1. (1) (2 points) Which x can make the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n-1}}$ converge? select all that applies.

A. $x=-1$ B. $x=1$ C. $x=2$ D. $x=5$ E. $x=7$

(2) (2 points) For any number x which can make the series converge, find $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n-1}}$. (Express the sum in terms of x.)

Solution.
(1) **Solution 1.** Use the Ratio Test:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|\frac{(x-2)^{n+1}}{3^n}|}{|\frac{(x-2)^n}{3^{n-1}}|} = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{|x-2|^n} \cdot \frac{3^{n-1}}{3^n} = |\frac{x-2}{3}|$$

By the ratio test, the series absolutely converges when $L<1$. Solving $|\frac{x-2}{3}|<1$ we get

$$\begin{aligned} |x-2| &< 3 \\ x-2<3 &\text{ and } x-2>-3 \\ x<5 &\text{ and } x>-1 \end{aligned}$$

i.e.

$$-1<x<5$$

Also, plug in the endpoint $x=-1$: $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^{n-1}} = \sum_{n=0}^{\infty}(-1)(-1)^n$ diverges.

Plug in the endpoint $x=5$: $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{3^n}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{1}{3}$ diverges. Therefore, the series converges if and only if

$$-1<x<5$$

Solution 2. Use gemetric series. The first serveral terms of the geometric series are:

$$\frac{(x-2)^0}{3^{-1}}, \frac{(x-2)}{3^0}, \frac{(x-2)^2}{3^1}, \frac{(x-2)^3}{3^2} \cdots$$

The ratio of the geometric series is $\frac{(x-2)}{3}$. Since **Geometric Series converges if and only if the absolute value of ratio is less than 1**, we require

$$|\frac{(x-2)}{3}|<1$$

i.e.

$$\begin{aligned} |(x-2)| &< 3 \\ x-2<3 &\text{ and } x-2>-3 \\ x<5 &\text{ and } x>-1 \end{aligned}$$

i.e.

$$1<x<5$$

Therefore, in (1), B, C are correct.
(2) **The sum of geomeric series is**

$$\begin{aligned} \textbf{first term} \times \frac{\mathbf{1}}{\mathbf{1-Ratio}} &= \frac{(x-2)^0}{3^{-1}} \cdot \frac{1}{1-\frac{(x-2)}{3}} \\ &= \frac{1}{(\frac{1}{3})} \cdot \frac{3}{3-(x-2)} \\ &= \frac{9}{5-x} \end{aligned}$$

2. (3 points) Determine whether the following series

$$\sum_{n=1}^{\infty}(-1)^n\frac{4n^2+1}{5^n}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution. First check each term $a_n \to 0$ as $n \to \infty$, since 5^n is much larger than $4n^2+1$.
To test absolute convergence, we use the Ratio test, for then

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{\frac{4(n+1)^2+1}{5^{n+1}}}{\frac{4n^2+1}{5^n}}=\lim_{n\to\infty}\frac{4(n+1)^2+1}{4n^2+1}\cdot\frac{1}{5}=1\cdot\frac{1}{5}=\frac{1}{5}<1$$

Consequently, **by the Ratio test, the given series is absolutely convergent.**

3. (3 points) Determine whether the following series

$$\sum_{k=2}^{\infty}(-1)^k\frac{\ln(k)}{\sqrt{k}}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution.
The given series is an alternating series

$$\sum_{k=2}^{\infty}(-1)^k f(k) \text{ with } f(x)=\frac{ln(x)}{\sqrt{x}}$$

For this series to be absolutely convergent, the series

$$\sum_{k=2}^{\infty}|a_k|=\sum_{k=2}^{\infty}\frac{ln(k)}{\sqrt{k}}$$

has to be convergent. However, notice that $\ln(k)\to\infty$ as $k\to\infty$,

$$\frac{\ln(k)}{\sqrt{k}}\geq\frac{1}{\sqrt{k}} \text{ for all } k\geq 3$$

, while $\sum_{k=2}^{\infty}\frac{1}{\sqrt{k}}$ is divergent by the p-series test with $p=\frac{1}{2}$. **By the comparison test, the given series is not absolutely convergent.**

On the other hand, since $\ln k$ is much smaller than $\sqrt{k}$ with k large enough, $f(k)$ decreases and goes to 0 as $k\to\infty$. Rigorously, you can verify it by taking derivative and L'Hospital's rule:

$$f'(x)=\frac{\frac{1}{x}\sqrt{x}-\ln(x)\frac{1}{2}\frac{1}{\sqrt{x}}}{(\sqrt{x})^2}=\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x}}\ln(x)\frac{1}{2}}{x}=\frac{2-\ln(x)}{2x\sqrt{x}}<0 \text{ for all large } x.$$

So $f(k)\geq f(k+1)$, for large k.
By L'Hospital's Rule,

$$\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

Thus $f(k)\to 0$ as $k\to\infty$, so **the Alternating Series Test applies. Consequently, the given series is conditionally convergent.**