compare it with the FBSDE one proposed herein. We conclude in section 7. For the reader's convenience, all proofs are deferred to the appendix.

2 Background results

We introduce the incomplete market and review fundamental concepts and results in forward performance criteria, recalling their definition and the ill-posed SPDE that governs their evolution. We also review their convex dual and the corresponding SPDE.

2.1 The incomplete market

Let $W = (W^1, W^2)$ be a two-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The market consists of one riskless asset, taken to be numeraire and offering with zero interest rate, and a stock whose (discounted) price solves

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t^1, \quad t \geq 0, \quad S_0 = S > 0.$$

The coefficients $\mu \in \mathbb{R}$ and $\sigma > 0$ are $\mathbb{F}$-progressively measurable processes and the market price of risk $\theta := \frac{\mu}{\sigma}$ is assumed to be uniformly bounded.

The market is considered incomplete because the dimension of noise is higher than the number of stocks. Models with such incompleteness can be readily extended to multidimensional cases for both the traded assets and the sources of imcompleteness (see, for example, [1, 17, 28, 40]), but for simplicity, we keep the dimensionality low.

Let process $\tilde{\pi}$ be the amount invested in the stock. Assume that it is self-financing and let $\pi := \tilde{\pi}\sigma$. Then, the wealth SDE is given by

$$dX_t^\pi = \pi_t \left(\theta_t dt + dW_t^1\right), \quad t \geq 0, \quad X_0^\pi = x \in \mathbb{R}. \tag{2.1}$$

For each $T > 0$, we denote by $\mathcal{A}_{[t,T]}$ the set of admissible trading strategies in horizon $[t, T]$, $0 \leq t \leq T$, defined as

$$\mathcal{A}_{[t,T]} := \left\{\pi \in \mathbb{L}^2_{BMO}\left[t, T\right], \pi_s \in \mathbb{R}, \, t \leq s \leq T\right\},$$

where

$$\mathbb{L}^2_{BMO}\left[t, T\right] := \left\{ (\pi_s)_{t\leq s\leq T} : \pi \text{ is } \mathbb{F}\text{-progressively measurable and } \mathbb{E}\left[\int_\tau^T |\pi_r|^2 \, dr \middle| \mathcal{F}_\tau\right] \leq C, \, \mathbb{P}\text{-a.s.,} \right.$$
$$\left. \text{for some constant } C > 0 \text{ and all } \mathbb{F}\text{-stopping times } \tau \text{ valued in } [t, T] \right\}.$$

We denote by $\mathcal{A} := \cup_{t\geq 0}\mathcal{A}_{[0,t]}$ the set of admissible trading strategies for all $t \geq 0$.

This market model admits multiple state price densities (or equivalent martingale measures), which we parameterize via a process q on $[t, T]$, $0 \leq t \leq T < \infty$. Specifically, these price densities are represented as

$$M_s^{t,q} = \exp\left(-\int_t^s \left(\theta_r dW_r^1 + q_r dW_r^2\right) - \frac{1}{2}\int_t^s \left(|\theta_r|^2 + |q_r|^2\right) dr\right), \quad 0 \leq t \leq s \leq T,$$

where $q \in \mathcal{Q}_{[t,T]}$ with

$$\mathcal{Q}_{[t,T]} := \left\{q \in \mathbb{L}^2_{BMO}\left[t, T\right], q_s \in \mathbb{R}, \, t \leq s \leq T\right\}, \quad 0 \leq t \leq T.$$

Therefore, for $q \in \mathcal{Q}_{[t,T]}$, the corresponding state price density process satisfies the SDE

$$dM_s^{t,q} = -M_s^{t,q}\left(\theta_s dW_s^1 + q_s dW_s^2\right), \quad 0 \le t \le s \le T, \tag{2.2}$$

with initial condition $M_t^{t,q} = 1$, $0 \le t \le T$. Moreover, $M_s^{t,q}$ is a true $\mathbb{F}$-martingale since θ is uniformly bounded and $q \in \mathcal{Q}_{[t,T]}$ (see [36]). We also have the following property, which implies that the wealth process is a true $\mathbb{F}$-martingale under each probability measure transformed by $q \in \mathcal{Q}_{[t,T]}$.

Lemma 2.1 *For each $T > 0$, $0 \le t \le T$, $\pi \in \mathcal{A}_{[t,T]}$ and $q \in \mathcal{Q}_{[t,T]}$, the process*

$$M_s^{t,q} \cdot \int_t^s \pi_r\left(\theta_r dr + dW_r^1\right), \quad t \le s \le T,$$

is a true $\mathbb{F}$-martingale on $[t,T]$.

2.2 Forward performance criteria and their convex dual representation

We recall the definition of forward performance processes and their SPDE representation introduced by Musiela and Zariphopoulou in [41, 42] and established in [45], respectively. We also recall their convex dual representation introduced by El Karoui and Mrad in [23].

For $t \ge 0$, let $L^0(\mathcal{F}_t)$ denote the space of $\mathcal{F}_t$-measurable random variables. Similarly, let $L^p(\mathcal{F}_t)$, $p \ge 1$, be the space of L^p-integrable $\mathcal{F}_t$-measurable random variables and $L^\infty(\mathcal{F}_t)$ the space of bounded $\mathcal{F}_t$-measurable random variables.

Definition 2.2 *Let $\mathbb{D} = [0,\infty) \times \mathbb{R}$. A process $U(t,x)$, $(t,x) \in \mathbb{D}$, is called a forward performance process if*

(i) for each $x \in \mathbb{R}$, $U(t,x)$ is $\mathbb{F}$-progressively measurable,

(ii) for each $t \ge 0$, the mapping $x \mapsto U(t,x)$ is strictly increasing and strictly concave,

(iii) for any bounded $\mathbb{F}$-stopping time τ and $0 \le t \le \tau$,

$$U(t,\xi) = \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,\tau]}} \mathbb{E}\left[U\left(\tau, \xi + \int_t^\tau \pi_r\left(\theta_r dr + dW_r^1\right)\right)\middle| \mathcal{F}_t\right] \text{ for any } \xi \in \cap_{p\ge 1} L^p(\mathcal{F}_t). \tag{2.3}$$

Here we choose $\xi \in \cap_{p\ge 1} L^p(\mathcal{F}_t)$ to ensure that all its moments are well-defined, a property needed for the related BMO processes and admissible trading strategies we will be using herein. We will frequently refer to (2.3) as the *self-generation* property (see [52]).

In [45], the forward performance process U was shown to be associated with the so-called forward performance SPDE

$$dU(t,x) = \beta(t,x)\,dt + \alpha^\top(t,x)\,dW_t, \quad (t,x) \in \mathbb{D}, \tag{2.4}$$

with the drift β given by

$$\beta(t,x) = \frac{1}{2}\frac{\left|U_x(t,x)\,\theta_t + \alpha_x^1(t,x)\right|^2}{U_{xx}(t,x)},$$

and the volatility $\alpha = (\alpha^1, \alpha^2)$ being an $\mathbb{R}^2$-valued $\mathbb{F}$-progressively measurable process. We recall that, contrary to the classical framework of maximal expected utility where the volatility is endogenously determined from the dynamics in the Itô-Ventzel expansion of the value function process, the forward volatility coefficient $\alpha(t,x)$ in the forward approach is an independent modeling input.

The optimal control process is, in turn, expressed in terms of the solution to the above SPDE. Specifically, if there exists a strong solution to

$$dX_t^* = -\frac{U_x\left(t, X_t^*\right)\theta_t + \alpha_x^1\left(t, X_t^*\right)}{U_{xx}\left(t, X_t^*\right)}\left(\theta_t dt + dW_t^1\right), \quad t \geq 0, \tag{2.5}$$

and the feedback control

$$\pi_t^* := -\frac{U_x\left(t, X_t^*\right)\theta_t + \alpha_x^1\left(t, X_t^*\right)}{U_{xx}\left(t, X_t^*\right)}, \quad t \geq 0, \tag{2.6}$$

is in the admissible set $\mathcal{A}$, then π^* is optimal.

Throughout, we assume that $U(t,x)$ solves the SPDE (2.4) for a volatility process $\alpha(t,x)$, and SDE (2.5) admits a strong solution X^*. Furthermore, we will assume the following conditions hold.

Assumption 2.3 *(i) The progressive random field $U(t,x)$, $(t,x) \in \mathbb{D}$, is a $\mathcal{K}_{loc}^{2,\delta}$-semimartingale for some $\delta \in (0,1]$ and $U \in \mathcal{C}^3$ (see Appendix A).*

(ii) There exist positive constants C_l, C_u and C_α such that, a.s.,

$$0 < C_l \leq -\frac{U_x(t,x)}{U_{xx}(t,x)} \leq C_u \quad \text{and} \quad \left|\frac{\alpha_x^i(t,x)}{U_{xx}(t,x)}\right| \leq C_\alpha, \ (t,x) \in \mathbb{D}, \quad i = 1,2.$$

Remark 2.4 *In Appendix A, we provide the differential rule and Itô-Ventzel formula as well as the spaces of regular random fields. According to Theorem A.1, Assumption 2.3 (i) guarantees that the differential rule for U holds, namely,*

$$dU_x(t,x) = \beta_x(t,x)\,dt + \alpha_x^\top(t,x)\,dW_t, \quad (t,x) \in \mathbb{D}, \tag{2.7}$$

where

$$\begin{aligned}\beta_x(t,x) = \left(U_x(t,x)\theta_t + \alpha_x^1(t,x)\right)\theta_t + \frac{\left(U_x(t,x)\theta_t + \alpha_x^1(t,x)\right)\alpha_{xx}^1(t,x)}{U_{xx}(t,x)} \\ - \frac{1}{2}\frac{\left|U_x(t,x)\theta_t + \alpha_x^1(t,x)\right|^2 U_{xxx}(t,x)}{\left|U_{xx}(t,x)\right|^2}.\end{aligned} \tag{2.8}$$

Furthermore, the Itô-Ventzel formula can be applied to both $U(t, X_t^\pi)$ and $U_x(t, X_t^\pi)$.

Remark 2.5 *In Assumption 2.3 (ii), C_u provides an upper bound for the risk tolerance $-U_x/U_{xx}$, $1/C_l$ provides an upper bound for the risk aversion $-U_{xx}/U_x$. We emphasize that Assumption 2.3 (ii) is necessary to ensure that the optimal control processes remain within the admissible set. However, this condition can be further relaxed, for instance, by allowing the optimal control processes to belong to $\mathbb{L}_{BMO}^2[t,T]$.*

Next, we recall the convex dual of the process $U(t,x)$, introduced in [23]. To this end, for a given forward performance process $U(t,x)$, its convex conjugate $\tilde{U}(t,z)$ is defined as the Fenchel-Legendre transform of $-U(t,-x)$, namely,

$$\tilde{U}(t,z) := \sup_{x \in \mathbb{R}}\left(U(t,x) - xz\right), \quad (t,z) \in \mathbb{D}^+, \tag{2.9}$$

where $\mathbb{D}^+ = [0,\infty) \times \mathbb{R}^+$.

One may readily verify that $\tilde{U}(t,z)$ satisfies:

(i) For each $t \geq 0$, the mapping $z \mapsto \tilde{U}(t,z)$ is strictly convex and $\tilde{U} \in \mathcal{C}^3$.

(ii) The inverse of the marginal utility U_x is the negative of the marginal of the conjugate utility $\tilde{U}_z$:

$$\begin{aligned} -\tilde{U}_z\left(t, U_x\left(t,x\right)\right) &= x, \qquad (t,x) \in \mathbb{D}, \\ U_x\left(t, -\tilde{U}_z\left(t,z\right)\right) &= z, \qquad (t,z) \in \mathbb{D}^+. \end{aligned} \tag{2.10}$$

(iii) The bidual relation

$$U\left(t,x\right) = \inf_{z>0}\left(\tilde{U}\left(t,z\right) + xz\right), \quad (t,x) \in \mathbb{D}, \tag{2.11}$$

holds.

(iv) For $(t,x) \in \mathbb{D}$ and $(t,z) \in \mathbb{D}^+$,

$$U\left(t,x\right) = \tilde{U}\left(t, U_x\left(t,x\right)\right) + xU_x\left(t,x\right), \quad \tilde{U}\left(t,z\right) = U\left(t, -\tilde{U}_z\left(t,z\right)\right) + z\tilde{U}_z\left(t,z\right), \tag{2.12}$$

$$\tilde{U}_{zz}\left(t, U_x\left(t,x\right)\right) = -\frac{1}{U_{xx}\left(t,x\right)}, \quad U_{xx}\left(t, -\tilde{U}_z\left(t,z\right)\right) = -\frac{1}{\tilde{U}_{zz}\left(t,z\right)}, \tag{2.13}$$

and

$$\tilde{U}_{zzz}\left(t, U_x\left(t,x\right)\right) = \frac{U_{xxx}\left(t,x\right)}{\left(U_{xx}\left(t,x\right)\right)^3}, \quad U_{xxx}\left(t, -\tilde{U}_z\left(t,z\right)\right) = -\frac{\tilde{U}_{zzz}\left(t,z\right)}{\left(\tilde{U}_{zz}\left(t,z\right)\right)^3}. \tag{2.14}$$

As the analysis herein shows, the process $\tilde{U}_z(t,z)$ is a key quantity in establishing the FBSDE representations in Section 3. It turns out that stronger regularity assumptions are needed for $\tilde{U}$ and $\tilde{U}_z$, introduced below; see [21, 23] for similar assumptions.

Assumption 2.6 *The progressive random field $\tilde{U}(t,z)$, $(t,z) \in \mathbb{D}^+$, is a $\mathcal{K}^{2,\delta}_{loc}$-semimartingale for some $\delta \in (0,1]$.*

Recalling Theorem 1.5 in [23], we obtain the dynamics of $\tilde{U}_z$,

$$d\tilde{U}_z\left(t,z\right) = \tilde{\beta}_z\left(t,z\right)dt + \tilde{\alpha}_z^\top\left(t,z\right)dW_t, \quad (t,z) \in \mathbb{D}^+, \tag{2.15}$$

where

$$\begin{aligned} \tilde{\beta}_z\left(t,z\right) =& \frac{\beta_x\left(t, -\tilde{U}_z\left(t,z\right)\right)}{U_{xx}\left(t, -\tilde{U}_z\left(t,z\right)\right)} + \frac{1}{2}\frac{U_{xxx}\left(t, -\tilde{U}_z\left(t,z\right)\right)\left|\alpha_x\left(t, -\tilde{U}_z\left(t,z\right)\right)\right|^2}{\left(U_{xx}\left(t, -\tilde{U}_z\left(t,z\right)\right)\right)^3} \\ & - \frac{\alpha_x^\top\left(t, -\tilde{U}_z\left(t,z\right)\right)\alpha_{xx}\left(t, -\tilde{U}_z\left(t,z\right)\right)}{\left|U_{xx}\left(t, -\tilde{U}_z\left(t,z\right)\right)\right|^2} \\ =& -\tilde{U}_{zz}\left(t,z\right)z\left|\theta_t\right|^2 + \tilde{\alpha}^1_{zz}\left(t,z\right)z\theta_t + \tilde{\alpha}^1_z\left(t,z\right)\theta_t - \frac{1}{2}\tilde{U}_{zzz}\left(t,z\right)\left|z\theta_t\right|^2 \\ & - \frac{1}{2}\frac{\tilde{U}_{zzz}\left(t,z\right)\left|\tilde{\alpha}^2_z\left(t,z\right)\right|^2}{\left|\tilde{U}_{zz}\left(t,z\right)\right|^2} + \frac{\tilde{\alpha}^2_z\left(t,z\right)\tilde{\alpha}^2_{zz}\left(t,z\right)}{\tilde{U}_{zz}\left(t,z\right)}, \end{aligned} \tag{2.16}$$

and

$$\tilde{\alpha}_z(t,z) = \frac{\alpha_x\left(t,-\tilde{U}_z(t,z)\right)}{U_{xx}\left(t,-\tilde{U}_z(t,z)\right)} = -\tilde{U}_{zz}(t,z)\,\alpha_x\left(t,-\tilde{U}_z(t,z)\right). \tag{2.17}$$

We conclude by stressing that it is not immediately clear whether the convex conjugate $\tilde{U}(t,z)$ satisfies a self-generation type property like (2.3) for the primal forward process $U(t,x)$. This question was studied in [52]. Therein, the author proved the equivalence between the self-generation properties of $U(t,x)$ and $\tilde{U}(t,z)$ and, in turn, provided a dual characterization of the forward criterion. In this paper, the FBSDE approach we develop can be also used to establish similar results for forward processes, see Proposition 3.7.

3 Forward performance criteria with random endowment: an FBSDE approach

We start by choosing an arbitrary time $T>0$ at which we introduce a random endowment, represented as a random variable $P \in L^\infty(\mathcal{F}_T)$. We then introduce the *primal forward problem with random endowment*

$$u^P(t,\xi;T) = \operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T,\xi+\int_t^T \pi_r\left(\theta_r dr + dW_r^1\right)+P\right)\middle|\mathcal{F}_t\right], \quad 0\le t\le T,\ \xi\in\cap_{p\ge1}L^p(\mathcal{F}_t). \tag{3.1}$$

The associated *dual forward problem with random endowment* is then given by

$$\tilde{u}^P(t,\eta;T) = \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T,\eta M_T^{t,q}\right)+\eta M_T^{t,q}P\middle|\mathcal{F}_t\right], \quad 0\le t\le T,\ \eta\in L^{0,+}(\mathcal{F}_t), \tag{3.2}$$

where $\tilde{U}$ is the convex conjugate of U as (2.9), and $L^{0,+}(\mathcal{F}_t)$ denotes the space of positive $\mathcal{F}_t$-measurable random variables.

A conventional approach to solving the primal and dual problems (3.1) and (3.2) is to first characterize their value functions, which are, in general, expected to solve related backward SPDEs due to the underlying non-Markovian model. The corresponding optimal controls for (3.1) and (3.2) are then expressed in terms of the solutions to these SPDEs. However, as we demonstrate in Section 6, their derivation is formal and their solvability is far from clear.

In this section, we take a different approach by directly characterizing the optimal control processes for both the primal and dual problems using an FBSDE approach. By substituting the optimal controls into (3.1) and (3.2), we can in turn obtain the value functions. Our main contributions are the derivation of two FBSDEs: the primal FBSDE (3.3) and the dual FBSDE (3.16), both of which provide a means to characterize the solutions for both the primal and dual problems. These two FBSDEs are closely connected, forming a convex dual relationship, similar to what (3.1) and (3.2) represent.

3.1 Optimal policy characterization using the primal FBSDE

The first main result is the derivation of both necessary and sufficient conditions for the optimal control process for the primal problem (3.1) through the FBSDE below, which we will refer as the

primal FBSDE. Specifically, consider the FBSDE, for $0 \le t \le s \le T$,

$$\begin{cases} X_s^P = \xi - \int_t^s \left(\dfrac{U_x\left(r, X_r^P + Y_r^P\right)\theta_r + \alpha_x^1\left(r, X_r^P + Y_r^P\right)}{U_{xx}\left(r, X_r^P + Y_r^P\right)} + Z_r^{P,1} \right)\left(\theta_r dr + dW_r^1\right), \\ Y_s^P = P + \int_s^T \left(-Z_r^{P,1}\theta_r + \dfrac{1}{2}\dfrac{U_{xxx}\left(r, X_r^P + Y_r^P\right)}{U_{xx}\left(r, X_r^P + Y_r^P\right)}\left|Z_r^{P,2}\right|^2 + \dfrac{\alpha_{xx}^2\left(r, X_r^P + Y_r^P\right)}{U_{xx}\left(r, X_r^P + Y_r^P\right)} Z_r^{P,2} \right) dr \\ \qquad - \int_s^T \left(Z_r^P\right)^\top dW_r, \end{cases} \tag{3.3}$$

with initial-terminal condition (ξ, P) for $\xi \in \cap_{p\ge 1} L^p(\mathcal{F}_t)$ and $P \in L^\infty(\mathcal{F}_T)$. Here we use the superscript P to highlight the dependence of the random endowment.

Theorem 3.1 *Let $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and $\xi \in \cap_{p\ge 1} L^p(\mathcal{F}_t)$, $0 \le t \le T$.*

(i) Assume $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ is an optimal control process of the primal problem (3.1) and denote

$$X_s^{*,P} := \xi + \int_t^s \pi_r^{*,P}(\theta_r dr + dW_r^1), \quad t \le s \le T. \tag{3.4}$$

Furthermore, assume that

$$\mathbb{E}[|U_x(T, X_T^{*,P} + P)|^2] < \infty, \tag{3.5}$$

and

$$\begin{gathered} \tfrac{1}{\varepsilon}\left|U\left(T, \xi + \textstyle\int_t^T \left(\pi_s^{*,P} + \varepsilon\pi_s\right)\left(\theta_s ds + dW_s^1\right) + P\right) - U\left(T, X_T^{*,P} + P\right)\right| \\ \text{is uniformly integrable in } \varepsilon \in (0,1) \text{ for any bounded } \pi \in \mathcal{A}_{[t,T]}, \end{gathered} \tag{3.6}$$

Then, there exists a triplet (X^P, Y^P, Z^P) satisfying FBSDE (3.3). Furthermore, $\pi^{*,P}$ can be represented as

$$\pi_s^{*,P} = -\frac{U_x\left(s, X_s^P + Y_s^P\right)\theta_s + \alpha_x^1\left(s, X_s^P + Y_s^P\right)}{U_{xx}\left(s, X_s^P + Y_s^P\right)} - Z_s^{P,1}, \quad t \le s \le T, \tag{3.7}$$

and $X_s^{*,P} = X_s^P$, $t \le s \le T$.

(ii) Suppose (X^P, Y^P, Z^P) is a solution to FBSDE (3.3) satisfying $Z^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1, 2$. Then, the control process $\pi^{*,P}$ defined by (3.7) is optimal for the primal problem (3.1) and $X_s^{*,P} = X_s^P$, $t \le s \le T$. Specifically, $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ and

$$u^P(t, \xi; T) = \mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) + P\right)\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[U\left(T, X_T^P + P\right)\middle|\, \mathcal{F}_t\right]. \tag{3.8}$$

Remark 3.2 *Condition (3.6) is not needed when $U(t,x)$ satisfies*

$$U_x(t, x+y) \le C U_x(t,x)(1 + e^{ky}), \quad t \ge 0, \quad x, y \in \mathbb{R},$$

for some constants $C > 0$ and $k \in \mathbb{R}$. Indeed, under this additional growth condition and following a similar argument as in [28, Section 3], one can verify that condition (3.5) implies condition (3.6).

Lemma 3.3 *Suppose* (X^P, Y^P, Z^P) *is a solution to FBSDE (3.3) satisfying* $Z^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1, 2$. *Define*

$$q_s^{*,P} := -\frac{U_{xx}\left(s, X_s^P + Y_s^P\right) Z_s^{P,2} + \alpha_x^2\left(s, X_s^P + Y_s^P\right)}{U_x\left(s, X_s^P + Y_s^P\right)}, \quad 0 \le t \le s \le T. \tag{3.9}$$

Then $q^{*,P} \in \mathcal{Q}_{[t,T]}$ *and the corresponding state price density is given by*

$$M_s^{t,q^{*,P}} = \frac{U_x\left(s, X_s^P + Y_s^P\right)}{U_x\left(t, \xi + Y_t^P\right)}, \quad 0 \le t \le s \le T.$$

Furthermore, $M^{t,q^{*,P}}$ *is a true* $\mathbb{F}$*-martingale.*

The proof follows easily by Assumption 2.3 (ii) and the fact that

$$dU_x\left(s, X_s^P + Y_s^P\right) = -U_x\left(s, X_s^P + Y_s^P\right)\left(\theta_s dW_s^1 + q_s^{*,P} dW_s^2\right), \quad 0 \le t \le s \le T, \tag{3.10}$$

where we use FBSDE (3.3) and the dynamics of U_x in (2.7). Comparing with the SDE for the state price density in (2.2), we have thus identified $M^{t,q^{*,P}}$ as a candidate process for the optimal state price density.

The above lemma is used to verify that FBSDE (3.3) serves as a *sufficient* condition for the primal problem (3.1) in Theorem 3.1 (ii) above, and also for the dual problem (3.2) in Theorem 3.5 below. We also mention that process $q^{*,P}$ depends on the solution of primal FBSDE (3.3) with initial-terminal condition (ξ, P), and so is the state price density $M^{t,q^{*,P}}$.

Remark 3.4 *When* $P \equiv 0$, *equation (3.8) reduces to*

$$u^0(t,\xi;T) = \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_r\left(\theta_r dr + dW_r^1\right)\right) \middle| \mathcal{F}_t\right] = U(t,\xi). \tag{3.11}$$

It then follows directly that the triplet $(X^0, Y^0, Z^0) = (X^*, 0, 0)$, *with* X^* *satisfying (2.5) with initial condition* $X_t^* = \xi$, *solves FBSDE (3.3). Thus, Theorem 3.1 (ii) yields that the control process*

$$\pi_s^{*,0} = -\frac{U_x\left(s, X_s^*\right)\theta_s + \alpha_x^1\left(s, X_s^*\right)}{U_{xx}\left(s, X_s^*\right)}, \quad 0 \le t \le s \le T,$$

is optimal and $X^{*,0} = X^0 = X^*$, *where* $X^{*,0}$ *solves (2.1) with* $\pi^{*,0}$ *above being used. As expected,* $\pi^{*,0}$ *coincides with the optimal process derived in (2.6). Furthermore, (3.11) aligns with the self-generation property (2.3).*

3.1.1 Primal FBSDE and the dual problem

We demonstrate that the primal FBSDE (3.3) provides a solution to the dual problem (3.2).

Theorem 3.5 *Let* $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ *and* $\xi \in \cap_{p \ge 1} L^p(\mathcal{F}_t)$, $0 \le t \le T$. *Suppose* (X^P, Y^P, Z^P) *is a solution to FBSDE (3.3) on* $[t,T]$ *with initial-terminal condition* (ξ, P), *satisfying* $Z^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1, 2$. *Let*

$$\hat{\eta} := U_x\left(t, \xi + Y_t^P\right). \tag{3.12}$$

Then,

(i) the control process $q^{*,P}$ defined in (3.9) is optimal for the dual problem (3.2), namely, $q^{*,P} \in \mathcal{Q}_{[t,T]}$ and

$$\begin{aligned}\tilde{u}^P(t,\hat{\eta};T) &= \mathbb{E}\left[\tilde{U}\left(T,\hat{\eta}M_T^{t,q^{*,P}}\right) + \hat{\eta}M_T^{t,q^{*,P}}P\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\tilde{U}\left(T,U_x\left(T,X_T^P+P\right)\right) + U_x\left(T,X_T^P+P\right)P\middle|\mathcal{F}_t\right],\end{aligned}$$

(ii) the bidual relation

$$u^P(t,\xi;T) = \operatorname*{essinf}_{\eta\in L^{0,+}(\mathcal{F}_t)}\left(\tilde{u}^P(t,\eta;T)+\xi\eta\right) = \tilde{u}^P(t,\hat{\eta};T)+\xi\hat{\eta}$$

holds.

Remark 3.6 *The reverse direction of Theorem 3.5 amounts to the case of the initial state η of the density process being arbitrary, but the initial wealth $\hat{\xi}$ depending on η. More precisely, let $\eta \in L^{0,+}(\mathcal{F}_t)$. Then, if (X^P, Y^P, Z^P) is a solution to the FBSDE (3.3) on $[t,T]$ with initial-terminal condition $(\hat{\xi},P)$, satisfying $Z^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$ for $i=1,2$, and*

$$\hat{\xi} = -\tilde{U}_z(t,\eta) - Y_t^P,$$

with $\hat{\xi} \in \cap_{p\geq 1}L^p(\mathcal{F}_t)$, the control process $q^{*,P}$, defined in (3.9) and depending on $\hat{\xi}$, is optimal for the dual problem $\tilde{u}^P(t,\eta;T)$ in (3.2). Furthermore, the bidual relation

$$\tilde{u}^P(t,\eta;T) = \operatorname*{esssup}_{\xi\in\cap_{p\geq 1}L^p(\mathcal{F}_t)}\left(u^P(t,\xi;T)-\xi\eta\right) = u^P(t,\hat{\xi};T)-\hat{\xi}\eta$$

holds. This result follows directly from Theorem 3.5, using similar arguments.

Next, using the above primal FBSDE characterization, we re-establish the self-generation property of $\tilde{U}$, first proven in [52].

Proposition 3.7 *For any $0\leq t\leq T$ and $\eta \in L^{0,+}(\mathcal{F}_t)$ satisfying $\hat{\xi} := -\tilde{U}_z(t,\eta) \in \cap_{p\geq 1}L^p(\mathcal{F}_t)$,*

$$\tilde{U}(t,\eta) = \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}}\mathbb{E}\left[\tilde{U}\left(T,\eta M_T^{t,q}\right)\middle|\mathcal{F}_t\right], \tag{3.13}$$

and

$$\tilde{U}(t,\eta) = \mathbb{E}\left[\tilde{U}\left(T,\eta M_T^{t,q^*}\right)\middle|\mathcal{F}_t\right], \tag{3.14}$$

where $q^* \in \mathcal{Q}_{[t,T]}$ is given by

$$q_s^* := -\frac{\alpha_x^2(s,X_s^*)}{U_x(s,X_s^*)}, \quad t\leq s\leq T, \tag{3.15}$$

with X^* satisfying (2.5) with initial condition $X_t^* = \hat{\xi}$.

This result follows by choosing $P\equiv 0$ in Remark 3.6. Specifically, by Remark 3.4, we know that $(X^0,Y^0,Z^0) = (X^*,0,0)$ solves FBSDE (3.3) on $[t,T]$ with initial-terminal condition $(\hat{\xi},0)$, and q^* in (3.15) is bounded according to Assumption 2.3 (ii). By Remark 3.6, if $\hat{\xi}\in\cap_{p\geq 1}L^p(\mathcal{F}_t)$, then q^* is optimal for $\tilde{u}^0(t,\eta;T)$, i.e.,

$$\tilde{u}^0(t,\eta;T) = \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}}\mathbb{E}\left[\tilde{U}\left(T,\eta M_T^{t,q}\right)\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\tilde{U}\left(T,\eta M_T^{t,q^*}\right)\middle|\mathcal{F}_t\right].$$

Moreover,

$$\tilde{u}^0(t,\eta;T) = \operatorname*{esssup}_{\xi\in\cap_{p\geq 1}L^p(\mathcal{F}_t)} \left(u^0(t,\xi;T) - \xi\eta\right) = u^0\left(t,\hat{\xi};T\right) - \hat{\xi}\eta.$$

Note that $u^0(t,\xi;T) = U(t,\xi)$ for any $\xi \in \cap_{p\geq 1}L^p(\mathcal{F}_t)$ according to Remark 3.4. It follows that

$$\tilde{u}^0(t,\eta;T) = U\left(t,\hat{\xi}\right) - \hat{\xi}\eta = \tilde{U}(t,\eta),$$

which proves Proposition 3.7.

3.2 Optimal state price density characterization using the dual FBSDE

Having formulated FBSDE (3.3) for the primal problem, we revert to the dual problem and derive an analogous FBSDE, given in (3.16) below. With a slight abuse of terminology, we will be refering to (3.16) as the *dual FBSDE*.

To this end, consider the FBSDE, for $0 \leq t \leq s \leq T$,

$$\begin{cases} D_s^P = \eta - \displaystyle\int_t^s \left(D_r^P \theta_r dW_r^1 + \frac{\tilde{Z}_r^{P,2} + \tilde{\alpha}_z^2\left(r, D_r^P\right)}{\tilde{U}_{zz}\left(r, D_r^P\right)} dW_r^2 \right), \\ \tilde{Y}_s^P = P + \displaystyle\int_s^T \left(\frac{1}{2}\frac{\tilde{U}_{zzz}\left(r, D_r^P\right)\left|\tilde{Z}_r^{P,2}\right|^2}{\left|\tilde{U}_{zz}\left(r, D_r^P\right)\right|^2} - \tilde{Z}_r^{P,1}\theta_r - \tilde{Z}_r^{P,2}\left(\frac{\tilde{\alpha}_{zz}^2\left(r, D_r^P\right)}{\tilde{U}_{zz}\left(r, D_r^P\right)} - \frac{\tilde{U}_{zzz}\left(r, D_r^P\right)\tilde{\alpha}_z^2\left(r, D_r^P\right)}{\left|\tilde{U}_{zz}\left(r, D_r^P\right)\right|^2} \right) \right) dr \\ \qquad - \displaystyle\int_s^T \left(\tilde{Z}_r^P\right)^\top dW_r, \end{cases} \tag{3.16}$$

with initial-terminal condition (η, P) for $\eta \in L^{0,+}(\mathcal{F}_t)$, $P \in L^\infty(\mathcal{F}_T)$.

We use this FBSDE to characterize the optimal control process of the dual problem (3.2). Specifically, we first derive a necessary condition for the optimal density process and, in turn, demonstrate that it also yields a sufficient optimality condition.

Theorem 3.8 *Let $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \leq t \leq T$.*

(i) Assume $q^{*,P} \in \mathcal{Q}_{[t,T]}$ is an optimal control process of the dual problem (3.2). Furthermore, assume that

$$\mathbb{E}[|(\tilde{U}_z(T, \eta M_T^{t,q^{*,P}}) + P)\eta M_T^{t,q^{*,P}}|^2] < \infty,$$

and

$$\tfrac{1}{\varepsilon}\left|\tilde{U}\left(T, \eta M_T^{t,q^{*,P}+\varepsilon q}\right) - \tilde{U}\left(T, \eta M_T^{t,q^{*,P}}\right)\right|$$

is uniformly integrable in $\varepsilon \in (0,1)$ for any bounded $q \in \mathcal{Q}_{[t,T]}$.

Then there exists a triplet $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ satisfying FBSDE (3.16). Furthermore, $q^{*,P}$ can be represented as

$$q_s^{*,P} = \frac{\tilde{Z}_s^{P,2} + \tilde{\alpha}_z^2\left(s, D_s^P\right)}{D_s^P \tilde{U}_{zz}\left(s, D_s^P\right)}, \quad t \leq s \leq T, \tag{3.17}$$

and $\eta M_s^{t,q^{*,P}} = D_s^P$, $t \leq s \leq T$.

(ii) Suppose $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ *is a solution to FBSDE (3.16) satisfying* $D^P > 0$ *and* $\tilde{Z}^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$. *Then, the control process* $q^{*,P}$ *defined by (3.17) is optimal for the dual problem (3.2) and* $\eta M_s^{t,q^{*,P}} = D_s^P$, $t \le s \le T$. *Specifically,* $q^{*,P} \in \mathcal{Q}_{[t,T]}$ *and*

$$\tilde{u}^P(t,\eta;T) = \mathbb{E}\left[\tilde{U}\left(T, \eta M_T^{t,q^{*,P}}\right) + \eta M_T^{t,q^{*,P}} P \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\tilde{U}\left(T, D_T^P\right) + D_T^P P \middle| \mathcal{F}_t\right].$$

Lemma 3.9 *Suppose* $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ *is a solution to FBSDE (3.16) satisfying* $D^P > 0$ *a.s. and* $\tilde{Z}^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$. *Define*

$$\pi_s^{*,P} := \tilde{U}_{zz}\left(s, D_s^P\right) D_s^P \theta_s - \tilde{\alpha}_z^1\left(s, D_s^P\right) - \tilde{Z}_s^{P,1}, \quad 0 \le t \le s \le T. \tag{3.18}$$

Then $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ *and*

$$d\left(\tilde{U}_z\left(s, D_s^P\right) + \tilde{Y}_s^P\right) = -\pi_s^{*,P}\left(\theta_s ds + dW_s^1\right), \quad 0 \le t \le s \le T. \tag{3.19}$$

The proof follows easily from Assumption 2.3 (ii) and by combining FBSDE (3.16) with the dynamics of $\tilde{U}_z$ in (2.15). From (3.19), $\pi^{*,P}$ can be identified as a candidate process for the optimal policy of the primal problem.

This lemma is used to verify that FBSDE (3.16) serves as a sufficient condition for the dual problem (3.2) in Theorem 3.8 (ii) above and also for the primal problem (3.1) in Theorem 3.10 below. We note that, as expected, process $\pi^{*,P}$ depends on the solution of the dual FBSDE (3.16) with initial-terminal condition (η, P).

3.2.1 Dual FBSDE and the primal problem

Corresponding to the results in Section 3.1.1, the dual FBSDE (3.16) can be also used to characterize the solution of the primal problem (3.1).

Theorem 3.10 *Let* $T > 0, P \in L^\infty(\mathcal{F}_T)$ *and* $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \le t \le T$. *Suppose* $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ *is a solution to FBSDE (3.16) on* $[t,T]$ *with initial-terminal condition* (η, P) *satisfying* $D^P > 0$ *and* $\tilde{Z}^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$. *Define*

$$\hat{\xi} := -\tilde{U}_z(t,\eta) - \tilde{Y}_t^P. \tag{3.20}$$

Then, if $\hat{\xi} \in \cap_{p\ge 1} L^p(\mathcal{F}_t)$,

(i) the control process $\pi^{*,P}$ *defined in (3.18) is optimal for the primal problem (3.1), namely,* $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ *and*

$$\begin{aligned} u^P\left(t,\hat{\xi};T\right) &= \mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T^P\right)\right) \middle| \mathcal{F}_t\right], \end{aligned}$$

(ii) the bidual relation

$$\tilde{u}^P(t,\eta;T) = \operatorname*{esssup}_{\xi \in \cap_{p\ge 1} L^p(\mathcal{F}_t)} \left(u^P(t,\xi;T) - \xi\eta\right) = u^P\left(t,\hat{\xi};T\right) - \hat{\xi}\eta$$

holds.

Remark 3.11 *Similarly to Remark 3.6, the reverse direction of Theorem 3.10 amounts to the case of the initial wealth ξ being arbitrary, but the initial state $\hat{\eta}$ specified as a particular class of initial states for the density process, depending on ξ. More precisely, for $\xi \in \cap_{p\geq 1} L^p(\mathcal{F}_t)$, suppose $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ is a solution to the FBSDE (3.16) on $[t,T]$ with the initial-terminal condition $(\hat{\eta}, P)$, satisfying $D^P > 0$, $\tilde{Z}^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$ for $i = 1,2$, and*

$$\hat{\eta} = U_x\left(t, \xi + \tilde{Y}^P_t\right) \in L^{0,+}(\mathcal{F}_t).$$

Then, the control process $\pi^{*,P}$, defined in (3.18) and depending on $\hat{\eta}$, is optimal for the primal problem $u^P(t,\xi;T)$ in (3.1). Furthermore, the bidual relation

$$u^P(t,\xi;T) = \operatorname*{essinf}_{\eta \in L^{0,+}(\mathcal{F}_t)} \left(\tilde{u}^P(t,\eta;T) + \xi\eta\right) = \tilde{u}^P(t,\hat{\eta};T) + \xi\hat{\eta}$$

holds.

3.3 Relations between the primal and dual FBSDEs

We show that the primal FBSDE (3.3) and the dual FBSDE (3.16) form a convex dual pair. Their relationship is a direct analogue to the convex duality between the primal problem (3.1) and the dual problem (3.2).

Proposition 3.12 *Let $T > 0$ and $P \in L^\infty(\mathcal{F}_T)$.*

(i) For $\eta \in L^{0,+}(\mathcal{F}_t)$, $0 \leq t \leq T$, suppose $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ is a solution to FBSDE (3.16) on $[t,T]$ with initial-terminal condition (η, P), satisfying $D^P > 0$. Let $\hat{\xi}$ be defined in (3.20). Then, if $\hat{\xi} \in \cap_{p\geq 1} L^p(\mathcal{F}_t)$, the triplet (X^P, Y^P, Z^P) given by

$$X^P_s := -\tilde{U}_z\left(s, D^P_s\right) - \tilde{Y}^P_s, \quad Y^P_s := \tilde{Y}^P_s, \quad Z^P_s := \tilde{Z}^P_s, \quad t \leq s \leq T,$$

is a solution to FBSDE (3.3) on $[t,T]$ with initial-terminal condition $(\hat{\xi}, P)$.

(ii) For $\xi \in \cap_{p\geq 1} L^p(\mathcal{F}_t)$, $0 \leq t \leq T$, suppose (X^P, Y^P, Z^P) is a solution to FBSDE (3.3) on $[t,T]$ with initial-terminal condition (ξ, P). Then, for $\hat{\eta}$ defined in (3.12), the triplet $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ given by

$$D^P_s := U_x\left(s, X^P_s + Y^P_s\right), \quad \tilde{Y}^P_s := Y^P_s, \quad \tilde{Z}^P_s := Z^P_s, \quad t \leq s \leq T,$$

is a solution to FBSDE (3.16) on $[t,T]$ with initial-terminal condition $(\hat{\eta}, P)$.

The proof follows easily by (3.10) and (3.19) using the duality relations (2.10), (2.13), (2.14) and (2.17).

Remark 3.13 *When both the primal and dual FBSDEs both admit a unique solution, there is a bijection between their solutions. We refer the reader to Section 4 for some popular cases of solvable FBSDEs, for which existence and uniqueness hold.*

3.4 Maturity independence of the value functions in forward performance criteria with random endowment

We set the ground for the upcoming notion of forward OCE by pointing out the fundamental maturity-independence property of the value functions $u^P(t,\xi;T)$ and $\tilde{u}^P(t,\eta;T)$. Recall the primal problem

(3.1), rewritten below for convenience, in a slightly different form and with some abuse of notation, $u^P(t,\xi;T) = u^{P_T}(t,\xi;T)$, where, for $0 \le t \le T \le T'$,

$$u^{P_T}(t,\xi;T') = \operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T']}} \mathbb{E}\left[U\left(T', \xi + \int_t^{T'} \pi_r\left(\theta_r dr + dW_r^1\right) + P_T\right)\middle|\mathcal{F}_t\right]. \tag{3.21}$$

We essentially parameterize the value function by the time T that the random endowment P_T arrives as well as by T' at which we set the forward performance criterion. Problem (3.21) can be then thought as a classical expected utility maximization problem in $[t,T']$ with terminal random utility $U(T',x)$ and random endowment P_T received at time T. We now claim that $u^{P_T}(t,\xi;T')$ is independent of T' in the sense that the horizon-invariance/maturity-independence property

$$u^{P_T}(t,\xi;T) = u^{P_T}(t,\xi;T')$$

holds. Note that the above property cannot hold for any $T' > T$ in the classical setting due to the pre-chosen, arbitrary but fixed, terminal horizon at which the (static) utility is allocated. Under the self-generation properties of $\tilde{U}$, we have the following general result.

Proposition 3.14 *Let $0 < T \le T'$, $P_T \in L^\infty(\mathcal{F}_T)$ and $0 \le t \le T$. Then, for $\eta \in L^{0,+}(\mathcal{F}_t)$,*

$$\tilde{u}^{P_T}(t,\eta;T) = \tilde{u}^{P_T}(t,\eta;T'). \tag{3.22}$$

If, furthermore, FBSDE (3.3) admits a solution $(X^{P_T}, Y^{P_T}, Z^{P_T})$ on $[t,T]$ with initial-terminal condition (ξ, P_T), $\xi \in \cap_{p\ge1} L^p(\mathcal{F}_t)$, satisfying $Z^{P_T,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$, then

$$u^{P_T}(t,\xi;T) = u^{P_T}(t,\xi;T'). \tag{3.23}$$

Reverting to $u^P(t,\xi;T)$, i.e., when $T = T'$, we see that the dependence on T is generated exclusively by the fact that this is the arrival time of the random endowment. This then motives us to introduce the following notion of *endowment maturity* which will be useful in building a universal framework across all times. Consider random endowments in the general space L defined by

$$L := \cup_{T\ge0} L^\infty(\mathcal{F}_T).$$

For any random endowment $P \in L$, we define its *maturity* by

$$T_P := \inf\{T \ge 0 : P \text{ is } \mathcal{F}_T\text{-measurable}\}. \tag{3.24}$$

In turn, the maturity independent value functions u^P and $\tilde{u}^P$ can be defined as

$$\begin{aligned} u^P(t,\xi) &:= u^P(t,\xi;T_P) \\ &= \operatorname*{esssup}_{\pi\in\mathcal{A}_{[t,T_P]}} \mathbb{E}\left[U\left(T_P, \xi + \int_t^{T_P} \pi_r\left(\theta_r dr + dW_r^1\right) + P\right)\middle|\mathcal{F}_t\right], \quad 0 \le t \le T_P,\ \xi \in \cap_{p\ge1} L^p(\mathcal{F}_t), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \tilde{u}^P(t,\eta) &:= \tilde{u}^P(t,\eta;T_P) \\ &= \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T_P]}} \mathbb{E}\left[\tilde{U}\left(T_P, \eta M^{t,q}_{T_P}\right) + \eta M^{t,q}_{T_P} P\middle|\mathcal{F}_t\right], \quad 0 \le t \le T_P,\ \eta \in L^{0,+}(\mathcal{F}_t). \end{aligned} \tag{3.26}$$

The above property essentially allows us to define the value functions $u^P(t,\xi)$ and $\tilde{u}^P(t,\eta)$ no matter what the maturity of the random endowment is. We stress that this does *not* imply that $u^P(t,\xi)$ and $\tilde{u}^P(t,\eta)$ do not depend on T_P, the time when the endowment arrives, an obviously wrong statement. Rather, it expresses how problems (3.25) and (3.26) can be well-defined for *all* times and *all* endowments using the flexible forward performance framework. Once more, note that in the classical framework this cannot be done as the entire optimization problem is tied down to the a priori chosen terminal horizon. We comment further on this in Section 7.

This "maturity-independent" construction was first developed in [53] and, later in [17] for forward entropic risk measures. Herein, it will play a fundamental role in the new notion of forward optimized certainty equivalent we introduce and develop in Section 5.

4 Solving the primal and dual FBSDEs with random endowment

We consider representative examples which give rise to solvable primal and dual FBSDEs.

4.1 Complete market

We start with the complete market case, where both the primal FBSDE (3.3) and the dual FBSDE (3.16) simplify considerably. In particular, the dual FBSDE (3.16) admits a closed-form solution. To this end, assume that the Brownian motion W is one-dimensional and that $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by W. Hence, the controlled wealth state equation (2.1) becomes

$$dX_t^\pi = \pi_t \left(\theta_t dt + dW_t\right), \quad t \geq 0, \; X_0^\pi = x \in \mathbb{R}.$$

Due to market completeness, there exists a *unique* state price density process given by

$$\mathcal{E}\left(-\int \theta dW\right)_t = \exp\left(-\tfrac{1}{2}\int_0^t |\theta_r|^2 \, dr - \int_0^t \theta_r dW_r\right), \quad t \geq 0,$$

where we recall that the market price of risk θ is a bounded $\mathbb{F}$-progressively measurable process.

For the primal FBSDE (3.3), since $W^2 \equiv 0$, the solution component $Z^{P,2}$ vanishes and thus (3.3) reduces to a classical BSDE, namely, for $0 \leq t \leq s \leq T$,

$$\begin{cases} X_s^P = \xi - \displaystyle\int_t^s \left(\frac{U_x\left(r, X_r^P + Y_r^P\right)\theta_r + \alpha_x\left(r, X_r^P + Y_r^P\right)}{U_{xx}\left(r, X_r^P + Y_r^P\right)} + Z_r^P \right)\left(\theta_r dr + dW_r\right), \\ Y_s^P = P - \displaystyle\int_s^T Z_r^P \theta_r dr - \int_s^T Z_r^P dW_r. \end{cases} \tag{4.1}$$

We easily see that it admits a unique solution (Y^P, Z^P) and that Y^P is represented, for $t \leq s \leq T$, as

$$Y_s^P = \mathbb{E}\left[\mathcal{E}\left(-\int \theta dW\right)_T^s P \middle| \mathcal{F}_s\right] = \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_s\right] \quad \text{with} \quad \mathcal{E}\left(-\int \theta dW\right)_T^s := \frac{\mathcal{E}(-\int \theta dW)_T}{\mathcal{E}(-\int \theta dW)_s}, \tag{4.2}$$

where $\mathbb{Q}^0$ is the unique equivalent martingale measure given by

$$\left.\frac{d\mathbb{Q}^0}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(-\int \theta dW\right)_T.$$

We also have that $Z^P \in \mathbb{L}^2_{BMO}[t,T]$, as it follows by a priori estimates for (4.1) (see, for example, [51, Section 7.2]). Hence, if X^P is the solution to the SDE in (4.1) with Y^P in (4.2) and $Z^P_s = \frac{d\langle Y^P, W\rangle_s}{ds}$, $t \le s \le T$, we obtain, by Theorem 3.1 (ii), that the control process

$$\pi^{*,P}_s := -\frac{U_x\left(s, X^P_s + Y^P_s\right)\theta_s + \alpha_x\left(s, X^P_s + Y^P_s\right)}{U_{xx}\left(s, X^P_s + Y^P_s\right)} - Z^P_s, \quad t \le s \le T,$$

is optimal for the primal problem (3.1). Furthermore, by Remark 3.6, we have that $q^{*,P} \equiv 0$ is the optimal control process for the dual problem (3.2). Note that although the forward equation and the backward equation in the primal FBSDE (3.3) become *decoupled*, as shown in (4.1), with (Y^P, Z^P) given explicitly, the solvability of the SDE in (4.1) is not straightforward, This is where the dual formulation proves helpful.

Since the market is complete, the dual problem is, as expected, much simpler. Indeed, for the dual FBSDE (3.16), the solution component $\tilde{Z}^{P,2}$ vanishes, and thus (3.16) reduces, for $0 \le t \le s \le T$, to

$$\begin{cases} D_s = \eta - \displaystyle\int_t^s D_r\theta_r dW_r, \\ \tilde{Y}^P_s = P - \displaystyle\int_s^T \tilde{Z}^P_r\theta_r dr - \int_s^T \tilde{Z}^P_r dW_r. \end{cases} \tag{4.3}$$

Note here D is independent of P. It is then straightforward to check that there exists a unique solution given by

$$D_s = \eta\mathcal{E}\left(-\int\theta dW\right)^t_s, \quad \tilde{Y}^P_s = Y^P_s = \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_s\right], \quad t \le s \le T,$$

and $\tilde{Z}^P = Z^P \in \mathbb{L}^2_{BMO}[t,T]$. Therefore, Theorem 3.8 (ii) yields that $q^{*,P} \equiv 0$ is the optimal control process for the dual problem (3.2).

We now solve the SDE in (4.1) using the dual formulation. For any $\xi \in \cap_{p\ge1}L^p(\mathcal{F}_t)$, Remark 3.11 yields that the control process

$$\pi^{*,P}_s := \tilde{U}_{zz}\left(s, \hat{\eta}\mathcal{E}\left(-\int\theta dW\right)^t_s\right)\hat{\eta}\mathcal{E}\left(-\int\theta dW\right)^t_s\theta_s - \tilde{\alpha}_z\left(s, \hat{\eta}\mathcal{E}\left(-\int\theta dW\right)^t_s\right) - \tilde{Z}^P_s, \quad 0 \le t \le s \le T,$$

with $\hat{\eta} := U_x(t, \xi + \mathbb{E}^{\mathbb{Q}^0}[P|\mathcal{F}_t])$, is optimal for the primal problem (3.1). Furthermore, we deduce, using Proposition 3.12, that the optimal wealth $X^{*,P}$ is given by

$$X^{*,P}_s = -\tilde{U}_z\left(s, U_x\left(t, \xi + \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_t\right]\right)\mathcal{E}\left(-\int\theta dW\right)^t_s\right) - \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_s\right], \quad 0 \le t \le s \le T,$$

solves the SDE in (4.1) with initial condition $X^{*,P}_t = \xi$.

Finally, we derive an explicit expression for the value function $\tilde{u}^P$ of the dual problem. Recall that, $\tilde{U}$ satisfies the self-generation property (3.13)-(3.14), namely, $\tilde{U}(t,\eta) = \mathbb{E}[\tilde{U}(T, D_T)|\mathcal{F}_t]$. Furthermore, we can easily check by (4.3) that $D\tilde{Y}^P$ is a true $\mathbb{F}$-martingale. Thus, we obtain an explicit form of the value function of the dual problem by Theorem 3.8 (ii):

$$\tilde{u}^P(t,\eta;T) = \mathbb{E}\left[\tilde{U}(T, D_T) + D_T P|\mathcal{F}_t\right] = \tilde{U}(t,\eta) + \eta\tilde{Y}^P_t = \tilde{U}(t,\eta) + \eta\mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_t\right]. \tag{4.4}$$

4.2 Stochastic factor model and exponential forward criteria

We analyze the popular exponential forward performance criteria in an incomplete market driven by a single stochastic factor. In the absence of random endowment, this class was studied in [40] where

an ergodic BSDE was developed. A by-product of the results below is recovery of the results in [40] and [17] as special cases.

To this end, we recall the single stochastic factor model, as in [40] and [17], which assumes that the discounted stock price process follows

$$\frac{dS_t}{S_t} = \mu\left(V_t\right) dt + \sigma\left(V_t\right) dW_t^1, \quad S_0 = S > 0,$$

where μ and $\sigma > 0$ are deterministic functions. The stochastic factor solves

$$dV_t = l\left(V_t\right)dt + \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right), \quad t \geq 0, \quad V_0 = v \in \mathbb{R}, \quad \rho \in [0,1]. \tag{4.5}$$

The controlled wealth state equation (2.1) then becomes

$$dX_t^\pi = \pi_t \left(\theta\left(V_t\right) ds + dW_t^1\right), \quad t \geq 0, \;\; X_0^\pi = x \in \mathbb{R}, \tag{4.6}$$

with $\theta\left(v\right) := \frac{\mu(v)}{\sigma(v)}$. We assume that, for $l, \theta : \mathbb{R} \to \mathbb{R}$,

(i) there exists a large enough constant $C > 0$ such that

$$\left(l\left(v\right) - l\left(v'\right)\right)\left(v - v'\right) \leq -C\left|v - v'\right|^2,$$

(ii) θ is uniformly bounded and Lipschitz continuous.

Due to the homothetic property of the exponential forward performance process and the single stochastic factor Markovian setup, SPDE (2.4) simplifies to an ergodic BSDE, derived in [40]. Indeed, consider the ergodic BSDE

$$dY_t^e = \left(\theta\left(V_t\right) Z_t^{e,1} + \frac{1}{2}\left|\theta\left(V_t\right)\right|^2 - \frac{1}{2}\left|Z_t^{e,2}\right|^2 + \lambda t\right) dt + \left(Z_t^e\right)^\top dW_t, \tag{4.7}$$

for which we easily deduce that it admits a unique Markovian solution $(Y_t^e, Z_t^e, \lambda) = (y^e(V_t), z^e(V_t), \lambda)$, $t \geq 0$, where $y^e : \mathbb{R} \to \mathbb{R}$ has at most linear growth and $z^e : \mathbb{R} \to \mathbb{R}^2$ is bounded. Then, the process given by

$$U\left(t,x\right) = -e^{-\gamma x + Y_t^e - \lambda t}, \quad (t,x) \in \mathbb{D}, \quad \gamma > 0, \tag{4.8}$$

is an exponential forward performance process. It provides a solution to SPDE (2.4) with drift and volatility choice as

$$\beta\left(t,x\right) = \frac{1}{2} U\left(t,x\right)\left|\theta\left(V_t\right) + Z_t^{e,1}\right|^2 \quad \text{and} \quad \alpha^i\left(t,x\right) = U\left(t,x\right) Z_t^{e,i}, \; i = 1,2. \tag{4.9}$$

Moreover, the optimal control process π^* for (2.3) is given by

$$\pi_t^* = \frac{\theta\left(V_t\right) + Z_t^{e,1}}{\gamma}, \quad t \geq 0.$$

It follows easily that U in (4.8) satisfies both conditions in Assumption 2.3. The ergodic BSDE (4.7) can be also used to characterize the convex conjugate of U, which takes the form

$$\tilde{U}\left(t,z\right) = -\frac{z}{\gamma} + \frac{z}{\gamma}\ln\frac{z}{\gamma} - \frac{z}{\gamma}\left(Y_t^e - \lambda t\right), \quad (t,z) \in \mathbb{D}^+. \tag{4.10}$$

In turn, the primal problem (3.1) and the dual problem (3.2) simplify, respectively, to, for $0 \leq t \leq T$,

$$u^P(t,\xi;T) = \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[-e^{-\gamma\left(\xi+\int_t^T \pi_r\left(\theta(V_r)dr+dW_r^1\right)+P\right)+Y_T^e-\lambda T}\middle|\mathcal{F}_t\right] = e^{-\gamma\xi}u^P(t,0;T), \tag{4.11}$$

and

$$\begin{aligned}\tilde{u}^P(t,\eta;T) = &\operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[-\frac{\eta M_T^{t,q}}{\gamma}+\frac{\eta M_T^{t,q}}{\gamma}\ln\frac{\eta M_T^{t,q}}{\gamma}-\frac{\eta M_T^{t,q}}{\gamma}\left(Y_T^e-\lambda T\right)+\eta M_T^{t,q}P\middle|\mathcal{F}_t\right] \\ =&\eta\tilde{u}^P(t,1;T)+\frac{1}{\gamma}\eta\ln\eta.\end{aligned} \tag{4.12}$$

For this class of exponential forward performance processes, the forward and backward equations in both the primal FBSDE (3.3) and the dual FBSDE (3.16) become *decoupled*. Specifically, (3.3) reduces, for $0 \leq t \leq s \leq T$, to

$$\begin{cases} X_s^P = \xi + \displaystyle\int_t^s \left(\frac{\theta(V_r)+Z_r^{e,1}}{\gamma} - Z_r^{P,1}\right)\left(\theta(V_r)\,dr + dW_r^1\right), \\ Y_s^P = P - \displaystyle\int_s^T \left(Z_r^{P,1}\theta(V_r) + \frac{\gamma}{2}\left|Z_r^{P,2}\right|^2 - Z_r^{e,2}Z_r^{P,2}\right)dr - \int_s^T \left(Z_r^P\right)^\top dW_r. \end{cases} \tag{4.13}$$

Recalling Theorem 3.2 in [17], there exists a unique solution (Y^P, Z^P) with Y^P being bounded and $Z^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1,2$. We note that, in the exponential preference case, unbounded random endowment P can also be accommodated, provided it satisfies $\mathbb{E}[e^{\gamma' P}] < \infty$ for some $\gamma' > \gamma$ (see [13, 31]).

By Theorem 3.1 (ii), the optimal control process for $u^P(t,\xi;T)$, $0 \leq t \leq T$, is given by

$$\pi_s^{*,P} = \frac{\theta(V_s)+Z_s^{e,1}}{\gamma} - Z_s^{P,1}, \quad t \leq s \leq T. \tag{4.14}$$

Moreover, using Remark 3.6, we also derive the optimal control process for the dual problem (3.2),

$$q_s^{*,P} = \gamma Z_s^{P,2} - Z_s^{e,2}, \quad t \leq s \leq T. \tag{4.15}$$

The dual FBSDE (3.16) reduces to the following decoupled FBSDE, for $0 \leq t \leq s \leq T$,

$$\begin{cases} D_s^P = \eta - \displaystyle\int_s^T D_r^P\left(\theta(V_r)\,dW_r^1 + \left(\gamma\tilde{Z}_r^{P,2} - Z_r^{e,2}\right)dW_r^2\right), \\ \tilde{Y}_s^P = P - \displaystyle\int_s^T \left(\tilde{Z}_r^{P,1}\theta(V_r) + \frac{\gamma}{2}\left|\tilde{Z}_r^{P,2}\right|^2 - Z_r^{e,2}\tilde{Z}_r^{P,2}\right)dr - \int_s^T \left(\tilde{Z}_r^P\right)^\top dW_r. \end{cases}$$

Comparing with the BSDE in (4.13), we notice that $(\tilde{Y}^P, \tilde{Z}^P) = (Y^P, Z^P)$ by the uniqueness of the solution. Then, from Theorem 3.8, the optimal control process $q^{*,P}$ for (3.2) is given by

$$q_s^{*,P} = \gamma\tilde{Z}_s^{P,2} - Z_s^{e,2}, \quad 0 \leq t \leq s \leq T,$$

which aligns with (4.15). Moreover, from Remark 3.11, the optimal control process $\pi^{*,P}$ for (3.1) is given by

$$\pi_s^{*,P} = \tfrac{\theta(V_s)+Z_s^{e,1}}{\gamma} - \tilde{Z}_s^{P,1}, \quad 0 \leq t \leq s \leq T,$$

which aligns with (4.14).

Finally, we derive the explicit forms of the value functions $u^P(t,\xi;T)$ and $\tilde{u}^P(t,\eta;T)$, $0 \le t \le T$, using the explicit representations of U in (4.8) and $\tilde{U}$ in (4.10). For this, we first, observe that $U(t,x) = -\frac{1}{\gamma}U_x(t,x)$. In turn, from Theorem 3.1 (ii), we have

$$u^P(t,\xi;T) = \mathbb{E}\left[U\left(T, X_T^P + P\right)\middle|\mathcal{F}_t\right] = -\frac{1}{\gamma}\mathbb{E}\left[U_x\left(T, X_T^P + P\right)\middle|\mathcal{F}_t\right].$$

It follows by Lemma 3.3 that $U_x(s, X_s^P + Y_s^P)$, $t \le s \le T$, is a true $\mathbb{F}$-martingale, which implies that

$$u^P(t,\xi;T) = -\frac{1}{\gamma}U_x\left(t, \xi + Y_t^P\right) = U\left(t, \xi + Y_t^P\right).$$

Next, noting that $\tilde{U}(t,z) = -\frac{z}{\gamma} + z\tilde{U}_z(t,z)$, we obtain by Theorem 3.10 (ii) that,

$$\begin{aligned}\tilde{u}^P(t,\eta;T) &= \mathbb{E}\left[\tilde{U}\left(T, D_T^P\right) + D_T^P P\right] = -\frac{\eta}{\gamma} + \mathbb{E}\left[D_T^P\left(\tilde{U}_z\left(T, D_T^P\right) + P\right)\right] \\ &= -\frac{\eta}{\gamma} + \eta\left(\tilde{U}(t,\eta) + \tilde{Y}_t^P\right) = \tilde{U}(t,\eta) + \eta\tilde{Y}_t^P. \end{aligned} \tag{4.16}$$

Comparing (4.16) to (4.4), we observe that the last term in (4.4), $\mathbb{E}^{\mathbb{Q}^0}[P|\mathcal{F}_t]$, is now replaced by $\tilde{Y}_t^P$.

4.3 Stochastic factor model and general forward performance criteria: decoupling field method

By employing the decoupling field method used in [26], we establish the solvability of the primal FBSDE (3.3) for general forward performance criteria in single stochastic factor models. The concepts of the decoupling field are provided in Appendix B, and we refer to [25, 26] for further background on this method.

Consider the stochastic factor model introduced in previous subsection. As shown in [45, Section 6.2.4], the forward performance process is associated with the (ill-posed) forward HJB equation

$$u_t - \frac{1}{2}\frac{|u_x\theta(v) + \rho u_{xv}|^2}{u_{xx}} + \frac{1}{2}u_{vv} + l(v)u_v = 0, \quad t \ge 0, \quad v, x \in \mathbb{R}, \tag{4.17}$$

with a suitable initial condition $u(0,v,x)$. If u solves (4.17), then the process $U(t,x) := u(t,V_t,x)$ satisfies the forward performance SPDE (2.4) with the choice of forward volatility

$$\alpha^1(t,x) = \rho u_v(t,V_t,x) \quad \text{and} \quad \alpha^2(t,x) = \sqrt{1-\rho^2}u_v(t,V_t,x).$$

The forward HJB equation (4.17) was investigated in [47, 46] where specific examples are provided.

For simplicity, we introduce the auxiliary notation

$$\varphi := \frac{u_x}{u_{xx}}, \ \phi^1 := \frac{u_{xxx}}{u_{xx}}, \ \phi^2 := \frac{u_{xxv}}{u_{xx}} \quad \text{and} \quad \psi := \frac{u_x\theta(v) + \rho u_{xv}}{u_{xx}}.$$

To align with Assumption 2.3, we introduce the following conditions on the function $u(t,v,x)$.

Assumption 4.1 *(i) $u(t,v,x) \in C^{1,2,4}([0,\infty)\times\mathbb{R}\times\mathbb{R})$ and, for each $t \ge 0$ and $v \in \mathbb{R}$, the mapping $x \mapsto u(t,v,x)$ is strictly increasing and strictly concave.*

(ii) There exist positive constants C_l, C_u and C_α such that $0 < C_l \le -\varphi \le C_u$ and $|\frac{u_{xv}}{u_{xx}}| \le C_\alpha$.

Let $T > 0$ be arbitrary but fixed. We consider a random endowment of the form $P(V_T, X_T^\pi)$, where $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is deterministic. In this case, for $0 \le s \le T$, the primal FBSDE (3.3) takes the form

$$\begin{cases} X_s^P = x - \int_0^s \left(\psi\left(r, V_r, X_r^P + Y_r^P\right) + Z_r^{P,1}\right)\left(\theta\left(V_r\right) dr + dW_r^1\right), \\ Y_s^P = P\left(V_T, X_T^P\right) - \int_s^T \left(Z_r^P\right)^\top dW_r \\ \qquad + \int_s^T \left(-Z_r^{P,1}\theta\left(V_r\right) + \frac{1}{2}\phi^1\left(r, V_r, X_r^P + Y_r^P\right)\left|Z_r^{P,2}\right|^2 + \sqrt{1-\rho^2}\phi^2\left(r, V_r, X_r^P + Y_r^P\right) Z_r^{P,2}\right) dr, \end{cases} \tag{4.18}$$

where the superscript P represents the function appearing the terminal condition. We introduce the following additional assumptions.

Assumption 4.2 *(i) The functions $\phi^i, \phi_x^i, \phi_v^i$, $i = 1, 2$, and $\frac{u_{xvv}}{u_{xx}}$ are uniformly bounded, and $\phi_x^1 < 0$.*

(ii) P is uniformly bounded and Lipschitz continuous in both components with $L_{P,x} < 1$, where $L_{P,x}$ denotes the Lipschitz constant of $P(v, x)$ with respect to the second component (see (B.3)).

(iii) θ and l are uniformly bounded, differentiable in v, and with bounded derivatives.

Remark 4.3 *We provide a simple example of u satisfying the above conditions, based on [26, Lemma 1.2]. Let $\theta = 0$ and consider $u(t, v, x)$ in the form $u\left(t, v, x\right) = \tilde{u}\left(x\right) + F\left(t, v\right)$ with*

$$\tilde{u}\left(x\right) = -\int_x^\infty \int_y^\infty e^{-\kappa(z)} dz dy,$$

where $\kappa \in C^2(\mathbb{R})$ satisfies $0 < \delta_l \le \kappa', \kappa'' \le \delta_u$ for some constants δ_l, δ_u. Suppose that F solves PDE

$$F_t + \frac{1}{2}F_{vv} + l\left(v\right)F_v = 0,$$

where $\frac{F_v}{F}, \frac{F_{vv}}{F}$ are uniformly bounded. Then, by ([26, Lemma 1.2]), it follows that all the above assumptions on u are satisfied. The solvability of the PDE for F has been studied in detail in [46] using Widder's theorem.

We consider following cases:
Case 1. $\phi_x^2 = 0$ (ϕ_x^2 denotes the partial derivative of ϕ^2 with respect to x),
Case 2. $\phi_x^2 \neq 0$, $\phi_x^1 \le -\delta$ for some $\delta > 0$. Furthermore, $T < \frac{1}{K} \ln \frac{2}{1+L_{P,x}}$, where constant K satisfies $\frac{1-\rho^2}{2}|\phi_x^2|^2 \le -K\phi_x^1$.

Theorem 4.4 *Suppose that Assumptions 4.1 and 4.2 hold. Then, under either Case 1 or Case 2, the primal FBSDE (4.18) admits a unique solution (X^P, Y^P, Z^P) with Z^P being uniformly bounded.*

Using the above result, we may, in turn, obtain the existence and uniqueness of the associated dual FBSDE in the Markovian form, following the results in Proposition 3.12.

5 Forward optimized certainty equivalent (Forward OCE)

The concept of optimized certainty equivalent (OCE) was first introduced by Ben-Tal and Teboulle in [8] and yields a valuation criterion rooted in expected utility theory. It is a *static* criterion, defined through the optimization problem,

$$S(P) := \sup_{r\in\mathbb{R}} \left(\mathbb{E}\left[u\left(P-r\right)\right]+r\right), \tag{5.1}$$

where u is the utility of the investor and P is a claim/random endowment. Essentially, the OCE represents the optimal split of funds if the investor has the option to choose to spend a portion r of the random endowment, and receive the present value of this r plus the expected utility of $P-r$.

Note that the optimization in (5.1) balances the *expected utility satisfaction from the uncertain payoff*, $P-r$, with the *certainty of receiving* r. Since both terms are considered unitless, their sum represents the combination of the expected utility from $P-r$ units of wealth and the present value of r units of wealth. This interpretation is consistent with the dual representation in (5.2), where the first term corresponds to the expected value of P, and the second term represents the penalty function, both of which are also unitless.

One of the key results established by Ben-Tal and Teboulle in their followup work [9] is the dual representation for OCE, which we recall next. Let $\tilde{u}$ be the convex conjugate of u, $\tilde{u}(z) := \sup_{x\in\mathbb{R}}(u(x)-xz)$. Then, the OCE of P admits the dual representation,

$$S(P) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}^{\mathbb{Q}}[P] + I_{\tilde{u}}(\mathbb{Q},\mathbb{P})\right), \tag{5.2}$$

where $\mathcal{Q}$ is the set of probability measures equivalent to $\mathbb{P}$, and $I_{\tilde{u}}(\mathbb{Q},\mathbb{P})$ is the associated penalty function, given by

$$I_{\tilde{u}}(\mathbb{Q},\mathbb{P}) := \mathbb{E}\left[\tilde{u}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right].$$

In the context of risk measures, the quantity $\rho(P) := -S(P)$ turns out to be a convex risk measure for P and admits the dual representation

$$\rho(P) = \sup_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}^{\mathbb{Q}}[-P] - I_{\tilde{u}}(\mathbb{Q},\mathbb{P})\right),$$

also known as a divergence risk measure (see [15] amd [5]). See also [7, 32] for its numerical computational aspects.

A challenging problem, which we attempt to address herein, is how to produce a genuinely dynamic extension of OCE in (5.1). Specifically, we are interested in building a valuation framework that is viable for all claims and all maturities and, furthermore, yields time-consistent OCE.

We face several difficulties here, both conceptual and technical. Firstly, it is not clear how the static valuation rule (5.1) should be modified across times. The obvious choice to merely replace the utility u by a utility, say U, at a given future horizon T, will not work as it will generate, to say the least, similar difficulties with the ones we face in the random endowment and the indifference valuation settings. Among others, such a framework will neither allow for valuation beyond T, nor will allow for adaptive model revision. One may simply propose to work in an infinite horizon to avoid horizon limitations. However, this will require to put very stringent constraints on a pre-chosen market model in $[0,\infty)$ to maintain the consistency as explained in the introduction.

As mentioned in the Introduction, efforts have been made to build dynamic extensions of OCE. Recently, Backhoff-Veraguas et al. (see [10] and [11]) proposed a dynamic version working with the convex dual and, in addition, introduced an additional variable to guarantee time-consistency. This approach can be compared with the convex dual representation of dynamic risk measures in [20]. However, the valuation machanism remains bound to the fixed horizon T as we mentioned earlier.

5.1 Interpretation of OCE via convex dual of utility maximization with random endowment

Before we introduce the forward OCE, we provide a key observation which, to the best of our knowledge, has not been employed so far. It interprets the existing static OCE *via the value function of the convex dual of a utility maximization problem with random endowment.*

To this end, we observe that the OCE of a random endowment $P \in L^\infty(\mathcal{F}_T)$ can be rewritten as

$$S(P) = \sup_{x\in\mathbb{R}} \left(\mathbb{E}\left[u\left(x+P\right)\right] - x\right) \quad \text{with} \quad x = -r. \tag{5.3}$$

This provides a new interpretation of OCE: the value $S(P)$ represents the optimal allocation of wealth when the investor decides to deposit x into a savings account at present. Consequently, at maturity, she will receive an additional amount x along with the endowment P, and the corresponding value is the expected utility of $x + P$.

When P is negative, this interpretation adjusts slightly. In this case, P can be seen as a liability or loss. The investor is effectively considering how much to deposit into the savings account to offset this negative endowment. The OCE still indicates the optimal allocation of wealth, but it now reflects the investor's strategy to minimize the impact of the negative endowment by determining how much additional capital x is needed to achieve a level of utility that compensates for the adverse effects of P.

Let us then consider the utility maximization problem with the random endowment P,

$$v^P(x) = \sup_{X_T\in L^0(\mathcal{F}_T)} \mathbb{E}\left[u\left(X_T+P\right)\right],$$

under the constraint $\mathbb{E}^{\mathbb{Q}}[X_T] \le x$ for all equivalent martingale measures $\mathbb{Q}$, and X_T being the terminal wealth generated by trading strategies with initial wealth $x \in \mathbb{R}$. If we assume that, for any $X_T \in L^0(\mathcal{F}_T)$,

$$X_T \text{ is independent of } P \text{ and } \mathbb{E}[X_T] = x, \text{ i.e., } \mathbb{P} \text{ itself is a martingale measure,} \tag{5.4}$$

then, by Jensen's inequality, we have

$$\mathbb{E}\left[u\left(X_T+P\right)\right] = \mathbb{E}\left[\mathbb{E}\left[u\left(X_T+p\right)\right]\big|_{p=P}\right] \le \mathbb{E}\left[u\left(\mathbb{E}\left[X_T+p\right]\right)\big|_{p=P}\right] = \mathbb{E}\left[u\left(x+P\right)\right].$$

In other words, we obtain that the optimal wealth is given by $X_T^* = x$, and the value function is given by

$$v^P(x) = \mathbb{E}\left[u\left(x+P\right)\right].$$

Therefore, $S(P)$ can be expressed as the convex conjugate of v^P at value 1,

$$S(P) = \sup_{x\in\mathbb{R}} \left(v^P(x) - x\cdot 1\right) =: \left(v^P\right)^*(1), \tag{5.5}$$

where $(v^P)^*$ represents the convex conjugate of the value function v^P. In other words, $S(P)$ can be seen as the convex conjugate, evaluated at value 1, of a utility maximization problem involving P as the random endowment in an auxiliary financial market whose driving noise is orthogonal to this random endowment. In this setting, the optimal policy involves holding only the riskless asset, so that $X_T^* = x$.

5.2 Forward OCE

We are now ready to introduce the novel concept of forward OCE, which, to the best of our knowledge, is new. The definition makes full use of the key observation above for the representation (5.3) and (5.5) of the OCE and employs the suitable primal and dual forward counterparts of the static utility in (5.1) and (5.2).

In the following definition of forward OCE, we will make use of an auxiliary variable η, which can be naturally interpreted as a deflator, serving to convert nominal values of current wealth into real values. A similar concept has been explored in [10], albeit without the context of utility maximization. Denote the set of deflators at time $t \geq 0$ by

$$\mathcal{H}_t := \{\eta \in L^{0,+}(\mathcal{F}_t),\, \mathbb{E}[\eta] = 1\}.$$

Definition 5.1 *Let $T > 0$ be arbitrary and fixed.*

(i) For $0 \leq t \leq T$ and $\eta \in \mathcal{H}_t$, the forward OCE of $P \in L^\infty(\mathcal{F}_T)$ at time t is defined by

$$\begin{aligned} F(t, P; \eta, T) &:= \operatorname*{esssup}_{\xi \in \cap_{p \geq 1} L^p(\mathcal{F}_t)} \left(u^P(t, \xi; T) - \xi\eta\right) \\ &= \operatorname*{esssup}_{\xi \in \cap_{p \geq 1} L^p(\mathcal{F}_t)} \left(\operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_r \left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t \right] - \xi\eta \right). \end{aligned}$$

(ii) In particular, at $t = 0$,

$$F(0, P; 1, T) = \sup_{x \in \mathbb{R}} \left(\sup_{\pi \in \mathcal{A}_{[0,T]}} \mathbb{E}\left[U\left(T, x + \int_0^T \pi_r \left(\theta_r dr + dW_r^1\right) + P\right)\right] - x \right).$$

The forward OCE represents the result of an optimal dynamic allocation of funds. An investor faces the choice of saving a portion of their current wealth ξ at time t in addition to the random endowment P at time T. The consequence of this decision is a sacrifice in the real value due to reduced spending, which amounts to $\xi\eta$. Thus, the forward OCE at time t mirrors the optimal balance between maximizing the forward performance process from the saving amount of ξ at time t alongside the random endowment P at time T and reducing spending due to this saving choice. In the newly defined framework for forward OCE, we introduce an associated auxiliary financial market where the underlying asset can be utilized to partially hedge the risk associated with P. In contrast, in the classical OCE framework, the driving noise in this auxiliary market is assumed to be orthogonal to P.

Similar to the classical OCE, we also obtain the following dual representation for the forward OCE. The proof follows easily from Theorems 3.8 (ii) and 3.10, and is, then, omitted.

Theorem 5.2 *Let $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and $\eta \in \mathcal{H}_t$, $0 \leq t \leq T$. Suppose $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ is a solution to the dual FBSDE (3.16) on $[t, T]$ with initial-terminal condition (η, P) satisfying $D^P > 0$ and $\tilde{Z}^{P,i} \in \mathbb{L}^2_{BMO}[t, T]$, $i = 1, 2$. Then, the forward OCE of P at time t admits the dual representation*

$$F(t, P; \eta, T) = \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T, \eta M_T^{t,q}\right) + \eta M_T^{t,q} P \middle| \mathcal{F}_t\right].$$

The corresponding optimal control $q^{*,P}$ is given by

$$q_s^{*,P} := \frac{\tilde{Z}_s^{P,2} + \tilde{\alpha}_z^2\left(s, D_s^P\right)}{D_s^P \tilde{U}_{zz}\left(s, D_s^P\right)}, \quad t \leq s \leq T.$$

In light of Proposition 3.14, we obtain directly the *maturity-independence* property of the forward OCE under the self-generation property of $\tilde{U}$.

Proposition 5.3 *Let $0 < T \leq T'$, $P \in L^{\infty}(\mathcal{F}_T)$ and $\eta \in \mathcal{H}_t$, $0 \leq t \leq T$. Then*

$$F(t, P; \eta, T) = F(t, P; \eta, T').$$

5.2.1 Connection with the classical (static) OCE

We discuss how the forward OCE relates to the OCE. Firstly, note that the two assumptions in (5.4) play a crucial role in transitioning from the perspective of utility maximization to the classical OCE definition. These assumptions involve the orthogonality between the market noise and the random endowment, as well as the exclusive use of the riskless asset as the optimal policy.

In the context of the forward performance framework, we need to consider the counterparts of (5.4), which ultimately lead to the recovery of classical OCE. Specifically, we assume the following conditions in the forward performance framework:

(i) the market price of risk $\theta \equiv 0$,

(ii) the random endowment $P \in L^{\infty}(\mathcal{F}_T^2)$, where $\mathbb{F}^2 = (\mathcal{F}_t^2)_{t \geq 0}$ is the natural filtration of W^2,

(iii) the volatility component $\alpha^1 \equiv 0$ in the forward performance SPDE (2.4).

Under the above assumptions, the dynamics of the forward performance process U reduce to $dU(t,x) = \alpha^2(t,x)\, dW_t^2$. Additionally, the value function in (3.1) takes the form:

$$u^P(t, \xi; T) = \operatorname*{esssup}_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_r dW_r^1 + P \right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T, \quad \xi \in \cap_{p \geq 1} L^p(\mathcal{F}_t).$$

From Theorem 3.1 (ii), we deduce that if $(Y^P, Z^{P,2})$ is a solution to the BSDE

$$Y_s^P = P + \int_s^T \left(\frac{1}{2} \frac{U_{xxx}(r, \xi + Y_r^P)}{U_{xx}(r, \xi + Y_r^P)} \left| Z_r^{P,2} \right|^2 + \frac{\alpha_{xx}^2(r, \xi + Y_r^P)}{U_{xx}(r, \xi + Y_r^P)} Z_r^{P,2} \right) dr - \int_s^T Z_r^{P,2} dW_r^2, \quad 0 \leq t \leq s \leq T,$$

namely, $(\xi, Y^P, (0, Z^{P,2}))$ solves FBSDE (3.3) under assumptions (i)-(iii), then the control process $\pi^{*,P} = 0$ is optimal for $u^P(t, \xi; T)$. Thus, $u^P(t, \xi; T) = \mathbb{E}[U(T, \xi + P) | \mathcal{F}_t]$, and the forward OCE of P at time t is given by

$$F(t, P; \eta, T) = \operatorname*{esssup}_{\xi \in \cap_{p \geq 1} L^p(\mathcal{F}_t)} \left(\mathbb{E}[U(T, \xi + P) | \mathcal{F}_t] - \xi \eta \right).$$

In particular, at $t = 0$, we have

$$F(0, P; 1, T) = \sup_{x \in \mathbb{R}} \left(\mathbb{E}[U(T, x + P)] - x \right) = \sup_{r \in \mathbb{R}} \left(\mathbb{E}[U(T, P - r)] + r \right).$$

Thus, if, furthermore, the volatility component $\alpha^2 \equiv 0$, then $U(t,x) \equiv u(x)$ and the forward OCE coincides with the classical OCE, $F(0, P; 1, T) = S(P)$.

Under assumptions (i)-(iii), we have a more explicit form of the dual representation,

$$F(t, P; \eta, T) = \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T, \eta \mathcal{E}\left(-\int q dW^2 \right)_T^t \right) + \eta \mathcal{E}\left(-\int q dW^2 \right)_T^t P \middle| \mathcal{F}_t \right],$$

where

$$\mathcal{E}\left(-\int qdW^2\right)_T^t := \frac{\mathcal{E}\left(-\int qdW^2\right)_T}{\mathcal{E}\left(-\int qdW^2\right)_t} \text{ with } \mathcal{E}\left(-\int qdW^2\right)_t := \exp\left(-\tfrac{1}{2}\int_0^t |q_r|^2\, dr - \int_0^t q_r dW_r^2\right).$$

In particular, at $t = 0$, we have

$$F\left(0, P; 1, T\right) = \inf_{q\in\mathcal{Q}_{[0,T]}} \mathbb{E}\left[\tilde{U}\left(T, \mathcal{E}\left(-\int qdW^2\right)_T\right) + \mathcal{E}\left(-\int qdW^2\right)_T P\right].$$

5.2.2 Properties of the forward OCE

We conclude this section by presenting several key properties of the forward OCE. The proofs follow easily from the properties of the forward performance process. To simplify and unify the presentation, we work with a slight modification of the forward OCE through normalization. The *normalized* forward OCE, denoted by $\tilde{F}(t, P; \eta, T)$, is defined as

$$\tilde{F}\left(t, P; \eta, T\right) := F\left(t, P; \eta, T\right) - F\left(t, 0; \eta, T\right), \tag{5.6}$$

so that $\tilde{F}\left(t, 0; \eta, T\right) = 0$.

Proposition 5.4 *Let $T > 0$ be arbitrary and fixed, and $\eta \in \mathcal{H}_t$, $0 \le t \le T$. Then the (normalized) forward OCE has the following properties:*

(i) Monotonicity: for $P^i \in L^\infty(\mathcal{F}_T)$, $i = 1, 2$ and $P^1 \ge P^2$,

$$F(t, P^1; \eta, T) \ge F(t, P^2; \eta, T).$$

(ii) Cash invariance: for $P \in L^\infty(\mathcal{F}_T)$ and $c \in L^\infty(\mathcal{F}_t)$,

$$F(t, P + c; \eta, T) = F(t, P; \eta, T) + \eta c.$$

(iii) Concavity: for $P^i \in L^\infty(\mathcal{F}_T)$, $i = 1, 2$, and for $\lambda \in L^\infty(\mathcal{F}_t)$ valued in $(0, 1)$,

$$\lambda F\left(t, P^1; \eta, T\right) + (1 - \lambda) F\left(t, P^2; \eta, T\right) \le F\left(t, P^\lambda; \eta, T\right),$$

where $P^\lambda := \lambda P^1 + (1 - \lambda)P^2$.

(iv) Replication invariance: for $P \in L^\infty(\mathcal{F}_T)$ and for any $\pi \in \mathcal{A}_{[t,T]}$,

$$F\left(t, P + \int_t^T \pi_r\left(\theta_r dr + dW_r^1\right); \eta, T\right) = F\left(t, P; \eta, T\right).$$

(v) Positivity: for nonnegative $P \in L^\infty(\mathcal{F}_T)$, $\tilde{F}(t, P; \eta, T) \ge 0$.

(vi) Constancy: for $c \in L^\infty(\mathcal{F}_t)$,

$$\tilde{F}(t, c; \eta, T) = \eta c.$$

Remark 5.5 *The normalized forward OCE $\tilde{F}$ defined in (5.6) satisfies Proposition 5.3 and Proposition 5.4 (i)-(iv).*

Remark 5.6 *In the case of complete markets following the discussion at the end of Section 4.1, we obtain by (4.4) that*

$$\tilde{F}\left(t, P; \eta, T\right) = \tilde{u}^P\left(t, \eta; T\right) - \tilde{u}^0\left(t, \eta; T\right) = \eta \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_t\right],$$

where $\mathbb{Q}^0$ is the unique equivalent martingale measure. Particularly, when $\eta = 1$, the normalized forward OCE coincides with the complete market price, given by $\tilde{F}\left(t, P; 1, T\right) = \mathbb{E}^{\mathbb{Q}^0}\left[P|\mathcal{F}_t\right]$.

5.3 Forward OCE under exponential forward performance criteria

Building on the ergodic BSDE representation for the exponential forward performance process introduced in [40], Chong et al. [17] extended this framework to study exponential forward utility maximization with random endowment and its application to forward entropic risk measures. In this section, based on the results in Sections 4.2 and 5.2, we focus on the *normalized* forward OCE defined in (5.6), with the exponential forward performance process given by (4.8). We further demonstrate its direct connection to the negative of the forward entropic risk measure introduced in [17].

For $T > 0$, $P \in L^\infty(\mathcal{F}_T)$ and stochastic deflator $\eta \in \mathcal{H}_t$, $0 \le t \le T$, the definition of the normalized forward OCE in (5.6) and Theorem 5.2 yield that

$$\tilde{F}(t, P; \eta, T) = \tilde{u}^P(t, \eta; T) - \tilde{u}^0(t, \eta; T) = \tilde{u}^P(t, \eta; T) - \tilde{U}(t, \eta),$$

where we used that $\tilde{u}^0(t, \eta; T) = \tilde{U}(t, \eta)$, as it follows from Proposition 3.7.

On one hand, as shown in (4.16), the normalized exponenital forward OCE in terms of the solution component Y^P of FBSDE (4.13) is given by

$$\tilde{F}(t, P; \eta, T) = \eta Y_t^P. \tag{5.7}$$

By choosing a deterministic deflator $\eta = 1$, we observe that (5.7) for the normalized exponential forward OCE corresponds to the negative of the forward entropic risk measure,

$$\tilde{F}(t, P; 1, T) = Y_t^P = -\rho(t, P; T),$$

where the last inequality follows from [17, Theorem 3.2], and ρ is defined as the utility indifference price of P, namely,

$$U(t, x) = u^P(t, x + \rho(t, P; T)),$$

as stated in [17, Definition 3.1]. This result further implies that, under the exponential case, the forward OCE coincides with the forward indifference price. Notably, this is the only case in utility indifference pricing that satisfies the cash invariance property.

On the other hand, using the dual representation in (4.12) yields

$$\begin{aligned}\tilde{F}(t, P; \eta, T) &= \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}\left[\eta M_T^{t,q}\left(\frac{\ln M_T^{t,q}}{\gamma} - \frac{Y_T^e - \lambda T}{\gamma} + P\right)\middle| \mathcal{F}_t\right] + \frac{\eta}{\gamma}(Y_t^e - \lambda t) \\ &= \eta \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}^{\mathbb{Q}^q}\left[\frac{\ln M_T^{t,q}}{\gamma} - \frac{Y_T^e - \lambda T}{\gamma} + P\middle| \mathcal{F}_t\right] + \frac{\eta}{\gamma}(Y_t^e - \lambda t),\end{aligned} \tag{5.8}$$

where $\frac{d\mathbb{Q}^q}{d\mathbb{P}}|_{\mathcal{F}_T} = M_T^{t,q}$. Note that

$$\ln M_T^{t,q} = \frac{1}{2}\int_t^T \left(|\theta(V_r)|^2 + |q_r|^2\right) dr - \int_t^T \theta(V_r)\, dW_r^{1,\theta} - \int_t^T q_r dW_r^{2,q},$$

with $(dW_r^{1,\theta}, dW_r^{2,q}) := (dW_r^1 + \theta(V_r)\, dr, dW_r^2 + q_r dr)$, and, in accordance with the ergodic BSDE (4.7), we have

$$Y_T^e - \lambda T = Y_t^e - \lambda t + \int_t^T \left(\frac{1}{2}|\theta(V_r)|^2 - \frac{1}{2}\left|Z_r^{e,2}\right|^2 - Z_r^{e,2} q_r\right) dr + \int_t^T Z_r^{e,1} dW_r^{1,\theta} + \int_t^T Z_r^{e,2} dW_r^{2,q}.$$

As a result, the dual representation (5.8) can be expressed more explicitly as

$$\tilde{F}(t, P; \eta, T) = \eta \operatorname*{essinf}_{q \in \mathcal{Q}_{[t,T]}} \mathbb{E}^{\mathbb{Q}^q}\left[P + \frac{1}{2\gamma}\int_t^T \left|Z_r^{e,2} + q_r\right|^2 dr\middle| \mathcal{F}_t\right], \tag{5.9}$$

which is the negative of the dual representation for the forward entropic risk measure established in [17, Theorem 3.5] for $\eta = 1$.

Remark 5.7 *In [3], the author also studied the forward indifference pricing under the exponential forward performance process investigated in [52]. Based on the same representation of the exponential forward performance process, the corresponding forward OCE can be derived directly. Notably, this forward OCE coincides with the explicit expression provided in [3, (3.4)], which represents the exponential forward indifference price.*

6 Backward SPDEs for forward performance criteria with random endowment

To conclude, we briefly discuss how to tackle the primal problem (3.1) and the dual problem (3.2) by directly characterizing their value functions in terms of the solutions of backward SPDEs. We also explore how to use these backward SPDE solutions to construct the solutions of the primal FBSDE (3.3) and dual FBSDE (3.16). Since the solvability of the corresponding backward SPDEs is an open problem, the discussion that follows is formal. Nevertheless, we find that building this connection is useful.

6.1 Formal derivation of backward SPDEs

For a given $T > 0$ and random endowment $P \in L^\infty(\mathcal{F}_T)$, we recall the value function of the primal problem (3.1),

$$u^P(t,x;T) = \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}\left[U\left(T, X_T^\pi + P\right) \middle| \mathcal{F}_t, X_t^\pi = x\right], \quad 0 \le t \le T, \quad x \in \mathbb{R}. \tag{6.1}$$

Assume that

$$du^P(t,x;T) = b^P(t,x;T)\,dt + \left(a^P(t,x;T)\right)^\top dW_t,$$

and the Itô-Ventzel formula yields

$$\begin{aligned} du^P(t,X_t^\pi;T) = &\left(b^P(t,X_t^\pi;T) + \left(u_x^P(t,X_t^\pi;T)\,\theta_t + a_x^{P,1}(t,X_t^\pi;T)\right)\pi_t + \frac{1}{2}u_{xx}^P(t,X_t^\pi;T)\,|\pi_t|^2\right)dt \\ &+ \left(a^{P,1}(t,X_t^\pi;T) + u_x^P(t,X_t^\pi;T)\,\pi_t\right)dW_t^1 + a^{P,2}(t,X_t^\pi;T)\,dW_t^2. \end{aligned}$$

Following the standard martingale optimality condition (as described, for example, in [40]), we select the drift $b^P(t,x;T)$ to ensure that $u^P(t,X_t^\pi;T)$ is a supermartingale for any $\pi \in \mathcal{A}_{[t,T]}$ and a martingale for an optimal control process. This yields that the drift must of the form

$$\begin{aligned} b^P(t,x;T) &= -\sup_{\pi \in \mathbb{R}^n}\left(\frac{1}{2}u_{xx}^P(t,x;T)\,|\pi|^2 + \left(u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)\right)\pi\right) \\ &= \frac{1}{2}\frac{\left|u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)\right|^2}{u_{xx}^P(t,x;T)}, \end{aligned}$$

with the above optimum given by

$$\pi^{*,P}(t,x;T) = -\frac{u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)}{u_{xx}^P(t,x;T)}.$$

Thus, u^P is expected to satisfy the backward SPDE

$$du^P(t,x;T) = \frac{1}{2}\frac{\left|u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)\right|^2}{u_{xx}^P(t,x;T)}dt + \left(a^P(t,x;T)\right)^\top dW_t \tag{6.2}$$

with terminal condition $u^P(T,x;T) = U(T,x+P)$. Note that, different from the SPDE characterization (2.4) of the forward performance U where the volatility is a model input, (6.2) is a *backward* SPDE with the volatility a^P being part of the solution.

Differentiating (6.2) in x yields

$$du_x^P(t,x;T) = b_x^P(t,x;T)\,dt + \left(a_x^P(t,x;T)\right)^\top dW_t, \quad 0 \le t \le T, \quad x \in \mathbb{R}, \tag{6.3}$$

with terminal condition $u_x^P(T,x;T) = U_x(T,x+P)$, where

$$\begin{aligned} b_x^P(t,x;T) = &\left(u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)\right)\theta_t + \frac{\left(u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)\right)a_{xx}^{P,1}(t,x;T)}{u_{xx}^P(t,x;T)} \\ &- \frac{1}{2}\frac{\left|u_x^P(t,x;T)\,\theta_t + a_x^{P,1}(t,x;T)\right|^2 u_{xxx}^P(t,x;T)}{\left|u_{xx}^P(t,x;T)\right|^2}. \end{aligned}$$

Next, we define the convex conjugate of u^P,

$$\tilde{u}^P(t,z;T) := \sup_{x\in\mathbb{R}}\left(u^P(t,x;T) - xz\right), \quad 0 \le t \le T, \quad z > 0.$$

We know that u^P and $\tilde{u}^P$ satisfy the analogous relations as U and $\tilde{U}$ do. As in Section 2.2, we have the dual relation

$$u_x^P\left(t, -\tilde{u}_z^P(t,z;T);T\right) = z, \tag{6.4}$$

from which we can also deduce the dynamics of $\tilde{u}_z^P(t,z;T)$, similarly to $\tilde{U}_z(t,z)$ in (2.15).

6.2 Backward SPDEs and FBSDEs

We will use the backward SPDE (6.3) satisfied by u_x^P to construct a solution of the primal FBSDE (3.3) and an optimal control for the primal problem (3.1) as well as the corresponding quantities for the dual problem (3.2).

Proposition 6.1 *Let w be a solution to the backward SPDE*

$$\begin{cases} dw(t,x) = h(t,x)\,dt + d^\top(t,x)\,dW_t, \quad 0 \le t \le T, \quad x \in \mathbb{R}, \\ w(T,x) = U_x(T,x+P), \end{cases} \tag{6.5}$$

where

$$h(t,x) = \left(w(t,x)\,\theta_t + d^1(t,x)\right)\left(\theta_t + \frac{d_x^1(t,x)}{w_x(t,x)}\right) - \frac{1}{2}\frac{\left|w(t,x)\,\theta_t + d^1(t,x)\right|^2 w_{xx}(t,x)}{\left|w_x(t,x)\right|^2},$$

and assume that $w(t,x)$ is a progressive $\mathcal{K}_{loc}^{1,\delta}$-semimartingale random field and $w \in \mathcal{C}^2$. Then, for $0 \le t \le T$, the following assertions hold.

(i) For $\xi \in \cap_{p\geq 1} L^p(\mathcal{F}_t)$, the triplet (X^P, Y^P, Z^P) given by, for $t \leq s \leq T$,

$$X_s^P := \xi - \int_t^s \frac{w\left(r, X_r^P\right)\theta_r + d^1\left(r, X_r^P\right)}{w_x\left(r, X_r^P\right)}\left(\theta_r dr + dW_r^1\right), \tag{6.6}$$

$$Y_s^P := -\tilde{U}_z\left(s, w\left(s, X_s^P\right)\right) - X_s^P,$$

and

$$\begin{aligned} Z_s^{P,1} &:= \frac{w\left(s, X_s^P\right)\theta_s + d^1\left(s, X_s^P\right)}{w_x\left(s, X_s^P\right)} - \tilde{\alpha}_z^1\left(s, w\left(s, X_s^P\right)\right) + \tilde{U}_{zz}\left(s, w\left(s, X_s^P\right)\right) w\left(s, X_s^P\right)\theta_s, \\ Z_s^{P,2} &:= -\tilde{\alpha}_z^2\left(s, w\left(s, X_s^P\right)\right) - d^2\left(s, X_s^P\right)\tilde{U}_{zz}\left(s, w\left(s, X_s^P\right)\right), \end{aligned} \tag{6.7}$$

satisfies FBSDE (3.3) on $[t,T]$ with initial-terminal condition (ξ, P).

(ii) If, furthermore, the processes

$$\frac{w\left(s, X_s^P\right)\theta_s + d^1\left(s, X_s^P\right)}{w_x\left(s, X_s^P\right)} \in \mathbb{L}^2_{BMO}[t,T] \quad \text{and} \quad d^2\left(s, X_s^P\right)\tilde{U}_{zz}\left(s, w\left(s, X_s^P\right)\right) \in \mathbb{L}^2_{BMO}[t,T],$$

then, $Z^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1, 2$, and $\pi^{*,P}$ given by

$$\pi_s^{*,P} := -\frac{w\left(s, X_s^P\right)\theta_s + d^1\left(s, X_s^P\right)}{w_x\left(s, X_s^P\right)}, \quad t \leq s \leq T,$$

is optimal for $u^P(t, \xi; T)$. Respectively, the process $q^{*,P}$ given by

$$q_s^{*,P} := -\frac{d^2\left(s, X_s^P\right)}{w\left(s, X_s^P\right)}, \quad t \leq s \leq T, \tag{6.8}$$

is optimal for $\tilde{u}^P(t, \hat{\eta}; T)$ with $\hat{\eta} := w(t, \xi)$. Furthermore, for any $\eta \in L^{0,+}(\mathcal{F}_t)$ such that $\hat{\xi} := w^{-1}(t, \eta) \in \cap_{p\geq 1} L^p(\mathcal{F}_t)$, the control process $q^{*,P}$ defined by (6.8) with X^P satisfying $X_t^P = \hat{\xi}$ is optimal for $\tilde{u}^P(t, \eta; T)$.

It is easy to verify (i) by the Itô-Ventzel formula (A.2) and the duality relations between U and $\tilde{U}$ given in Section 2.2. Part (ii) can be obtained by Theorems 3.1 (ii), 3.5 and Remark 3.6.

Similarly to Proposition 6.1, we also have the following results from the dual backward SPDE.

Proposition 6.2 *Let $\tilde{w}$ be a solution to the backward SPDE*

$$\begin{cases} d\tilde{w}(t,z) = \tilde{h}(t,z)\,dt + \tilde{d}^\top(t,z)\,dW_t, \quad 0 \leq t \leq T, \quad z > 0, \\ \tilde{w}(T,z) = \tilde{U}_z(T,z) + P, \end{cases} \tag{6.9}$$

with

$$\begin{aligned} \tilde{h}(t,z) = &-\tilde{w}_z(t,z)\,z\,|\theta_t|^2 + \tilde{d}_z^1(t,z)\,z\theta_t + \tilde{d}^1(t,z)\,\theta_t - \frac{1}{2}\tilde{w}_{zz}(t,z)\,|z\theta_t|^2 \\ &- \frac{1}{2}\frac{\tilde{w}_{zz}(t,z)}{|\tilde{w}_z(t,z)|^2}\left|\tilde{d}^2(t,z)\right|^2 + \frac{\tilde{d}^2(t,z)\,\tilde{d}_z^2(t,z)}{\tilde{w}_z(t,z)}, \end{aligned}$$

and assume that $\tilde{w}(t,z)$ is a progressive $\mathcal{K}^{1,\delta}_{loc}$-semimartingale random field and $\tilde{w} \in \mathcal{C}^2$. Then, for $0 \leq t \leq T$, the following assertions hold.

(i) For $\eta \in L^{0,+}(\mathcal{F}_t)$, the triplet $(D^P, \tilde{Y}^P, \tilde{Z}^P)$ given by, for $t \le s \le T$,

$$D_s^P := \eta - \int_t^s \left(D_r^P \theta_r dW_r^1 + \frac{\tilde{d}^2\left(r, D_r^P\right)}{\tilde{w}_z\left(r, D_r^P\right)} dW_r^2 \right),$$

$$\tilde{Y}_s^P := \tilde{w}\left(s, D_s^P\right) - \tilde{U}_z\left(s, D_s^P\right),$$

and

$$\tilde{Z}_s^{P,1} := \left(\tilde{w}_z\left(s, D_s^P\right) - \tilde{U}_{zz}\left(s, D_s^P\right) \right) D_s^P \theta_s - \tilde{\alpha}_z^1\left(s, D_s^P\right),$$
$$\tilde{Z}_s^{P,2} := \left(\tilde{w}_z\left(s, D_s^P\right) - \tilde{U}_{zz}\left(s, D_s^P\right) \right) \frac{\tilde{d}^2\left(s, D_s^P\right)}{\tilde{w}_z\left(s, D_s^P\right)} - \tilde{a}_z^2\left(s, D_s^P\right),$$

satisfies FBSDE (3.16) on $[t,T]$ with initial-terminal condition (η, P).

(ii) If, furthermore, the processes

$$\tilde{w}_z(s, D_s^P) D_s^P \in \mathbb{L}^2_{BMO}[t,T] \quad \text{and} \quad \frac{\left(\tilde{w}_z\left(s, D_s^P\right) - \tilde{U}_{zz}\left(s, D_s^P\right) \right) \tilde{d}^2\left(s, D_s^P\right)}{\tilde{w}_z\left(s, D_s^P\right)} \in \mathbb{L}^2_{BMO}[t,T],$$

then, $\tilde{Z}^{P,i} \in \mathbb{L}^2_{BMO}[t,T]$, $i = 1, 2$, and $q^{*,P}$ given by

$$q_s^{*,P} := \frac{\tilde{d}^2\left(s, D_s^P\right)}{\tilde{w}_z\left(s, D_s^P\right) D_s^P}, \quad t \le s \le T,$$

is optimal for $\tilde{u}^P(t, \eta; T)$. Respectively, the process $\pi^{*,P}$ defined by

$$\pi_s^{*,P} := -\tilde{w}_z\left(s, D_s^P\right) D_s^P \theta_s, \quad t \le s \le T, \tag{6.10}$$

is optimal for $u^P(t, \hat{\xi}; T)$ with $\hat{\xi} := -\tilde{w}(t, \eta)$. Furthermore, for any $\xi \in \cap_{p \ge 1} L^p(\mathcal{F}_t)$ and $\hat{\eta} := \tilde{w}^{-1}(t, -\xi)$, the control process $\pi^{*,P}$ defined by (6.10) with D^P satisfying $D_t^P = \hat{\eta}$ is optimal for $u^P(t, \xi; T)$.

Remark 6.3 *Let w and $\tilde{w}$ be the solutions to backward SPDEs (6.5) and (6.9), respectively, with enough regularity. Applying the Itô-Ventzel formula to these two equations, one can directly verify that, for $0 \le t \le T$, $x \in \mathbb{R}$ and $z > 0$,*

$$-\tilde{w}\left(t, w\left(t, x\right)\right) = x \quad \text{and} \quad w\left(t, -\tilde{w}\left(t, z\right)\right) = z, \tag{6.11}$$

which gives the relation between the primal and dual backward SPDEs.

7 Conclusions

In this paper, we extended the notion of forward performance criteria to settings with random endowment in incomplete markets and studied the related stochastic optimization problems. For this, we developed a new methodology by directly studying the candidate optimal control processes for both the primal and dual forward problems. We constructed two new system of FBSDEs and established necessary and sufficient conditions for optimality, and various equivalences between the primal and dual problems. This new approach is general and complements the existing one based on backward SPDEs for the related value functions.

Building on these results, we introduced and developed the novel concept of forward optimized certainty equivalent, which offers a genuinely dynamic valuation mechanism that accommodates progressively adaptive market model updates, stochastic risk preferences, and incoming claims with arbitrary maturities. We, also, considered representative examples for both forward performance criteria with random endowment and forward OCE. In particular, for the case of exponential forward performance criteria, we investigated the connection of forward OCE with the forward entropic risk measure.

Appendices

A Differential rule and Itô-Ventzel formula

We recall various results we use herein for the derivatives of processes and the related Itô-Ventzel formula, stating in detail the underlying regality assumptions. We refer the reader to Kunita [37] for further details of regularity and properties of random fields, and also to [21, 23] for applications of some of these results to the SPDE associated with forward performance processes.

We begin with background definitions for *regular random fields*. Calligraphic notations are used for the set of random fields.

For $\mathbb{H} = \mathbb{D}, \mathbb{D}^+$, let $G(t,x)$, $(t,x) \in \mathbb{H}$, be a progressive random field. Let $m \geq 0$ be an integer and $\delta \in (0,1]$.

- G is called $\mathcal{C}^m$-regular if G is m-times continuously differentiable in x for any t a.s. We denote this by $G \in \mathcal{C}^m$ and the j^{th}-derivative of G in x by $G_x^{(j)}$ for $1 \leq j \leq m$.
- G is called $\mathcal{C}^{m,\delta}$-regular if G is m-times continuously differentiable in x with $G_x^{(m)}$ being δ-Hölder in x for any t a.s. We denote this by $G \in \mathcal{C}^{m,\delta}$.
- G is called $\mathcal{K}_{loc}^{m,\delta}$-regular if $G \in \mathcal{C}^{m,\delta}$ and, for any compact set $K \subset \mathbb{R}$ (resp. $\mathbb{R}^+$) when $\mathbb{H} = \mathbb{D}$ (resp. $\mathbb{D}^+$) and any $T > 0$,
$$\int_0^T \|G\|_{m,\delta:K}(t)\, dt < \infty, \text{ a.s.},$$
where the random K-seminorm is defined by
$$\|G\|_{m,\delta:K}(t) := \sup_{x\in K} \frac{|G(t,x)|}{1+|x|} + \sum_{j=1}^{m} \sup_{x\in K} |G_x^{(j)}(t,x)| + \sup_{\substack{x,y\in K \\ x\neq y}} \frac{|G_x^{(m)}(t,x) - G_x^{(m)}(t,y)|}{|x-y|^\delta}.$$
We denote this by $G \in \mathcal{K}_{loc}^{m,\delta}$.
- G is called $\bar{\mathcal{K}}_{loc}^{m,\delta}$-regular if $G \in \mathcal{C}^{m,\delta}$ and, for any compact set $K \subset \mathbb{R}$ (resp. $\mathbb{R}^+$) when $\mathbb{H} = \mathbb{D}$ (resp. $\mathbb{D}^+$) and any $T > 0$,
$$\int_0^T \|G\|^2_{m,\delta:K}(t)\, dt < \infty, \text{ a.s.}$$
We denote this by $G \in \bar{\mathcal{K}}_{loc}^{m,\delta}$.

Next, for $\mathbb{H} = \mathbb{D}, \mathbb{D}^+$, let $F(t,x)$, $(t,x) \in \mathbb{H}$, be a progressive Itô semimartingale random field with dynamics given by
$$dF(t,x) = \phi(t,x)\, dt + \psi(t,x)^\top dW_t, \quad (t,x) \in \mathbb{H}, \tag{A.1}$$
where $W = (W^1, W^2)$ is a two-dimensional Brownian motion, $\phi(t,x)$ and $\psi(t,x) = (\psi^1(t,x), \psi^2(t,x))^\top$ are progressive random fields valued in $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Let $m \geq 0$ be an integer and $\delta \in (0,1]$.

- F is called a $\mathcal{K}^{m,\delta}_{loc}$-semimartingale if $F(0,x)$ is m-times continuously differentiable in x with $F^{(m)}_x(0,x)$ being δ-Hölder in x a.s. and

$$\int_0^t \phi(s,x)\,ds \in \mathcal{K}^{m,\delta}_{loc}, \quad \int_0^t \psi(s,x)^\top dW_s \in \bar{\mathcal{K}}^{m,\delta}_{loc}.$$

Theorem A.1 (Differential Rule and Itô-Ventzel Formula) *Let $F(t,x)$, $(t,x) \in \mathbb{H}$, be a progressive Itô semimartingale random field with dynamics as in (A.1).*

(i) If F is a $\mathcal{K}^{m,\delta}_{loc}$-semimartingale for some integer $m \geq 0$ and $\delta \in (0,1]$, then $(\phi,\psi) \in \mathcal{K}^{m,\varepsilon}_{loc} \times \bar{\mathcal{K}}^{m,\varepsilon}_{loc}$ for any $\varepsilon < \delta$. Conversely, if $(\phi,\psi) \in \mathcal{K}^{m,\delta}_{loc} \times \bar{\mathcal{K}}^{m,\delta}_{loc}$ for some $m \geq 0$ and $\delta \in (0,1]$, then F is a $\mathcal{K}^{m,\varepsilon}_{loc}$-semimartingale for any $\varepsilon < \delta$.

(ii) If F is a $\mathcal{K}^{m,\delta}_{loc}$-semimartingale for some integer $m \geq 1$ and $\delta \in (0,1]$, then the differential rule holds:

$$dF_x(t,x) = \phi_x(t,x)\,dt + \psi_x(t,x)^\top dW_t.$$

(iii) If $F \in \mathcal{C}^2$ and it is a $\mathcal{K}^{1,\delta}_{loc}$-semimartingale for some $\delta \in (0,1]$, then the Itô-Ventzel formula holds. Specifically, for any Itô process X valued in $\mathbb{H}$, the process $F(t,X_t)$, $t \geq 0$, is a continuous Itô semimartingale satisfying

$$\begin{aligned} F(t,X_t) = F(0,X_0) &+ \int_0^t \phi(s,X_s)\,ds + \int_0^t \psi(s,X_s)^\top dW_s \\ &+ \int_0^t F_x(s,X_s)\,dX_s + \frac{1}{2}\int_0^t F_{xx}(s,X_s)\,d\langle X\rangle_s + \int_0^t \psi_x(s,X_s)^\top d\langle W,X\rangle_s. \end{aligned} \tag{A.2}$$

B Decoupling field method

We present background results that we use to establish the solvability of the FBSDE derived in Section 4.3. Specifically, we recall the notion of a decoupling field and some of the general results established in [26]. To this end, consider a general FBSDE, for $0 \leq s \leq T$ and $x \in \mathbb{R}^2$,

$$\begin{cases} X_s = x + \displaystyle\int_0^s \mu\left(r, X_r, Y_r, Z_r\right) dr + \int_0^s \sigma\left(r, X_r, Y_r, Z_r\right)^\top dW_r, \\ Y_s = H\left(X_T\right) - \displaystyle\int_s^T f\left(r, X_r, Y_r, Z_r\right) dr - \int_s^T Z_r^\top dW_r, \end{cases} \tag{B.1}$$

where $\mu, \sigma, f : [0,T] \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \mathbb{R}^{2\times 2}, \mathbb{R}$ and $H : \mathbb{R}^2 \to \mathbb{R}$ are deterministic measurable functions.

Definition B.1 *Let T be arbitrary and fixed, and $0 \leq t \leq T$. A function $w : [t,T] \times \mathbb{R}^2 \to \mathbb{R}$ with $w(T,x) = H(x)$ is called a Markovian decoupling field for FBSDE (B.1) on $[t,T]$ if, for all $t \leq t_1 \leq t_2 \leq T$ and any $\mathcal{F}_{t_1}$-measurable $X_{t_1} : \Omega \to \mathbb{R}^2$, there exist progressively measurable processes (X,Y,Z) on $[t_1,t_2]$ such that $\|Z\|_\infty < \infty$ and*

$$\begin{cases} X_s = X_{t_1} + \displaystyle\int_{t_1}^s \mu\left(r, X_r, Y_r, Z_r\right) dr + \int_{t_1}^s \sigma\left(r, X_r, Y_r, Z_r\right)^\top dW_r, \\ Y_s = Y_{t_2} - \displaystyle\int_s^{t_2} f\left(r, X_r, Y_r, Z_r\right) dr - \int_s^{t_2} Z_r^\top dW_r, \\ Y_s = w\left(s, X_s\right), \end{cases} \tag{B.2}$$

hold a.s. for all $t_1 \leq s \leq t_2$.

Notation B.2 *For a measurable function $F : \mathbb{R}^n \to \mathbb{R}^m$, we define*

$$L_{F,x} := \inf \{L \geq 0 : |F(x) - F(x')| \leq L|x - x'| \text{ for all } x, x' \in \mathbb{R}^n\}, \tag{B.3}$$

with the convention $\inf \varnothing := \infty$. Here, we use x in $L_{F,x}$ to denote the argument of the function $F(x)$.

Definition B.3 *Let T be arbitrary and fixed, and $0 \leq t \leq T$. Let w be a Markovian decoupling field for FBSDE (B.1) on $[t, T]$.*

(i) w is weakly regular if $L_{w,x} < \frac{1}{L_{\sigma,z}}$ and $\sup_{t \leq s \leq T} |w(s, 0)| < \infty$.

(ii) w is strongly regular if it is weakly regular, and for all $t \leq t_1 \leq t_2 \leq T$, the processes (X, Y, Z) in (B.2) satisfy the conditions:

(a) they are unique for each constant $X_{t_1} = x \in \mathbb{R}^2$,

(b) for all $x \in \mathbb{R}^2$,

$$\sup_{t_1 \leq s \leq t_2} \mathbb{E}_{t_1,\infty} \left[|X_s|^2 \right] + \sup_{t_1 \leq s \leq t_2} \mathbb{E}_{t_1,\infty} \left[|Y_s|^2 \right] < \infty,$$

where $\mathbb{E}_{t_1,\infty}[\cdot] := \operatorname{esssup} \mathbb{E}[\,\cdot\,|\mathcal{F}_{t_1}]$,

(c) they are measurable and weakly differentiable with respect to $x \in \mathbb{R}^2$ such that

$$\operatorname*{esssup}_{x \in \mathbb{R}^2} \sup_{\zeta \in S^1} \sup_{t_1 \leq s \leq t_2} \mathbb{E}_{t_1,\infty} \left[|\partial_x X_s|_\zeta^2 \right] < \infty,$$

$$\operatorname*{esssup}_{x \in \mathbb{R}^2} \sup_{\zeta \in S^1} \sup_{t_1 \leq s \leq t_2} \mathbb{E}_{t_1,\infty} \left[|\partial_x Y_s|_\zeta^2 \right] < \infty,$$

$$\operatorname*{esssup}_{x \in \mathbb{R}^2} \sup_{\zeta \in S^1} \mathbb{E}_{t_1,\infty} \left[\int_{t_1}^{t_2} |\partial_x Z_s|_\zeta^2 \, ds \right] < \infty,$$

where $S^1 := \{z \in \mathbb{R}^2 : |z| = 1\}$, and $|\partial_x X_s|_\zeta = |\langle \partial_x X_s, \zeta \rangle|$ is the magnitude of the directional derivative in direction ζ.

For the definition and properties of weak derivatives of processes with respect to their initial values, we refer to [25].

Definition B.4 *Let w be a Markovian decoupling field for FBSDE (B.1). We call w controlled in z if there exists a constant $C > 0$ such that, for all $t \leq t_1 \leq t_2 \leq T$ and all initial values $X_{t_1} \in \mathcal{F}_{t_1}$, the corresponding processes (X, Y, Z) in (B.2) satisfy $||Z||_\infty \leq C$.*

Definition B.5 *The FBSDE (B.1) satisfies the modified local Lipschitz condition (MLLC) if*

(i) μ, σ, f are Lipschitz continuous in x, y, z on $[0, T] \times \mathbb{R}^2 \times \mathbb{R} \times B$ with $B \subset \mathbb{R}^2$ being an arbitrary bounded set,

(ii) $\|\mu(\cdot, 0, 0, 0)\|_\infty, \|\sigma(\cdot, 0, 0, 0)\|_\infty, \|f(\cdot, 0, 0, 0)\|_\infty < \infty$,

(iii) $L_{\sigma,z} < \infty$ and $L_{H,x} < \frac{1}{L_{\sigma,z}}$.

Notation B.6 *For the FBSDE (B.1), let $I_{\max}^M \subset [0, T]$ be the union of all intervals $[t, T] \subset [0, T]$, $0 \leq t \leq T$, such that there exists a weakly regular Markovian decoupling field w on $[t, T]$.*

Theorem B.7 *Let FBSDE (B.1) satisfy MLLC. Then, there exists a unique weakly regular Markovian decoupling field u on $I^M_{\max}$, which is strongly regular, controlled in z and continuous. Furthermore, it is either the case $I^M_{\max} = [0, T]$ or $I^M_{\max} = (t^M_{\min}, T]$, with $0 \le t^M_{\min} < T$.*

To prove the well-posedness of the FBSDE (3.3), we recall the abstract result in [26, Section 3]. For $\varepsilon > 0$, consider the FBSDE

$$\begin{cases} \tilde{X}_s = \tilde{x} + \displaystyle\int_t^s \frac{1}{\varepsilon}\tilde{\mu}\left(r, \varepsilon\tilde{X}_r\right) dr + \frac{1}{\varepsilon}\int_t^s \left(\rho dW^1_r + \sqrt{1-\rho^2} dW^2_r\right), \ \rho \in (0,1), \\ \bar{X}_s = \bar{x} + \displaystyle\int_t^s \left(\bar{\sigma}\left(r, \varepsilon\tilde{X}_r, \bar{X}_r + Y_r, Z^1_r\right)\right) dW^1_r, \\ Y_s = H\left(\varepsilon\tilde{X}_T, \bar{X}_T\right) - \displaystyle\int_s^T f\left(r, \varepsilon\tilde{X}_r, \bar{X}_r + Y_r, Z^2_r\right) dr - \int_s^T Z^1_r dW^1_r - \int_s^T Z^2_r dW^2_r, \end{cases} \tag{B.4}$$

with $0 \le t \le s \le T$, $\tilde{x}, \bar{x} \in \mathbb{R}$, and the coefficients satisfying

(i) $\tilde{\mu} : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable and Lipschitz continuous in the second component, and $\|\tilde{\mu}(\cdot, 0)\|_\infty < \infty$,

(ii) $\bar{\sigma} : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable and Lipschitz continuous in the last three components, and $\|\bar{\sigma}(\cdot, \cdot, \cdot, 0)\|_\infty < \infty$,

(iii) $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable and Lipschitz continuous in the last three components on all sets of the form $[0,T] \times \mathbb{R} \times \mathbb{R} \times B$ with $B \subset \mathbb{R}$ being an arbitrary bounded set, and $\|f(\cdot, 0, 0, 0)\|_\infty < \infty$,

(iv) $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in both components, and $L_{H,\bar{x}} < \frac{1}{L_{\bar{\sigma},z^1}}$.

As shown in [26], under the above conditions, the FBSDE (B.4) satisfies the MLLC for some $\varepsilon > 0$ small enough.

Finally, to establish the global well-posedness of (B.4), we also introduce additional assumptions:

(v) f is differentiable in $(\tilde{x}, \bar{x} + y, z^2)$, $\|f(\cdot, \cdot, \cdot, 0)\|_\infty < \infty$ and there exists some constant $C > 0$ such that

$$|f_{\tilde{x}}| \le C\left(1 + \left|z^2\right|^2\right), \quad |f_{\bar{x}+y}| \le C\left(1 + \left|z^2\right|^2\right), \quad \text{and} \quad |f_z| \le C\left(1 + \left|z^2\right|\right),$$

(vi) $\bar{\sigma}$ is differentiable in $(\tilde{x}, \bar{x} + y, z^1)$ with bounded derivatives and $L_{\bar{\sigma},z^1} \le 1$,

(vii) $\|H\|_\infty < \infty$, and $L_{H,\bar{x}} < 1$.

Theorem B.8 ([26, Theorem 3.3]) *Let assumptions (i)-(vii) hold and, furthermore, assume that there exist constants $K \in [0, \frac{1}{L_{\bar{\sigma},z^1}})$ and $\varepsilon_0 \in (0, \infty)$ such that, for any $\varepsilon \in (0, \varepsilon_0]$ and any $0 \le t \le T$, if a weakly regular Markovian decoupling field $w^\varepsilon : [t,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ associated with FBSDE (B.4) exists, it also satisfies $\sup_{t \le s \le T} \|w^\varepsilon_{\tilde{x}}(s, \cdot, \cdot)\|_\infty < K$. Then, there exists $\varepsilon > 0$ such that $I^M_{\max} = [0, T]$ for FBSDE (B.4).*

Remark B.9 *We note that herein we assume that $|f_{\tilde{x}}| \le C(1 + |z^2|^2)$ which is weaker than the assumption in Theorem 3.3 in [26]. However, this condition does not affect the results of Theorem B.8 due to the uniform boundedness of the Z-component. More precisely, in the proof of Theorem 3.3 in [26], $f_{\tilde{x}}$ only appears in the representation of the derivative process $\tilde{R}$ (see p.2676) and is used to establish the uniform bound for $\tilde{R}$ (see the middle of p.2677). On the other hand, Z is uniformly bounded by the definition of decoupling field and this bound is independent of both t_1 and ε, by the argument in the middle of p.2685. Therefore, the estimate of $\tilde{R}$ will not change and, hence, Theorem B.8 still holds under this weaker assumption.*

C Proofs

C.1 Proof of Lemma 2.1

Let $X_s^{t,\pi} := \int_t^s \pi_r \left(\theta_r dr + dW_r^1\right)$, $t \le s \le T$. Itô's formula yields that $X^{t,\pi} M^{t,q}$ is a local martingale, given by

$$d\left(X_s^{t,\pi} M_s^{t,q}\right) = \left(-X_s^{t,\pi} M_s^{t,q} \theta_s + M_s^{t,q} \pi_s\right) dW_s^1 - X_s^{t,\pi} M_s^{t,q} q_s dW_s^2.$$

We observe that for $n > 2$, and using that $\pi \in \mathbb{L}^2_{BMO}[0,T]$, the energy inequality implies that

$$\mathbb{E}\left[\left(\int_0^T |\pi_r|^2 \, dr\right)^{\frac{n}{2}}\right] \le (\frac{n}{2})!||\int_0^{\cdot} \pi_r dW_r^1||^n_{BMO} < \infty. \tag{C.1}$$

In turn, using the Burkholder-Davis-Gundy inequality and the uniform boundedness of θ, we deduce

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le s\le T} \left|X_s^{t,\pi}\right|^n\right] &\le C\mathbb{E}\left[\sup_{t\le s\le T}\left(\left|\int_t^s \pi_r\theta_r dr\right|^n + \left|\int_t^s \pi_r dW_r^1\right|^n\right)\right] \\ &\le C\mathbb{E}\left[\left(\int_t^T |\pi_r\theta_r| \, dr\right)^n\right] + C\mathbb{E}\left[\left(\int_t^T |\pi_r|^2 \, dr\right)^{\frac{n}{2}}\right] < \infty. \end{aligned} \tag{C.2}$$

Next, we show that $\int_t^s X_r^{t,\pi} M_r^{t,q} q_r dW_r^2$, $t \le s \le T$, is a true $\mathbb{F}$-martingale. For this, we first observe that the stochastic exponential $M^{t,q}$ is a uniformly integrable martingale due to the boundedness of θ and the fact that $q \in \mathbb{L}^2_{BMO}[t,T]$. Using Doob's inequality and the reverse Hölder's inequality, we deduce

$$\mathbb{E}\left[\sup_{t\le s\le T} \left|M_s^{t,q}\right|^{p_0}\right] \le \left(\frac{p_0}{p_0-1}\right)^{p_0} \mathbb{E}\left[|M_T^{t,q}|^{p_0}\right] < \infty,$$

for some $p_0 > 1$. Using once more the Burkholder-Davis-Gundy inequality gives

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le s\le T}\left|\int_t^s X_r^{t,\pi} M_r^{t,q} q_r dW_r^2\right|\right] &\le C\mathbb{E}\left[\sup_{t\le s\le T}\left|M_s^{t,q}\right|\left(\int_t^T \left|X_r^{t,\pi} q_r\right|^2 dr\right)^{\frac{1}{2}}\right] \\ &\le C\mathbb{E}\left[\sup_{t\le s\le T}\left|M_s^{t,q}\right|^{p_0}\right] + C\mathbb{E}\left[\left(\int_t^T \left|X_r^{t,\pi} q_r\right|^2 dr\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right].\end{aligned}$$

Moreover, inequality (C.2) and that $q \in \mathbb{L}^2_{BMO}[t,T]$ yield

$$\begin{aligned}\mathbb{E}\left[\left(\int_t^T \left|X_r^{t,\pi} q_r\right|^2 dr\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right] &\le \mathbb{E}\left[\sup_{t\le s\le T}\left|X_s^{t,\pi}\right|^{\frac{p_0}{p_0-1}}\left(\int_t^T |q_r|^2 \, dr\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right] \\ &\le \mathbb{E}\left[\sup_{t\le s\le T}\left|X_s^{t,\pi}\right|^{\frac{2p_0}{p_0-1}}\right] + \mathbb{E}\left[\left(\int_t^T |q_r|^2 \, dr\right)^{\frac{p_0}{p_0-1}}\right] < \infty,\end{aligned}$$

and we conclude. Using similar arguments, we deduce that $\int_t^s (-X_r^{t,\pi} M_r^{t,q}\theta_r + M_r^{t,q}\pi_r) dW_r^1$, $t \le s \le T$, is also a true $\mathbb{F}$-martingale.

C.2 Proof of Theorem 3.1

In this proof, we omit the superscript P for the primal FBSDE for notational simplicity.

Part (i). Step 1. Since $\pi^{*,P}$ is optimal, we must have, for any bounded $\pi \in \mathcal{A}_{[t,T]}$ and $\varepsilon \in (0,1)$, that a.s.,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[U\left(T, X_T^{\varepsilon,\pi} + P\right) - U\left(T, X_T^* + P\right) \middle| \mathcal{F}_t \right] \leq 0, \tag{C.3}$$

where $X_T^{\varepsilon,\pi} := \xi + \int_t^T (\pi_r^{*,P} + \varepsilon \pi_r)(\theta_r dr + dW_r^1)$ and X_T^* as in (2.5). By the fundamental theorem of calculus and the chain rule, we can write

$$\frac{1}{\varepsilon}\left[U\left(T, X_T^{\varepsilon,\pi} + P\right) - U\left(T, X_T^* + P\right) \right] = \int_0^1 U_x\left(T, X_T^{\delta\varepsilon,\pi} + P\right) d\delta \int_t^T \pi_r \left(\theta_r dr + dW_r^1\right).$$

Substituting this into the left-hand side of (C.3) and applying the uniform integrability assumption along with the dominated convergence theorem, we obtain

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \to 0} \mathbb{E}\left[\int_0^1 U_x\left(T, X_T^{\delta\varepsilon,\pi} + P\right) d\delta \int_t^T \pi_r \left(\theta_r dr + dW_r^1\right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \int_t^T \pi_r \left(\theta_r dr + dW_r^1\right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Replacing π by $-\pi$ implies that $0 \leq \mathbb{E}[U_x\left(T, X_T^* + P\right) \int_t^T \pi_r \left(\theta_r dr + dW_r^1\right) | \mathcal{F}_t]$, and, thus,

$$0 = \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \int_t^T \pi_r \left(\theta_r dr + dW_r^1\right) \middle| \mathcal{F}_t \right] = \mathbb{E}\left[N_T A_T^\pi \middle| \mathcal{F}_t \right], \tag{C.4}$$

where, for $0 \leq t \leq s \leq T$,

$$N_s := \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \middle| \mathcal{F}_s \right] \quad \text{and} \quad A_s^\pi := \int_t^s \pi_r \left(\theta_r dr + dW_r^1\right). \tag{C.5}$$

By the martingale representation theorem, there exists a square integrable density process, say $\varphi = (\varphi^1, \varphi^2)$ on $[t,T]$, such that

$$dN_s = \varphi_s^\top dW_s, \quad 0 \leq t \leq s \leq T. \tag{C.6}$$

Step 2. We define an $\mathbb{F}$-progressively measurable process Y on $[t,T]$ such that

$$U_x\left(s, X_s^* + Y_s\right) = N_s = \mathbb{E}\left[U_x\left(T, X_T^* + P\right) \middle| \mathcal{F}_s \right], \quad 0 \leq t \leq s \leq T. \tag{C.7}$$

In other words, $Y_s := U_x^{-1}\left(s, N_s\right) - X_s^*$, $t \leq s \leq T$, and $Y_T = P$. From the dual relation (2.10), we have that

$$Y_s = -\tilde{U}_z\left(s, N_s\right) - X_s^*, \quad t \leq s \leq T. \tag{C.8}$$

In turn, applying the Itô-Ventzel formula to $\tilde{U}_z$ in (2.15) with N as in (C.6), we obtain

$$\begin{aligned} dY_s = &\left(-\tilde{\beta}_z\left(s, N_s\right) - \frac{1}{2}\tilde{U}_{zzz}\left(s, N_s\right)|\varphi_s|^2 - \tilde{\alpha}_{zz}^\top\left(s, N_s\right)\varphi_s - \pi_s^{*,P}\theta_s \right) ds \\ &+ \left(-\tilde{\alpha}_z^1\left(s, N_s\right) - \tilde{U}_{zz}\left(s, N_s\right)\varphi_s^1 - \pi_s^{*,P} \right) dW_s^1 + \left(-\tilde{\alpha}_z^2\left(s, N_s\right) - \tilde{U}_{zz}\left(s, N_s\right)\varphi_s^2 \right) dW_s^2. \end{aligned} \tag{C.9}$$

Recalling (2.13) and (2.14), we have

$$\tilde{U}_{zz}\left(s, N_s\right) = -\frac{1}{U_{xx}\left(s, X_s^* + Y_s\right)} \quad \text{and} \quad \tilde{U}_{zzz}\left(s, N_s\right) = \frac{U_{xxx}\left(s, X_s^* + Y_s\right)}{\left(U_{xx}\left(s, X_s^* + Y_s\right)\right)^3}. \tag{C.10}$$

We also have by (2.17) that

$$\tilde{\alpha}_z\left(s, N_s\right) = \frac{\alpha_x\left(s, X_s^* + Y_s\right)}{U_{xx}\left(s, X_s^* + Y_s\right)}, \tag{C.11}$$

$$\tilde{\alpha}_{zz}\left(s, N_s\right) = -\frac{U_{xxx}\left(s, X_s^* + Y_s\right)\alpha_x\left(s, X_s^* + Y_s\right)}{\left(U_{xx}\left(s, X_s^* + Y_s\right)\right)^3} + \frac{\alpha_{xx}\left(s, X_s^* + Y_s\right)}{\left|U_{xx}\left(s, X_s^* + Y_s\right)\right|^2}, \tag{C.12}$$

and by (2.16) that

$$\begin{aligned}\tilde{\beta}_z\left(s, N_s\right) =& \frac{\beta_x\left(s, X_s^* + Y_s\right)}{U_{xx}\left(s, X_s^* + Y_s\right)} + \frac{1}{2}\frac{U_{xxx}\left(s, X_s^* + Y_s\right)\left|\alpha_x\left(s, X_s^* + Y_s\right)\right|^2}{\left(U_{xx}\left(s, X_s^* + Y_s\right)\right)^3} \\ & - \frac{\alpha_x^\top\left(s, X_s^* + Y_s\right)\alpha_{xx}\left(s, X_s^* + Y_s\right)}{\left|U_{xx}\left(s, X_s^* + Y_s\right)\right|^2},\end{aligned} \tag{C.13}$$

with $\beta_x(t,x)$ given in (2.8). Combining (C.9)-(C.13) yields that

$$dY_s = -f\left(s, X_s^*, Y_s, Z_s\right)ds + Z_s^\top dW_s, \quad t \le s \le T, \tag{C.14}$$

where

$$\begin{aligned}f\left(s,x,y,z\right) :=& \frac{\beta_x\left(s, x+y\right)}{U_{xx}\left(s, x+y\right)} + \frac{1}{2}\frac{U_{xxx}\left(s, x+y\right)\left|\varphi_s - \alpha_x\left(s, x+y\right)\right|^2}{\left(U_{xx}\left(s, x+y\right)\right)^3} \\ & + \frac{\alpha_{xx}^\top\left(s, x+y\right)\left(\varphi_s - \alpha_x\left(s, x+y\right)\right)}{\left|U_{xx}\left(s, x+y\right)\right|^2} + \pi_s^{*,P}\theta_s\end{aligned} \tag{C.15}$$

and

$$Z_s^1 := \frac{\varphi_s^1 - \alpha_x^1\left(s, X_s^* + Y_s\right)}{U_{xx}\left(s, X_s^* + Y_s\right)} - \pi_s^{*,P} \quad \text{and} \quad Z_s^2 := \frac{\varphi_s^2 - \alpha_x^2\left(s, X_s^* + Y_s\right)}{U_{xx}\left(s, X_s^* + Y_s\right)}. \tag{C.16}$$

Thus, the processes $\varphi = (\varphi^1, \varphi^2)$ in (C.6) must satisfy

$$\begin{aligned}\varphi_s^1 &= \alpha_x^1\left(s, X_s^* + Y_s\right) + U_{xx}\left(s, X_s^* + Y_s\right)\pi_s^{*,P} + U_{xx}\left(s, X_s^* + Y_s\right)Z_s^1, \\ \varphi_s^2 &= \alpha_x^2\left(s, X_s^* + Y_s\right) + U_{xx}\left(s, X_s^* + Y_s\right)Z_s^2.\end{aligned} \tag{C.17}$$

Therefore, substituting the representation of φ into (C.15), we obtain that the drift f in BSDE (C.14) is given by

$$\begin{aligned}f\left(s,x,y,z\right) =& \frac{\beta_x\left(s, x+y\right)}{U_{xx}\left(s, x+y\right)} + \frac{1}{2}\frac{U_{xxx}\left(s, x+y\right)\left(\left|z^1 + \pi_s^{*,P}\right|^2 + \left|z^2\right|^2\right)}{U_{xx}\left(s, x+y\right)} \\ & + \frac{\alpha_{xx}^1\left(s, x+y\right)\left(z^1 + \pi_s^{*,P}\right) + \alpha_{xx}^2\left(s, x+y\right)z^2}{U_{xx}\left(s, x+y\right)} + \pi_s^{*,P}\theta_s.\end{aligned} \tag{C.18}$$

Step 3. Recalling N in (C.6) with φ as in (C.17) and A^π in (C.5), we obtain

$$\begin{aligned}d(N_s A_s^\pi) =& \pi_s\left(N_s\theta_s + \alpha_x^1\left(s, X_s^* + Y_s\right) + U_{xx}\left(s, X_s^* + Y_s\right)\pi_s^{*,P} + U_{xx}\left(s, X_s^* + Y_s\right)Z_s^1\right)ds \\ & + \left(A_s^\pi\varphi_s^1 + N_s\pi_s\right)dW_s^1 + A_s^\pi\varphi_s^2 dW_s^2.\end{aligned}$$

By Lemma C.1, which, to ease the presentation, is provided right after this proof, we have that $0 \leq \int_t^s \left(A_r^\pi \varphi_r^1 + N_r \pi_r\right) dW_r^1 + \int_t^s A_r^\pi \varphi_r^2 dW_r^2$, $0 \leq t \leq s \leq T$, is a true $\mathbb{F}$-martingale. Then, integrating both sides from t to T and taking conditional expectation, we obtain using (C.4) that

$$\mathbb{E}\left[\int_t^T \pi_s \left(N_s \theta_s + \alpha_x^1 \left(s, X_s^* + Y_s\right) + U_{xx}\left(s, X_s^* + Y_s\right) \pi_s^{*,P} + U_{xx}\left(s, X_s^* + Y_s\right) Z_s^1\right) ds \middle| \mathcal{F}_t\right] = 0.$$

Since π is an arbitrary admissible control process, we deduce that, for $0 \leq t \leq s \leq T$, it must be that at the optimum

$$\pi_s^{*,P} = -\frac{N_s \theta_s + \alpha_x^1\left(s, X_s^* + Y_s\right)}{U_{xx}\left(s, X_s^* + Y_s\right)} - Z_s^1 = -\frac{U_x\left(s, X_s^* + Y_s\right) \theta_s + \alpha_x^1\left(s, X_s^* + Y_s\right)}{U_{xx}\left(s, X_s^* + Y_s\right)} - Z_s^1. \tag{C.19}$$

Step 4. Combining (C.18), (2.8) and (C.19) yields

$$f(s,x,y,z) = -z^1 \theta_s + \frac{\frac{1}{2} U_{xxx}\left(s, x+y\right) \left|z^2\right|^2 + \alpha_{xx}^2\left(s, x+y\right) z^2}{U_{xx}\left(s, x+y\right)}.$$

In turn, we easily deduce the BSDE

$$dY_s = -f\left(s, X_s^*, Y_s, Z_s\right) ds + Z_s^\top dW_s, \quad Y_T = P.$$

Meanwhile, the process X^*, defined in (3.4) and associated with the optimal control $\pi^{*,P}$ from (C.19), provides the forward component of the FBSDE (3.3).

Part (ii). If (X, Y, Z) satisfies FBSDE (3.3), it follows easily that $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ using Assumption 2.3 (ii) and the fact that $Z^1 \in \mathbb{L}^2_{BMO}[t,T]$. Moreover, we have that

$$X_s = \xi + \int_t^s \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) = X_s^*, \quad 0 \leq t \leq s \leq T.$$

For any $\pi \in \mathcal{A}_{[t,T]}$, denote $X_s^\pi := \xi + \int_t^s \pi_r \left(\theta_r dr + dW_r^1\right)$. Next, we show that, for any $\pi \in \mathcal{A}_{[t,T]}$,

$$\mathbb{E}\left[U\left(T, X_T^\pi + P\right) \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[U\left(T, X_T + P\right) \middle| \mathcal{F}_t\right], \tag{C.20}$$

where X is given by the forward component of the FBSDE (3.3). To this end, working as in Lemma 3.3, we introduce a probability measure

$$\left.\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right|_{\mathcal{F}_T} = M_T^{t,q^{*,P}} = \frac{U_x\left(T, X_T + P\right)}{U_x\left(t, \xi + Y_t\right)},$$

where $M^{t,q^{*,P}}$ is the optimal state price density process. Then, Girsanov's theorem yields that $\left(dW_s^{*,1}, dW_s^{*,2}\right) := \left(dW_s^1 + \theta_s ds, dW_s^2 + q_s^{*,P} ds\right)$ is a Brownian motion under $\mathbb{Q}^*$. Using the spatial concavity of U, we deduce

$$\begin{aligned}
&\mathbb{E}\left[U\left(T, X_T^\pi + P\right) \middle| \mathcal{F}_t\right] - \mathbb{E}\left[U\left(T, X_T + P\right) \middle| \mathcal{F}_t\right] \\
\leq &\mathbb{E}\left[U_x\left(T, X_T + P\right)\left(X_T^\pi - X_T\right) \middle| \mathcal{F}_t\right] \\
= &U_x\left(t, \xi + Y_t\right) \mathbb{E}^{\mathbb{Q}^*}\left[\int_t^T \left(\pi_r - \pi_r^{*,P}\right) dW_r^{*,1} \middle| \mathcal{F}_t\right].
\end{aligned} \tag{C.21}$$

Applying the reverse Hölder inequality to $M^{t,q^{*,P}}$, we obtain that there exists a constant $p_0 > 1$ such that $\mathbb{E}\left[\left|M_T^{t,q^{*,P}}\right|^{p_0}\right] < \infty$. Moreover, noting that $\pi - \pi^{*,P} \in \mathbb{L}^2_{\mathrm{BMO}}[t,T]$ and invoking the energy inequality (C.1), we deduce that

$$\mathbb{E}\left[\left(\int_t^T \left|\pi_r - \pi_r^{*,P}\right|^2 dr\right)^{\frac{n}{2}}\right] < \infty \quad \text{for all } n > 2.$$

In turn, the Burkholder-Davis-Gundy inequality yields

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^*}\left[\sup_{t\le s\le T}\left|\int_t^s \left(\pi_r - \pi_r^{*,P}\right) dW_r^{*,1}\right|\right] &\le C\mathbb{E}^{\mathbb{Q}^*}\left[\left(\int_t^T \left|\pi_r - \pi_r^{*,P}\right|^2 dr\right)^{\frac{1}{2}}\right] \\ &\le C\mathbb{E}\left[\left|M_T^{t,q^{*,P}}\right|^{p_0}\right] + C\mathbb{E}\left[\left(\int_t^T \left|\pi_r - \pi_r^{*,P}\right|^2 dr\right)^{\frac{1}{2}\frac{p_0}{p_0-1}}\right] < \infty,\end{aligned}$$

which further implies that the right hand side of (C.21) is zero and (C.20) follows.

Lemma C.1 *For any $\pi \in \mathcal{A}_{[t,T]}$, the process*

$$\int_t^s \left(A_r^\pi \varphi_r^1 + N_r \pi_r\right) dW_r^1 + \int_t^s A_r^\pi \varphi_r^2 dW_r^2, \quad t \le s \le T,$$

is a true $\mathbb{F}$-martingale on $[t,T]$.

Proof. By (C.2), we know that $\mathbb{E}[\sup_{t\le s\le T}|A_s^\pi|^2] < \infty$. The Burkholder-Davis-Gundy inequality yields

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le s\le T}\left|\int_t^s A_r^\pi \varphi_r^i dW_r^i\right|\right] &\le C\mathbb{E}\left[\left(\int_t^T \left|A_r^\pi \varphi_r^i\right|^2 dr\right)^{\frac{1}{2}}\right] \\ &\le C\mathbb{E}\left[\sup_{t\le s\le T}|A_s^\pi|^2\right] + C\mathbb{E}\left[\int_t^T \left|\varphi_r^i\right|^2 dr\right] < \infty.\end{aligned}$$

Moreover, by the assumption of square integrability of $U_x(T, X_T^* + P)$ and Doob's L^p-inequality, we deduce that

$$\mathbb{E}\left[\sup_{t\le s\le T}|N_s|^2\right] \le C\mathbb{E}\left[\left|U_x\left(T, X_T^* + P\right)\right|^2\right] < \infty.$$

Using similar arguments, we obtain that $\mathbb{E}[\sup_{t\le s\le T}|\int_t^s N_r\pi_r dW_r^1|] < \infty$. ∎

C.3 Proof of Theorem 3.5

Part (i). By Lemma 3.3, we have that $q^{*,P} \in \mathcal{Q}_{[t,T]}$. Next, we show that, for any $q \in \mathcal{Q}_{[t,T]}$,

$$\mathbb{E}\left[\tilde{U}\left(T, \hat{\eta}M_T^{t,q^{*,P}}\right) + \hat{\eta}M_T^{t,q^{*,P}}P \middle| \mathcal{F}_t\right] \le \mathbb{E}\left[\tilde{U}\left(T, \hat{\eta}M_T^{t,q}\right) + \hat{\eta}M_T^{t,q}P \middle| \mathcal{F}_t\right]. \tag{C.22}$$

Note that by Lemma 3.3 and the definition of $\hat{\eta}$ in (3.12),

$$\hat{\eta}M_T^{t,q^{*,P}} = U_x\left(T, X_T^P + P\right). \tag{C.23}$$

From part (ii) in Theorem 3.1, we have that $X_T^P = \xi + \int_t^T \pi_r^{*,P}(\theta_r dr + dW_r^1)$ with $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ as in (3.7). It follows from (2.12), (C.23) and Lemma 2.1 that

$$\begin{aligned}\mathbb{E}\left[\tilde{U}\left(T, \hat{\eta} M_T^{t,q^{*,P}}\right) + \hat{\eta} M_T^{t,q^{*,P}} P \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\tilde{U}\left(T, U_x\left(T, X_T^P + P\right)\right) + U_x\left(T, X_T^P + P\right) P \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, X_T^P + P\right) - U_x\left(T, X_T^P + P\right) X_T^P \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, X_T^P + P\right) \middle| \mathcal{F}_t\right] - \xi\hat{\eta}. \end{aligned} \tag{C.24}$$

On the other hand, for any $q \in \mathcal{Q}_{[t,T]}$, by (2.9) and Lemma 2.1, we easily deduce that

$$\begin{aligned}\mathbb{E}\left[\tilde{U}\left(T, \hat{\eta} M_T^{t,q}\right) + \hat{\eta} M_T^{t,q} P \middle| \mathcal{F}_t\right] &\geq \mathbb{E}\left[U\left(T, X_T^P + P\right) - \hat{\eta} M_T^{t,q} X_T^P \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, X_T^P + P\right) \middle| \mathcal{F}_t\right] - \xi\hat{\eta},\end{aligned}$$

which together with (C.24) proves (C.22).

Part (ii). From the bidual relation (2.11) and Lemma 2.1, we have, for any $\pi \in \mathcal{A}_{[t,T]}$, $q \in \mathcal{Q}_{[t,T]}$, $\xi \in \cap_{p\geq 1} L^p(\mathcal{F}_t)$ and $\eta \in L^{0,+}(\mathcal{F}_t)$,

$$\begin{aligned}&\mathbb{E}\left[U\left(T, \xi + \int_t^T \pi_r\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\tilde{U}\left(T, \eta M_T^{t,q}\right) + \eta M_T^{t,q}\left(\xi + \int_t^T \pi_r\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\tilde{U}\left(T, \eta M_T^{t,q}\right) + \eta M_T^{t,q} P \middle| \mathcal{F}_t\right] + \xi\eta. \end{aligned} \tag{C.25}$$

Thus, due to the arbitrariness of π, q and η, we obtain

$$u^P(t, \xi; T) \leq \operatorname*{essinf}_{\eta \in L^{0,+}(\mathcal{F}_t)} \left(\tilde{u}^P(t, \eta; T) + \xi\eta\right). \tag{C.26}$$

On the other hand, for $q^{*,P}$ defined in (3.9), using Theorem 3.1 (ii), equality (C.24) and part (i) of this theorem gives

$$\begin{aligned}u^P(t, \xi; T) &= \mathbb{E}\left[U\left(T, X_T^P + P\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\tilde{U}\left(T, \hat{\eta} M_T^{t,q^{*,P}}\right) + \hat{\eta} M_T^{t,q^{*,P}} P \middle| \mathcal{F}_t\right] + \xi\hat{\eta} = \tilde{u}^P(t, \hat{\eta}; T) + \xi\hat{\eta}, \end{aligned} \tag{C.27}$$

which together with (C.26) proves the desired result.

C.4 Proof of Theorem 3.8

In this proof, we omit the superscript P for the dual FBSDE for notational simplicity.

Part (i). Step 1. Since $q^{*,P}$ is optimal, for any bounded $q \in \mathcal{Q}_{[t,T]}$ and $\varepsilon \in (0,1)$, we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}\left[\left(\tilde{U}\left(T, M_T^{\varepsilon,q}\right) + M_T^{\varepsilon,q} P\right) - \left(\tilde{U}\left(T, M_T^*\right) + M_T^* P\right) \middle| \mathcal{F}_t\right] \geq 0, \tag{C.28}$$

where $M^{\varepsilon,q} := \eta M^{t,q^{*,P}+\varepsilon q}$ and $M^* := \eta M^{t,q^{*,P}}$. Note that

$$\frac{1}{\varepsilon}\left(\left(\tilde{U}\left(T, M_T^{\varepsilon,q}\right) + M_T^{\varepsilon,q} P\right) - \left(\tilde{U}\left(T, M_T^*\right) + M_T^* P\right)\right) = \frac{1}{\varepsilon}\int_0^1 \left(\tilde{U}_z\left(T, M_T^{\delta\varepsilon,q}\right) + P\right) \partial_\delta M_T^{\delta\varepsilon,q} d\delta$$

and

$$\partial_\delta M_T^{\delta\varepsilon,q} = M_T^{\delta\varepsilon,q}\left(-\int_t^T \varepsilon q_r dW_r^2 - \int_t^T \left(q_r^{*,P} + \delta\varepsilon q_r\right)\varepsilon q_r dr\right).$$

Therefore, by the uniformly integrability assumption and dominated convergence, (C.28) implies that

$$\begin{aligned}0 &\le \lim_{\varepsilon\to 0}\mathbb{E}\left[\int_0^1 \left(\tilde{U}_z\left(T, M_T^{\delta\varepsilon,q}\right) + P\right) M_T^{\delta\varepsilon,q}\left(-\int_t^T q_r dW_r^2 - \int_t^T \left(q_r^{*,P} + \delta\varepsilon q_r\right) q_r dr\right) d\delta \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\left(\tilde{U}_z\left(T, M_T^*\right) + P\right) M_T^*\left(-\int_t^T q_r dW_r^2 - \int_t^T q_r^{*,P} q_r dr\right)\middle| \mathcal{F}_t\right].\end{aligned}$$

Replacing q by $-q$, we obtain that

$$0 = \mathbb{E}\left[\left(\tilde{U}_z\left(T, M_T^*\right) + P\right) M_T^*\left(\int_t^T q_r dW_r^2 + \int_t^T q_r^{*,P} q_r dr\right)\middle| \mathcal{F}_t\right] = \mathbb{E}\left[\Gamma_T H_T^q \middle| \mathcal{F}_t\right], \tag{C.29}$$

where, for $0 \le t \le s \le T$, the processes Γ and H^q are defined as

$$\Gamma_s := \mathbb{E}\left[\left(\tilde{U}_z\left(T, M_T^*\right) + P\right) M_T^*\middle| \mathcal{F}_s\right] \quad \text{and} \quad H_s^q := \int_t^s q_r dW_r^2 + \int_t^s q_r^{*,P} q_r dr. \tag{C.30}$$

From the martingale representation theorem, there exists a square integrable density process $\psi = (\psi^1, \psi^2)$ on $[t,T]$ such that

$$d\Gamma_s = \psi_s^\top dW_s, \quad 0 \le t \le s \le T. \tag{C.31}$$

Step 2. We introduce an $\mathbb{F}$-progressively measurable process $\tilde{Y}$ on $[t,T]$ such that

$$\left(\tilde{U}_z\left(s, M_s^*\right) + \tilde{Y}_s\right) M_s^* = \Gamma_s = \mathbb{E}\left[\left(\tilde{U}_z\left(T, M_T^*\right) + P\right) M_T^*\middle| \mathcal{F}_s\right], \quad t \le s \le T.$$

In other words, $\tilde{Y}_s := \frac{\Gamma_s}{M_s^*} - \tilde{U}_z\left(s, M_s^*\right)$, $t \le s \le T$, and $\tilde{Y}_T = P$. Applying Itô's formula to $\tilde{U}_z$ in (2.15), Γ in (C.31) and to M^*, we obtain

$$\begin{aligned}d\tilde{Y}_s = &\left(\frac{\Gamma_s}{M_s^*}\left(|\theta_s|^2 + \left|q_s^{*,P}\right|^2\right) + \frac{\psi_s^1\theta_s + \psi_s^2 q_s^{*,P}}{M_s^*} - \tilde{\alpha}_z^1\left(s, M_s^*\right)\theta_s - \frac{\tilde{\alpha}_z^2\left(s, M_s^*\right)\tilde{\alpha}_{zz}^2\left(s, M_s^*\right)}{\tilde{U}_{zz}\left(s, M_s^*\right)}\right. \\ &+ \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* |\theta_s|^2 + \frac{1}{2}\tilde{U}_{zzz}\left(s, M_s^*\right)\frac{\left|\tilde{\alpha}_z^2\left(s, M_s^*\right)\right|^2}{\left|\tilde{U}_{zz}\left(s, M_s^*\right)\right|^2} - \frac{1}{2}\tilde{U}_{zzz}\left(s, M_s^*\right)\left|M_s^* q_s^{*,P}\right|^2 \\ &\left.+ \tilde{\alpha}_{zz}^2\left(s, M_s^*\right) M_s^* q_s^{*,P}\right) ds + \left(\frac{\Gamma_s\theta_s + \psi_s^1}{M_s^*} - \tilde{\alpha}_z^1\left(s, M_s^*\right) + \tilde{U}_{zz}\left(s, M_s^*\right) M_s^*\theta_s\right) dW_s^1 \\ &+ \left(\frac{\Gamma_s q_s^{*,P} + \psi_s^2}{M_s^*} - \tilde{\alpha}_z^2\left(s, M_s^*\right) + \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* q_s^{*,P}\right) dW_s^2.\end{aligned} \tag{C.32}$$

Next, let, for $0 \le t \le s \le T$,

$$\begin{aligned}\tilde{Z}_s^1 &:= \frac{\Gamma_s\theta_s + \psi_s^1}{M_s^*} - \tilde{\alpha}_z^1\left(s, M_s^*\right) + \tilde{U}_{zz}\left(s, M_s^*\right) M_s^*\theta_s, \\ \tilde{Z}_s^2 &:= \frac{\Gamma_s q_s^{*,P} + \psi_s^2}{M_s^*} - \tilde{\alpha}_z^2\left(s, M_s^*\right) + \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* q_s^{*,P}.\end{aligned} \tag{C.33}$$

It, then, follows that

$$\begin{aligned}\psi_s^1 &= \left(\tilde{Z}_s^1 + \tilde{\alpha}_z^1\left(s, M_s^*\right) - \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* \theta_s\right) M_s^* - \Gamma_s \theta_s,\\ \psi_s^2 &= \left(\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, M_s^*\right) - \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* q_s^{*,P}\right) M_s^* - \Gamma_s q_s^{*,P}.\end{aligned} \tag{C.34}$$

Substituting (C.34) into (C.32) yields

$$d\tilde{Y}_s = -\tilde{f}\left(s, M_s^*, \tilde{Y}_s, \tilde{Z}_s\right) ds + \tilde{Z}_s^\top dW_s, \quad 0 \le t \le s \le T, \tag{C.35}$$

where

$$\begin{aligned}\tilde{f}\left(s,d,y,z\right) :=& - z^1\theta_s - z^2 q_s^{*,P} - \tilde{\alpha}_z^2\left(s,d\right) q_s^{*,P} + \tilde{U}_{zz}\left(s,d\right) d\left|q_s^{*,P}\right|^2 + \frac{\tilde{\alpha}_z^2\left(s,d\right)\tilde{\alpha}_{zz}^2\left(s,d\right)}{\tilde{U}_{zz}\left(s,d\right)}\\ &- \frac{1}{2}\tilde{U}_{zzz}\left(s,d\right)\frac{\left|\tilde{\alpha}_z^2\left(s,d\right)\right|^2}{\left|\tilde{U}_{zz}\left(s,d\right)\right|^2} + \frac{1}{2}\tilde{U}_{zzz}\left(s,d\right)\left|dq_s^{*,P}\right|^2 - \tilde{\alpha}_{zz}^2\left(s,d\right) dq_s^{*,P}.\end{aligned} \tag{C.36}$$

Step 3. Applying Itô's formula to Γ in (C.31) with ψ as in (C.34) and H^q as in (C.30), we obtain that, for any bounded $q \in \mathcal{Q}_{[t,T]}$,

$$d\left(\Gamma_s H_s^q\right) = \; q_s\left(\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, M_s^*\right) - \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* q_s^{*,P}\right) M_s^* ds + H_s^q \psi_s^1 dW_s^1 + \left(\Gamma_s q_s + H_s^q \psi_s^2\right) dW_s^2.$$

From the definition of H^q in (C.30) and the Burkholder-Davis-Gundy inequality, we have that $\mathbb{E}[\sup_{t\le s\le T}|H_s^q|^2] < \infty$. In turn, applying similar arguments to the ones used to establish Lemma C.1, we deduce that $\int_t^s H_r^q \psi_r^1 dW_r^1 + \int_t^s \left(\Gamma_r q_r + H_r^q \psi_r^2\right) dW_r^2$, $0 \le t \le s \le T$, is a true $\mathbb{F}$-martingale.

Thus, (C.29) gives

$$\mathbb{E}\left[\int_t^T q_s\left(\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, M_s^*\right) - \tilde{U}_{zz}\left(s, M_s^*\right) M_s^* q_s^{*,P}\right) M_s^* ds \middle| \mathcal{F}_t\right] = 0,$$

which together with the arbitrariness of q implies that, for $0 \le t \le s \le T$,

$$q_s^{*,P} = \frac{\tilde{Z}_s^2 + \tilde{\alpha}_z^2\left(s, M_s^*\right)}{\tilde{U}_{zz}\left(s, M_s^*\right) M_s^*}. \tag{C.37}$$

Step 4. Combining (C.37) and (C.36) yields that $\tilde{f}$ in (C.36) is given by

$$\tilde{f}\left(s,d,y,z\right) = -z^1\theta_s + \frac{1}{2}\tilde{U}_{zzz}\left(s,d\right)\frac{\left|z^2\right|^2}{\left|\tilde{U}_{zz}\left(s,d\right)\right|^2} + \tilde{U}_{zzz}\left(s,d\right)\frac{\tilde{\alpha}_z^2\left(s,d\right) z^2}{\left|\tilde{U}_{zz}\left(s,d\right)\right|^2} - \frac{\tilde{\alpha}_{zz}^2\left(s,d\right) z^2}{\tilde{U}_{zz}\left(s,d\right)}.$$

Thus,

$$d\tilde{Y}_s = -\tilde{f}\left(s, D_s, \tilde{Y}_s, \tilde{Z}_s\right) ds + \tilde{Z}_s^\top dW_s, \quad \tilde{Y}_T = P,$$

and the process $M^* = \eta M^{t,q^{*,P}}$, with $q^{*,P}$ given in (C.37), provides the forward component of the FBSDE (3.16).

Part (ii). Using the duality relations (2.10) and (2.13), we rewrite $q^{*,P}$ as

$$q_s^{*,P} = -\frac{U_{xx}\left(s, -\tilde{U}_z\left(s, D_s\right)\right)}{U_x\left(s, -\tilde{U}_z\left(s, D_s\right)\right)}\left(\frac{\alpha_x^2\left(s, -\tilde{U}_z\left(s, D_s\right)\right)}{U_{xx}\left(s, -\tilde{U}_z\left(s, D_s\right)\right)} + \tilde{Z}_s^2\right),$$

which, together with Assumption 2.3 (ii) yields that $q^{*,P} \in \mathcal{Q}_{[t,T]}$.

Next, we show that, for any $q \in \mathcal{Q}_{[t,T]}$,

$$\mathbb{E}\left[\tilde{U}\left(T, D_T\right) + D_T P \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[\tilde{U}\left(T, \eta M_T^{t,q}\right) + \eta M_T^{t,q} P \middle| \mathcal{F}_t\right].$$

Indeed, from the definition of $q^{*,P}$, we have that $\eta M^{t,q^{*,P}} = D$. Furthermore, from Lemma 3.9 and Lemma 2.1, we have that $(\tilde{U}_z(s, D_s) + \tilde{Y}_s)D_s$ is a true $\mathbb{F}$-martingale. Thus, (2.12) implies

$$\begin{aligned}\mathbb{E}\left[\tilde{U}\left(T, D_T\right) + D_T P \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) + D_T\left(\tilde{U}_z\left(T, D_T\right) + P\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) \middle| \mathcal{F}_t\right] + \eta\left(\tilde{U}_z\left(t, \eta\right) + \tilde{Y}_t\right). \end{aligned} \tag{C.38}$$

On the other hand, from (2.9) and (3.19), we have

$$\begin{aligned}&\mathbb{E}\left[\tilde{U}\left(T, \eta M_T^{t,q}\right) + \eta M_T^{t,q} P \middle| \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) + \eta M_T^{t,q} \tilde{U}_z\left(T, D_T\right) + \eta M_T^{t,q} P \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) + \eta M_T^{t,q}\left(\tilde{U}_z\left(t, \eta\right) + \tilde{Y}_t - \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right)\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T\right)\right) \middle| \mathcal{F}_t\right] + \eta\left(\tilde{U}_z\left(t, \eta\right) + \tilde{Y}_t\right), \end{aligned} \tag{C.39}$$

where, in the last equality, we used Lemma 2.1 with π replaced by $\pi^{*,P}$, and $q \in \mathbb{L}^2_{\mathrm{BMO}}[t,T]$. Combining (C.38) and (C.39) we conclude.

C.5 Proof of Theorem 3.10

Part (i). From Lemma 3.9 we have that $\pi^{*,P} \in \mathcal{A}_{[t,T]}$ and

$$\mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] = \mathbb{E}\left[U\left(T, -\tilde{U}_z\left(T, D_T^P\right)\right) \middle| \mathcal{F}_t\right].$$

Next, we verify that, for any $\pi \in \mathcal{A}_{[t,T]}$,

$$\mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_r\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right]. \tag{C.40}$$

To this end, by the concavity of U, the equality (3.19) and the duality equalities (2.10), we obtain that

$$\begin{aligned}&\mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_r\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] - \mathbb{E}\left[U\left(T, \hat{\xi} + \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) + P\right) \middle| \mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[U_x\left(T, \hat{\xi} + \int_t^T \pi_r^{*,P}\left(\theta_r dr + dW_r^1\right) + P\right) \int_t^T \left(\pi_r - \pi_r^{*,P}\right)\left(\theta_r dr + dW_r^1\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U_x\left(T, -\tilde{U}_z\left(T, D_T^P\right)\right) \int_t^T \left(\pi_r - \pi_r^{*,P}\right)\left(\theta_r dr + dW_r^1\right) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[D_T^P \int_t^T \left(\pi_r - \pi_r^{*,P}\right)\left(\theta_r dr + dW_r^1\right) \middle| \mathcal{F}_t\right]. \end{aligned} \tag{C.41}$$

Note that, by Theorem 3.8 (ii), we have $D^P = \eta M^{t,q^{*,P}}$ with $q^{*,P} \in \mathcal{Q}_{[t,T]}$, and $\pi - \pi^{*,P} \in \mathcal{A}_{[t,T]}$. Thus, Lemma 2.1 gives $\mathbb{E}[D_T^P \int_t^T (\pi_r - \pi_r^{*,P}) (\theta_r dr + dW_r^1) | \mathcal{F}_t] = 0$, which, together with (C.41), establishes (C.40).

Part (ii). From the inequality (C.25), we obtain that

$$\tilde{u}^P(t,\eta;T) \geq \operatorname*{esssup}_{\xi \in \cap_{p\geq 1} L^p(\mathcal{F}_t)} (u^P(t,\xi;T) - \xi\eta).$$

On the other hand, using (i) of the theorem and the equalities (C.38), we have that

$$u^P\left(t,\hat{\xi};T\right) = \mathbb{E}\left[U\left(T,-\tilde{U}_z\left(T,D_T^P\right)\right)\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\tilde{U}\left(T,D_T^P\right) + D_T^P P\middle|\mathcal{F}_t\right] - \eta\hat{\xi},$$

which, combined with Theorem 3.8 (ii), yields that $u^P(t,\hat{\xi};T) = \tilde{u}^P(t,\eta;T) + \hat{\xi}\eta$.

C.6 Proof of Proposition 3.14

The key idea behind the proof of the maturity independence of the value function $\tilde{u}^P$ is to exploit the self-generation property of $\tilde{U}$.

For $0 \leq t \leq T \leq T'$ and any $q \in \mathcal{Q}_{[t,T']}$, we have

$$M_{T'}^{t,q} = M_T^{t,q} M_{T'}^{T,q} \quad \text{and} \quad \mathbb{E}[M_{T'}^{T,q} | \mathcal{F}_T] = 1,$$

since $q \in \mathbb{L}^2_{BMO}[t,T']$. Then, from (3.2) and the tower property of conditional expectation, we obtain

$$\begin{aligned}\tilde{u}^P(t,\eta;T') &= \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T']}} \mathbb{E}\left[\mathbb{E}\left[\tilde{U}\left(T',\eta M_T^{t,q} M_{T'}^{T,q}\right) + \eta M_T^{t,q} M_{T'}^{T,q} P\middle|\mathcal{F}_T\right]\middle|\mathcal{F}_t\right] \\ &= \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T']}} \mathbb{E}\left[\mathbb{E}\left[\tilde{U}\left(T',\eta M_T^{t,q} M_{T'}^{T,q}\right)\middle|\mathcal{F}_T\right] + \eta M_T^{t,q} P\middle|\mathcal{F}_t\right]. \end{aligned} \tag{C.42}$$

On the other hand, from Proposition 3.7, we have that, for any $q \in \mathcal{Q}_{[T,T']}$,

$$\mathbb{E}\left[\tilde{U}\left(T',\eta M_T^{t,q} M_{T'}^{T,q}\right)\middle|\mathcal{F}_T\right] \geq \tilde{U}\left(T,\eta M_T^{t,q}\right),$$

and, furthermore, that there exists $q^* \in \mathcal{Q}[T,T']$ such that

$$\tilde{U}\left(t,\eta M_T^{t,q}\right) = \mathbb{E}\left[\tilde{U}\left(T',\eta M_T^{t,q} M_{T'}^{T,q^*}\right)\middle|\mathcal{F}_T\right].$$

It, then, follows that

$$\tilde{u}^P(t,\eta;T') \geq \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(T,\eta M_T^{t,q}\right) + \eta M_T^{t,q} P\middle|\mathcal{F}_t\right],$$

and

$$\begin{aligned}\tilde{u}^P(t,\eta;T') &\leq \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\mathbb{E}\left[\tilde{U}\left(T',\eta M_T^{t,q} M_{T'}^{T,q^*}\right)\middle|\mathcal{F}_T\right] + \eta M_T^{t,q} P\middle|\mathcal{F}_t\right] \\ &= \operatorname*{essinf}_{q\in\mathcal{Q}_{[t,T]}} \mathbb{E}\left[\tilde{U}\left(t,\eta M_T^{t,q}\right) + \eta M_T^{t,q} P\middle|\mathcal{F}_t\right].\end{aligned}$$

Combining the above yields $\tilde{u}^P(t,\eta;T') = \tilde{u}^P(t,\eta;T)$.

To show the maturity independence of the value function u^P, we first observe that FBSDE (3.3) on $[t,T]$ with initial-terminal condition (ξ,P) can be extended to $[t,T']$. Indeed, if (X^P,Y^P,Z^P) is

a solution to FBSDE (3.3) on $[t,T]$ with initial-terminal condition (ξ,P), then $(\bar{X}^P,\bar{Y}^P,\bar{Z}^P)$ defined by

$$\begin{aligned}\bar{X}_s^P &:= X_s^P\mathbf{1}_{[t,T]}(s)+X_s^{T',P}\mathbf{1}_{(T,T']}(s),\\ \bar{Y}_s^P &:= Y_s^P\mathbf{1}_{[t,T]}(s)+P\mathbf{1}_{(T,T']}(s),\\ \bar{Z}_s^P &:= Z_s^P\mathbf{1}_{[t,T]}(s)+0\cdot\mathbf{1}_{(T,T']}(s),\end{aligned}$$

on $0\le t\le s\le T'$, is a solution to FBSDE (3.3) on $[t,T']$ with initial-terminal condition (ξ,P), where $X_s^{T',P}:=\tilde{X}_s^P-P$, $T\le s\le T'$, with $\tilde{X}^P$ satisfying SDE (2.5) on $[T,T']$ with initial condition $\tilde{X}_T^P=X_T^P+P$. Thus, using the first part of the proposition and Theorem 3.5 (ii), we obtain that

$$u^P(t,\xi;T')=\operatorname*{essinf}_{\eta\in L^{0,+}(\mathcal{F}_t)}\left(\tilde{u}^P(t,\eta;T')+\xi\eta\right)=\operatorname*{essinf}_{\eta\in L^{0,+}(\mathcal{F}_t)}\left(\tilde{u}^P(t,\eta;T)+\xi\eta\right)=u^P(t,\xi;T),$$

and we easily conclude.

C.7 Proof of Theorem 4.4

In line with [26, Theorem 3.4], the main steps are to verify all the conditions in Theorem B.8, which in turn yields the global existence and uniqueness of the FBSDE (4.18). For notational simplicity, we omit the superscript P from the FBSDE in the following.

Denote

$$B_s^1:=W_s^1+\int_0^s\theta(V_r)\,dr\quad\text{and}\quad B_s^2:=W_s^2,\quad 0\le s\le T.$$

Then, B is a Brownian motion under a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, by Girsanov's theorem. We note that B and W generate the same augmented filtration $\mathbb{F}=(\mathcal{F}_t)_{t\ge0}$.

Therefore, the aim is now to solve FBSDE (4.18) including the SDE for stochastic factor V in (4.5). To this end, we consider a FBSDE satisfying MLLC in Definition B.5. Namely, for sufficient small $\varepsilon>0$ and any $0\le t\le s\le T$,

$$\begin{cases}\tilde{V}_s=\tilde{v}+\displaystyle\int_t^s\frac{1}{\varepsilon}\left(l\left(\varepsilon\tilde{V}_r\right)-\rho\theta\left(\varepsilon\tilde{V}_r\right)\right)dr+\frac{1}{\varepsilon}\int_t^s\left(\rho dB_r^1+\sqrt{1-\rho^2}dB_r^2\right),\ \rho\in(0,1),\\ X_s=x-\displaystyle\int_t^s\left(\psi\left(r,\varepsilon\tilde{V}_r,X_r+Y_r\right)+Z_r^1\right)dB_r^1,\\ Y_s=P\left(\varepsilon\tilde{V}_T,X_T\right)+\displaystyle\int_s^T\left(\frac{1}{2}\phi^1\left(r,\varepsilon\tilde{V}_r,X_r+Y_r\right)\left|Z_r^2\right|^2+\sqrt{1-\rho^2}\phi^2\left(r,\varepsilon\tilde{V}_r,X_r+Y_r\right)Z_r^2\right)dr\\ \qquad-\displaystyle\int_s^T Z_r^1dB_r^1-\int_s^T Z_r^2dB_r^2.\end{cases}\tag{C.43}$$

This FBSDE is equivalent to the original one in that, if $(\tilde{V},X,Y,Z)$ solves FBSDE (C.43), then $V=\varepsilon\tilde{V}$ solves SDE (4.5) with a modified initial condition and (X,Y,Z) solves the original FBSDE (4.18). Therefore, it suffices to study FBSDE (C.43) which satisfies MLLC in Definition B.5. To align with the form of FBSDE (B.4), we use the notation

$$\begin{aligned}&H(v,x)=P(v,x),\quad \tilde{\mu}(r,v)=l(v)-\rho\theta(v),\\ &\bar{\sigma}\left(r,v,x+y,z^1\right)=-\psi(r,v,x+y)-z^1,\\ &f\left(r,v,x+y,z^2\right)=-\frac{1}{2}\phi^1(r,v,x+y)\left|z^2\right|^2-\sqrt{1-\rho^2}\phi^2(r,v,x+y)z^2,\end{aligned}$$

and $B, \tilde{V}, X$, respectively, play the role of $W, \tilde{X}, \bar{X}$.

Next, we verify that all the conditions of Theorem B.8 hold which will, in turn, yield that $I^M_{\max} = [0,T]$ and the global existence of the solution would follow.

To this end, we first note that all conditions listed before Theorem B.8 are satisfied. Indeed, the conditions for H and $\tilde{\mu}$ follow from Assumption 4.2 (ii) and (iii), and one can easily verify the conditions for $\bar{\sigma}$ and f by taking derivatives and using Assumption 4.1 (ii) and Assumption 4.2 (i).

Thus, the main task is to deduce a uniform bound of w_x for any weakly regular decoupling field $w : [t,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of (C.43). Similarly to the proof of [26, Theorem 3.4], the main tool is to use the dynamics of the process

$$\Phi_s := w_x \left(s, \tilde{V}_s, X_s\right),$$

which is bounded by $L^{-1}_{\bar{\sigma},z^1} = 1$ (but not uniformly) due to the weakly regularity. Note, however, that w is also strongly regular by Theorem B.7. Thus, differentiating FBSDE (C.43) with respect to x, using the chain rule (see [25, Lemma A.3.1]), we obtain that the weak derivatives of X and Y with respect to x are

$$\begin{aligned}
\partial_x X_s &= 1 + \int_t^s \left(\delta^{\bar{\sigma}}_{x+y} \left(\partial_x X_r + \partial_x Y_r\right) - \partial_x Z^1_r\right) dB^1_r, \\
\partial_x Y_s &= \partial_x P\left(\varepsilon \tilde{V}_T, X_T\right) - \int_s^T \left(\delta^f_{x+y} \left(\partial_x X_r + \partial_x Y_r\right) + \delta^f_z \partial_x Z^2_r\right) dr - \int_s^T \partial_x Z^1_r dB^1_r - \int_s^T \partial_x Z^2_r dB^2_r,
\end{aligned}$$

where

$$\begin{aligned}
\delta^{\bar{\sigma}}_{x+y} &:= -\psi_x \left(r, \varepsilon \tilde{V}_r, X_r + Y_r\right), \\
\delta^f_{x+y} &:= -\frac{1}{2} \phi^1_x \left(r, \varepsilon \tilde{V}_r, X_r + Y_r\right) \left|Z^2_r\right|^2 - \sqrt{1-\rho^2} \phi^2_x \left(r, \varepsilon \tilde{V}_r, X_r + Y_r\right) Z^2_r, \\
\delta^f_z &:= -\phi^1 \left(r, \varepsilon \tilde{V}_r, X_r + Y_r\right) Z^2_r - \sqrt{1-\rho^2} \phi^2 \left(r, \varepsilon \tilde{V}_r, X_r + Y_r\right).
\end{aligned} \tag{C.44}$$

Next, we observe that, by Assumption 4.1, Assumption 4.2 and the uniform boundedness of Z (see Remark B.9), $\delta^{\bar{\sigma}}_{x+y}$, δ^f_{x+y} and δ^f_z are all uniformly bounded independently of t and ε.

Let τ, $t \le \tau \le T$, be any stopping time such that $\partial_x X > 0$ a.e. on $[t,\tau]$. Using the chain rule in [25, Lemma A.3.1], we have

$$\partial_x Y_s = \partial_x \left(w\left(s, \tilde{V}_s, X_s\right)\right) = \Phi_s \partial_x X_s, \quad 0 \le t \le s \le \tau,$$

i.e., $\Phi_s = \frac{\partial_x Y_s}{\partial_x X_s}$. Applying Itô's formula yields that

$$d\left(\frac{1}{\partial_x X_r}\right) = -\frac{1}{\partial_x X_r} \left(\delta^{\bar{\sigma}}_{x+y} \left(1 + \Phi_r\right) - \frac{\partial_x Z^1_r}{\partial_x X_r}\right) dB^1_r + \frac{1}{\partial_x X_r} \left|\delta^{\bar{\sigma}}_{x+y} \left(1 + \Phi_r\right) - \frac{\partial_x Z^1_r}{\partial_x X_r}\right|^2 dr,$$

and, thus,

$$\begin{aligned}
d\Phi_r &= \left(\frac{\partial_x Z^1_r}{\partial_x X_r} - \Phi_r \delta^{\bar{\sigma}}_{x+y}\right) \left(1 + \Phi_r\right) dB^1_r + \frac{\partial_x Z^2_r}{\partial_x X_r} dB^2_r \\
&+ \left(\Phi_r \left|\delta^{\bar{\sigma}}_{x+y} \left(1 + \Phi_r\right) - \frac{\partial_x Z^1_r}{\partial_x X_r}\right|^2 + \delta^f_{x+y} \left(1 + \Phi_r\right) + \delta^f_z \frac{\partial_x Z^2_r}{\partial_x X_r} - \frac{\partial_x Z^1_r}{\partial_x X_r} \left(\delta^{\bar{\sigma}}_{x+y} \left(1 + \Phi_r\right) - \frac{\partial_x Z^1_r}{\partial_x X_r}\right)\right) dr.
\end{aligned}$$

Let,

$$\hat{Z}^1_r := \left(\frac{\partial_x Z^1_r}{\partial_x X_r} - \Phi_r \delta^{\bar{\sigma}}_{x+y}\right) \left(1 + \Phi_r\right) \quad \text{and} \quad \hat{Z}^2_r := \frac{\partial_x Z^2_r}{\partial_x X_r}, \quad s \le r \le \tau.$$

Then, we obtain that $(\Phi, \hat{Z})$ solves

$$\begin{aligned}\Phi_s =&\Phi_\tau - \int_s^\tau \left(\delta^f_{x+y}\left(1+\Phi_r\right) + \delta^f_z \hat{Z}^2_r - \hat{Z}^1_r\left(\delta^{\bar{\sigma}}_{x+y}\left(1+\Phi_r\right) - \frac{\hat{Z}^1_r + \Phi_r \delta^{\bar{\sigma}}_{x+y}\left(1+\Phi_r\right)}{1+\Phi_r}\right)\right) dr \\ &- \int_s^\tau \hat{Z}^\top_r dB_r,\end{aligned} \tag{C.45}$$

which is a quadratic BSDE. According to [25, Theorem A.1.11], and using that Φ is uniformly bounded and $\hat{Z}$ is well-defined in L^2 by strong regularity, we have that $\hat{Z}$ is a BMO process. Using a similar argument as in [26, Lemma A.1], it can be shown that $\tau = T$ in (C.45). In turn, applying Itô's formula gives

$$\ln\left(1+\Phi_s\right) = \ln\left(1+\Phi_T\right) - \int_s^T \left(\delta^f_{z^2}\bar{Z}^2_r + \delta^f_{x+y} - \delta^{\bar{\sigma}}_{x+y}\bar{Z}^1_r + \frac{1}{2}\left|\bar{Z}^1_r\right|^2 - \frac{1}{2}\left|\bar{Z}^2_r\right|^2\right) dr - \int_s^T \bar{Z}^\top_r dB_r, \tag{C.46}$$

where $\bar{Z}^i := \frac{\hat{Z}^i}{1+\Phi}, i = 1, 2$. Using once more [25, Theorem A.1.11], we obtain that $\bar{Z}^i$ is also a BMO process.

Next, we use (C.46) to obtain the upper and lower bounds for w_x.

Upper bound: Rewrite (C.46) as

$$\begin{aligned}\ln\left(1+\Phi_s\right) =& \ln\left(1+\Phi_T\right) - \int_s^T \delta^f_{x+y} dr \\ &- \int_s^T \bar{Z}^1_r \left(dB^1_r + \left(\frac{1}{2}\bar{Z}^1_r - \delta^{\bar{\sigma}}_{x+y}\right) dr\right) - \int_s^T \bar{Z}^2_r \left(dB^2_r + \left(\delta^f_{z^2} - \frac{1}{2}\bar{Z}^2_r\right) dr\right) \\ :=& \ln\left(1+\Phi_T\right) - \int_s^T \delta^f_{x+y} dr - \int_s^T \bar{Z}^\top_r d\tilde{B}_r,\end{aligned}$$

where

$$d\tilde{B}^1_r := dB^1_r + \left(\frac{1}{2}\bar{Z}^1_r - \delta^{\bar{\sigma}}_{x+y}\right) dr, \quad d\tilde{B}^2_r := dB^2_r + \left(\delta^f_{z^2} - \frac{1}{2}\bar{Z}^2_r\right) dr,$$

and $\frac{1}{2}\bar{Z}^1_r - \delta^{\bar{\sigma}}_{x+y}$ and $\delta^f_{z^2} - \frac{1}{2}\bar{Z}^2_r$ are both BMO process. Using Girsanov's theorem for BMO processes, we have that $\tilde{B}$ is Brownian motion under an equivalent probability measure $\tilde{\mathbb{Q}}$. Therefore,

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln\left(1+\Phi_s\right)\right] = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln\left(1+\Phi_T\right) - \int_s^T \delta^f_{x+y} dr\right]. \tag{C.47}$$

Recall the definition of δ^f_{x+y} in (C.44), where $\phi^1_x < 0$ from Assumption 4.2 (i). Noting that δ^f_{x+y} can be in fact seen as a quadratic function of Z^2, we have

$$\delta^f_{x+y} \geq \frac{1-\rho^2}{2} \frac{\left|\phi^2_x\left(r, \varepsilon\tilde{V}_r, X_r + Y_r\right)\right|^2}{\phi^1_x\left(r, \varepsilon\tilde{V}_r, X_r + Y_r\right)}. \tag{C.48}$$

Next, we look at the following distinct cases.

Case 1. $\phi^2_x = 0$. Then, it is obvious that $\delta^f_{x+y} \geq 0$ and, thus,

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln\left(1+\Phi_s\right)\right] \leq \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln\left(1+\Phi_T\right)\right] \leq \ln\left(1+L_{P,x}\right).$$

In particular, $\ln(1+w_x(t,\tilde{v},x)) \leq \ln\left(1+L_{P,x}\right)$. It follows that $w_x(t,\tilde{v},x) \leq L_{P,x} < 1$.

Case 2. $\phi_x^2 \neq 0$. Since $\frac{1-\rho^2}{2}\frac{|\phi_x^2|^2}{\phi_x^1} \geq -K$, we obtain from (C.48) and (C.47) that

$$\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln\left(1+\Phi_s\right)\right] \leq \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\ln\left(1+\Phi_T\right)\right] + KT \leq \ln\left(1+L_{P,x}\right) + KT,$$

and, in particular,

$$\ln(1+w_x(t,\tilde{v},x)) \leq \ln\left(1+L_{P,x}\right) + KT.$$

Therefore,

$$w_x(t,\tilde{v},x) \leq \left(1+L_{P,x}\right)e^{KT} - 1 < 1.$$

In summary, under either Case 1 or Case 2, $w_x(t,\tilde{v},x)$ is uniformly bounded by 1 from above.

Lower bound: Using (C.46) yields

$$\begin{aligned}
\ln\left(1+\Phi_s\right) =& \ln\left(1+\Phi_T\right) - \int_s^T \left(\delta_{x+y}^f - \frac{1}{2}\left|\bar{Z}_r^1\right|^2 - \frac{1}{2}\left|\bar{Z}_r^2\right|^2\right) dr \\
&- \int_s^T \bar{Z}_r^1 \left(dB_r^1 + \left(\bar{Z}_r^1 - \delta_{x+y}^{\bar{\sigma}}\right) dr\right) - \int_s^T \bar{Z}_r^2 \left(dB_r^2 + \delta_{z^2}^f dr\right) \\
:=& \ln\left(1+\Phi_T\right) - \int_s^T \left(\delta_{x+y}^f - \frac{1}{2}\left|\bar{Z}_r^1\right|^2 - \frac{1}{2}\left|\bar{Z}_r^2\right|^2\right) dr - \int_s^T \bar{Z}_r^\top d\hat{B}_r,
\end{aligned}$$

where $\hat{B}$ defined by

$$\begin{aligned}
d\hat{B}_r^1 &:= dB_r^1 + \left(\bar{Z}_r^1 - \delta_{x+y}^{\bar{\sigma}}\right) dr = dB_r^1 + \left(\bar{Z}_r^1 + \theta\varphi_x + \rho\left(\phi^2 - \frac{u_{xv}}{u_{xx}}\phi^1\right)\right) dr, \\
d\hat{B}_r^2 &:= dB_r^2 + \delta_{z^2}^f dr = dB_r^2 - \left(\phi^1 Z_r^2 + \sqrt{1-\rho^2}\phi^2\right) dr,
\end{aligned}$$

is a Brownian motion under an equivalent probability measure $\hat{\mathbb{Q}}$ due to the fact that $\bar{Z}^1 - \delta_{x+y}^{\bar{\sigma}}$ and $\delta_{z^2}^f$ are BMO processes. It, then, follows that

$$\mathbb{E}^{\hat{\mathbb{Q}}}\left[\ln\left(1+\Phi_s\right)\right] = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\ln\left(1+\Phi_T\right) - \int_s^T \left(\delta_{x+y}^f - \frac{1}{2}\left|\bar{Z}_r^1\right|^2 - \frac{1}{2}\left|\bar{Z}_r^2\right|^2\right) dr\right]. \tag{C.49}$$

Next, we obtain a uniform upper bound for δ_{x+y}^f in (C.44). To this end, applying Itô's formula to $\varphi(r, \varepsilon\tilde{V}_r, X_r + Y_r)$, we deduce, by using the definitions of $\hat{B}$ and of φ, ϕ^i and ψ, that,

$$\begin{aligned}
&d\varphi\left(r, \varepsilon\tilde{V}_r, X_r + Y_r\right) \\
&= \left(-\frac{1}{2}\varphi\phi_x^1\left|Z_r^2\right|^2 - \sqrt{1-\rho^2}\varphi\phi_x^2 Z_r^2 + \left(\varphi_x\psi - \rho\varphi_v\right)\bar{Z}_r^1\right. \\
&\quad \left. + \varphi_t + \varphi_v\left(l - \rho\theta\right) + \frac{1}{2}\varphi_{vv} + \frac{1}{2}\varphi_{xx}\left|\psi\right|^2 + \psi\left(\varphi_x\psi_x - \rho\varphi_{xv}\right) + \varphi_v\left(-\rho\psi_x + \left(1-\rho^2\right)\phi^2\right)\right) dr \\
&\quad + \left(\rho\varphi_v - \varphi_x\psi\right) d\hat{B}_r^1 + \left(\sqrt{1-\rho^2}\varphi_v + \varphi_x Z_r^2\right) d\hat{B}_r^2 \\
&=: \left(-\frac{1}{2}\varphi\phi_x^1\left|Z_r^2\right|^2 - \sqrt{1-\rho^2}\varphi\phi_x^2 Z_r^2 + \beta_r\bar{Z}_r^1 + \alpha_r\right) dr + \left(\rho\varphi_v - \varphi_x\psi\right) d\hat{B}_r^1 + \left(\sqrt{1-\rho^2}\varphi_v + \varphi_x Z_r^2\right) d\hat{B}_r^2,
\end{aligned}$$

where

$$\begin{aligned}
\beta_r &:= \varphi_x\psi - \rho\varphi_v, \\
\alpha_r &:= \varphi_t + \varphi_v\left(l - \rho\theta\right) + \frac{1}{2}\varphi_{vv} + \frac{1}{2}\varphi_{xx}\left|\psi\right|^2 + \psi\left(\varphi_x\psi_x - \rho\varphi_{xv}\right) + \varphi_v\left(-\rho\psi_x + \left(1-\rho^2\right)\phi^2\right).
\end{aligned}$$

Above, we omitted arguments $(r, \varepsilon\tilde{V}_r, X_r + Y_r)$ for simplicity. From Assumptions 4.1 and 4.2, we deduce that $\alpha, \beta, \varphi\phi_x^1, \varphi\phi_x^2$ and $\varphi_v, \varphi_x, \psi$ are uniformly bounded. Thus, taking expectation on both sides above yields

$$\begin{aligned}
&\mathbb{E}^{\hat{\mathbb{Q}}}\left[\varphi\left(s, \varepsilon\tilde{V}_s, X_s + Y_s\right)\right] \\
=&\mathbb{E}^{\hat{\mathbb{Q}}}\left[\varphi\left(T, \varepsilon\tilde{V}_T, X_T + Y_T\right)\right] - \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \left(-\frac{1}{2}\varphi\phi_x^1 \left|Z_r^2\right|^2 - \sqrt{1-\rho^2}\varphi\phi_x^2 Z_r^2 + \beta_r \bar{Z}_r^1 + \alpha_r\right) dr\right] \\
=&\mathbb{E}^{\hat{\mathbb{Q}}}\left[\varphi\left(T, \varepsilon\tilde{V}_T, X_T + Y_T\right)\right] + \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T (-\varphi)\, \delta_{x+y}^f dr\right] - \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \left(\beta_r \bar{Z}_r^1 + \alpha_r\right) dr\right]. \qquad \text{(C.50)}
\end{aligned}$$

It follows by Assumption 4.1 (ii) that

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \delta_{x+y}^f dr\right] &= \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \delta_{x+y}^f \mathbf{1}_{\delta_{x+y}^f < 0} dr\right] + \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \delta_{x+y}^f \mathbf{1}_{\delta_{x+y}^f \geq 0} dr\right] \\
&\leq \frac{1}{C_u}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T (-\varphi)\, \delta_{x+y}^f \mathbf{1}_{\delta_{x+y}^f < 0} dr\right] + \frac{1}{C_l}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T (-\varphi)\, \delta_{x+y}^f \mathbf{1}_{\delta_{x+y}^f \geq 0} dr\right] \\
&= \frac{1}{C_l}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T (-\varphi)\, \delta_{x+y}^f dr\right] - \frac{C_u - C_l}{C_l C_u}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T (-\varphi)\, \delta_{x+y}^f \mathbf{1}_{\delta_{x+y}^f < 0} dr\right],
\end{aligned}$$

which, together with (C.48), gives

$$\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \delta_{x+y}^f dr\right] \leq \frac{1}{C_l}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T (-\varphi)\, \delta_{x+y}^f dr\right] + \frac{C_u - C_l}{C_l} KT,$$

for both Case 1 and Case 2. Combining the above with (C.50), we deduce

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \delta_{x+y}^f dr\right] &\leq \frac{1}{C_l}\left(C_u + T\left\|\alpha\right\|_\infty\right) + \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \frac{\beta_r}{C_l}\bar{Z}_r^1 dr\right] + \frac{C_u - C_l}{C_l} KT \\
&:= \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \frac{\beta_r}{C_l}\bar{Z}_r^1 dr\right] + K_b.
\end{aligned}$$

Thus (C.49) implies

$$\begin{aligned}
\mathbb{E}^{\hat{\mathbb{Q}}}\left[\ln\left(1 + \Phi_s\right)\right] &\geq \mathbb{E}^{\hat{\mathbb{Q}}}\left[\ln\left(1 + \Phi_T\right)\right] - \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \delta_{x+y}^f dr\right] + \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \frac{1}{2}\left|\bar{Z}_r^1\right|^2 dr\right] \\
&\geq \ln\left(1 - L_{H,x}\right) - \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \frac{\beta_r}{C_l}\bar{Z}_r^1 dr\right] + \mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \frac{1}{2}\left|\bar{Z}_r^1\right|^2 dr\right] - K_b \\
&\geq \ln\left(1 - L_{H,x}\right) - \frac{1}{2}\mathbb{E}^{\hat{\mathbb{Q}}}\left[\int_s^T \left|\frac{\beta_r}{C_l}\right|^2 dr\right] - K_b \\
&\geq \ln\left(1 - L_{H,x}\right) - \frac{T}{2\left|C_l\right|^2}\left\|\beta\right\|_\infty^2 - K_b =: -C_b,
\end{aligned}$$

where $C_b > 0$ is independent of both ε and time. It, then, follows that $\ln\left(1 + w_x\left(t, \tilde{v}, x\right)\right) \geq -C_b$, and, in turn,

$$w_x\left(t, \tilde{v}, x\right) \geq e^{-C_b} - 1 > -1.$$

To conclude, we note that w_x can be bounded uniformly by 1 and, thus, we can apply Theorem B.8 to obtain the global existence result. The uniqueness can be proved by Lemma 2.5 in [26] and is omitted.

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