Selected Old Quest Problems for Differential Equations

9.3

For the differential equation

$$5\frac{dy}{dx} + \frac{5}{xy^4} = 0, (x, y > 0),$$

(1) Find the general solution.
(2) If $y(1) = 2$, find the particular solution.
(3) Find the value of $y(e)$.

9.4

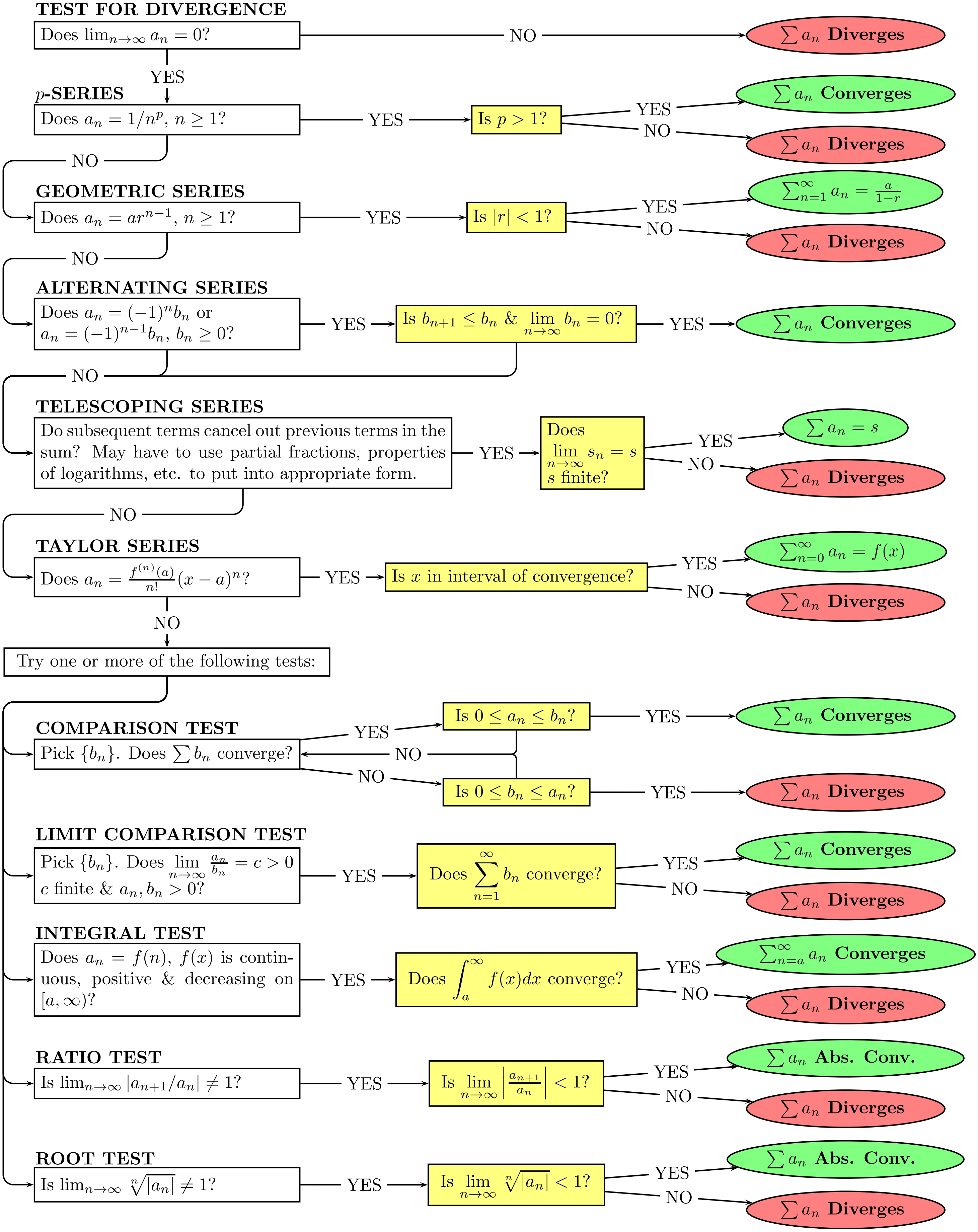
1.A drug becomes ineffective at a rate proportional to the amount still present. In other words,

$$\frac{dP}{dt} = kP \text{ for some } k < 0.$$

Half of the drug is effective at exactly $t = 13$ day.
(1) Find k.
(2) What is t when the drug has 90% ineffective ingredients? (i.e.$P(t) = 10\%P(0)$)

2.*(Set up an equation by yourself)
Scientists began studying the elk population in Yellowstone Park in 1990 when there were 600 elk. They determined that t years after the study began the population size,$P(t)$, was increasing at a rate proportional to $1500 - P(t)$.
Given that the population was 1300 in year 2000,
(1) Set up a differential equation for $P(t)$;
(2) Using the given values of elk population, find the particular solution to differential equation;
(3) Estimate the size of the elk population in year 2010 (need to use a calculator).

Series Convergence/Divergence Flow Chart



1 Selected Old Quest Problems for Series

1.1 11.2

1. If the n-th partial sum S_n of an infinite series

$$S_n = 8 - \frac{n}{3^n}$$

find a_n. Also, does the series $\sum_0^\infty a_n$converge? What does it converge to?
Answer: $a_0 = S_0 = 8$, $a_n = S_n - S_{n-1} = \frac{2n-3}{3^n}$ for $n \geq 1$.
$\sum_0^\infty a_n = \lim_{n\to\infty} S_n = 8$.
2. Determine whether the **sequence** converges:

$$a_n = \frac{n + (-3)^n}{5^n}$$

Does the **series** converge?

$$\sum_{n=0}^{\infty} \frac{n + (-3)^n}{5^n}$$

Answer: Both yes. (Hint: $\sum \frac{(-3)^n}{5^n}$ converges. For $\sum \frac{n}{5^n}$, use comparison test/ratio test to show it converges.)

1.2 11.3

Determine whether the series converges:

$$\sum_{k=0}^{\infty} \frac{1}{k(\ln(2k))^2}$$

Answer: Yes.

1.3 11.4

1. Determine whether the series converges:

$$\sum_{n=0}^{\infty} \frac{\sin(n)}{n^2}$$

Answer: Yes ($|\sin(n)| \leq 1$)
2. Determine whether the series converges. If converges, conditionally or absolutely:

$$\sum_{n=0}^{\infty} (-1)^n \sin(\frac{1}{3n})$$

Answer: converges conditionally. (For $\sum \sin(\frac{1}{3n})$, compare to $\sum \frac{1}{3n}$.)

1.4 11.5.4

Determine whether the series

$$\sum_{n=0}^{\infty} \frac{4}{\sqrt{n+2}} \cos(n\pi)$$

converges or diverges.
Answer: converges.

1.5 11.6.4

Decide whether the series

$$\sum_{n=1}^{\infty} 2^n \left(\frac{n-2}{n} \right)^{n^2}$$

converges or diverges.
Answer: converges.

1.6 11.7.5

Determine which, if either, of the series

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{\sqrt[3]{n}}$

2. $\sum_{k=3}^{\infty} \frac{(-1)^k}{2k \ln(k+3)}$

are conditionally convergent.
Answer: Both.

1.7 11.8.2

If the series

$$\sum_{n=0}^{\infty} c_n 4^n$$

is convergent, which of the following statements must be true without further restrictions on c_n.

1. $\sum_{n=0}^{\infty} c_n(-4)^n$ is convergent

2. $\sum_{n=0}^{\infty} c_n(-4)^n$ is divergent

3. $\sum_{n=0}^{\infty} c_n(-3)^n$ is convergent

4. $\sum_{n=0}^{\infty} c_n(-3)^n$ is divergent

Answer: 3 is true.

1.8 11.9.4

Find a power series representation for the function

$$f(x) = \ln(7 - x)$$

Answer:

$$f(x) = \ln(7) - \sum_{n=1}^{\infty} \frac{x^n}{n7^n}$$

We can either use the known power series representation

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

or the fact that

$$\begin{aligned}
\ln(1 - x) &= -\int_0^x \frac{1}{1-s} ds \\
&= -\int_0^x \left(\sum_{n=0}^{\int} s^n \right) ds \\
&= -\sum_{n=0}^{\int} \int_0^x s^n ds \\
&= -\sum_{n=0}^{\int} \frac{x^n}{n} ds
\end{aligned}$$

For then by properties of logs,

$$f(x) = \ln(7)\left(1 - \frac{1}{7}x\right) = \ln(7) - \left(1 - \frac{1}{7}x\right)$$

so that

$$f(x) = \ln(7) - \sum_{n=1}^{\infty} \frac{x^n}{n7^n}$$

2 Some facts that may be helpful

2.1

For a constant $c > 0$

$$\lim_{n\to\infty} \sqrt[n]{c} = \lim_{n\to\infty} c^{1/n} = 1$$

2.2

$$\lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x$$

$$\lim_{n\to\infty}(1-\frac{x}{n})^n = e^{-x}$$

For any fixed number x.

2.2

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

$$\lim_{x\to 0}\frac{\tan x}{x} = 1$$

So you will have, for example,

$$\lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{(\frac{1}{n})} = 1$$

2.3

When n is large,

$$ln(n) \ll n \ll e^n \ll n! \ll n^n$$

(You can replace $ln(n)$ by any $\log_a(n)$ $(a > 1)$ and replace e^n by any a^n, $a > 1$. For example, you can say $\log_2(n) < n < (1.01)^n$.)

2.4

$$(n+1)! = (n+1)\cdot n!$$

$$(2(n+1))! = (2n+2)\cdot(2n+1)\cdot(2n)\cdots 1 = (2n+2)\cdot(2n+1)\cdot(2n)!$$

Note:

$$(2n)! \neq 2\cdot n!$$

2.5

Be familiar with properties of exponential functions and logarithms.