

## Selected Old Quest Problems for Differential Equations

### 9.3

For the differential equation

$$5\frac{dy}{dx} + \frac{5}{xy^4} = 0, (x, y > 0),$$

- (1) Find the general solution.
- (2) If  $y(1) = 2$ , find the particular solution.
- (3) Find the value of  $y(e)$ .

### 9.4

1. A drug becomes ineffective at a rate proportional to the amount still present. In other words,

$$\frac{dP}{dt} = kP \text{ for some } k < 0.$$

Half of the drug is effective at exactly  $t = 13$  day.

- (1) Find  $k$ .
- (2) What is  $t$  when the drug has 90% ineffective ingredients? (i.e.  $P(t) = 10\%P(0)$ )

2. \*(Set up an equation by yourself)

Scientists began studying the elk population in Yellowstone Park in 1990 when there were 600 elk. They determined that  $t$  years after the study began the population size,  $P(t)$ , was increasing at a rate proportional to  $1500 - P(t)$ .

Given that the population was 1300 in year 2000,

- (1) Set up a differential equation for  $P(t)$ ;
- (2) Using the given values of elk population, find the particular solution to differential equation;
- (3) Estimate the size of the elk population in year 2010 (need to use a calculator).

# SERIES CONVERGENCE/DIVERGENCE FLOW CHART

## TEST FOR DIVERGENCE

Does  $\lim_{n \rightarrow \infty} a_n = 0$ ?

NO

$\sum a_n$  Diverges

YES

## p-SERIES

Does  $a_n = 1/n^p, n \geq 1$ ?

YES

Is  $p > 1$ ?

YES

$\sum a_n$  Converges

NO

$\sum a_n$  Diverges

NO

## GEOMETRIC SERIES

Does  $a_n = ar^{n-1}, n \geq 1$ ?

YES

Is  $|r| < 1$ ?

YES

$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$

NO

$\sum a_n$  Diverges

NO

## ALTERNATING SERIES

Does  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n-1} b_n, b_n \geq 0$ ?

YES

Is  $b_{n+1} \leq b_n$  &  $\lim_{n \rightarrow \infty} b_n = 0$ ?

YES

$\sum a_n$  Converges

NO

## TELESCOPING SERIES

Do subsequent terms cancel out previous terms in the sum? May have to use partial fractions, properties of logarithms, etc. to put into appropriate form.

YES

Does  $\lim_{n \rightarrow \infty} s_n = s$  finite?

YES

$\sum a_n = s$

NO

$\sum a_n$  Diverges

NO

## TAYLOR SERIES

Does  $a_n = \frac{f^{(n)}(a)}{n!} (x-a)^n$ ?

YES

Is  $x$  in interval of convergence?

YES

$\sum_{n=0}^{\infty} a_n = f(x)$

NO

$\sum a_n$  Diverges

NO

Try one or more of the following tests:

## COMPARISON TEST

Pick  $\{b_n\}$ . Does  $\sum b_n$  converge?

YES

Is  $0 \leq a_n \leq b_n$ ?

YES

$\sum a_n$  Converges

NO

Is  $0 \leq b_n \leq a_n$ ?

YES

$\sum a_n$  Diverges

NO

## LIMIT COMPARISON TEST

Pick  $\{b_n\}$ . Does  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$  finite &  $a_n, b_n > 0$ ?

YES

Does  $\sum_{n=1}^{\infty} b_n$  converge?

YES

$\sum a_n$  Converges

NO

$\sum a_n$  Diverges

## INTEGRAL TEST

Does  $a_n = f(n), f(x)$  is continuous, positive & decreasing on  $[a, \infty)$ ?

YES

Does  $\int_a^{\infty} f(x) dx$  converge?

YES

$\sum_{n=a}^{\infty} a_n$  Converges

NO

$\sum a_n$  Diverges

## RATIO TEST

Is  $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| \neq 1$ ?

YES

Is  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ ?

YES

$\sum a_n$  Abs. Conv.

NO

$\sum a_n$  Diverges

## ROOT TEST

Is  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \neq 1$ ?

YES

Is  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ ?

YES

$\sum a_n$  Abs. Conv.

NO

$\sum a_n$  Diverges

# 1 Selected Old Quest Problems for Series

## 1.1 11.2

1. If the  $n$ -th partial sum  $S_n$  of an infinite series

$$S_n = 8 - \frac{n}{3^n}$$

find  $a_n$ . Also, does the series  $\sum_0^\infty a_n$  converge? What does it converge to?

Answer:  $a_0 = S_0 = 8$ ,  $a_n = S_n - S_{n-1} = \frac{2n-3}{3^n}$  for  $n \geq 1$ .

$\sum_0^\infty a_n = \lim_{n \rightarrow \infty} S_n = 8$ .

2. Determine whether the **sequence** converges:

$$a_n = \frac{n + (-3)^n}{5^n}$$

Does the **series** converge?

$$\sum_{n=0}^{\infty} \frac{n + (-3)^n}{5^n}$$

Answer: Both yes. (Hint:  $\sum \frac{(-3)^n}{5^n}$  converges. For  $\sum \frac{n}{5^n}$ , use comparison test/ratio test to show it converges.)

## 1.2 11.3

Determine whether the series converges:

$$\sum_{k=0}^{\infty} \frac{1}{k(\ln(2k))^2}$$

Answer: Yes.

## 1.3 11.4

1. Determine whether the series converges:

$$\sum_{n=0}^{\infty} \frac{\sin(n)}{n^2}$$

Answer: Yes ( $|\sin(n)| \leq 1$ )

2. Determine whether the series converges. If converges, conditionally or absolutely:

$$\sum_{n=0}^{\infty} (-1)^n \sin\left(\frac{1}{3n}\right)$$

Answer: converges conditionally. (For  $\sum \sin(\frac{1}{3n})$ , compare to  $\sum \frac{1}{3n}$ .)

### 1.4 11.5.4

Determine whether the series

$$\sum_{n=0}^{\infty} \frac{4}{\sqrt{n+2}} \cos(n\pi)$$

converges or diverges.

Answer: converges.

### 1.5 11.6.4

Decide whether the series

$$\sum_{n=1}^{\infty} 2^n \left( \frac{n-2}{n} \right)^{n^2}$$

converges or diverges.

Answer: converges.

### 1.6 11.7.5

Determine which, if either, of the series

1.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{\sqrt[3]{n}}$
2.  $\sum_{k=3}^{\infty} \frac{(-1)^k}{2k \ln(k+3)}$

are conditionally convergent.

Answer: Both.

### 1.7 11.8.2

If the series

$$\sum_{n=0}^{\infty} c_n 4^n$$

is convergent, which of the following statements must be true without further restrictions on  $c_n$ .

1.  $\sum_{n=0}^{\infty} c_n (-4)^n$  is convergent
2.  $\sum_{n=0}^{\infty} c_n (-4)^n$  is divergent
3.  $\sum_{n=0}^{\infty} c_n (-3)^n$  is convergent
4.  $\sum_{n=0}^{\infty} c_n (-3)^n$  is divergent

Answer: 3 is true.

## 1.8 11.9.4

Find a power series representation for the function

$$f(x) = \ln(7 - x)$$

Answer:

$$f(x) = \ln(7) - \sum_{n=1}^{\infty} \frac{x^n}{n7^n}$$

We can either use the known power series representation

$$\ln(1 - x) = - \sum_{n=1}^{\infty} \frac{x^n}{n}$$

or the fact that

$$\begin{aligned} \ln(1 - x) &= - \int_0^x \frac{1}{1 - s} ds \\ &= - \int_0^x \left( \sum_{n=0}^{\infty} s^n \right) ds \\ &= - \sum_{n=0}^{\infty} \int_0^x s^n ds \\ &= - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \end{aligned}$$

For then by properties of logs,

$$f(x) = \ln(7) \left( 1 - \frac{1}{7}x \right) = \ln(7) - \left( 1 - \frac{1}{7}x \right)$$

so that

$$f(x) = \ln(7) - \sum_{n=1}^{\infty} \frac{x^n}{n7^n}$$

## 2 Some facts that may be helpful

### 2.1

For a constant  $c > 0$

$$\lim_{n \rightarrow \infty} \sqrt[n]{c} = \lim_{n \rightarrow \infty} c^{1/n} = 1$$

### 2.2

$$\lim_{n \rightarrow \infty} \left( 1 + \frac{x}{n} \right)^n = e^x$$

$$\lim_{n \rightarrow \infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$$

For any fixed number  $x$ .

## 2.2

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

So you will have, for example,

$$\lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$$

## 2.3

When  $n$  is large,

$$\ln(n) \ll n \ll e^n \ll n! \ll n^n$$

(You can replace  $\ln(n)$  by any  $\log_a(n)$  ( $a > 1$ ) and replace  $e^n$  by any  $a^n$ ,  $a > 1$ . For example, you can say  $\log_2(n) < n < (1.01)^n$ .)

## 2.4

$$(n+1)! = (n+1) \cdot n!$$

$$(2(n+1))! = (2n+2) \cdot (2n+1) \cdot (2n) \cdots 1 = (2n+2) \cdot (2n+1) \cdot (2n)!$$

Note:

$$(2n)! \neq 2 \cdot n!$$

## 2.5

Be familiar with properties of exponential functions and logarithms.