

$$14. f(x, y, z) = \frac{z^2}{x^2 - y^2}.$$

$$15. f(x, y, z) = -\frac{z^2}{\sqrt{x^2 - y^2}}.$$

$$16. f(x, y, z) = \frac{z}{x - y}.$$

$$17. f(x, y) = \frac{2}{\sqrt{9 - (x^2 + y^2)}}.$$

$$18. f(x, y, z) = \frac{\sqrt{1 - x^2} + \sqrt{4 - y^2}}{1 + \sqrt{9 - z^2}}.$$

$$19. f(x, y, z) = \ln(x + 2y + 3z).$$

$$20. f(x, y, z) = e^{\sqrt{4 - (x^2 + y^2 + z^2)}}.$$

## 16.2...A BRIEF CATALOG OF THE QUADRIC SURFACES; PROJECTIONS

We represented functions of one variable by curves in the  $xy$ -plane. We will represent functions of two variables by surfaces in three-space.

In this section we'll examine in a systematic manner the surfaces given by equations of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Hx + Iy + Jz + K = 0.$$

Such surfaces are called *quadric surfaces*.

By suitable translations and rotations of the coordinate axes we can simplify such equations and thereby show that the nondegenerate† quadrics fall into nine distinct types:

1. the ellipsoid
2. the hyperboloid of one sheet
3. the hyperboloid of two sheets
4. the quadric cone
5. the elliptic paraboloid
6. the hyperbolic paraboloid
7. the parabolic cylinder
8. the elliptic cylinder
9. the hyperbolic cylinder.

As you go on with calculus, you'll encounter these surfaces time and time again. Here we give you a picture of each one, together with its equation in standard form

† We are excluding such degenerate quadrics as

$$1 + x^2 + y^2 = -z^2 \quad \text{and} \quad x^2 + y^2 = -z^2.$$

The first one has no points and the second consists of only one point, the origin.

and some information about its special properties. These are some of the things to look for:

- (a) the *intercepts* (the points at which the surface intersects the coordinate axes)
- (b) the *traces* (the intersections with the coordinate planes)
- (c) the *sections* (the intersections with planes in general)
- (d) the *center* (some quadrics have a center; some do not)
- (e) *symmetry*
- (f) *boundedness, unboundedness.*

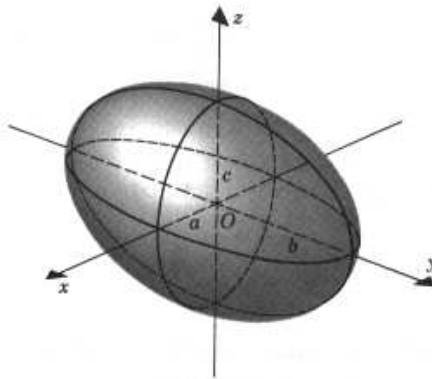


FIGURE 16.2.1

### 1. The Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (\text{Figure 16.2.1})$$

The ellipsoid is centered at the origin and is symmetric about the three coordinate planes. It intersects the coordinate axes at six points:  $(\pm a, 0, 0)$ ,  $(0, \pm b, 0)$ ,  $(0, 0, \pm c)$ . These points are called the *vertices*. The surface is bounded, being contained in the rectangular solid:  $|x| \leq a$ ,  $|y| \leq b$ ,  $|z| \leq c$ .† All three traces are ellipses; thus, for example, the trace in the  $xy$ -plane (set  $z = 0$ ) is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

All sections parallel to the coordinate planes are also ellipses; for example, taking  $y = y_0$  we have

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1 - \frac{y_0^2}{b^2}.$$

This ellipse is the intersection of the ellipsoid with the plane  $y = y_0$ . The numbers  $a$ ,  $b$ ,  $c$  are called the *semiaxes* of the ellipsoid. If two of the semiaxes are equal, then we

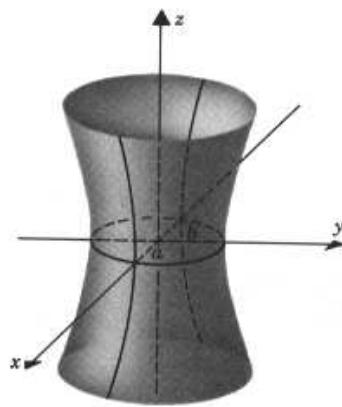
† Throughout this section we take  $a$ ,  $b$ ,  $c$  as positive.

have an *ellipsoid of revolution*. (If, for example,  $a = c$ , then all sections parallel to the  $xz$ -plane are circles and the surface can be obtained by revolving the trace in the  $xy$ -plane about the  $y$ -axis.) If all three semiaxes are equal, the surface is a *sphere*.  $\square$

## 2. The Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(Figure 16.2.2)



hyperboloid of one sheet

FIGURE 16.2.2

The surface is unbounded. It is centered at the origin and is symmetric about the three coordinate planes. The surface intersects the coordinate axes at four points:  $(\pm a, 0, 0)$ ,  $(0, \pm b, 0)$ . The trace in the  $xy$ -plane (set  $z = 0$ ) is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

All sections parallel to the  $xy$ -plane are ellipses. The trace in the  $xz$ -plane (set  $y = 0$ ) is the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1,$$

and the trace in the  $yz$ -plane (set  $x = 0$ ) is the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

All sections parallel to the  $xz$ -plane or  $yz$ -plane are hyperbolas. If  $a = b$ , then all sections parallel to the  $xy$ -plane are circles and we have a *hyperboloid of revolution*.  $\square$

### 3. The Hyperboloid of Two Sheets

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1. \quad (\text{Figure 16.2.3})$$

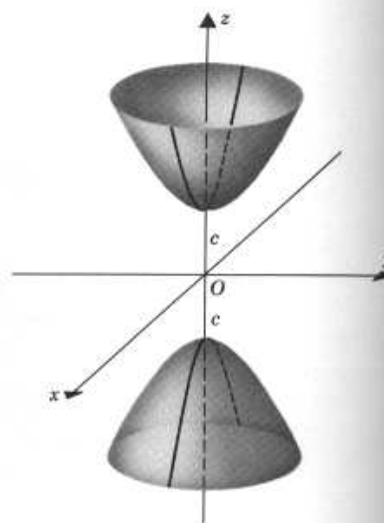
The surface intersects the coordinate axes only at the two vertices  $(0, 0, \pm c)$ . The surface consists of two parts: one for which  $z \geq c$ , another for which  $z \leq -c$ . Each of these parts is unbounded. All sections parallel to the  $xy$ -plane are ellipses: for  $z = z_0$  with  $|z_0| \geq c$ , we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0^2}{c^2} - 1.$$

Sections parallel to the other coordinate planes are hyperbolas; for example, for  $y = y_0$  we have

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1 + \frac{y_0^2}{b^2}.$$

The surface is symmetric about the three coordinate planes and is centered at the origin.  $\square$



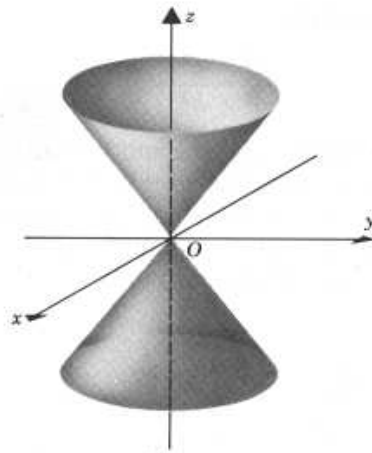
hyperboloid of two sheets

FIGURE 16.2.3

### 4. The Quadric Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2. \quad (\text{Figure 16.2.4})$$

The surface intersects the coordinate axes only at the origin. The surface is unbounded. Once again there is symmetry about the three coordinate planes. The trace in the  $xz$ -plane is a pair of intersecting lines:  $z = \pm x/a$ . The trace in the  $yz$ -plane is also a pair of intersecting lines:  $z = \pm y/b$ . The trace in the  $xy$ -plane is just the origin. Sections parallel to the  $xy$ -plane are ellipses. If  $a = b$ , these sections are circles and we have what is commonly called a *double circular cone* or simply a *cone*. The upper and lower portions of the cone are called *nappes*.  $\square$



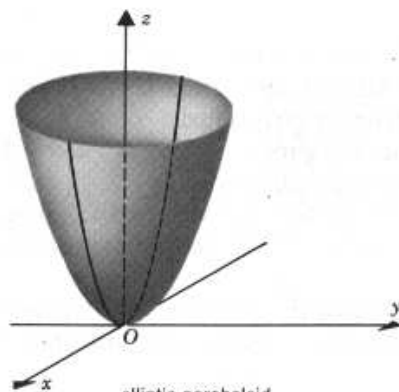
quadric cone  
FIGURE 16.2.4

We come now to the *paraboloids*. The equations in standard form will involve  $x^2$  and  $y^2$ , but then  $z$  instead of  $z^2$ .

### 5. The Elliptic Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z.$$

(Figure 16.2.5)



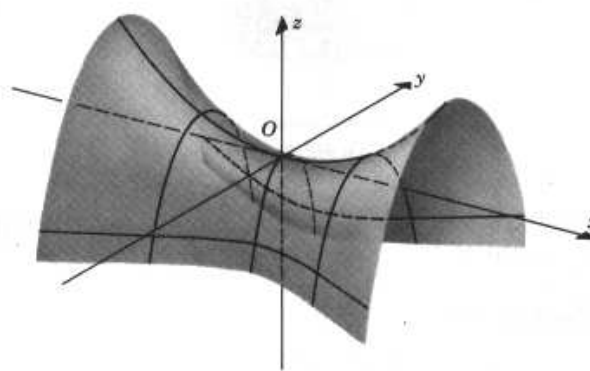
elliptic paraboloid  
FIGURE 16.2.5

The surface does not extend below the  $xy$ -plane. Otherwise it is unbounded. The origin is called the *vertex*. Sections parallel to the  $xy$ -plane are ellipses; sections parallel to the other coordinate planes are parabolas. Hence the term “elliptic paraboloid.” The surface is symmetric about the  $xz$ -plane and about the  $yz$ -plane. It is also symmetric about the  $z$ -axis. If  $a = b$ , then the surface is a *paraboloid of revolution*.  $\square$

## 6. The Hyperbolic Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z. \quad (\text{Figure 16.2.6})$$

Here there is symmetry about the  $xz$ -plane and  $yz$ -plane. Sections parallel to the  $xy$ -plane are hyperbolas; sections parallel to the other coordinate planes are parabolas. Hence the term “hyperbolic paraboloid.” The origin is a minimum point for the trace in the  $xz$ -plane but a maximum point for the trace in the  $yz$ -plane. The origin is called a *minimax* or *saddle point* of the surface.  $\square$



hyperbolic paraboloid  
FIGURE 16.2.6

The rest of the quadric surfaces are *cylinders*. The term deserves definition. Take any plane curve  $C$ . All the lines through  $C$  that are perpendicular to the plane of  $C$  form a surface. Such a surface is called a *cylinder*, the *cylinder with base curve*  $C$ . The perpendicular lines are known as the *generators* of the cylinder.

If the base curve lies in the  $xy$ -plane (or in a plane parallel to the  $xy$ -plane), then the generators of the cylinder are parallel to the  $z$ -axis. In such a case the equation of the cylinder involves only  $x$  and  $y$ . The  $z$ -coordinate is left unrestricted; it can take on all values.

There are three types of quadric cylinders. We will give you their equations in standard form: base curve in the  $xy$ -plane, generators parallel to the  $z$ -axis.

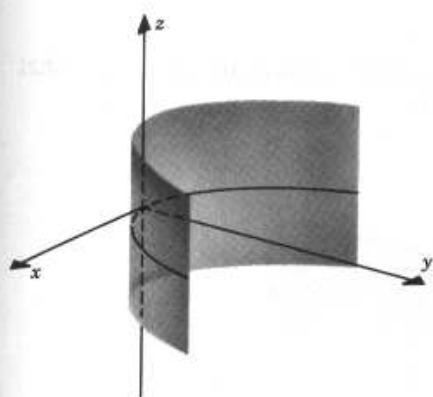
## 7. The Parabolic Cylinder

$$x^2 = 4cy. \quad (\text{Figure 16.2.7})$$

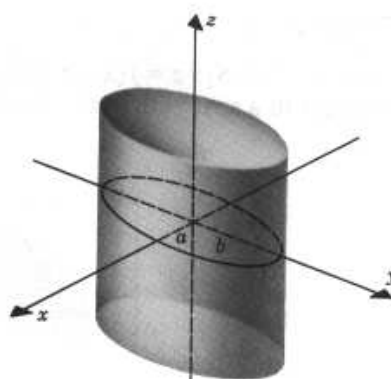
This surface is formed by all lines that pass through the parabola  $x^2 = 4cy$  and are perpendicular to the  $xy$ -plane.  $\square$

## 8. The Elliptic Cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (\text{Figure 16.2.8})$$



parabolic cylinder  
FIGURE 16.2.7

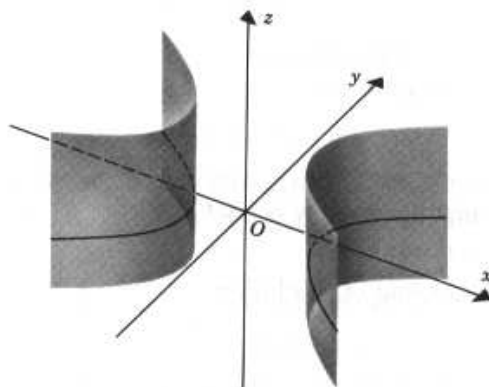


elliptic cylinder  
FIGURE 16.2.8

The surface is formed by all lines that pass through the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and are perpendicular to the  $xy$ -plane. If  $a = b$ , we have the common *right circular cylinder*. □



hyperbolic cylinder  
FIGURE 16.2.9

### 9. The Hyperbolic Cylinder

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (\text{Figure 16.2.9})$$

The surface has two parts, each generated by a branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \square$$