Intro & Motiv.

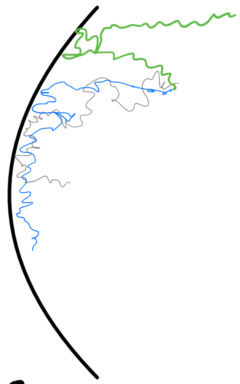
Simulation used in so-called milestoning algorithm:

$$dX_t = b\,dt + \sigma\, dW_t$$

& eval

$$I = \int_0^t f(s, X_s)\,ds + g(t, X_t)$$

- existing: Euler-Maruyama or Milstone ...



animation fig ①

Algorithm (E-M)

$$X_{n+1} = X_n + b(t_n, X_n)\cdot \Delta t + \sigma(t_n, X_n)\cdot \Delta W_t$$

- problem: how to handle BC?

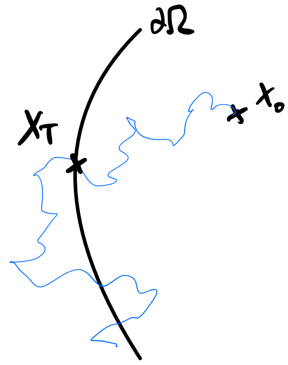
SDE w/ absorbing BC

to absorb means the process "stops" when reaching the boundary.

let $\tau := \inf\{t : X_t \in \partial\Omega\}$ be a stopping time.

then to simulate X_t $(0 \le t \le \tau)$,

we modify the (E-M) algorithm:



Alg (E-M w/ stopping time)

- propose $X_{n+1} = \cdots$ as in E-M

- determine
  + whether the process has stopped $\mathbb{P}(t_n \le \tau < t_{n+1} \mid X_n, X_{n+1})$
  + if so, where shall we placed X_τ $\quad X_\tau \mid (t_n \le \tau < t_{n+1}, X_n, X_{n+1})$

- a few assumptions to make life easier:
  + b, σ constant
  + $\partial\Omega$ is a half-plane

  not too unreasonable when $\Delta t \ll 1$

Roadmap: ① CMG theorem : b is not important

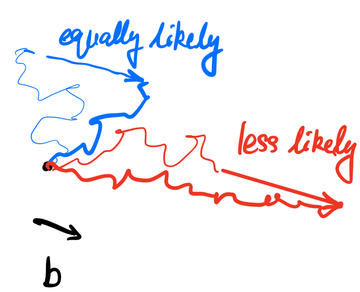
② apply rotation transform to reduce to a 1d prob.

③ deal with $X_\tau \mid (\tau < t_{n+1}, X_{n+1})$

① $\boxed{\text{Cameron-Martin-Girsanov thm.}}$

if. B_t is a BM under $\mathbb{P}$, then $\tilde{B}_t := B_t + b \cdot t$ is a BM under $\mathbb{Q}$ given $\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathcal{E}(\mathcal{B})_t = e^{\mathcal{B}_t - \frac{1}{2}\langle \mathcal{B}, \mathcal{B} \rangle_t}$

where $\mathcal{B}_t := b \cdot \tilde{B}_t$



Cor. $\mathbb{E}^{\mathbb{P}}[Y_t \mid X_t] = \mathbb{E}^{\mathbb{Q}}[Y_t \mid X_t]$

given Y_t adapted, $\mathbb{P}$ original prob & $\mathbb{Q}$ the modified prob

s.t. X_t is a BM.

② Further reduction: consider W_t a standard BM. that starts from 0 & stopping time $\tau := \inf\{t : W_t \geq u\}$ given $u>0$.

Prop. Given $v<u$ & $0 \leq t \leq T$, joint density of (τ, W_T):

$$q(\tau=t, W_T=v) = \frac{u \cdot \exp\left(\frac{Tu^2-2tuv+tv^2}{2t^2-2tT}\right)}{2\pi\sqrt{t^3(T-t)}}$$

Pf. $\mathbb{Q}(\tau \leq t, W_T \leq v)$

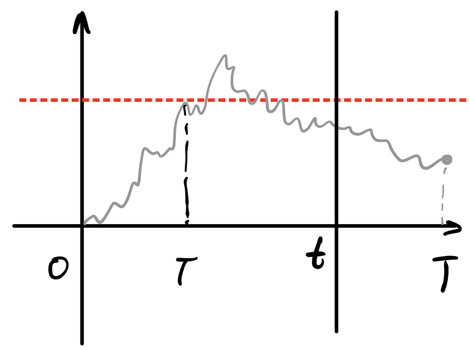
must have hit

$= \mathbb{Q}(\cancel{\tau \leq t}, W_T \leq v, W_t > u)$

$+ \mathbb{Q}(\tau \leq t, \underline{W_T \leq v, W_t \leq u})$ $\rightarrow \mathbb{Q}(W_T \geq 2u-v, W_t > u)$

$W_T \geq (2u-v), W_t > u$

by reflection principle.



Def. $IG(\mu, \lambda)$ p.d.f. $:= \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]$

Cor. Conditioned on $v<u$ & $\tau \leq T$, ς follows $IG\left(\frac{u}{u-v}, \frac{u^2}{T}\right)$

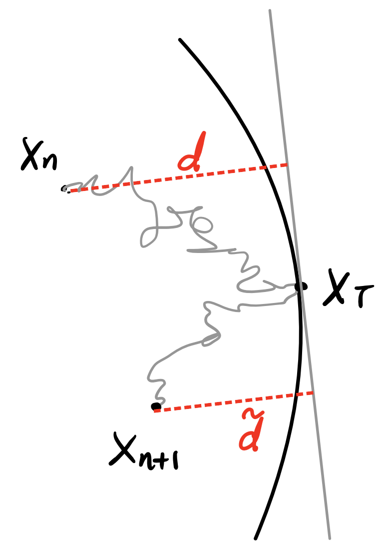
$\frac{\tau}{T-\tau}$

* illustration ??

Cor. Similarly, when $v > u$, one have $\zeta \sim IG\left(\frac{u}{v-u}, \frac{u^2}{T}\right)$

- Then the hitting prob. can be obtain as $Q(\tau \leq T \mid W_T = v)$

Cor. $P_{cross} = \begin{cases} 1 & v \geq u \\ e^{\frac{2u(v-u)}{T}} & v < u \end{cases}$



or in the numerical scheme

$$e^{-\frac{2d\tilde{d}^+}{\Delta t}}$$

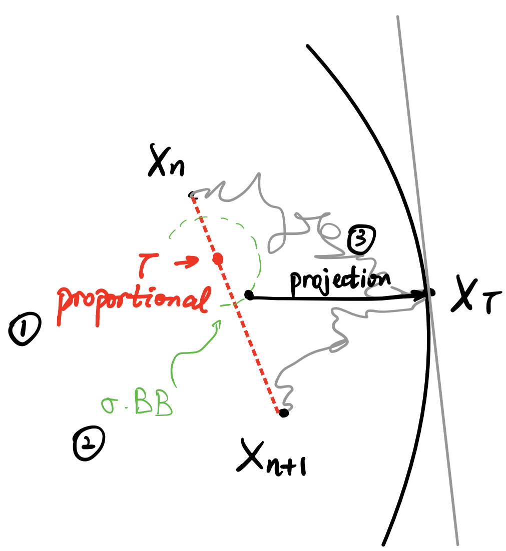
③ <u>Prop.</u> $X_\tau \mid \tau = t$ shares the same law as

$$X_n + P\left[\frac{t-t_n}{\Delta t}(X_{n+1} - X_n) + \sigma \xi_t\right]$$

where ξ_t BB,

$P := \sigma\,[I - a_1 a_1^T]\sigma^{-1}$

proj. mat., $a_1 := \widehat{\sigma^T a}$



- Reflected process

adding a local time to force process into Ω:

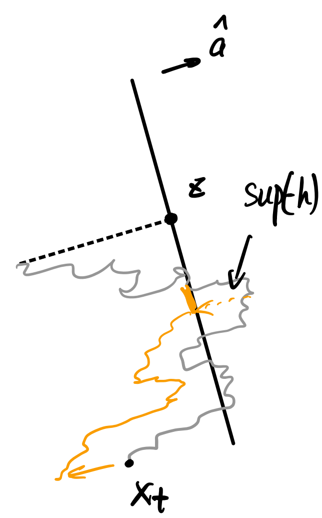
$$dY_t = b\,dt + \sigma\,dB_t + \gamma\,dk_t$$

where k_t increases only when $Y_t \in \partial\Omega$.

Theory is quite fishy about the existence of k_t, but we state that

Prop. Assuming $\sigma = I$, define $h_t := \hat{a}\cdot(z - X_t)$

then $$k_t = \max\{0, \sup_{0\le s\le t}(-h_s)\}$$



Cor. Introducing $m_t := \sup_{0\le s\le t} w_s$, then

$$k_t = \max\{0, m_t - u\}$$

Prop. Conditional density.

$$q(m_T = \mu \mid w_T = v) = \frac{2(2\mu - v)}{T}\exp\left(\frac{-2\mu^2 + 2\mu v}{T}\right)\mathbb{1}_{v^+<\mu}$$

Proof: Assuming $\mu > 0, \mu > v$ or o.w. trivial. By reflection principle,

$$\mathbb{Q}(W_T \le v,\ m_T \ge \mu) = \mathbb{Q}(W_T \le v,\ \overset{\swarrow\ \inf(\tau : w_t \ge \mu)}{T_\mu} \le T)$$

$$= \mathbb{Q}(W_T \ge 2\mu - v,\ T_\mu \le T)$$

$$= \mathbb{Q}(W_T \ge 2\mu - v) \quad \cdots \qquad \square$$

Def. Truncated Rayleigh Distribution tRayleigh(l, σ^2) if.

$$p(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2 - l^2}{2\sigma^2}\right)$$

Cor. $p(x) = \dfrac{\frac{2\mu - v}{2}}{T/4} \exp\left(-\dfrac{\left(\frac{2\mu - v}{2}\right)^2 - \left(\frac{v}{2}\right)^2}{T/4}\right) \Rightarrow m_T \sim \text{tRayleigh}\left(\frac{W_T}{2}, T/4\right)$