1. (5 pts) Find the slope in the **y-direction** at the point $P(2,3,f(2,3))$ on the graph of f when

$$f(x,y) = \arctan(\frac{x}{y})$$

Solution.

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{x}{y}\right) \\ &= \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) \\ &= -\frac{x}{y^2+x^2}\end{aligned}$$

Plug in $x=2$, $y=3$:

$$\frac{\partial f}{\partial y}(2,3) \text{ (or write as } f_y(2,3) \text{)} = -\frac{2}{3^2+2^2} = -\frac{2}{13}$$

2. (5 pts) Find the volume of the solid lying under the surface

$$z = \frac{y}{x(x^2+1)}$$

and above the rectange

$$D = \{(x,y) : 1 \le x \le 2, 0 \le y \le 2\}$$

Solution 1.

$$\begin{aligned}\int\int_D \frac{y}{x(x^2+1)} dA &= \int_1^2 \int_0^2 \frac{y}{x(x^2+1)} dy dx \\ &= \int_1^2 \left[\frac{1}{x(x^2+1)} \cdot \frac{y^2}{2}\bigg|_{y=0}^{y=2} dy\right] dx \\ &= \int_1^2 \frac{2}{x(x^2+1)} dx\end{aligned}$$

Use partial fraction decomposition:

$$\frac{2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

And solve for A, B, C:

$$\begin{aligned}2 &= A(x^2+1) + (Bx+C)x \\ 2 &= (A+B)x^2 + Cx + A\end{aligned}$$

Then $A=2$, $B=-2$, $C=0$ can make the above equation hold. (Or: plug in $x = 0, 1, -1$.)
After partial fraction decomposition, you can check whether it is correct by the common denominator:

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{2}{x} + \frac{(-2)x+0}{x^2+1} = \frac{2(x^2+1) - 2x \cdot x}{x(x^2+1)} = \frac{2x^2+2-2x^2}{x(x^2+1)} = \frac{2}{x(x^2+1)}$$

So our decomposition is correct.

$$\begin{aligned}
\int\int_D \frac{y}{x(x^2+1)} dA &= \int_1^2 \frac{2}{x(x^2+1)} dx \\
&= \int_1^2 \frac{2}{x} - \frac{2x}{x^2+1} dx \\
&= 2\ln|x|\Big|_1^2 - \int_1^2 \frac{2x}{x^2+1} dx \\
&\quad \text{(do substitution } u = x^2 \text{or } x^2+1) \\
&= 2\ln|x| - \ln|x^2+1|\Big|_1^2 \\
&= 2\ln 2 - \ln 5 - (2\ln 1 - \ln 2) \\
&= 3\ln 2 - \ln 5 \\
&= \ln\left(\frac{8}{5}\right)
\end{aligned}$$

You can also integrate x first, which will give you the same answer:

$$\begin{aligned}
\int\int_D \frac{y}{x(x^2+1)} dA &= \int_0^2 \int_1^2 \frac{y}{x(x^2+1)} dx dy \\
&= \int_1^2 y \left[\int_1^2 \frac{1}{x(x^2+1)} dx\right] dy \\
&= \int_1^2 y \left[\left(\ln|x| - \frac{1}{2}\ln|x^2+1|\right)\Big|_{x=1}^{x=2}\right] dy \\
&= \int_1^2 y \left[\frac{3}{2}\ln 2 - \frac{1}{2}\ln 5\right] dy \\
&= 2\left(\frac{3}{2}\ln 2 - \frac{1}{2}\ln 5\right) \\
&= 3\ln 2 - \ln 5
\end{aligned}$$